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Abstract The superspace formalism for  $\mathcal{N} = 1$  supergravity in four dimensions is a powerful geometric setting to engineer off-shell supergravity-matter theories, including higher-derivative couplings. This review provides a unified description of the three superspace approaches to  $\mathcal{N} = 1$  conformal supergravity: (i) conformal superspace; (ii) U(1) superspace; and (iii) the Grimm-Wess-Zumino formalism. The prepotential formulation for the latter is discussed. We briefly describe the known off-shell formulations for Poincaré and anti-de Sitter supergravity theories as conformal supergravity coupled to certain compensators. As simple applications of the formalism, we present the superfield equations of motion for various off-shell formulations for pure Poincaré and anti-de Sitter supergravity, and show that every solution of these equations is also a solution of the equations of motion for conformal supergravity.

#### Keywords

Superconformal symmetry, Supergravity, Superspace

Dedicated to the creators of superfield supergravity

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## **1** Introduction

Soon after the discovery of  $\mathcal{N} = 1$  supergravity in four spacetime dimensions [1, 2] (and subsequent construction of the supersymmetric cosmological term [3]) several off-shell formulations for this theory, with different sets of auxiliary fields, were developed. These include the non-minimal [4, 5, 6], old minimal [7, 8, 9, 10] and new minimal [11, 12] supergravity theories. The most general matter couplings are offered by the old minimal formulation [13, 14]. All interactions constructed in the framework of the new minimal as well as the non-minimal formulations are particular cases of those that can be realised within the old minimal theory [14].

Traditionally, Refs. [9, 10] are credited with the discovery of old minimal supergravity, see e.g. [15], since they have played a fundamental role in the development of supergravity. In fact, this off-shell theory was constructed for the first time using superfield techniques in an unpublished 1977 work by Siegel [7] (which was difficult to digest at the time) and then re-discovered by Wess and Zumino [8], shortly before the publication of [9, 10]. It was explicitly shown [6, 16, 17] that the component reduction of the superfield action for supergravity proposed in [8] coincides with the component actions given in [9, 10]. It was also explicitly demonstrated [6, 18] that the Wess-Zumino action [8] is equivalent to the one proposed in [7]. These are just simple examples of the power of superspace approaches to supergravity.

Einstein's theory of gravity can be described using a Weyl invariant extension of the Einstein-Hilbert action by means of a compensating scalar field [19, 20]. In other words, ordinary gravity can be thought of as conformal gravity coupled to a compensator. As a generalisation of this idea, Poincaré supergravity can be realised as a locally superconformal invariant theory of supergravity [21] in which the Weyl multiplet of conformal supergravity [22, 23] is coupled to a compensating scalar multiplet [13, 14].<sup>2</sup> It turns out that all known off-shell formulations for Poincaré and anti-de Sitter (AdS) supergravity theories can be recast as conformal supergravity coupled to certain compensating multiplets. Different choices of a compensator correspond to different off-shell formulations for supergravity. This superconformal setting has been developed in the conventional component approach [24, 25, 26, 27] under the name "superconformal tensor calculus" and has proved to be truly useful in order to formulate general two-derivative supergravity-matter systems and to study their dynamical properties, see [28] for a review. In our opinion, it becomes especially powerful within superspace formulations for supergravity, which (i) provide remarkably compact expressions for general supergravity-matter actions; (ii) make manifest the geometric properties of such theories; and, most importantly, (iii) offer unique tools to generate higher-derivative couplings in matter-coupled supergravity.

There are three fully fledged approaches to describe  $\mathcal{N} = 1$  conformal supergravity in superspace: (i) the Grimm-Wess-Zumino (GWZ) formalism [29] extending the Wess-Zumino formulation for on-shell supergravity [30]; (ii) the so-called U(1) superspace proposed by Howe [31]; and (iii) the  $\mathcal{N} = 1$  conformal superspace approach developed by Butter [32]. Conformal superspace is an ultimate formula-

<sup>&</sup>lt;sup>2</sup> A similar idea was put forward earlier in [7].

tion for conformal supergravity in the sense that any different off-shell formulation is either equivalent to it or is obtained from it by partially fixing the gauge freedom. In particular, U(1) superspace can be obtained from a partial gauge fixing of the gauge group in conformal superspace. The  $\mathcal{N} = 1$  superconformal tensor calculus reviewed in [28, 33] is also a gauged fixed version of conformal superspace as demonstrated in [32]. Recently, new supertwistor formulations were discovered for conformal supergravity theories in diverse dimensions [34]. In the four-dimensional  $\mathcal{N} = 1$  case, the supertwistor formulation is expected to be related to conformal superspace, however relevant technical details have not yet been worked out in the literature.

Due to space limitations, in this review we are not able to discuss many important aspects of  $\mathcal{N} = 1$  supergravity and its matter couplings. We apologise for the unavoidable omissions and missing references.

Our two-component spinor notation and conventions follow [35], and are similar to those adopted in [17]. The only difference is that the spinor Lorentz generators  $(\sigma_{ab})_{\alpha}{}^{\beta}$  and  $(\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}}$  used in [35] have an extra minus sign as compared with [17], specifically  $\sigma_{ab} = -\frac{1}{4}(\sigma_a \tilde{\sigma}_b - \sigma_b \tilde{\sigma}_a)$  and  $\tilde{\sigma}_{ab} = -\frac{1}{4}(\tilde{\sigma}_a \sigma_b - \tilde{\sigma}_b \sigma_a)$ .

# 2 Rigid and local superconformal transformations

In this section we first review the structure of rigid superconformal transformations in Minkowski superspace  $\mathbb{M}^{4|4}$ . Then we introduce local superconformal transformations and describe the multiplet of conformal supergravity, following the approach due to Ogievetsky and Sokatchev [36]. It should be pointed out that the superconformal transformations in  $\mathbb{M}^{4|4}$  were first studied by Sohnius [37]. Our presentation follows [35].

#### 2.1 Rigid superconformal transformations

We denote by  $z^A = (x^a, \theta^\alpha, \bar{\theta}_{\dot{\alpha}})$  the Cartesian coordinates for Minkowski superspace  $\mathbb{M}^{4|4}$ , and use the notation  $D_A = (\partial_a, D_\alpha, \bar{D}^{\dot{\alpha}})$  for the superspace covariant derivatives. The only non-trivial graded commutation relation is

$$\{D_{\alpha}, \bar{D}_{\dot{\alpha}}\} = -2\mathrm{i}(\sigma^b)_{\alpha\dot{\alpha}}\partial_b = -2\mathrm{i}\partial_{\alpha\dot{\alpha}}. \qquad (2.1)$$

An infinitesimal superconformal transformation  $z^A \to z^A + \delta z^A$ , with  $\delta z^A = \xi z^A = \left(\xi^a + i(\xi \sigma^a \bar{\theta} - \theta \sigma^a \bar{\xi}), \xi^\alpha, \bar{\xi}_{\dot{\alpha}}\right)$ , is generated by a *conformal Killing supervector field* 

$$\xi = \overline{\xi} = \xi^b \partial_b + \xi^\beta D_\beta + \overline{\xi}_{\dot{\beta}} \overline{D}^\beta .$$
(2.2)

The defining property of  $\xi$  is that it takes every chiral superfield  $\Phi$  to a chiral one,

$$\bar{D}^{\dot{\alpha}}\Phi = 0 \longrightarrow \bar{D}^{\dot{\alpha}}(\xi\Phi) = 0.$$
 (2.3)

This condition implies the relations

$$\bar{D}^{\dot{\alpha}}\xi^{\beta} = 0 , \qquad \bar{D}^{\dot{\alpha}}\xi^{\dot{\beta}\beta} = 4i\varepsilon^{\dot{\alpha}\dot{\beta}}\xi^{\beta} \implies \xi^{\alpha} = -\frac{i}{8}\bar{D}_{\dot{\alpha}}\xi^{\dot{\alpha}\alpha} \qquad (2.4)$$

and their complex conjugates, and therefore

$$\bar{D}_{(\alpha}\xi_{\beta)\dot{\beta}} = 0 , \qquad \bar{D}_{(\dot{\alpha}}\xi_{\beta\dot{\beta}}) = 0 \implies \partial_{(\alpha(\dot{\alpha}}\xi_{\beta)\dot{\beta})} = 0 .$$
(2.5)

It follows that

$$[\xi, D_{\alpha}] = -(D_{\alpha}\xi^{\beta})D_{\beta} = -K_{\alpha}{}^{\beta}[\xi]D_{\beta} - \left(\bar{\sigma}[\xi] - \frac{1}{2}\sigma[\xi]\right)D_{\alpha} .$$
(2.6)

Here we have introduced chiral Lorentz ( $K_{\beta\gamma}[\xi] = K_{\gamma\beta}[\xi]$ ) and super-Weyl ( $\sigma[\xi]$ ) parameters defined by

$$K_{\alpha\beta}[\xi] = D_{(\alpha}\xi_{\beta)} , \qquad \bar{D}_{\dot{\gamma}}K_{\alpha\beta}[\xi] = 0 , \qquad (2.7a)$$

$$\sigma[\xi] = \frac{1}{3} (D_{\alpha}\xi^{\alpha} + 2\bar{D}^{\dot{\alpha}}\bar{\xi}_{\dot{\alpha}}) , \qquad \bar{D}_{\dot{\gamma}}\sigma[\xi] = 0 .$$
 (2.7b)

We recall that the Lorentz parameters with vector and spinor indices are related to each other as follows:  $K^{bc}[\xi] = (\sigma^{bc})_{\beta\gamma} K^{\beta\gamma}[\xi] - (\tilde{\sigma}^{bc})_{\dot{\beta}\dot{\gamma}} \bar{K}^{\dot{\beta}\dot{\gamma}}[\xi]$ . The most general conformal Killing supervector field has the form

$$\xi^{\dot{\alpha}\alpha}_{+} = a^{\dot{\alpha}\alpha} + \frac{1}{2}(\sigma + \bar{\sigma})y^{\dot{\alpha}\alpha} + \bar{K}^{\dot{\alpha}}{}_{\dot{\beta}}y^{\dot{\beta}\alpha} + y^{\dot{\alpha}\beta}K_{\beta}{}^{\alpha} - y^{\dot{\alpha}\beta}b_{\beta\dot{\beta}}y^{\dot{\beta}\alpha} + 4\mathrm{i}\,\bar{\epsilon}^{\dot{\alpha}}\theta^{\alpha} - 4y^{\dot{\alpha}\beta}\eta_{\beta}\theta^{\alpha} , \qquad (2.8a)$$

$$\xi^{\alpha} = \varepsilon^{\alpha} + \left(\bar{\sigma} - \frac{1}{2}\sigma\right)\theta^{\alpha} + \theta^{\beta}K_{\beta}^{\alpha} - \theta^{\beta}b_{\beta\dot{\beta}}y^{\dot{\beta}\alpha} - i\bar{\eta}_{\dot{\beta}}y^{\dot{\beta}\alpha} + 2\theta^{2}\eta^{\alpha} , \quad (2.8b)$$

where we have introduced the complex four-vector

$$\xi^a_+ = \xi^a + 2i\xi\sigma^a\bar{\theta} , \qquad \bar{\xi}^a = \xi^a , \qquad (2.9)$$

along with the complex bosonic coordinates  $y^a = x^a + i\theta\sigma^a\bar{\theta}$  of the chiral subspace of  $\mathbb{M}^{4|4}$ . The constant bosonic parameters in (2.8) correspond to the spacetime translation  $(a^{\dot{\alpha}\alpha})$ , Lorentz transformation  $(K_{\beta}{}^{\alpha}, \bar{K}^{\dot{\alpha}}{}_{\dot{\beta}})$ , special conformal transformation  $(b_{\alpha\beta})$ , and combined scale and *R*-symmetry transformations  $(\sigma = \tau - \frac{2}{3}i\varphi)$ . The constant fermionic parameters in (2.8) correspond to the Q-supersymmetry ( $\varepsilon^{\alpha}$ ) and S-supersymmetry ( $\eta_{\alpha}$ ) transformations. The constant parameters  $K_{\alpha\beta}$  and  $\sigma$  are obtained from  $K_{\alpha\beta}[\xi]$  and  $\sigma[\xi]$ , respectively, by setting  $z^A = 0$ .

It is convenient to introduce a condensed notation for the superconformal parameters

$$\lambda^{\tilde{a}} = (a^{A}, K^{ab}, \tau, \varphi, b_{A}) , \quad a^{A} := (a^{a}, \varepsilon^{\alpha}, \bar{\varepsilon}_{\dot{\alpha}}) , \quad b_{A} := (b_{a}, \eta_{\alpha}, \bar{\eta}^{\dot{\alpha}}) , \quad (2.10)$$

as well as for the generators of the superconformal group

$$X_{\tilde{a}} = (P_A, M_{ab}, \mathbb{D}, \mathbb{Y}, K^A) , \quad P_A := (P_a, Q_\alpha, \bar{Q}^{\dot{\alpha}}) , \quad K^A := (K^a, S^\alpha, \bar{S}_{\dot{\alpha}}) .$$
(2.11)

The general conformal Killing supervector field on  $\mathbb{C}^{4|2}$ ,

$$\xi = \xi^a_+(y,\theta)\frac{\partial}{\partial y^a} + \xi^\alpha(y,\theta)\frac{\partial}{\partial \theta^\alpha} \equiv \xi^a_+\partial/\partial y^a + \xi^\alpha\partial_\alpha , \qquad (2.12)$$

may be written in the form:

$$\xi = \lambda^{\tilde{a}} \xi_{\tilde{a}}(X) = a^A \xi_A(P) + \frac{1}{2} K^{ab} \xi_{ab}(M) + \tau \xi(\mathbb{D}) + \mathbf{i}\varphi \xi(\mathbb{Y}) + b_A \xi^A(K) .$$
(2.13)

We read off the relevant supervector fields:

$$\xi_{a}(P) = \partial/\partial y^{a}, \quad \xi_{\alpha}(P) = \partial_{\alpha}, \quad \tilde{\xi}^{\dot{\alpha}}(P) = -2\mathrm{i}(\tilde{\sigma}^{c}\theta)^{\dot{\alpha}}\partial/\partial y^{c}, \quad (2.14a)$$

$$\xi_{ab}(M) = y_a \partial / \partial y^b - y_b \partial / \partial y^a + (\theta \sigma_{ab})^\gamma \partial_\gamma , \qquad (2.14b)$$

$$\xi(\mathbb{D}) = y^c \partial / \partial y^c + \frac{1}{2} \theta^{\gamma} \partial_{\gamma}, \quad \xi(\mathbb{Y}) = \theta^{\gamma} \partial_{\gamma}, \quad (2.14c)$$

$$\xi^{a}(K) = 2y^{a}y^{c}\partial/\partial y^{c} - y^{2}\partial/\partial y_{a} - (\theta\sigma^{a}\tilde{\sigma}^{c})^{\gamma}y_{c}\partial_{\gamma}, \qquad (2.14d)$$

$$\xi^{\alpha}(K) = 2(\theta \sigma^{c} \tilde{\sigma}^{d})^{\alpha} y_{d} \partial / \partial y^{c} - 2\theta^{2} \varepsilon^{\alpha \gamma} \partial_{\gamma}, \qquad (2.14e)$$

$$\xi_{\dot{\alpha}}(K) = \mathbf{i}(\sigma^c)^{\gamma}{}_{\dot{\alpha}} y_c \partial_{\gamma} \,. \tag{2.14f}$$

Making use of the above operators, we derive the graded commutation relations for the superconformal algebra,  $[X_{\tilde{a}}, X_{\tilde{b}}] = -f_{\tilde{a}\tilde{b}}{}^{\tilde{c}}X_{\tilde{c}}$ , keeping in mind the relation

$$\boldsymbol{\xi} = \boldsymbol{\lambda}^{\tilde{a}} \boldsymbol{\xi}_{\tilde{a}}(X) \rightarrow \boldsymbol{\delta}_{\boldsymbol{\xi}} = \boldsymbol{\lambda}^{\tilde{a}} \boldsymbol{X}_{\tilde{a}} , \qquad \begin{bmatrix} \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \end{bmatrix} \rightarrow -\begin{bmatrix} \boldsymbol{\delta}_{\boldsymbol{\xi}_1}, \boldsymbol{\delta}_{\boldsymbol{\xi}_2} \end{bmatrix} .$$
(2.15)

We start with the commutation relations for the conformal algebra:

$$[M_{ab}, M_{cd}] = 2\eta_{c[a}M_{b]d} - 2\eta_{d[a}M_{b]c} , \qquad (2.16a)$$

$$[M_{ab}, P_c] = 2\eta_{c[a}P_{b]} , \qquad [\mathbb{D}, P_a] = P_a , \qquad (2.16b)$$

$$[M_{ab}, K_c] = 2\eta_{c[a}K_{b]} , \qquad [\mathbb{D}, K_a] = -K_a , \qquad (2.16c)$$

$$[K_a, P_b] = 2\eta_{ab}\mathbb{D} + 2M_{ab} . \tag{2.16d}$$

The *R*-symmetry generator  $\mathbb{Y}$  commutes with all the generators of the conformal group. The superconformal algebra is obtained by extending the translation generator to  $P_A$  and the special conformal generator to  $K^A$ . The commutation relations involving the *Q*-supersymmetry generators with the bosonic ones are:

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$$\begin{bmatrix} M_{ab}, Q_{\gamma} \end{bmatrix} = (\sigma_{ab})_{\gamma}{}^{\delta}Q_{\delta} , \quad \begin{bmatrix} M_{ab}, \bar{Q}^{\dot{\gamma}} \end{bmatrix} = (\tilde{\sigma}_{ab})^{\dot{\gamma}}{}_{\dot{\delta}}\bar{Q}^{\delta} , \qquad (2.17a)$$

$$\left[\mathbb{D}, Q_{\alpha}\right] = \frac{1}{2} Q_{\alpha} , \quad \left[\mathbb{D}, \bar{Q}^{\dot{\alpha}}\right] = \frac{1}{2} \bar{Q}^{\dot{\alpha}} , \qquad (2.17b)$$

$$\begin{bmatrix} \mathbb{Y}, Q_{\alpha} \end{bmatrix} = Q_{\alpha} , \quad \begin{bmatrix} \mathbb{Y}, \bar{Q}^{\dot{\alpha}} \end{bmatrix} = -\bar{Q}^{\dot{\alpha}} , \qquad (2.17c)$$

$$\left[K^{a}, \mathcal{Q}_{\beta}\right] = -\mathrm{i}(\sigma^{a})_{\beta}{}^{\dot{\beta}}\bar{S}_{\dot{\beta}} , \quad \left[K^{a}, \bar{\mathcal{Q}}^{\dot{\beta}}\right] = -\mathrm{i}(\sigma^{a})^{\dot{\beta}}{}_{\beta}S^{\beta} . \tag{2.17d}$$

The commutation relations involving the S-supersymmetry generators with the bosonic operators are:

$$\begin{bmatrix} M_{ab}, S^{\gamma} \end{bmatrix} = -(\boldsymbol{\sigma}_{ab})_{\beta}{}^{\gamma}S^{\beta} , \quad \begin{bmatrix} M_{ab}, \bar{S}_{\dot{\gamma}} \end{bmatrix} = -(\tilde{\boldsymbol{\sigma}}_{ab})^{\dot{\beta}}{}_{\dot{\gamma}}\bar{S}_{\dot{\beta}} , \qquad (2.18a)$$

$$\left[\mathbb{D}, S^{\alpha}\right] = -\frac{1}{2}S^{\alpha} , \quad \left[\mathbb{D}, \bar{S}_{\dot{\alpha}}\right] = -\frac{1}{2}\bar{S}_{\dot{\alpha}} , \qquad (2.18b)$$

$$\left[\mathbb{Y}, S^{\alpha}\right] = -S^{\alpha} , \quad \left[\mathbb{Y}, S_{\dot{\alpha}}\right] = S_{\dot{\alpha}} , \qquad (2.18c)$$

$$\left[S^{\alpha}, P_{b}\right] = \mathrm{i}(\sigma_{b})^{\alpha}{}_{\dot{\beta}}\bar{Q}^{\beta} , \quad \left[\bar{S}_{\dot{\alpha}}, P_{b}\right] = \mathrm{i}(\sigma_{b})_{\dot{\alpha}}{}^{\beta}Q_{\beta} . \tag{2.18d}$$

Finally, the anti-commutation relations of the fermionic generators are:

$$\{Q_{\alpha}, \bar{Q}^{\dot{\alpha}}\} = -2\mathrm{i}(\sigma^{b})_{\alpha}{}^{\dot{\alpha}}P_{b} = -2\mathrm{i}P_{\alpha}{}^{\dot{\alpha}}, \qquad (2.19a)$$

$$\{S^{\alpha}, \bar{S}_{\dot{\alpha}}\} = 2\mathbf{i}(\sigma^{b})^{\alpha}{}_{\dot{\alpha}}K_{b} = 2\mathbf{i}K^{\alpha}{}_{\dot{\alpha}}, \qquad (2.19b)$$

$$\{S^{\alpha}, Q_{\beta}\} = 2\delta^{\alpha}_{\beta} \mathbb{D} - 4M^{\alpha}_{\ \beta} - 3\delta^{\alpha}_{\beta} \mathbb{Y} , \qquad (2.19c)$$

$$\{\bar{S}_{\dot{\alpha}}, \bar{Q}^{\dot{\beta}}\} = 2\delta^{\beta}_{\dot{\alpha}}\mathbb{D} + 4\bar{M}_{\dot{\alpha}}{}^{\dot{\beta}} + 3\delta^{\beta}_{\dot{\alpha}}\mathbb{Y} , \qquad (2.19d)$$

where  $M_{\alpha\beta} = \frac{1}{2} (\sigma^{ab})_{\alpha\beta} M_{ab}$  and  $\bar{M}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2} (\tilde{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}} M_{ab}$ . Note that all remaining (anti-)commutators not explicitly listed above vanish identically.

The graded commutation relations (2.16) - (2.19) constitute the  $\mathcal{N} = 1$  superconformal algebra,  $\mathfrak{su}(2,2|1)$ . Its generators obey the graded Jacobi identity

$$(-1)^{\varepsilon_{\tilde{a}}\varepsilon_{\tilde{c}}}[X_{\tilde{a}}, [X_{\tilde{b}}, X_{\tilde{c}}]\} + (\text{two cycles}) = 0, \qquad (2.20)$$

where  $\varepsilon_{\tilde{a}} = \varepsilon(X_{\tilde{a}})$  is the Grassmann parity of the generator  $X_{\tilde{a}}$ . Making use of  $[X_{\tilde{a}}, X_{\tilde{b}}] = -f_{\tilde{a}\tilde{b}}{}^{\tilde{c}}X_{\tilde{c}}$ , the Jacobi identities are equivalently written as

$$f_{[\tilde{a}\tilde{b}}{}^{\tilde{d}}f_{|\tilde{d}|\tilde{c}\}}{}^{\tilde{e}} = 0.$$
 (2.21)

It remains to discuss superconformal transformation laws for superfields. Here we restrict our discussion to *primary superfields*. Given a conformal Killing supervector field  $\xi$ , the corresponding infinitesimal superconformal transformation acts on a primary tensor superfield U (with suppressed indices) by the rule

$$\delta_{\xi}U = \mathscr{K}[\xi]U, \qquad \mathscr{K}[\xi] = \xi + \frac{1}{2}K^{ab}[\xi]M_{ab} + p\sigma[\xi] + q\bar{\sigma}[\xi]. \quad (2.22)$$

Here the parameters p and q are related to the dimension (Weyl weight) w and  $U(1)_R$  charge c of U as follows: w = p + q and  $p - q = -\frac{3}{2}c$ . The Lorentz generators  $M_{ab}$  in (2.22) act on the indices of U. The commutation relation for these matrices differs by overall sign from (2.16a). This is due to the fact that, in conformal (super)gravity, subsequent transformations are applied as follows:  $\delta_{\xi_2} \delta_{\xi_1} U = \delta_{\xi_2} \mathscr{K}[\xi_1]U = \mathscr{K}[\xi_1]\delta_{\xi_2}U = \mathscr{K}[\xi_1]\mathscr{K}[\xi_2]U$ , see [28] for more details.

## 2.2 Superconformal transformations and complex geometry

Minkowski superspace  $\mathbb{M}^{4|4}$  is embedded in the so-called chiral superspace  $\mathbb{C}^{4|2}$ , parametrised by complex coordinates  $y^a$  and  $\theta^{\alpha}$ , as the real surface

$$\frac{1}{2}(y^a - \bar{y}^a) = i\theta\sigma^a\bar{\theta} , \qquad \frac{1}{2}(y^a + \bar{y}^a) = x^a .$$
 (2.23)

This is a special member of a family of real superspaces  $\mathscr{M}^{4|4}(\mathscr{H})$  embedded in  $\mathbb{C}^{4|2}$  by the rule

$$\frac{1}{2}(y^a - \bar{y}^a) = i\mathcal{H}^a(x, \theta, \bar{\theta}) , \qquad \frac{1}{2}(y^a + \bar{y}^a) = x^a , \qquad (2.24)$$

where the four real bosonic functions  $\mathscr{H}^a(x, \theta, \overline{\theta})$  may be arbitrary. With respect to the super-Poincaré transformations

$$\delta y^{a} = a^{a} - K^{a}{}_{b}y^{b} + 2i\theta\sigma^{a}\bar{\epsilon} , \qquad \delta\theta^{\alpha} = \epsilon^{\alpha} + \frac{1}{2}K^{bc}(\theta\sigma_{bc})^{\alpha} , \qquad (2.25)$$

 $\mathscr{H}^{a}(z) - \theta \sigma^{a} \bar{\theta}$  proves to be a vector superfield. What is special about Minkowski superspace,  $\mathscr{M}^{4|4}(\theta \sigma \bar{\theta})$ , is the fact that (2.23) is the unique surface of the type  $\mathscr{M}^{4|4}(\mathscr{H})$  which is invariant under the super-Poincaré transformations (2.25).

It turns out that the superconformal transformations (2.8) are the most general holomorphic transformations on  $\mathbb{C}^{4|2}$  of the form

$$\delta y^a = \lambda^a(y,\theta) , \quad \delta \theta^\alpha = \lambda^\alpha(y,\theta) , \qquad (2.26)$$

which leave invariant the superspace  $\mathbb{M}^{4|4}$  defined by (2.23). This remarkable result indicates that (i) arbitrary holomorphic transformations (2.26) should be interpreted as local superconformal ones; and (ii)  $\mathscr{H}^a(x, \theta, \overline{\theta})$  should be used to describe conformal supergravity, a supersymmetric extension of conformal gravity.

# 2.3 Local superconformal transformations

Following [36], the gauge group of conformal supergravity is postulated to be the supergroup of holomorphic reparametrisations of  $\mathbb{C}^{4|2}$ 

$$y^m \to y'^m = f^m(y,\theta) , \quad \theta^\mu \to \theta'^\mu = f^\mu(y,\theta) , \quad \text{Ber}\left(\frac{\partial(y',\theta')}{\partial(y,\theta)}\right) \neq 0 .$$
 (2.27)

In curved superspace, we distinguish between curved and flat-space indices. Latin and Greek letters from the middle of each alphabet are used for curved-space indices. Letters from the beginning of each alphabet denote flat-space indices.

In practise, it suffices to work with infinitesimal holomorphic transformations,

$$y^m \to y'^m = y^m - \lambda^m(y, \theta), \quad \theta^\mu \to \theta'^\mu = \theta^\mu - \lambda^\mu(y, \theta).$$
 (2.28)

When restricted to  $\mathcal{M}^{4|4}(\mathcal{H})$ , this transformation acts as follows

$$x^m \to x'^m = x^m - \frac{1}{2}\lambda^m(x + i\mathscr{H}, \theta) - \frac{1}{2}\bar{\lambda}^m(x - i\mathscr{H}, \bar{\theta})$$
, (2.29a)

$$\theta^{\mu} \to \theta^{\prime \mu} = \theta^{\mu} - \lambda^{\mu} (x + i\mathscr{H}, \theta) ,$$
(2.29b)

as well as

$$\mathscr{H}^{\prime m}(x',\theta',\bar{\theta}') = \mathscr{H}^{m}(x,\theta,\bar{\theta}) + \frac{\mathrm{i}}{2}\lambda^{m}(x+\mathrm{i}\mathscr{H},\theta) - \frac{\mathrm{i}}{2}\bar{\lambda}^{m}(x-\mathrm{i}\mathscr{H},\bar{\theta}) . (2.29\mathrm{c})$$

From here it follows that  $\delta \mathscr{H}^m = \mathscr{H}^{lm}(x, \theta, \overline{\theta}) - \mathscr{H}^m(x, \theta, \overline{\theta})$  is given by

$$\delta \mathscr{H}^{m} = \frac{\mathrm{i}}{2} (\lambda^{m} - \bar{\lambda}^{m}) + \left(\frac{1}{2} (\lambda^{n} + \bar{\lambda}^{n}) \partial_{n} + \lambda^{\mu} \partial_{\mu} + \bar{\lambda}_{\mu} \partial^{\mu}\right) \mathscr{H}^{m} ,$$
  
$$\lambda^{m} = \lambda^{m} (x + \mathrm{i} \mathscr{H}, \theta) , \quad \lambda^{\mu} = \lambda^{\mu} (x + \mathrm{i} \mathscr{H}, \theta) .$$
(2.30)

This is the gauge transformation law of  $\mathscr{H}^m$ .

Making use of the gauge freedom for  $\mathscr{H}^m$  allows one to choose a gauge condition

$$\mathscr{H}^{m}(x,\theta,\bar{\theta}) = \theta \sigma^{a} \bar{\theta} e_{a}{}^{m}(x) - \mathrm{i} \bar{\theta}^{2} \theta^{\alpha} \Psi^{m}_{\alpha}(x) + \mathrm{i} \theta^{2} \bar{\theta}_{\dot{\alpha}} \bar{\Psi}^{m\dot{\alpha}}(x) + \theta^{2} \bar{\theta}^{2} \Big( A^{m}(x) - \frac{1}{4} e_{a}{}^{m}(x) \varepsilon^{abcd} \omega_{bcd}(x) \Big) .$$
(2.31)

Here  $\omega_{abc} = -\omega_{acb} = e_a{}^m \omega_{mbc}$  is the torsion-free Lorentz connection associated with the vielbein  $e^a = dx^m e_m{}^a$  and its dual frame field  $e_a = e_a{}^m \partial_m$ :

$$\boldsymbol{\omega}_{abc} = -\frac{1}{2} \left( \mathscr{C}_{bca} + \mathscr{C}_{acb} - \mathscr{C}_{abc} \right) , \qquad \left[ \boldsymbol{e}_a, \boldsymbol{e}_b \right] = \mathscr{C}_{ab}{}^c \boldsymbol{e}_c . \tag{2.32}$$

The residual gauge freedom, which preserves the condition (2.31), is given by

$$\lambda^{m}(\theta) = \mathfrak{a}^{m} + 2\mathrm{i}\theta\sigma^{a}\bar{\varepsilon}e_{a}^{m} - 2\theta^{2}\bar{\varepsilon}\bar{\Psi}^{m}, \qquad \bar{\mathfrak{a}}^{m} = \mathfrak{a}^{m}, \qquad (2.33a)$$

$$\lambda^{\alpha}(\theta) = \varepsilon^{\alpha} + \theta^{\alpha} \left(\frac{1}{2}\tau + \mathrm{i}\varphi\right) + \theta^{\beta} K_{\beta}^{\alpha} + \theta^{2} \left[\eta^{\alpha} - \frac{1}{2} (\bar{\varepsilon} \tilde{\sigma}^{a})^{\alpha} \left(\mathrm{i}\omega^{b}{}_{ba} + \frac{1}{2} \varepsilon^{abcd} \omega_{bcd}\right)\right], \qquad (2.33b)$$

with  $K_{\alpha}{}^{\beta} = \frac{1}{2}K^{ab}(\sigma_{ab})_{\alpha}{}^{\beta}$ . Keeping in mind the structure of conformal Killing supervector fields (2.8), we can give the following interpretations to the gauge parameters (2.33). The bosonic parameters correspond to the general coordinate ( $\alpha^{m}$ ), local Lorentz ( $K_{\alpha}{}^{\beta}$ ), Weyl ( $\tau$ ) and local chiral ( $\varphi$ ) transformations. The fermionic parameters correspond to the local *Q*-supersymmetry ( $\varepsilon^{\alpha}$ ) and *S*-supersymmetry ( $\eta^{\alpha}$ ) transformations.

The superfield gauge transformation (2.30) allows us to work out the transformation laws of the component fields in (2.31). Choosing  $a^m \neq 0$  and switching off the other parameters in (2.33) gives

$$\delta_{\mathfrak{a}} e_{a} = \begin{bmatrix} \mathfrak{a}, e_{a} \end{bmatrix}, \quad \delta_{\mathfrak{a}} \Psi_{\alpha} = \begin{bmatrix} \mathfrak{a}, \Psi_{\alpha} \end{bmatrix}, \quad \delta_{\mathfrak{a}} A = \begin{bmatrix} \mathfrak{a}, A \end{bmatrix}, \quad (2.34)$$

where we have introduced the first-order operators  $\mathfrak{a} = \mathfrak{a}^m \partial_m$ ,  $\Psi_{\alpha} = \Psi_{\alpha}^m \partial_m$  and  $A = A^m \partial_m$ . Next, choosing  $K_{\alpha}{}^{\beta} \neq 0$  and switching off the other parameters in (2.33) gives

$$\delta_{K}e_{a} = K_{a}^{\ b}e_{b} , \quad \delta_{K}\Psi_{\alpha} = K_{\alpha}^{\ \beta}\Psi_{\beta} , \quad \delta_{K}A = 0 .$$
(2.35)

The transformation laws (2.34) and (2.35) allow us to interpret the field  $e_a^m$  as the inverse vielbein. They also show that  $\Psi_{\alpha}^m$  transforms as a world vector and a Weyl spinor, while  $A^m$  is a vector field. Next, choosing  $\frac{1}{2}\tau + i\varphi \neq 0$  and switching off the other parameters in (2.33) gives the Weyl ( $\tau$ ) and local chiral ( $\varphi$ ) transformations

$$\delta_{\tau} e_a{}^m = \tau e_a{}^m , \quad \delta_{\tau} \Psi_{\alpha}^m = \frac{3}{2} \tau \Psi_{\alpha}^m , \quad \delta_{\tau} A_m = 0 ; \qquad (2.36)$$

$$\delta_{\varphi} e_a{}^m = 0 , \quad \delta_{\varphi} \Psi^m_{\alpha} = -i \varphi \Psi^m_{\alpha} , \quad \delta_{\varphi} A_m = \partial_m \varphi .$$
 (2.37)

Here we have introduced the one-form  $A_m = g_{mn}A^n$ , where  $g_{mn}(x) = e_m{}^a(x)e_n{}^b(x)\eta_{ab}$  is the Lorentzian metric associated with the vielbein  $e_m{}^a$ . It follows that  $A_m$  is the gauge field for the chiral U(1)<sub>R</sub> group. Since  $\mathscr{H}^m$  contains the inverse vielbein at the component level, it is called the *gravitational superfield*.

It remains to consider local supersymmetry transformations. Choosing  $\eta^{\alpha} \neq 0$  and switching off the other parameters in (2.33) gives the S-supersymmetry transformation laws

$$\delta_{\eta} e_a{}^m = 0 , \quad \delta_{\eta} \Psi_{\alpha}^m = e_a{}^m (\sigma^a \bar{\eta})_{\alpha} , \quad \delta_{\eta} A^m = i \left( \bar{\eta} \bar{\Psi}^m - \eta \Psi^m \right) .$$
(2.38)

Finally, for the Q-supersymmetry transformation we obtain

$$\delta_{\varepsilon} e_a^{\ m} = \mathrm{i} \left( \Psi^m \sigma_a \bar{\varepsilon} - \varepsilon \sigma_a \bar{\Psi}^m \right) \,, \tag{2.39a}$$

$$\delta_{\varepsilon} \Psi^{m}_{\alpha} = -(\sigma^{a} \tilde{\sigma}^{b} \nabla_{a} \varepsilon)_{\alpha} e_{b}^{\ m} + 2\mathrm{i} A^{m} \varepsilon_{\alpha} , \qquad (2.39\mathrm{b})$$

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$$\delta_{\varepsilon}A^{m} = -\frac{\mathrm{i}}{4}e_{a}^{m}\varepsilon^{abcd}\nabla_{b}\left(\varepsilon\sigma_{c}\bar{\Psi}^{n} - \Psi^{n}\sigma_{c}\bar{\varepsilon}\right)e_{nd} - \frac{1}{2}\nabla_{a}\left(\varepsilon\sigma^{a}\bar{\Psi}^{m} + \Psi^{m}\sigma^{a}\bar{\varepsilon}\right) + \left(\nabla_{n}\varepsilon\sigma^{a}\bar{\Psi}^{n} + \Psi^{n}\sigma^{a}\nabla_{n}\bar{\varepsilon}\right)e_{a}^{m}.$$
(2.39c)

Here  $\nabla_n$  and  $\nabla_a = e_a{}^n \nabla_n$  are standard torsion-free covariant derivatives, in particular

$$\nabla_n e_a^{\ m} = 0 , \qquad \nabla_n \Psi_\alpha^m = \partial_n \Psi_\alpha^m - \frac{1}{2} \omega_n^{\ bc} (\sigma_{bc})_\alpha^{\ \beta} \Psi_\beta^m + \Gamma_{nr}^m \Psi_\alpha^r , \qquad (2.40)$$

where the Lorentz connection is given by (2.32) and  $\Gamma_{nr}^{m}$  denotes the Christoffel symbols.

The *S* and *Q*-supersymmetry transformation laws can be rewritten in a more convenient and familiar form if the dynamical fields  $e_a^m$ ,  $\Psi_\alpha^m$  and *A* are replaced with  $e_m^a$ ,  $\Psi_{m\alpha} = g_{mn}\Psi_\alpha^n$  and

$$\mathfrak{A}_m = A_m - \frac{1}{2}\Psi^n \sigma_m \bar{\Psi}_n + \frac{1}{2} \left(\Psi_m \sigma_n \bar{\Psi}^n + \Psi^n \sigma_n \bar{\Psi}_m\right) + \frac{i}{8} e_m{}^a \varepsilon_{abcd} \Psi^b \sigma^c \bar{\Psi}^d . \quad (2.41)$$

Then the S-supersymmetry transformation turns into

η

$$\delta_{\eta} e_m{}^a = 0 , \quad \delta_{\eta} \Psi_{m\alpha} = (\sigma_m \bar{\eta})_{\alpha} , \quad \delta_{\eta} \mathfrak{A}_m = \frac{3}{4} \mathrm{i} \left( \eta \Psi_m - \bar{\eta} \bar{\Psi}_m \right) .$$
 (2.42)

It turns out that the simplest version of the Q-supersymmetry transformation corresponds to the variation

$$\hat{\delta}_{\varepsilon} = \delta_{\varepsilon} + \delta_{\eta(\varepsilon)} + \delta_{K(\varepsilon)} + \delta_{\varphi(\varepsilon)} , \qquad (2.43a)$$

$${}^{\alpha}(\varepsilon) = -\mathrm{i}(\nabla_{b}\bar{\varepsilon}\tilde{\sigma}^{b})^{\alpha} - \varepsilon^{\alpha}\bar{\Psi}^{n}\bar{\Psi}_{n} + \frac{1}{2}(\bar{\varepsilon}\tilde{\sigma}_{a})^{\alpha}\Psi^{n}\sigma^{a}\bar{\Psi}_{n}$$

$${}^{\mathrm{i}}(\bar{\varepsilon}\tilde{\sigma}^{\alpha})^{\alpha}\sigma^{abcd}\Psi^{n}\sigma^{a}\bar{\Psi}_{n} \qquad (2.421)$$

$$-\frac{1}{4} (\bar{\varepsilon} \tilde{\sigma}_a)^{\alpha} \varepsilon^{abcd} \Psi_b \sigma_c \bar{\Psi}_d , \qquad (2.43b)$$

$$K_{\alpha\beta}(\varepsilon) = \frac{1}{2} \left[ (\sigma_n \bar{\Psi}^n)_{\alpha} \varepsilon_{\beta} + (\sigma_n \bar{\Psi}^n)_{\beta} \varepsilon_{\alpha} - (\sigma_n \bar{\varepsilon})_{\alpha} \Psi_{\beta}^n - (\sigma_n \bar{\varepsilon})_{\beta} \Psi_{\alpha}^n \right], (2.43c)$$

$$\varphi(\varepsilon) = \frac{1}{2} \left( \Psi^n \sigma_n \bar{\varepsilon} + \varepsilon \sigma_n \bar{\Psi}^n \right) \,. \tag{2.43d}$$

Then we end up with the following transformation laws

$$\hat{\delta}_{\varepsilon} e_m{}^a = i \left( \varepsilon \sigma^a \bar{\Psi}_m - \Psi_m \sigma^a \bar{\varepsilon} \right) , \qquad (2.44a)$$

$$\hat{\delta}_{\varepsilon}\Psi_{m} = 2\hat{\nabla}_{m}\varepsilon = 2(\nabla_{m} - \frac{1}{2}\hat{\omega}_{m}{}^{bc}\sigma_{bc} + i\mathfrak{A}_{m})\varepsilon , \qquad (2.44b)$$

$$\hat{\delta}_{\varepsilon}\mathfrak{A}_{m} = \frac{1}{2}\varepsilon\sigma^{n}\left(\hat{\nabla}_{m}\bar{\Psi}_{n} - \hat{\nabla}_{n}\bar{\Psi}_{m}\right) - \frac{1}{2}\left(\hat{\nabla}_{m}\Psi_{n} - \hat{\nabla}_{n}\Psi_{m}\right)\sigma^{n}\bar{\varepsilon} -\frac{\mathrm{i}}{4}g_{mn}\varepsilon^{nijk}\left(\hat{\nabla}_{i}\Psi_{j}\sigma_{k}\bar{\varepsilon} - \varepsilon\sigma_{k}\hat{\nabla}_{i}\bar{\Psi}_{j}\right), \qquad (2.44c)$$

where we have introduced the covariant derivative with torsion

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$$\hat{\nabla}_m = \nabla_m - \frac{1}{2}\hat{\omega}_m{}^{bc}M_{bc} - \mathrm{i}w\mathfrak{A}_m , \qquad (2.45a)$$

$$\hat{\omega}_{cab} = -\frac{1}{2} (\hat{C}_{abc} + \hat{C}_{acb} - \hat{C}_{bca}) , \quad \hat{C}_{abc} = \frac{1}{2} (\Psi_a \sigma_c \bar{\Psi}_b - \Psi_b \sigma_c \bar{\Psi}_a) , \quad (2.45b)$$

with *w* being the  $U(1)_R$  charge of a field  $\Upsilon$  with the  $U(1)_R$  transformation law  $\Upsilon \to e^{iw\varphi}\Upsilon$ .

It follows from the above analysis that the gauge fields  $\{e_m{}^a, \Psi_{m\alpha}, \bar{\Psi}_m{}^{\dot{\alpha}}, \mathfrak{A}_m\}$  form a multiplet under the local *S* and *Q*-supersymmetry transformations. It will be referred to as the *reduced Weyl multiplet*<sup>3</sup> of conformal supergravity since it is a supersymmetric generalisation of conformal gravity in which the gauge group includes the Weyl transformations  $e_m{}^a(x) \to e^{-\tau(x)}e_m{}^a(x)$ .

#### **3** Conformal superspace

In the previous section we have reviewed a simple approach to obtain the reduced Weyl multiplet of conformal supergravity from superspace. In this setting, conformal supergravity is described by the gravitational superfield  $\mathcal{H}^m$ , which defines the embedding of curved superspace  $\mathcal{M}^{4|4}(\mathcal{H})$  in the chiral superspace  $\mathbb{C}^{4|2}$ , eq. (2.24). Although this approach is elegant and geometric, it is not covariant and does not offer powerful tools to construct manifestly gauge-invariant supergravity actions and to engineer general couplings of supergravity to matter. Such tools are provided by the so-called conformal superspace approach [32], which is reviewed in the present section.

#### 3.1 Gauging the superconformal algebra in superspace

Conformal superspace is a gauge theory of the superconformal algebra. It can be identified with a pair  $(\mathcal{M}^{4|4}, \nabla)$ . Here  $\mathcal{M}^{4|4}$  denotes a supermanifold parametrised by local coordinates  $z^{M} = (x^{m}, \theta^{\mu}, \bar{\theta}_{\mu})$ , and  $\nabla$  is a covariant derivative associated with the superconformal algebra. We recall that the generators  $X_{\tilde{a}}$  of the superconformal algebra are given by eq. (2.11). They can be grouped in two disjoint subsets,

$$X_{\tilde{a}} = (P_A, X_{\underline{a}}) , \qquad X_{\underline{a}} = (M_{ab}, \mathbb{D}, \mathbb{Y}, K^A) , \qquad (3.1)$$

each of which constitutes a superalgebra:

$$[P_A, P_B] = -f_{AB}{}^C P_C , \qquad (3.2a)$$

$$[X_{\underline{a}}, X_{\underline{b}}] = -f_{\underline{a}\underline{b}}{}^{\underline{c}} X_{\underline{c}} , \qquad (3.2b)$$

<sup>&</sup>lt;sup>3</sup> The Weyl multiplet also includes a dilatation gauge field,  $b_m$ , but this proves to describe purely gauge degrees of freedom, see e.g. [28, 33] for reviews and subsection 3.4 below.

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$$[X_{\underline{a}}, P_B] = -f_{\underline{a}B}{}^{\underline{c}}X_{\underline{c}} - f_{\underline{a}B}{}^{\underline{C}}P_C .$$
(3.2c)

Here the structure constants  $f_{AB}{}^C$  contain only one non-zero component, which is  $f_{\alpha}{}^{\dot{\beta}c} = 2i(\sigma^c)_{\alpha}{}^{\dot{\beta}}$ .

In order to define the covariant derivatives,  $\nabla_A = (\nabla_a, \nabla_\alpha, \bar{\nabla}^{\dot{\alpha}})$ , we associate with each generator  $X_{\underline{a}} = (M_{ab}, \mathbb{D}, \mathbb{Y}, K^A) = (M_{ab}, \mathbb{D}, \mathbb{Y}, K^a, S^{\alpha}, \bar{S}_{\dot{\alpha}})$  a connection one-form  $\omega^{\underline{a}} = (\Omega^{ab}, \overline{B}, \Phi, \mathfrak{F}_A) = (\Omega^{ab}, B, \Phi, \mathfrak{F}_a, \mathfrak{F}_a, \mathfrak{F}_a^{\dot{\alpha}}) = dz^M \omega_M{}^a$ , and with  $P_A$ a supervielbein one-form  $E^A = (E^a, E^{\alpha}, \bar{E}_{\dot{\alpha}}) = dz^M E_M{}^A$  (the latter will be often referred to as the vielbein). It is assumed that the supermatrix  $E_M{}^A$  is nonsingular,  $E := \operatorname{Ber}(E_M{}^A) \neq 0$ , and hence there exists a unique inverse supervielbein. The latter is given by the supervector fields  $E_A = E_A{}^M(z)\partial_M$ , with  $\partial_M = \partial/\partial z^M$ , which constitute a new basis for the tangent space at each point  $z^M \in \mathcal{M}^{4|4}$ . The supermatrices  $E_A{}^M$  and  $E_M{}^A$  satisfy the properties  $E_A{}^M E_M{}^B = \delta_A{}^B$  and  $E_M{}^A E_A{}^N = \delta_M{}^N$ . With respect to the basis  $E^A$ , the connection is expressed as  $\omega^{\underline{a}} = E^B \omega_B{}^{\underline{a}}$ , where  $\omega_B{}^{\underline{a}} = E_B{}^M \omega_M{}^{\underline{a}}$ . The *covariant derivatives* are given by

$$\nabla_A = E_A - \omega_A {}^{\underline{b}} X_{\underline{b}} = E_A - \frac{1}{2} \Omega_A {}^{bc} M_{bc} - B_A \mathbb{D} - \mathrm{i} \Phi_A \mathbb{Y} - \mathfrak{F}_{AB} K^B .$$
(3.3)

They can be recast in terms of one-forms

$$\nabla = \mathbf{d} - \boldsymbol{\omega}^{\underline{a}} X_{\underline{a}} , \quad \nabla = E^A \nabla_A . \tag{3.4}$$

The translation generators  $P_B$  do not show up in (3.3) and (3.4). It is assumed that the operators  $\nabla_A$  replace  $P_A$  and obey the graded commutation relations

$$[X_{\underline{b}}, \nabla_A] = -f_{\underline{b}A}{}^C \nabla_C - f_{\underline{b}A}{}^{\underline{c}} X_{\underline{c}} , \qquad (3.5)$$

compare with (3.2c). In particular, the algebra of  $K^A$  with  $\nabla_B$  is given by

$$[K^a, \nabla_b] = 2\delta^a_b \mathbb{D} + 2M^a_{\ b} , \qquad (3.6a)$$

$$\{S^{\alpha}, \nabla_{\beta}\} = \delta^{\alpha}_{\beta} \left(2\mathbb{D} - 3\mathbb{Y}\right) - 4M^{\alpha}{}_{\beta} , \qquad (3.6b)$$

$$\{\bar{S}_{\dot{\alpha}}, \bar{\nabla}^{\beta}\} = \delta^{\beta}_{\dot{\alpha}} (2\mathbb{D} + 3\mathbb{Y}) + 4\bar{M}_{\dot{\alpha}}{}^{\beta} , \qquad (3.6c)$$

$$[K^{a},\nabla_{\beta}] = -\mathrm{i}(\sigma^{a})_{\beta}{}^{\beta}\bar{S}_{\dot{\beta}} , \qquad [K^{a},\bar{\nabla}^{\beta}] = -\mathrm{i}(\sigma^{a})^{\beta}{}_{\beta}S^{\beta} , \qquad (3.6d)$$

$$[S^{\alpha}, \nabla_b] = \mathbf{i}(\sigma_b)^{\alpha}{}_{\dot{\beta}}\bar{\nabla}^{\dot{\beta}} , \qquad [\bar{S}_{\dot{\alpha}}, \nabla_b] = \mathbf{i}(\sigma_b)_{\dot{\alpha}}{}^{\beta}\nabla_{\beta} , \qquad (3.6e)$$

where all other graded commutators vanish.

By definition, the gauge group of conformal supergravity is generated by local transformations of the form

$$\boldsymbol{\delta}_{\mathscr{K}} \nabla_{A} = \left[ \mathscr{K}, \nabla_{A} \right], \tag{3.7a}$$

$$\mathscr{K} = \xi^B \nabla_B + \Lambda^{\underline{b}} X_{\underline{b}} = \xi^B \nabla_B + \frac{1}{2} K^{bc} M_{bc} + \Sigma \mathbb{D} + i\rho \mathbb{Y} + \Lambda_B K^B , \quad (3.7b)$$

where the gauge parameters satisfy natural reality conditions. In applying eq. (3.7), we interpret that

$$\nabla_A \xi^B := E_A \xi^B + \omega_A{}^c \xi^D f_{D\underline{c}}{}^B, \qquad (3.8a)$$

$$\nabla_A \Lambda^{\underline{b}} := E_A \Lambda^{\underline{b}} + \omega_A{}^{\underline{c}} \xi^D f_{D\underline{c}}{}^{\underline{b}} + \omega_A{}^{\underline{c}} \Lambda^{\underline{d}} f_{\underline{dc}}{}^{\underline{b}} .$$
(3.8b)

Then it follows from (3.7) that

$$\delta_{\mathscr{K}} E^{A} = \mathrm{d}\xi^{A} + E^{B} \Lambda^{\underline{c}} f_{\underline{c}B}{}^{A} + \omega^{\underline{b}} \xi^{C} f_{\underline{c}\underline{b}}{}^{A} + E^{B} \xi^{C} \mathscr{T}_{\underline{C}B}{}^{A} , \qquad (3.9a)$$

$$\delta_{\mathscr{H}}\omega^{\underline{a}} = \mathrm{d}\Lambda^{\underline{a}} + \omega^{\underline{b}}\Lambda^{\underline{c}}f_{\underline{c}\underline{b}}{}^{\underline{a}} + \omega^{\underline{b}}\xi^{C}f_{\underline{C}\underline{b}}{}^{\underline{a}} + E^{B}\Lambda^{\underline{c}}f_{\underline{c}\underline{B}}{}^{\underline{a}} + E^{B}\xi^{C}\mathscr{R}_{\underline{C}\underline{B}}{}^{\underline{a}} .$$
(3.9b)

Here we have made use of the graded commutation relations

$$[\nabla_A, \nabla_B\} = -\mathscr{T}_{AB}{}^C \nabla_C - \mathscr{R}_{AB}{}^c X_{\underline{c}} , \qquad (3.10)$$

where  $\mathcal{T}_{AB}{}^{C}$  and  $\mathcal{R}_{AB}{}^{\underline{c}}$  denote the torsion and the curvature, respectively. They can be recast in terms of two-forms

$$\mathscr{T}^{A} := \frac{1}{2} E^{C} \wedge E^{B} \mathscr{T}_{BC}{}^{A} = \mathrm{d} E^{A} - E^{C} \wedge \omega^{\underline{b}} f_{\underline{b}C}{}^{A} , \qquad (3.11a)$$

$$\mathscr{R}^{\underline{a}} := \frac{1}{2} E^{C} \wedge E^{B} \mathscr{R}_{BC}^{\underline{a}} = \mathrm{d}\omega^{\underline{a}} - E^{C} \wedge \omega^{\underline{b}} f_{\underline{b}C}^{\underline{a}} - \frac{1}{2} \omega^{\underline{c}} \wedge \omega^{\underline{b}} f_{\underline{b}\underline{c}}^{\underline{a}} .$$
(3.11b)

Making use of the graded Jacobi identity

$$0 = (-1)^{\varepsilon_{\underline{a}}\varepsilon_{C}} [X_{\underline{a}}, [\nabla_{B}, \nabla_{C}] \} + (\text{two cycles})$$
(3.12)

we derive the action of  $X_{\underline{a}}$  on the geometric objects

$$\begin{split} X_{\underline{a}}\mathscr{T}_{BC}{}^{D} &= -\left(-1\right)^{\varepsilon_{\underline{a}}(\varepsilon_{B}+\varepsilon_{C})}\mathscr{T}_{BC}{}^{E}f_{E\underline{a}}{}^{D} - 2f_{\underline{a}[B}{}^{E}\mathscr{T}_{|E|C\}}{}^{D} - 2f_{\underline{a}[B}{}^{\underline{e}}f_{|\underline{e}|C\}}{}^{D}, \quad (3.13a)\\ X_{\underline{a}}\mathscr{R}_{BC}{}^{\underline{d}} &= -\left(-1\right)^{\varepsilon_{\underline{a}}(\varepsilon_{B}+\varepsilon_{C})} \left(\mathscr{T}_{BC}{}^{E}f_{E\underline{a}}{}^{\underline{d}} + \mathscr{R}_{BC}{}^{\underline{e}}f_{\underline{e}\underline{a}}{}^{\underline{d}}\right) - 2f_{\underline{a}[B}{}^{E}\mathscr{R}_{|E|C\}}{}^{\underline{d}} \\ &- 2f_{\underline{a}[B}{}^{\underline{e}}f_{|\underline{e}|C\}}{}^{\underline{d}}. \quad (3.13b) \end{split}$$

The supergravity gauge group acts on a conformal tensor superfield U (with suppressed indices) as

$$\delta_{\mathscr{K}}U = \mathscr{K}U . \tag{3.14}$$

The torsion  $\mathscr{T}_{AB}{}^{C}$  and the curvature  $\mathscr{R}_{AB}{}^{c}$  are conformal tensor superfields, for which the action of the generators  $X_{\underline{a}}$  is specified by the relations (3.13). Of special significance are primary superfields. A tensor superfield U (with suppressed indices) is said to be *primary* if it is characterised by the properties

$$K^{A}U = 0, \quad \mathbb{D}U = wU, \quad \mathbb{Y}U = cU, \qquad (3.15)$$

for some real constants w and c which are called the dimension (or Weyl weight) and  $U(1)_R$  charge of U, respectively. It follows from (2.19b) that if a superfield is annihilated by the S-supersymmetry generators, then it is necessarily primary.

Let us summarise some important features of the gauging procedure. In curved superspace, the superconformal algebra (3.2) is replaced with

$$[X_{\underline{a}}, X_{\underline{b}}] = -f_{\underline{a}\underline{b}}{}^{\underline{c}}X_{\underline{c}} , \qquad (3.16a)$$

$$[X_{\underline{a}}, \nabla_B] = -f_{\underline{a}B}{}^C \nabla_C - f_{\underline{a}B}{}^{\underline{c}} X_{\underline{c}} , \qquad (3.16b)$$

$$[\nabla_A, \nabla_B] = -\mathscr{T}_{AB}{}^C \nabla_C - \mathscr{R}_{AB}{}^{\underline{C}} X_{\underline{C}} . \qquad (3.16c)$$

Here the torsion and curvature tensors obey Bianchi identities which follow from

$$0 = (-1)^{\varepsilon_A \varepsilon_C} [\nabla_A, [\nabla_B, \nabla_C] \} + (\text{two cycles}) .$$
(3.17)

Unlike (3.2), which is determined by the structure constants, the graded commutation relations (3.16) involve structure functions  $\mathcal{T}_{AB}{}^{C}$  and  $\mathcal{R}_{AB}{}^{\underline{c}}$ . Such an algebraic structure is sometimes called a *soft algebra*, see e.g. [28].

## 3.2 Conventional constraints for Weyl multiplet

The framework described in the previous subsection defines a geometric set-up to obtain a multiplet of conformal supergravity containing the metric. However, in general, the resulting multiplet is reducible. To obtain the irreducible multiplet described in section 2, it is necessary to impose constraints on the torsion and curvatures appearing in eq. (3.10). This is a standard task in geometric superspace approaches to supergravity, and it is pedagogically reviewed in [35, 13]. One beautiful feature of the construction of [32] is the simplicity of the superspace constraints needed to obtain the Weyl multiplet of conformal supergravity. In fact, to obtain a sufficient set of constraints, one requires the algebra (3.10) to have a Yang-Mills structure. Specifically, following [32] one imposes

$$\{\nabla_{\alpha}, \nabla_{\beta}\} = 0, \quad \{\bar{\nabla}_{\dot{\alpha}}, \bar{\nabla}_{\dot{\beta}}\} = 0, \quad \{\nabla_{\alpha}, \bar{\nabla}_{\dot{\alpha}}\} = -2i\nabla_{\alpha\dot{\alpha}}, \qquad (3.18a)$$

$$\left[\nabla_{\alpha}, \nabla_{\beta\dot{\beta}}\right] = 2i\varepsilon_{\alpha\beta}\tilde{\mathscr{W}}_{\dot{\beta}} , \quad \left[\bar{\nabla}_{\dot{\alpha}}, \nabla_{\beta\dot{\beta}}\right] = -2i\varepsilon_{\dot{\alpha}\dot{\beta}}\mathscr{W}_{\beta} , \qquad (3.18b)$$

where the operator  $\hat{\mathcal{W}}_{\dot{\alpha}}$  is the complex conjugate of  $\mathcal{W}_{\alpha}$ . The latter takes the form

$$\mathscr{W}_{\alpha} = \frac{1}{2} \mathscr{W}(M)_{\alpha}{}^{cd}M_{cd} + \mathscr{W}(\mathbb{D})_{\alpha}\mathbb{D} + \mathfrak{i}\mathscr{W}(\mathbb{Y})_{\alpha}\mathbb{Y} + \mathscr{W}(S)_{\alpha\gamma}S^{\gamma} + \mathscr{W}(S)_{\alpha}{}^{\dot{\gamma}}\bar{S}_{\dot{\gamma}} + \mathscr{W}(K)_{\alpha c}K^{c} .$$
(3.19)

Having imposed the constraints (3.18), the Bianchi identities (3.17) become nontrivial and now play the role of consistency conditions which may be used to determine the torsion and curvature. Their solution is as follows

$$\left[\nabla_{\alpha}, \nabla_{\beta\dot{\beta}}\right] = \mathrm{i}\varepsilon_{\alpha\beta} \left(2\bar{W}_{\dot{\beta}\dot{\gamma}\dot{\delta}}\bar{M}^{\dot{\gamma}\dot{\delta}} - \frac{1}{2}\bar{\nabla}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}\bar{S}^{\dot{\gamma}} + \frac{1}{2}\nabla_{\gamma}{}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}K^{\gamma\dot{\gamma}}\right), \quad (3.20a)$$

$$\left[\bar{\nabla}_{\dot{\alpha}}, \nabla_{\beta\dot{\beta}}\right] = -i\varepsilon_{\dot{\alpha}\dot{\beta}} \left(2W_{\beta}{}^{\gamma\delta}M_{\gamma\delta} + \frac{1}{2}\nabla^{\alpha}W_{\alpha\beta\gamma}S^{\gamma} + \frac{1}{2}\nabla^{\alpha\dot{\gamma}}W_{\alpha\beta}{}^{\gamma}K_{\gamma\dot{\gamma}}\right), \quad (3.20b)$$

$$\left[\nabla_{\alpha\dot{\alpha}}, \nabla_{\beta\dot{\beta}}\right] = \varepsilon_{\dot{\alpha}\dot{\beta}}\psi_{\alpha\beta} + \varepsilon_{\alpha\beta}\bar{\psi}_{\dot{\alpha}\dot{\beta}} , \qquad (3.20c)$$

where the symmetric bispinor operator  $\psi_{\alpha\beta}$  and its conjugate  $\bar{\psi}_{\dot{\alpha}\dot{\beta}}$  are given by

$$\begin{split} \psi_{\alpha\beta} &= W_{\alpha\beta}{}^{\gamma}\nabla_{\gamma} + \nabla^{\gamma}W_{\alpha\beta}{}^{\delta}M_{\gamma\delta} - \frac{1}{8}\nabla^{2}W_{\alpha\beta\gamma}S^{\gamma} + \frac{i}{2}\nabla^{\gamma\dot{\gamma}}W_{\alpha\beta\gamma}\bar{S}_{\dot{\gamma}} \\ &+ \frac{1}{4}\nabla^{\gamma\dot{\delta}}\nabla_{(\alpha}W_{\beta)\gamma}{}^{\delta}K_{\delta\dot{\delta}} + \frac{1}{2}\nabla^{\gamma}W_{\alpha\beta\gamma}\mathbb{D} - \frac{3}{4}\nabla^{\gamma}W_{\alpha\beta\gamma}\mathbb{Y} , \qquad (3.20d) \\ \bar{\psi}_{\dot{\alpha}\dot{\beta}} &= -\bar{W}_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}}\bar{\nabla}_{\dot{\gamma}} - \bar{\nabla}^{\dot{\gamma}}\bar{W}_{\dot{\alpha}\dot{\beta}}{}^{\dot{\delta}}\bar{M}_{\dot{\gamma}\dot{\delta}} + \frac{1}{8}\bar{\nabla}^{2}\bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}\bar{S}^{\dot{\gamma}} + \frac{i}{2}\nabla^{\gamma\dot{\gamma}}\bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}S_{\gamma} \end{split}$$

$$-\frac{1}{4}\nabla^{\delta\dot{\gamma}}\bar{\nabla}_{(\dot{\alpha}}\bar{W}_{\dot{\beta})\dot{\gamma}}{}^{\dot{\delta}}K_{\delta\dot{\delta}} - \frac{1}{2}\bar{\nabla}^{\dot{\gamma}}\bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}\mathbb{D} - \frac{3}{4}\bar{\nabla}^{\dot{\gamma}}\bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}\mathbb{Y}.$$
 (3.20e)

The structure of the conformal superspace algebra leads to highly non-trivial implications. In particular, eq. (3.6c) implies that primary covariantly chiral superfields can carry only undotted spinor indices. Given such a superfield,  $\phi_{\alpha(n)}$ , eq. (3.6c) further implies that the U(1)<sub>R</sub> charge of  $\phi_{\alpha(n)}$  is determined in terms of its dimension,

$$K^{B}\phi_{\alpha(n)} = 0 , \quad \bar{\nabla}^{\dot{\beta}}\phi_{\alpha(n)} = 0 , \quad \mathbb{D}\phi_{\alpha(n)} = w\phi_{\alpha(n)} \implies c = -\frac{2}{3}w . \quad (3.21)$$

There is a regular procedure to construct such constrained multiplets. Given a complex tensor superfield  $\psi_{\alpha(n)}$  with the superconformal properties

$$K^{B}\psi_{\alpha(n)} = 0 , \quad \mathbb{D}\psi_{\alpha(n)} = (w-1)\psi_{\alpha(n)} , \quad \mathbb{Y}\psi_{\alpha(n)} = 2\left(1-\frac{1}{3}w\right)\psi_{\alpha(n)} , \quad (3.22)$$

its descendant

$$\phi_{\alpha(n)} = -\frac{1}{4}\bar{\nabla}^2\psi_{\alpha(n)} \tag{3.23}$$

proves to be primary and covariantly chiral. Here  $\phi_{\alpha(n)}$  is invariant under gauge transformations of the form  $\delta \psi_{\alpha(n)} = \bar{\nabla}_{\dot{\beta}} \lambda_{\alpha(n)}{}^{\dot{\beta}}$ , where the gauge parameter  $\lambda_{\alpha(n)}{}^{\dot{\beta}}$  is primary.

We note that the conformal superspace algebra is expressed in terms of a single superfield  $W_{\alpha\beta\gamma} = W_{(\alpha\beta\gamma)}$ , its conjugate  $\bar{W}_{\alpha\beta\gamma}$ , and their covariant derivatives. This superfield defines an  $\mathcal{N} = 1$  extension of the Weyl tensor; it is known as the super-Weyl tensor. Further, it is a primary chiral superfield of dimension 3/2

$$K^{D}W_{\alpha\beta\gamma} = 0 , \quad \bar{\nabla}^{\dot{\beta}}W_{\alpha\beta\gamma} = 0 , \quad \mathbb{D}W_{\alpha\beta\gamma} = \frac{3}{2}W_{\alpha\beta\gamma} , \qquad (3.24)$$

and it obeys the Bianchi identity

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$$B_{\alpha\dot{\alpha}} := i\nabla^{\beta}{}_{\dot{\alpha}}\nabla^{\gamma}W_{\alpha\beta\gamma} = i\nabla_{\alpha}{}^{\beta}\bar{\nabla}^{\dot{\gamma}}\bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} = \bar{B}_{\alpha\dot{\alpha}} .$$
(3.25)

Here we have defined the superfield  $B_{\alpha\dot{\alpha}}$ , which is the  $\mathcal{N} = 1$  supersymmetric generalisation of the Bach tensor introduced in [38] (see also [39]). The super-Bach tensor,  $B_{\alpha\dot{\alpha}}$ , proves to be primary,  $K^B B_{\alpha\dot{\alpha}} = 0$ , carries weight 3,  $\mathbb{D}B_{\alpha\dot{\alpha}} = 3B_{\alpha\dot{\alpha}}$ , and satisfies the conservation equation

$$\nabla^{\alpha}B_{\alpha\dot{\alpha}} = 0 \quad \Longleftrightarrow \quad \bar{\nabla}^{\dot{\alpha}}B_{\alpha\dot{\alpha}} = 0.$$
(3.26)

At this stage, it is still necessary to show that the conformal superspace geometry described in this subsection indeed encodes the Weyl multiplet of conformal supergravity. We will demonstrate this in two different ways. First, we will describe the procedure of reducing the results of this section to their component field description. Secondly, we will prove that this geometry is equivalent to the U(1), and consequently the Grimm-Wess-Zumino, superspace descriptions of conformal supergravity [31, 29].

## 3.3 Superconformal action principles

In order to formulate locally superconformal field theories, an action principle is required. As in the rigid supersymmetric case, locally superconformal actions can be constructed in two different ways: either as integrals over the full superspace or over its chiral subspace. Here we consider separately these two options.

We look for a scalar superfield  $\mathcal{L}$  such that the action

$$S = \int \mathrm{d}^{4|4} z E \,\mathscr{L} \,, \qquad \mathrm{d}^{4|4} z := \mathrm{d}^4 x \,\mathrm{d}^2 \theta \,\mathrm{d}^2 \bar{\theta} \tag{3.27}$$

is locally superconformal. Performing a gauge transformation, eqs. (3.7) and (3.14), we arrive at the variation

$$\delta_{\mathscr{H}}S = \int \mathrm{d}^{4|4}z E\left((-1)^{\varepsilon_{A}}\left[\nabla_{A}(\xi^{A}\mathscr{L}) + \xi^{B}\mathscr{T}_{BA}{}^{A}\mathscr{L}\right] + \Lambda^{\underline{b}}\left[(-1)^{\varepsilon_{A}}f_{\underline{b}A}{}^{A}\mathscr{L} + X_{\underline{b}}\mathscr{L}\right]\right), \qquad (3.28)$$

which must vanish. Now, requiring the contributions containing  $\Lambda^{\underline{b}}$  to vanish gives

$$X_{\underline{b}}\mathscr{L} = -(-1)^{\varepsilon_A} f_{\underline{b}} {}^A \mathscr{L} \quad \Longleftrightarrow \quad K^B \mathscr{L} = 0 , \quad \mathbb{D}\mathscr{L} = 2\mathscr{L} , \quad \mathbb{Y}\mathscr{L} = 0 .$$
(3.29)

Once the conditions (3.29) are satisfied, it is readily seen that the remaining  $\xi$ -dependent contributions in (3.28) cancel out. In summary, given a primary real dimension-2 scalar Lagrangian  $\mathscr{L}$ , the action (3.27) is locally superconformal.

Given a primary chiral scalar Lagrangian  $\mathcal{L}_{c}$  of weight +3,

$$K^{B}\mathscr{L}_{c} = 0 , \quad \bar{\nabla}_{\dot{\alpha}}\mathscr{L}_{c} = 0 , \quad \mathbb{D}\mathscr{L}_{c} = 3\mathscr{L}_{c} , \qquad (3.30)$$

the chiral action

$$S_{\rm c} = \int \mathrm{d}^4 x \mathrm{d}^2 \theta \, \mathscr{E} \, \mathscr{L}_{\rm c} \tag{3.31}$$

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is locally superconformal. Here  $\mathscr{E}$  is a chiral density. The precise definition of  $\mathscr{E}$  requires the use of a prepotential formulation for supergravity, see subsection 5.5.

A different definition of  $S_c$  exists, which is based on the use of a complex superfield  $\Upsilon$  with the following superconformal properties (for some constant  $\Delta$ )

$$K^{B}\Upsilon = 0$$
,  $\mathbb{D}\Upsilon = (\Delta - 1)\Upsilon$ ,  $\mathbb{Y}\Upsilon = 2\left(1 - \frac{1}{3}\Delta\right)\Upsilon$ , (3.32)

such that  $\overline{\nabla}^2 \Upsilon$  is nowhere vanishing, that is  $(\overline{\nabla}^2 \Upsilon)^{-1}$  exists. Specifically, the chiral action may be identified with the functional

$$S_{\rm c} = -4 \int \mathrm{d}^{4|4} z E \, \frac{\Upsilon}{\bar{\nabla}^2 \Upsilon} \, \mathscr{L}_{\rm c} \,, \qquad (3.33)$$

which possesses the two fundamental properties: (i) it is locally superconformal under the conditions (3.30); and (ii) it is independent of  $\Upsilon$ ,

$$\delta_{\Gamma} \int d^{4|4} z E \frac{\Upsilon}{\bar{\nabla}^2 \Upsilon} \mathscr{L}_{c} = 0 , \qquad (3.34)$$

for an arbitrary variation  $\delta \Upsilon$ . Using the representation (3.33) for the chiral action (3.31), it holds that

$$\int d^{4|4} z E \mathscr{L} = \int d^4 x d^2 \theta \mathscr{E} \mathscr{L}_{\rm c} , \qquad \mathscr{L}_{\rm c} = -\frac{1}{4} \bar{\nabla}^2 \mathscr{L} . \tag{3.35}$$

This result can also be obtained using superspace normal coordinates [32], see also [40].

There is an alternative definition of the chiral action that follows from the superform approach to the construction of supersymmetric invariants [41, 42]. It is based on the use of the following super 4-form

$$\Xi_{4} = 2i\bar{E}_{\hat{\delta}} \wedge \bar{E}_{\dot{\gamma}} \wedge E^{b} \wedge E^{a} (\tilde{\sigma}_{ab})^{\dot{\gamma}\dot{\delta}} \mathscr{L}_{c} + \frac{1}{6} \varepsilon_{abcd} \bar{E}_{\dot{\delta}} \wedge E^{c} \wedge E^{b} \wedge E^{a} (\tilde{\sigma}^{d})^{\dot{\delta}\delta} \nabla_{\alpha} \mathscr{L}_{c} - \frac{1}{96} \varepsilon_{abcd} E^{d} \wedge E^{c} \wedge E^{b} \wedge E^{a} \nabla^{2} \mathscr{L}_{c} , \qquad (3.36)$$

which was constructed by Binétruy *et al.* [43] and independently by Gates *et al.* [42] in the GWZ superspace. This superform is closed,

$$d \Xi_4 = 0$$
. (3.37)

It proves to be primary<sup>4</sup>

$$K^B \Xi_4 = 0. (3.38)$$

The chiral action (3.31) can be recast as an integral of  $\Xi_4$  over a spacetime  $\mathcal{M}^4$ ,

$$S_{\rm c} = \int_{\mathscr{M}^4} \Xi_4 , \qquad (3.39a)$$

where  $\mathcal{M}^4$  is the bosonic body of the curved superspace  $\mathcal{M}^{4|4}$  obtained by switching off the Grassmann variables. It turns out that (3.39a) leads to the representation

$$S_{\rm c} = \int \mathrm{d}^4 x \, e \left( -\frac{1}{4} \nabla^2 + \frac{\mathrm{i}}{2} (\tilde{\sigma}^a)^{\dot{\alpha}\alpha} \bar{\Psi}_{a\dot{\alpha}} \nabla_\alpha - (\tilde{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}} \bar{\Psi}_{a\dot{\alpha}} \bar{\Psi}_{b\dot{\beta}} \right) \mathscr{L}_{\rm c} \Big|_{\theta=0} \tag{3.39b}$$

which is the simplest way to reduce the action from superfields to components. Here  $\bar{\Psi}_{a\dot{\alpha}} = e_a{}^m \bar{\Psi}_{m\dot{\alpha}}$  is the gravitino defined in (3.40).

# 3.4 Component reduction and the Weyl multiplet

Having formulated conformal superspace in the preceding subsections, it is instructive to utilise it in reproducing the results of section 2.3. Specifically, we will briefly describe the Weyl multiplet of conformal supergravity (see e.g. [28, 33] for pedagogical reviews) and *Q*-supersymmetry transformations of the corresponding fields. As described earlier, the former involves a set of gauge one-forms: the vielbein  $e_m{}^a$ , gravitino  $\Psi_m{}^\alpha$ ,  $U(1)_R$  gauge field  $\mathfrak{A}_m$  and dilatation gauge field  $b_m$ . Modulo purely gauge degrees of freedom, they may be shown to be the only independent geometric fields and arise as the lowest components of the superforms

$$e_m{}^a := E_m{}^a |, \qquad \Psi_m{}^a := 2E_m{}^a |, \qquad \mathfrak{A}_m := \Phi_m |, \qquad b_m := B_m |, \qquad (3.40)$$

where the bar projection of a superfield  $\Xi(x, \theta, \overline{\theta})$  is defined by  $\Xi | := \Xi |_{\theta = \overline{\theta} = 0}$ .

It remains to compute the *Q*-supersymmetry transformations of the fields (3.40) and show that they do indeed form the Weyl multiplet. By employing (3.9), their transformations when  $\mathscr{K}(\xi) = \xi^{\alpha} \nabla_{\alpha} + \bar{\xi}_{\dot{\alpha}} \bar{\nabla}^{\dot{\alpha}}$  may be shown to be:

$$\delta_{\mathscr{K}(\varepsilon)} e_m{}^a = \mathrm{i}(\varepsilon \sigma^a \bar{\Psi}_m - \Psi_m \sigma^a \bar{\varepsilon}) , \qquad (3.41a)$$

$$\delta_{\mathscr{K}(\varepsilon)}\Psi_m{}^{\alpha} = 2\hat{\nabla}_m \varepsilon^{\alpha} , \qquad (3.41b)$$

$$\delta_{\mathscr{K}(\varepsilon)}\mathfrak{A}_{m} = 3\mathrm{i}(\mathfrak{F}_{m}{}^{\alpha}|\varepsilon_{\alpha} - \bar{\mathfrak{F}}_{m\dot{\alpha}}|\bar{\varepsilon}^{\dot{\alpha}}), \qquad (3.41\mathrm{c})$$

$$\delta_{\mathscr{K}(\varepsilon)}b_m = 2(\mathfrak{F}_m{}^\alpha|\varepsilon_\alpha + \bar{\mathfrak{F}}_{m\dot{\alpha}}|\bar{\varepsilon}^{\dot{\alpha}}) , \qquad (3.41d)$$

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<sup>&</sup>lt;sup>4</sup> The superform may be degauged to the GWZ superspace described in the next section. Then the condition (3.38) is equivalent to the super-Weyl invariance of  $\Xi_4$ . The latter property was proved in [44].

where we have denoted  $\varepsilon^{\alpha} := \xi^{\alpha} |$  and  $\hat{\nabla}_m$  was defined in (2.45). Additionally, by a routine analysis of the spinor torsion two-form  $\mathscr{T}_{mn}^{\alpha} |$ , it may be shown that

$$(\sigma^{mn})^{\beta\gamma}\hat{\nabla}_{m}\Psi_{n}^{\alpha} + 2i\varepsilon^{\alpha(\beta}(\sigma^{m})^{\gamma)}{}_{\dot{\alpha}}\bar{\mathfrak{F}}_{m}^{\dot{\alpha}}| = W^{\alpha\beta\gamma}|, \qquad (3.42a)$$

which implies:

$$W_{\alpha\beta\gamma}| = (\sigma^{mn})_{(\alpha\beta}\hat{\nabla}_m \Psi_{n\gamma}) , \qquad (3.42b)$$

$$\mathfrak{F}_{m}{}^{\alpha}| = \frac{1}{3} \hat{\nabla}_{[m} \bar{\Psi}_{n]\dot{\alpha}} (\tilde{\sigma}^{n})^{\dot{\alpha}\alpha} - \frac{1}{12} g_{mn} \varepsilon^{nijk} \hat{\nabla}_{i} \bar{\Psi}_{j\dot{\alpha}} (\tilde{\sigma}_{k})^{\dot{\alpha}\alpha} . \tag{3.42c}$$

Then, upon inserting (3.42c) into (3.41), the transformations coincide with (2.44).

To conclude, it should be noted that the dilatation gauge field  $b_m$  describes purely gauge degrees of freedom. This may be seen by noting that, according to (3.9), it transforms algebraically when  $\mathcal{K}(\Lambda) = \Lambda_a K^a$ 

$$\delta_{\mathscr{K}(\Lambda)}b_m = -2\Lambda_m | . \tag{3.43}$$

Hence, we impose the gauge  $b_m = 0$  by fixing the special conformal gauge freedom. It should also be noted that the remaining fields appearing in (3.40) are inert under such transformations. Additionally, in order to preserve the gauge  $b_m = 0$ , each Q-supersymmetry transformation (3.41) must be accompanied with a compensating special conformal transformation (3.43) with  $\Lambda_m(\varepsilon) = \tilde{\mathfrak{F}}_m{}^{\alpha} |\varepsilon_{\alpha} + \tilde{\mathfrak{F}}_{m\dot{\alpha}}| \bar{\varepsilon}^{\dot{\alpha}}$ . As a result, we have shown that the fields  $\{e_m{}^a, \Psi_{m\alpha}, \bar{\Psi}_m{}^{\dot{\alpha}}, \mathfrak{A}_m\}$  do indeed constitute the reduced Weyl multiplet introduced in subsection 2.3.

#### 4 Other superspace formulations for conformal supergravity

As pointed out in section 1, conformal superspace is not the only superspace setting to describe conformal supergravity. The other most popular formulations are: (i) U(1) superspace [31]; and (ii) the GWZ superspace [29]. They differ by their structure groups, which are  $SL(2, \mathbb{C}) \times U(1)_R$  and  $SL(2, \mathbb{C})$ , respectively. Both of them can be derived from conformal superspace. Below we describe the relevant degauging procedures.

## 4.1 The U(1) superspace geometry

According to (3.7), under an infinitesimal special superconformal gauge transformation  $\mathcal{K} = \Lambda_B K^B$ , the dilatation connection transforms as follows

$$\delta_{\mathscr{K}}B_A = -2\Lambda_A \ . \tag{4.1}$$

Thus, it is possible to choose a gauge condition  $B_A = 0$ , which completely fixes the special superconformal gauge freedom.<sup>5</sup> As a result, the corresponding connection is no longer required for the covariance of  $\nabla_A$  under the residual gauge freedom and may be extracted from  $\nabla_A$ ,

$$\nabla_A = \mathfrak{D}_A - \mathfrak{F}_{AB} K^B \,. \tag{4.2}$$

Here the operator  $\mathfrak{D}_A$  involves only the Lorentz and  $U(1)_R$  connections.

The next step is to relate the special superconformal connection  $\mathfrak{F}_{AB}$  to the torsion tensor of U(1) superspace. To do this, it is necessary to make use of the relation

$$[\nabla_A, \nabla_B\} = [\mathfrak{D}_A, \mathfrak{D}_B\} - (\mathfrak{D}_A \mathfrak{F}_{BC} - (-1)^{AB} \mathfrak{D}_B \mathfrak{F}_{AC}) K^C - \mathfrak{F}_{AC} [K^C, \nabla_B] + (-1)^{AB} \mathfrak{F}_{BC} [K^C, \nabla_A] + (-1)^{BC} \mathfrak{F}_{AC} \mathfrak{F}_{BD} [K^D, K^C] .$$
(4.3)

In conjunction with (3.20), this relation leads to a set of consistency conditions that are equivalent to the Bianchi identities of U(1) superspace [31]. Their solution expresses the components of  $\mathfrak{F}_{AB}$  in terms of the torsion tensor of U(1) superspace. We will not provide a detailed analysis for this step and instead refer the reader to the proof in [32]. The outcome of the analysis is as follows:

$$\mathfrak{F}_{\alpha\beta} = \frac{1}{2} \varepsilon_{\alpha\beta} \bar{R} , \quad \bar{\mathfrak{F}}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2} \varepsilon_{\dot{\alpha}\dot{\beta}} R , \quad \mathfrak{F}_{\alpha\dot{\beta}} = -\bar{\mathfrak{F}}_{\dot{\beta}\alpha} = \frac{1}{4} G_{\alpha\dot{\beta}} , \quad (4.4a)$$

$$\mathfrak{F}_{\alpha,\beta\dot{\beta}} = -\frac{1}{4}\mathfrak{D}_{\alpha}G_{\beta\dot{\beta}} - \frac{1}{6}\varepsilon_{\alpha\beta}\bar{X}_{\dot{\beta}} = \mathfrak{F}_{\beta\dot{\beta},\alpha} , \qquad (4.4b)$$

$$\bar{\mathfrak{F}}_{\dot{\alpha},\beta\dot{\beta}} = \frac{1}{4}\bar{\mathfrak{D}}_{\alpha}G_{\beta\dot{\beta}} + \frac{1}{6}\varepsilon_{\dot{\alpha}\dot{\beta}}X_{\beta} = \mathfrak{F}_{\beta\dot{\beta},\alpha} , \qquad (4.4c)$$

$$\widetilde{\mathfrak{F}}_{\alpha\dot{\alpha},\beta\dot{\beta}} = -\frac{1}{8} \left[ \mathfrak{D}_{\alpha}, \widetilde{\mathfrak{D}}_{\dot{\alpha}} \right] G_{\beta\dot{\beta}} - \frac{1}{12} \varepsilon_{\dot{\alpha}\dot{\beta}} \mathfrak{D}_{\alpha} X_{\beta} + \frac{1}{12} \varepsilon_{\alpha\beta} \overline{\mathfrak{D}}_{\dot{\alpha}} \bar{X}_{\dot{\beta}} \\
+ \frac{1}{2} \varepsilon_{\alpha\beta} \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{R} R + \frac{1}{8} G_{\alpha\dot{\beta}} G_{\beta\dot{\alpha}} ,$$
(4.4d)

where *R* and  $X_{\alpha}$  are complex chiral

$$\bar{\mathfrak{D}}_{\dot{\alpha}}R = 0, \qquad \Im R = -2R, \qquad (4.5a)$$

$$\bar{\mathfrak{D}}_{\dot{\alpha}} X_{\alpha} = 0 , \qquad \mathbb{Y} X_{\alpha} = -X_{\alpha} , \qquad (4.5b)$$

while  $G_{\alpha\dot{\alpha}}$  is a real vector superfield. These are related via

$$X_{\alpha} = \mathfrak{D}_{\alpha} R - \bar{\mathfrak{D}}^{\dot{\alpha}} G_{\alpha \dot{\alpha}} . \tag{4.5c}$$

We now pause and comment on the geometry described by  $\mathfrak{D}_A$ . In particular, by employing (4.3) one arrives at the following anti-commutation relation

<sup>&</sup>lt;sup>5</sup> There is a class of residual gauge transformations preserving the gauge  $B_A = 0$ . These generate the super-Weyl transformations of U(1) superspace, see the next subsection.

$$\{\mathfrak{D}_{\alpha},\bar{\mathfrak{D}}_{\dot{\alpha}}\} = -2\mathrm{i}\mathfrak{D}_{\alpha\dot{\alpha}} - G^{\beta}{}_{\dot{\alpha}}M_{\alpha\beta} + G_{\alpha}{}^{\dot{\beta}}\bar{M}_{\dot{\alpha}\dot{\beta}} + \frac{3}{2}G_{\alpha\dot{\alpha}}\mathbb{Y}.$$
(4.6)

With the goal of simplicity in performing calculations within U(1) superspace, we prefer to work with a geometry where the right hand side of (4.6) contains no curvature-dependent terms. To this end, we perform the following redefinition

$$\mathfrak{D}_{\alpha\dot{\alpha}} = \mathscr{D}_{\alpha\dot{\alpha}} + \frac{\mathrm{i}}{2} G^{\beta}{}_{\dot{\alpha}} M_{\alpha\beta} - \frac{\mathrm{i}}{2} G_{\alpha}{}^{\dot{\beta}} \bar{M}_{\dot{\alpha}\dot{\beta}} - \frac{3\mathrm{i}}{4} G_{\alpha\dot{\alpha}} \mathbb{Y} , \qquad (4.7a)$$

$$\mathfrak{D}_{\alpha} = \mathscr{D}_{\alpha} , \qquad \bar{\mathfrak{D}}^{\dot{\alpha}} = \bar{\mathscr{D}}^{\dot{\alpha}} , \qquad (4.7b)$$

where  $\mathcal{D}_A$  takes the form

$$\mathcal{D}_{A} = E_{A} - \frac{1}{2}\hat{\Omega}_{A}{}^{bc}M_{bc} - i\,\hat{\Phi}_{A}\mathbb{Y}$$
$$= E_{A} - \hat{\Omega}_{A}{}^{\beta\gamma}M_{\beta\gamma} - \hat{\Omega}_{A}{}^{\beta\dot{\gamma}}\bar{M}_{\dot{\beta}\dot{\gamma}} - i\,\hat{\Phi}_{A}\mathbb{Y}.$$
(4.8)

Here we have attached a hat to each connection superfield to distinguish them from their cousins residing in the conformal covariant derivative  $\nabla_A$ . In what follows, these hats will be omitted.

Now it may be shown that the algebra obeyed by  $\mathcal{D}_A$  takes the form

$$\{\mathscr{D}_{\alpha}, \mathscr{D}_{\beta}\} = -4\bar{R}M_{\alpha\beta} , \qquad \{\bar{\mathscr{D}}_{\dot{\alpha}}, \bar{\mathscr{D}}_{\dot{\beta}}\} = 4R\bar{M}_{\dot{\alpha}\dot{\beta}} , \qquad (4.9a)$$

$$\{\mathscr{D}_{\alpha}, \mathscr{D}_{\dot{\alpha}}\} = -2i\mathscr{D}_{\alpha\dot{\alpha}} , \qquad (4.9b)$$

$$\begin{split} \left[\mathscr{D}_{\alpha},\mathscr{D}_{\beta\dot{\beta}}\right] &= \mathrm{i}\varepsilon_{\alpha\beta}\left(\bar{R}\bar{\mathscr{D}}_{\dot{\beta}} + G^{\gamma}{}_{\beta}\mathscr{D}_{\gamma} - (\mathscr{D}^{\gamma}G^{\delta}{}_{\dot{\beta}})M_{\gamma\delta} + 2\bar{W}_{\dot{\beta}}{}^{\gamma\delta}\bar{M}_{\dot{\gamma}\delta}\right) \\ &+ \mathrm{i}(\bar{\mathscr{D}}_{\beta}\bar{R})M_{\alpha\beta} - \frac{\mathrm{i}}{3}\varepsilon_{\alpha\beta}\bar{X}^{\dot{\gamma}}\bar{M}_{\dot{\gamma}\dot{\beta}} - \frac{\mathrm{i}}{2}\varepsilon_{\alpha\beta}\bar{X}_{\dot{\beta}}\mathbb{Y} , \end{split}$$
(4.9c)

$$\begin{bmatrix} \bar{\mathscr{D}}_{\dot{\alpha}}, \mathscr{D}_{\beta\dot{\beta}} \end{bmatrix} = -\mathrm{i}\varepsilon_{\alpha\dot{\beta}} \left( R \,\mathscr{D}_{\beta} + G_{\beta}{}^{\dot{\gamma}} \bar{\mathscr{D}}_{\dot{\gamma}} - (\bar{\mathscr{D}}^{\dot{\gamma}} G_{\beta}{}^{\dot{\delta}}) \bar{M}_{\dot{\gamma}\dot{\delta}} + 2W_{\beta}{}^{\gamma\delta} M_{\gamma\delta} \right) -\mathrm{i}(\mathscr{D}_{\beta}R) \bar{M}_{\alpha\dot{\beta}} + \frac{\mathrm{i}}{3}\varepsilon_{\alpha\dot{\beta}} X^{\gamma} M_{\gamma\beta} - \frac{\mathrm{i}}{2}\varepsilon_{\alpha\dot{\beta}} X_{\beta} \mathbb{Y} , \qquad (4.9d)$$

which lead to

$$\begin{split} \mathscr{D}_{\alpha\dot{\alpha}}, \mathscr{D}_{\beta\dot{\beta}} \bigg] &= \varepsilon_{\alpha\beta} \bar{\chi}_{\dot{\alpha}\dot{\beta}} + \varepsilon_{\dot{\alpha}\dot{\beta}} \chi_{\alpha\beta} , \qquad (4.9e) \\ \chi_{\alpha\beta} &= -\mathrm{i} G_{(\alpha}{}^{\dot{\gamma}} \mathscr{D}_{\beta)\dot{\gamma}} + \frac{1}{2} \mathscr{D}_{(\alpha} R \mathscr{D}_{\beta)} + \frac{1}{2} \mathscr{D}_{(\alpha} G_{\beta)}{}^{\dot{\gamma}} \bar{\mathscr{D}}_{\dot{\gamma}} + W_{\alpha\beta}{}^{\gamma} \mathscr{D}_{\gamma} \\ &+ \frac{1}{6} X_{(\alpha} \mathscr{D}_{\beta)} + \frac{1}{4} (\mathscr{D}^2 - 8R) \bar{R} M_{\alpha\beta} + \mathscr{D}_{(\alpha} W_{\beta)}{}^{\gamma\delta} M_{\gamma\delta} \\ &- \frac{1}{6} \mathscr{D}_{(\alpha} X^{\gamma} M_{\beta)\gamma} - \frac{1}{2} \mathscr{D}_{(\alpha} \bar{\mathscr{D}}^{\dot{\gamma}} G_{\beta)}{}^{\dot{\delta}} \bar{M}_{\dot{\gamma}\dot{\delta}} + \frac{1}{4} \mathscr{D}_{(\alpha} X_{\beta)} \mathbb{Y} , \quad (4.9f) \\ \bar{\chi}_{\dot{\alpha}\dot{\beta}} &= \mathrm{i} G^{\gamma}{}_{(\alpha} \mathscr{D}_{\gamma\dot{\beta})} - \frac{1}{2} \bar{\mathscr{D}}_{(\dot{\alpha}} \bar{R} \bar{\mathscr{D}}_{\dot{\beta})} - \frac{1}{2} \bar{\mathscr{D}}_{(\dot{\alpha}} G^{\gamma}{}_{\dot{\beta})} \mathscr{D}_{\gamma} - \bar{W}_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}} \bar{\mathscr{D}}_{\dot{\gamma}} \\ &- \frac{1}{6} \bar{X}_{(\dot{\alpha}} \bar{\mathscr{D}}_{\dot{\beta})} + \frac{1}{4} (\bar{\mathscr{D}}^2 - 8\bar{R}) R \bar{M}_{\dot{\alpha}\dot{\beta}} - \bar{\mathscr{D}}_{(\dot{\alpha}} \bar{W}_{\dot{\beta})}{}^{\dot{\gamma}\dot{\delta}} \bar{M}_{\dot{\gamma}\dot{\delta}} \end{split}$$

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$$+\frac{1}{6}\bar{\mathscr{D}}_{(\dot{\alpha}}\bar{X}^{\dot{\gamma}}\bar{M}_{\dot{\beta}})_{\dot{\gamma}}+\frac{1}{2}\bar{\mathscr{D}}_{(\dot{\alpha}}\mathscr{D}^{\gamma}G^{\delta}{}_{\dot{\beta}})M_{\gamma\delta}+\frac{1}{4}\bar{\mathscr{D}}_{(\dot{\alpha}}\bar{X}_{\dot{\beta}})^{\mathbb{Y}}.$$
 (4.9g)

These relations should be supplemented with the following Bianchi identities:

$$\mathscr{D}^{\alpha}X_{\alpha} = \bar{\mathscr{D}}_{\dot{\alpha}}\bar{X}^{\dot{\alpha}} , \qquad (4.10a)$$

$$\bar{\mathscr{D}}_{\dot{\alpha}}W_{\alpha\beta\gamma} = 0 , \qquad (4.10b)$$

$$\mathscr{D}^{\gamma}W_{\alpha\beta\gamma} = i\mathscr{D}_{(\alpha}{}^{\dot{\gamma}}G_{\beta)\dot{\gamma}} - \frac{1}{3}\mathscr{D}_{(\alpha}X_{\beta)} . \qquad (4.10c)$$

In particular, it should be noted that (4.10a) implies  $X_{\alpha}$  is the chiral field strength of a U(1) vector multiplet. The geometry described above is the U(1) superspace geometry [31, 13] in the form described in [45, 46].

To conclude our discussion of the U(1) superspace geometry, we make two comments. Firstly, one may check that degauging the relation (3.23) gives

$$\phi_{\alpha(n)} = -\frac{1}{4} \left( \bar{\mathscr{D}}^2 - 4R \right) \psi_{\alpha(n)} , \qquad \bar{\mathscr{D}}_{\dot{\beta}} \phi_{\alpha(n)} = 0 .$$

$$(4.11)$$

Secondly, integration by parts is remarkably simple in U(1) superspace:

$$\int d^{4|4} z E (-1)^{\varepsilon_A} \mathscr{D}_A \mathscr{V}^A = 0 , \qquad (4.12)$$

where  $\mathscr{V}^A$  is arbitrary.<sup>6</sup>

# 4.2 The super-Weyl transformations of U(1) superspace

In the previous subsection we made use of the special conformal gauge freedom to degauge from conformal to U(1) superspace. Here we will show that the residual dilatation symmetry manifests in the latter as super-Weyl transformations.

Specifically, to preserve the gauge  $B_A = 0$ , every local dilatation transformation with parameter  $\Sigma$  should be accompanied by a compensating special conformal one,  $\Lambda^B(\Sigma)$ 

$$\mathscr{K}(\Sigma) = \Lambda_B(\Sigma)K^B + \Sigma \mathbb{D} \quad \Longrightarrow \quad \delta_{\mathscr{K}(\Sigma)}B_A = 0.$$
 (4.13)

We then arrive at the following constraints

$$\Lambda_A(\Sigma) = \frac{1}{2} \nabla_A \Sigma . \qquad (4.14)$$

As a result, we define the following transformation

$$\delta_{\Sigma} \nabla_{A} = \delta_{\Sigma} \mathfrak{D}_{A} - \delta_{\Sigma} (\mathfrak{F}_{AB} K^{B}) = [\mathscr{K}(\Sigma), \nabla_{A}].$$
(4.15)

<sup>&</sup>lt;sup>6</sup> In conformal superspace, integration by parts requires special care [32].

By making use of (4.7) and (4.4), we arrive at the following transformation laws for the U(1) superspace covariant derivatives

$$\delta_{\Sigma} \mathscr{D}_{\alpha} = \frac{1}{2} \Sigma \mathscr{D}_{\alpha} + 2 \mathscr{D}^{\beta} \Sigma M_{\beta \alpha} - \frac{3}{2} \mathscr{D}_{\alpha} \Sigma \mathbb{Y} , \qquad (4.16a)$$

$$\delta_{\Sigma}\bar{\mathscr{D}}_{\dot{\alpha}} = \frac{1}{2}\Sigma\bar{\mathscr{D}}_{\dot{\alpha}} + 2\bar{\mathscr{D}}^{\dot{\beta}}\Sigma\bar{M}_{\dot{\beta}\dot{\alpha}} + \frac{3}{2}\bar{\mathscr{D}}_{\dot{\alpha}}\Sigma\mathbb{Y} , \qquad (4.16b)$$

$$\begin{split} \delta_{\Sigma} \mathscr{D}_{\alpha \dot{\alpha}} &= \Sigma \mathscr{D}_{\alpha \dot{\alpha}} + \mathrm{i} \mathscr{D}_{\alpha} \Sigma \bar{\mathscr{D}}_{\dot{\alpha}} + \mathrm{i} \bar{\mathscr{D}}_{\dot{\alpha}} \Sigma \mathscr{D}_{\alpha} + \mathrm{i} \bar{\mathscr{D}}_{\dot{\alpha}} \mathscr{D}^{\beta} \Sigma M_{\beta \alpha} \\ &+ \mathrm{i} \mathscr{D}_{\alpha} \bar{\mathscr{D}}^{\dot{\beta}} \Sigma \bar{M}_{\dot{\beta} \dot{\alpha}} + \frac{3}{4} \mathrm{i} \left[ \mathscr{D}_{\alpha}, \bar{\mathscr{D}}_{\dot{\alpha}} \right] \Sigma \mathbb{Y} , \end{split}$$
(4.16c)

while the torsion superfields arising from the degauged torsion  $\mathfrak{F}_{AB}$  transform as follows

$$\delta_{\Sigma} R = \Sigma R + \frac{1}{2} \bar{\mathscr{D}}^2 \Sigma , \qquad (4.17a)$$

$$\delta_{\Sigma} G_{\alpha \dot{\alpha}} = \Sigma G_{\alpha \dot{\alpha}} + [\mathscr{D}_{\alpha}, \bar{\mathscr{D}}_{\dot{\alpha}}] \Sigma , \qquad (4.17b)$$

$$\delta_{\Sigma} X_{\alpha} = \frac{3}{2} \Sigma X_{\alpha} - \frac{3}{2} (\bar{\mathscr{D}}^2 - 4R) \mathscr{D}_{\alpha} \Sigma . \qquad (4.17c)$$

Finally, as the super-Weyl tensor is conformally covariant, its transformation law is readily obtained via

$$\delta_{\Sigma} W_{\alpha\beta\gamma} = \left( \Lambda_B(\Sigma) K^B + \Sigma \mathbb{D} \right) W_{\alpha\beta\gamma} = \frac{3}{2} \Sigma W_{\alpha\beta\gamma} \,. \tag{4.17d}$$

The relations (4.16) and (4.17) give the super-Weyl transformations in U(1) superspace [13, 31] (see also [45, 46]). The conditions (3.15), which define a primary superfield U, are equivalent to the following

$$\delta_{\Sigma} U = w \Sigma U , \qquad \forall U = c U , \qquad (4.18)$$

in U(1) superspace.

The U(1) superspace formulation was fully developed in the book [13] in which various applications were also given.

#### 4.3 The Grimm-Wess-Zumino formulation

As pointed out in section 4.1, the covariantly chiral spinor  $X_{\alpha}$  is the field strength of an Abelian vector multiplet. It follows from (4.17c) that the super-Weyl gauge freedom allows us to choose the gauge

$$X_{\alpha} = 0. \qquad (4.19)$$

In this gauge the  $U(1)_R$  curvature vanishes, in accordance with (4.9), and therefore the  $U(1)_R$  connection may be gauged away,

$$\Phi_A = 0. \tag{4.20}$$

Then, the algebra of covariant derivatives (4.9) reduces to that describing the GWZ geometry [29].

Equation (4.17c) tells us that imposing the condition  $X_{\alpha} = 0$  does not fix completely the super-Weyl freedom. The residual transformations are generated by parameters of the form

$$\Sigma = \frac{1}{2} (\sigma + \bar{\sigma}) , \qquad \bar{\mathscr{D}}_{\dot{\alpha}} \sigma = 0 . \qquad (4.21)$$

However, in order to preserve the  $U(1)_R$  gauge  $\Phi_A = 0$ , every residual super-Weyl transformation (4.21) must be accompanied by a compensating  $U(1)_R$  transformation with

$$\rho = \frac{3}{4}i(\sigma - \bar{\sigma}) . \tag{4.22}$$

This leads to the transformation [47]

$$\delta_{\sigma} \mathscr{D}_{\alpha} = (\bar{\sigma} - \frac{1}{2}\sigma) \mathscr{D}_{\alpha} + (\mathscr{D}^{\beta}\sigma) M_{\alpha\beta} , \qquad (4.23a)$$

$$\delta_{\sigma}\bar{\mathscr{D}}_{\dot{\alpha}} = (\sigma - \frac{1}{2}\bar{\sigma})\bar{\mathscr{D}}_{\dot{\alpha}} + (\bar{\mathscr{D}}^{\dot{\beta}}\bar{\sigma})\bar{M}_{\dot{\alpha}\dot{\beta}} , \qquad (4.23b)$$

$$\delta_{\sigma} \mathscr{D}_{\alpha\dot{\alpha}} = \frac{1}{2} (\sigma + \bar{\sigma}) \mathscr{D}_{\alpha\dot{\alpha}} + \frac{i}{2} (\bar{\mathscr{D}}_{\dot{\alpha}} \bar{\sigma}) \mathscr{D}_{\alpha} + \frac{i}{2} (\mathscr{D}_{\alpha} \sigma) \bar{\mathscr{D}}_{\dot{\alpha}} + (\mathscr{D}^{\beta}{}_{\dot{\alpha}} \sigma) M_{\alpha\beta} + (\mathscr{D}_{\alpha}{}^{\dot{\beta}} \bar{\sigma}) \bar{M}_{\dot{\alpha}\dot{\beta}} .$$
(4.23c)

The torsion tensors transform as follows:

$$\delta_{\sigma}R = 2\sigma R + \frac{1}{4}(\bar{\mathscr{D}}^2 - 4R)\bar{\sigma} , \qquad (4.24a)$$

$$\delta_{\sigma}G_{\alpha\dot{\alpha}} = \frac{1}{2}(\sigma + \bar{\sigma})G_{\alpha\dot{\alpha}} + i\mathscr{D}_{\alpha\dot{\alpha}}(\sigma - \bar{\sigma}), \qquad (4.24b)$$

$$\delta_{\sigma} W_{\alpha\beta\gamma} = \frac{3}{2} \sigma W_{\alpha\beta\gamma} \,. \tag{4.24c}$$

The conditions (4.18), which define a primary superfield U, turn into

$$\delta_{\sigma}U = (p\sigma + q\bar{\sigma})U, \qquad p + q = w, \quad p - q = -\frac{3}{2}c, \qquad (4.25)$$

in the GWZ approach.

Let us fix a background curved superspace  $(\mathcal{M}^{4|4}, \mathcal{D})$ . A supervector field  $\xi = \xi^B E_B$  on this superspace is called *conformal Killing* if there exists a Lorentz parameter  $K^{bc}[\xi]$  and a super-Weyl chiral parameter  $\sigma[\xi]$  such that

$$\left[\xi^{B}\mathscr{D}_{B} + \frac{1}{2}K^{bc}[\xi]M_{bc}, \mathscr{D}_{A}\right] + \delta_{\sigma[\xi]}\mathscr{D}_{A} = 0.$$

$$(4.26)$$

In other words, the coordinate transformation generated by  $\xi$  is accompanied by certain Lorentz and super-Weyl transformations such that the superspace geometry does not change. It can be shown [35] that the equation (4.26) uniquely determines the spinor components of  $\xi^B = (\xi^b, \xi^\beta, \bar{\xi}_\beta)$  and the parameters  $K^{bc}[\xi]$  and  $\sigma[\xi]$  in terms of  $\xi^b$ , and the latter obeys the equation

$$\mathscr{D}_{(\alpha}\xi_{\beta)\dot{\beta}} = 0 \quad \Longleftrightarrow \quad \bar{\mathscr{D}}_{(\dot{\alpha}}\xi_{\beta\dot{\beta}}) = 0.$$
(4.27)

The set of all conformal Killing supervector fields on  $(\mathcal{M}^{4|4}, \mathcal{D})$  constitutes the superconformal algebra of  $(\mathcal{M}^{4|4}, \mathcal{D})$ . Given a super-Weyl invariant theory on  $(\mathcal{M}^{4|4}, \mathcal{D})$  described by primary superfields U, its action is invariant under the superconformal transformations

$$\delta_{\xi}U = \mathscr{K}[\xi]U, \quad \mathscr{K}[\xi] = \xi^{B}\mathscr{D}_{B} + \frac{1}{2}K^{bc}[\xi]M_{bc} + p\sigma[\xi] + q\bar{\sigma}[\xi], \quad (4.28)$$

for an arbitrary conformal Killing supervector field  $\xi$ . In the case that  $(\mathcal{M}^{4|4}, \mathcal{D})$  coincides with Minkowski superspace,  $(\mathbb{M}^{4|4}, D)$ , the superconformal Killing equation (4.26) is equivalent to (2.6) and the transformation law (4.28) to (2.22).

The GWZ formulation has been used in most applications of  $\mathcal{N} = 1$  superfield supergravity. It is reviewed in several textbook, see, e.g., [17, 35, 48].

#### **5** Supergravity prepotentials

The constraints on the GWZ geometry [8, 29] were solved by Siegel [18] in terms of unconstrained prepotentials. This solution was extended to non-minimal supergravity ( $n \neq -1/3,0$ ) by Gates and Siegel [6], and then to U(1) superspace in the book [13]. Here we review the original solution given in [18]. The prepotential description of conformal superspace was worked out in [32], and the interested reader is referred to the original publication.

The covariant derivatives have the form

$$\mathscr{D}_A = E_A - \frac{1}{2} \Omega_A{}^{bc} M_{bc} \tag{5.1}$$

and obey the graded commutation relations

$$\left[\mathscr{D}_{A},\mathscr{D}_{B}\right] = -\mathscr{T}_{AB}{}^{C}\mathscr{D}_{C} - \frac{1}{2}\mathscr{R}_{AB}{}^{cd}M_{cd} , \qquad (5.2)$$

where the torsion and curvature tensors are read off from (4.9) by setting  $X_{\alpha} = 0$ .

The gauge group of conformal supergravity is generated by the general coordinate ( $K^N$ ), local Lorentz ( $K^{bc}$ ) and super-Weyl ( $\sigma$  and  $\bar{\sigma}$ ) transformations. The

combined general coordinate and Local Lorentz transformation acts on  $\mathscr{D}_A$  and a tensor superfield  $\mathscr{T}$  (with suppressed indices) by the rule

$$\mathscr{D}'_{A} = \mathrm{e}^{\mathscr{K}} \mathscr{D}_{A} \mathrm{e}^{-\mathscr{K}} , \qquad \mathscr{T}' = \mathrm{e}^{\mathscr{K}} \mathscr{T} , \qquad \mathscr{K} = K^{N} \partial_{N} + \frac{1}{2} K^{bc} M_{bc} .$$
 (5.3)

The super-Weyl transformation of  $\mathscr{D}_A$  is given by eq. (4.23), with the parameter  $\sigma$  being covariantly chiral. Given a primary tensor superfield U, its super-Weyl transformation law is given by eq. (4.25).

#### 5.1 Spinor covariant derivatives

Nontrivial information is contained in the relations (4.9a) and (4.9b). First of all, the spinor components of the connection  $\Omega_A{}^{bc} = (\Omega_a{}^{bc}, \Omega_{\alpha}{}^{bc}, \bar{\Omega}{}^{\dot{\alpha}bc})$  are determined in terms of the anholonomy coefficients,  $C_{AB}{}^C$ , defined by

$$\left[E_A, E_B\right] = C_{AB}{}^C E_C \ . \tag{5.4}$$

In particular, for  $\frac{1}{2}\Omega_{\alpha}{}^{bc}M_{bc} = \Omega_{\alpha}{}^{\beta\gamma}M_{\beta\gamma} + \Omega_{\alpha}{}^{\dot{\beta}\dot{\gamma}}\bar{M}_{\dot{\beta}\dot{\gamma}}$  we obtain

$$\Omega_{\alpha\beta\gamma} = \frac{1}{2} \Big( C_{\alpha\beta\gamma} + C_{\alpha\gamma\beta} - C_{\beta\gamma\alpha} \Big) , \qquad \Omega_{\alpha\dot{\beta}\dot{\gamma}} = -C_{\alpha\dot{\beta}\dot{\gamma}} . \tag{5.5}$$

Secondly, since the curvature  $R_{\alpha\beta}{}^{\dot{\beta}\dot{\gamma}}\bar{M}_{\dot{\beta}\dot{\gamma}}$  vanishes, the connection  $\Omega_{\alpha}{}^{\dot{\beta}\dot{\gamma}}\bar{M}_{\dot{\beta}\dot{\gamma}}$  is flat,

$$\Omega_{\alpha}{}^{\dot{\beta}\dot{\gamma}}\bar{M}_{\dot{\beta}\dot{\gamma}} = -\bar{g}^{-1}E_{\alpha}\bar{g} , \qquad \bar{g} = \exp\left(\bar{L}^{\dot{\beta}\dot{\gamma}}\bar{M}_{\dot{\beta}\dot{\gamma}}\right) , \qquad (5.6a)$$

$$\bar{\Omega}_{\dot{\alpha}}{}^{\beta\gamma}M_{\beta\gamma} = -g^{-1}\bar{E}_{\dot{\alpha}}g , \qquad g = \exp\left(L^{\beta\gamma}M_{\beta\gamma}\right).$$
(5.6b)

It follows from (4.9a) that the spinor components  $E_{\alpha}$  of the inverse supervielbein  $E_A$  form a closed algebra in the sense that  $\{E_{\alpha}, E_{\beta}\} = C_{\alpha\beta}{}^{\gamma}E_{\gamma}$ . Then the Frobenius theorem implies that  $E_{\alpha}$  is a linear combination of coordinate supervector fields,

$$E_{\alpha} = F N_{\alpha}{}^{\mu} \hat{E}_{\mu} , \quad \hat{E}_{\mu} = e^{W} \partial_{\mu} e^{-W} , \quad W = W^{N} \partial_{N} , \qquad (5.7a)$$

$$\bar{E}_{\dot{\alpha}} = \bar{F}\bar{N}_{\dot{\alpha}}{}^{\dot{\mu}}\hat{\bar{E}}_{\dot{\mu}} , \quad \hat{\bar{E}}_{\dot{\mu}} = -e^{\bar{W}}\bar{\partial}_{\dot{\mu}}e^{-\bar{W}} , \quad \bar{W} = \bar{W}^N\partial_N .$$
(5.7b)

Here the matrix  $N = (N_{\alpha}^{\mu})$  is unimodular,  $N \in SL(2, \mathbb{C})$ , and the scalar *F* is nowhere vanishing. The complex supervector field  $W^N$  is unconstrained.

Consider a covariantly chiral tensor superfield  $\Psi_{\alpha_1...\alpha_n}$ . Making use of (5.6) and (5.7) gives

$$\bar{\mathscr{D}}_{\dot{\beta}}\Psi_{\alpha_{1}\dots\alpha_{n}}=0 \quad \Longleftrightarrow \quad \Psi_{\alpha_{1}\dots\alpha_{n}}(x,\theta,\bar{\theta})=g^{-1}\mathrm{e}^{\bar{W}}\hat{\Psi}_{\alpha_{1}\dots\alpha_{n}}(x,\theta) \;, \qquad (5.8)$$

where g is given by (5.6b)

Local Lorentz transformations correspond to setting  $K^N = 0$  in (5.3). They act only on the matrix N in (5.7a),

$$N' = \exp\left(\frac{1}{2}K^{ab}\sigma_{ab}\right)N, \qquad (5.9)$$

and therefore it it possible to choose

$$N = 1$$
. (5.10)

This gauge condition is useful for several applications, see below. Another useful gauge fixing of the local Lorentz symmetry is

$$\Omega_{\alpha}{}^{\beta\dot{\gamma}} = 0. \tag{5.11}$$

# 5.2 The $\Lambda$ gauge group

General coordinate transformations correspond to setting  $K^{bc} = 0$  in (5.3). They act on the building blocks in (5.7a) as follows:

$$F' = e^{K}F$$
,  $N' = e^{K}N$ ,  $e^{W'} = e^{K}e^{W}$ . (5.12)

Once the constraints on the torsion have been partially solved in terms of F, N and the complex unconstrained prepotential  $W^N$ , there may appear an additional gauge freedom. To uncover it, consider a covariantly chiral scalar superfield  $\Phi$ 

$$\bar{\mathscr{D}}_{\dot{\alpha}}\Phi = 0 \quad \Longleftrightarrow \quad \Phi(x,\theta,\bar{\theta}) = \mathrm{e}^{W}\hat{\Phi}(x,\theta) , \qquad \bar{\partial}_{\dot{\mu}}\hat{\Phi} = 0 .$$
 (5.13)

Its transformation law under (5.3) is

$$\Phi' = e^{K} \Phi \quad \iff \quad e^{\bar{W}'} = e^{K} e^{\bar{W}} , \quad \hat{\Phi}' = \hat{\Phi} .$$
 (5.14)

We can introduce a new gauge transformation defined by

$$\mathbf{e}^{\bar{W}'} = \mathbf{e}^{\bar{W}} \mathbf{e}^{-\Lambda} , \qquad \hat{\Phi}' = \mathbf{e}^{\Lambda} \hat{\Phi} = \exp\left(\lambda^n \partial_n + \lambda^\nu \partial_\nu\right) \hat{\Phi} , \qquad (5.15a)$$

$$\Lambda = \lambda^{N} \partial_{N} = \lambda^{n} \partial_{n} + \lambda^{\nu} \partial_{\nu} + \lambda_{\dot{\nu}} \bar{\partial}^{\dot{\nu}} , \quad \bar{\partial}_{\dot{\mu}} \lambda^{n} = 0 , \quad \bar{\partial}_{\dot{\mu}} \lambda^{\nu} = 0 , \quad (5.15b)$$

which does not change  $\Phi$  and which preserves the chirality of  $\hat{\Phi}$ . Of special significance is the fact that the spinor parameter  $\lambda_{\dot{v}}$  is unconstrained. It is obvious that the gauge transformations (5.15) form a group, which is known as the  $\Lambda$  gauge group. It turns out that the  $\Lambda$ -transformation of W,

$$\mathbf{e}^{W'} = \mathbf{e}^{W} \mathbf{e}^{-\bar{\Lambda}} , \quad \bar{\Lambda} = \bar{\lambda}^{n} \partial_{n} + \bar{\lambda}^{\nu} \partial_{\nu} + \bar{\lambda}_{\dot{\nu}} \bar{\partial}^{\dot{\nu}} , \qquad (5.16)$$

can be supplemented by certain transformations of F and N such that the supervector field  $E_{\alpha}$ , eq. (5.7a), does not change. In the infinitesimal case, since  $\delta \hat{E}_{\mu} = -e^{W}[\bar{\Lambda}, \partial_{\mu}]e^{-W}$ , these transformations are:

$$\delta F = -\frac{1}{2} F \partial_{\mu} \bar{\lambda}^{\mu} , \qquad \delta N_{\alpha}{}^{\mu} = -N_{\alpha}{}^{\nu} e^{W} \partial_{(\nu} \bar{\lambda}^{\mu)} . \qquad (5.17)$$

## 5.3 The gravitational superfield

Let us analyse the transformation of W under the K and  $\Lambda$  gauge groups,  $e^{W'} = e^{K}e^{W}e^{-\bar{\Lambda}}$ . in the infinitesimal case, this reduces to

$$\delta W = \delta W^M \partial_M = K - \bar{\Lambda} + O(W)$$
  
=  $(K^m - \bar{\lambda}^m) \partial_m + (K^\mu - \bar{\lambda}^\mu) \partial_\mu + (K_\mu - \bar{\lambda}_\mu) \bar{\partial}^{\dot{\mu}} + O(W)$ . (5.18)

Here the vector parameter  $K^m$  is real but otherwise unconstrained, and the spinor parameters  $K^{\mu}$  and  $\bar{\lambda}_{\mu}$  are unconstrained. Therefore it is possible to choose a gauge condition

$$W = -\mathrm{i}H , \qquad H = H^m \partial_m = \bar{H} . \tag{5.19}$$

A different gauge fixing is possible [49]. First one may gauge fix W to have no spinor components,  $W = W^n \partial_n$ , and then impose the additional condition

$$\exp\left(\bar{W}^{n}\partial_{n}\right)x^{m} = x^{m} + i\mathscr{H}^{m}(x,\theta,\bar{\theta}), \qquad \tilde{\mathscr{H}}^{m} = \mathscr{H}^{m}.$$
(5.20)

Given a covariantly chiral superfield (5.13), it holds that

$$\Phi(x,\theta,\bar{\theta}) = e^{W}\hat{\Phi}(x,\theta) = \hat{\Phi}(x+i\mathcal{H},\theta).$$
(5.21)

The residual gauge freedom, which preserves the condition (5.20) is determined by considering the variation

$$i\delta \mathscr{H}^{m} = \delta e^{\bar{W}} x^{m} = K e^{\bar{W}} x^{m} - e^{\bar{W}} \Lambda x^{m}$$
$$= K^{m} - e^{\bar{W}} \lambda^{m} + i K^{N} \partial_{N} \mathscr{H}^{m} . \qquad (5.22)$$

Here the right-hand side should be purely imaginary, hence  $K^m$  is expressed in terms of  $\lambda^m$  and  $\overline{\lambda}^m$  as follows

$$K^{m} = \frac{1}{2} e^{\bar{W}} \lambda^{m} + \frac{1}{2} e^{W} \bar{\lambda}^{m} = \frac{1}{2} \lambda^{m} (x + i\mathscr{H}, \theta) + \frac{1}{2} \bar{\lambda}^{m} (x - i\mathscr{H}, \bar{\theta}) , \quad (5.23)$$

and the variation  $\delta \mathscr{H}^m$  turns into

$$\delta \mathscr{H}^{m} = K^{N} \partial_{N} \mathscr{H}^{m} + \frac{\mathrm{i}}{2} \left( \lambda^{m} (x + \mathrm{i} \mathscr{H}, \theta) - \bar{\lambda}^{m} (x - \mathrm{i} \mathscr{H}, \bar{\theta}) \right).$$
(5.24)

It is also necessary to require  $\delta e^{\bar{W}} \theta^{\mu} = 0$  and  $\delta e^{\bar{W}} \bar{\theta}_{\mu} = 0$ , which gives

$$K^{\mu} = \lambda^{\mu}(x + i\mathscr{H}, \theta) , \quad \bar{K}_{\dot{\mu}} = \bar{\lambda}_{\dot{\mu}}(x - i\mathscr{H}, \bar{\theta}) , \qquad (5.25)$$

as well as  $\Lambda^{M} = (\lambda^{m}(x,\theta), \lambda^{\mu}(x,\theta), e^{-\bar{W}}e^{W}\bar{\lambda}_{\mu}(x,\bar{\theta}))$ . Substituting the obtained expressions for  $K^{\mu}$ ,  $K^{\mu}$  and  $\bar{K}_{\mu}$  into (5.22), we arrive at the gauge transformation law of the gravitational superfield, eq. (2.30).

# 5.4 Chiral prepotential

In order to uncover a remaining prepotential, we first analyse the structure of R, F and E. These objects are invariant under the local Lorentz transformations, and therefore we can compute them by imposing the gauge condition (5.10). In this gauge  $E_{\alpha} = F\hat{E}_{\alpha}$ ,  $C_{\alpha\beta}^{\gamma} = 2E_{(\alpha} \ln F \delta_{\beta})^{\gamma}$  and therefore the spinor connections are

$$\Omega_{\alpha\beta\gamma} = -2\varepsilon_{\alpha(\beta}E_{\gamma)}\ln F , \qquad \bar{\Omega}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} = -2\varepsilon_{\dot{\alpha}(\dot{\beta}}\bar{E}_{\dot{\gamma})}\ln\bar{F} .$$
(5.26)

Now we can evaluate the relation (4.9a) to end up with explicit expressions for the chiral torsion *R* and its conjugate  $\bar{R}$ :

$$\bar{R} = -\frac{1}{4}\hat{E}^{\mu}\hat{E}_{\mu}F^2 , \qquad R = -\frac{1}{4}\hat{E}_{\dot{\mu}}\hat{E}^{\dot{\mu}}\bar{F}^2 . \qquad (5.27)$$

Given a scalar superfield U, a short calculation gives

$$(\bar{\mathscr{D}}^2 - 4R)U = \hat{E}_{\mu}\hat{E}^{\mu}(\bar{F}^2 U) .$$
 (5.28)

In order to compute  $E^{-1} = \text{Ber}(E_A^M)$ , we introduce a semi-covariant frame

$$\hat{E}_A = (\hat{E}_a, \hat{E}_\alpha, \hat{\bar{E}}^{\dot{\alpha}}) = \hat{E}_A{}^M \partial_m , \qquad \hat{E}_a = -\frac{i}{4} (\tilde{\sigma}_a)^{\dot{\alpha}\alpha} \{ \hat{E}_\alpha, \hat{\bar{E}}_{\dot{\alpha}} \} , \qquad (5.29)$$

which is constructed in terms of the prepotential  $W^M$  and its conjugate, in accordance with (5.7). In the gauge (5.10), the inverse supervielbein  $E_A$  is related to  $\hat{E}_A$  as follows:

$$E_{\alpha} = F\hat{E}_{\alpha} , \qquad \bar{E}^{\dot{\alpha}} = \bar{F}\bar{E}^{\dot{\alpha}} , \qquad (5.30a)$$

$$E_{a} = F\bar{F}\hat{E}_{a} + \frac{1}{4}F(\tilde{\sigma}_{a})^{\dot{\alpha}\alpha} \left(\Omega_{\dot{\alpha}\alpha}{}^{\beta} - \delta_{\alpha}{}^{\beta}\bar{E}_{\dot{\alpha}}\ln F\right)\hat{E}_{\beta} + \frac{i}{4}\bar{F}(\tilde{\sigma}_{a})^{\dot{\alpha}\alpha} \left(\bar{\Omega}_{\alpha\dot{\alpha}}{}^{\dot{\beta}} - \delta_{\dot{\alpha}}{}^{\dot{\beta}}E_{\alpha}\ln\bar{F}\right)\hat{E}_{\dot{\beta}}.$$
 (5.30b)

It follows that

$$E^{-1} = F^2 \bar{F}^2 \hat{E}^{-1}$$
,  $\hat{E}^{-1} = \text{Ber}(\hat{E}_A^M)$ . (5.31)

So far, F appears to be unconstrained. However, it follows from the algebra of covariant derivatives that

$$(-1)^{\varepsilon_B} \mathscr{T}_{\alpha B}{}^B = 0 , \qquad (5.32)$$

while the direct evaluation of this structure gives (see, e.g., [35] for the technical details)

$$-(-1)^{\varepsilon_B}\mathscr{T}_{\alpha B}{}^B = E_{\alpha} \ln\left[E^{-1}F^2(1\cdot e^{\overleftarrow{W}})\right] = E_{\alpha} \ln\left[\hat{E}^{-1}\bar{F}^2F^4(1\cdot e^{\overleftarrow{W}})\right], \quad (5.33)$$

where the operator  $\stackrel{\leftarrow}{W}$  is defined by

$$U \stackrel{\leftarrow}{W} = (-1)^{\varepsilon_M} \partial_M (U W^M) \implies (U \cdot e^{\stackrel{\leftarrow}{W}}) = (1 \cdot e^{\stackrel{\leftarrow}{W}}) e^W U.$$
(5.34)

We conclude that  $\bar{\varphi}^{-3} := \bar{F}^2 F^4 \hat{E}^{-1} (1 \cdot e^{\overleftarrow{W}})$  is annihilated by  $E_{\alpha}$ . This result can be equivalently written as

$$\varphi^{-3} = F^2 \bar{F}^4 \hat{E}^{-1} (1 \cdot e^{\widetilde{W}}) , \qquad \bar{E}_{\dot{\alpha}} \varphi = 0 .$$
 (5.35)

By construction, the chiral superfield  $\varphi$  is nowhere vanishing. It follows that

$$F = \varphi^{1/2} \bar{\varphi}^{-1} (1 \cdot e^{\overleftarrow{W}})^{-1/3} (1 \cdot e^{\overleftarrow{W}})^{1/6} \hat{E}^{1/6} , \qquad (5.36)$$

and then eq. (5.31) gives

$$E = \bar{\varphi}\varphi \left[\hat{E}(1 \cdot e^{\overleftarrow{W}})(1 \cdot e^{\overleftarrow{W}})\right]^{1/3}.$$
(5.37)

The covariantly chiral superfield  $\varphi$  is called the chiral prepotential. It turns out that, modulo purely gauge degrees of freedom, the covariant derivatives are expressed in terms of  $W^M$ ,  $\varphi$  and their conjugates. These are the prepotentials for the GWZ superspace geometry. The transformation law of  $\varphi$  follows from (5.35)

$$\delta \varphi^3 = K^M \partial_M \varphi^3 + \varphi^3 e^{\bar{W}} (\partial_m \lambda^m - \partial_\mu \lambda^\mu) . \qquad (5.38)$$

If we represent the chiral prepotential in the form

$$\boldsymbol{\varphi} = \mathbf{e}^{\bar{W}} \hat{\boldsymbol{\varphi}} , \qquad \bar{\partial}_{\dot{\mu}} \hat{\boldsymbol{\varphi}} = 0 , \qquad (5.39)$$

then the transformation law (5.38) will turn into

$$\delta \hat{\varphi}^3 = \lambda^M \partial_M \hat{\varphi}^3 + \hat{\varphi}^3 (\partial_m \lambda^m - \partial_\mu \lambda^\mu) = \partial_m (\lambda^m \hat{\varphi}^3) - \partial_\mu (\lambda^\mu \hat{\varphi}^3) .$$
 (5.40)

This is the transformation law of a chiral density.

By construction, the prepotential  $W^M$  is invariant under the super-Weyl transformations (4.23). It is a short calculation to see that the chiral prepotential  $\varphi$  trans-

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forms as

$$\delta_{\sigma}\varphi = -\sigma\varphi \,. \tag{5.41}$$

In conformal supergravity, the super-Weyl transformations belong to the gauge group. Making use of the super-Weyl gauge freedom allows one to impose the gauge  $\varphi = 1$ . Therefore, the gravitational superfield is the only prepotential in conformal supergravity, modulo purely gauge degrees of freedom.

# 5.5 Chiral action

It follows from the above analysis that *E* can be written as  $E = \varphi^3 \bar{F}^2(1 \cdot e^{\bar{W}})$ . In conjunction with the identity (5.28), the chiral action (3.33) can be rewritten as follows

$$S_{\rm c} = -4 \int \mathrm{d}^4 x \mathrm{d}^2 \theta \, \mathrm{d}^2 \bar{\theta} \, \varphi^3 (1 \cdot \mathrm{e}^{\overleftarrow{\psi}}) \mathscr{L}_{\rm c} \, \frac{\bar{F}^2 \Upsilon}{\hat{E}_{\dot{\mu}} \hat{E}^{\dot{\mu}} (\bar{F}^2 \Upsilon)} \,, \tag{5.42}$$

We now recall the well-known result for a change of variable in superspace [49] (see [35] for a pedagogical derivation). Given a first-order differential operator  $K = K^N \partial_N$ , it holds that

$$z'^{M} = e^{K} z^{M} \implies \operatorname{Ber}(\partial_{M} z'^{N}) = (1 \cdot e^{\overline{K}}), \qquad (5.43)$$

and therefore  $\int dz' L(z') = \int dz (1 \cdot e^{\overleftarrow{K}}) e^{K} L(z)$ . As follows from (5.13), the covariantly chiral superfields depend on chiral variables  $\tilde{x}^{m}$  and  $\tilde{\theta}^{\mu}$ ,

$$\hat{E}_{\dot{\mu}}\Phi = 0 \implies \Phi(z) = \hat{\Phi}(\tilde{x}, \tilde{\theta}) , \qquad \tilde{z}^{M} = (\tilde{x}^{m}, \tilde{\theta}^{\mu}, \tilde{\tilde{\theta}}_{\dot{\mu}}) = e^{\tilde{W}} z^{M} .$$
(5.44)

In the variables  $\tilde{z}^M$ , the operator  $\hat{\bar{E}}^{\dot{\mu}}$  becomes a partial derivative,  $\hat{\bar{E}}^{\dot{\mu}} = \partial/\partial \tilde{\bar{\theta}}_{\dot{\mu}}$ . Now, making use of (5.43) in (5.42) leads to the following simple result

$$S_{\rm c} = \int \mathrm{d}^4 x \mathrm{d}^2 \theta \,\hat{\varphi}^3 \hat{\mathscr{L}}_{\rm c} \,. \tag{5.45}$$

Here we have denoted  $\tilde{x}^m$  and  $\tilde{\theta}^{\mu}$  simply as  $x^m$  and  $\theta^{\mu}$ . This result shows that the chiral integration measure in (3.31) is

$$\mathscr{E} = \varphi^3 , \qquad (5.46)$$

and this interpretation agrees with the transformation law (5.40).

#### 6 Matter multiplets in conformal supergravity

In this section we introduce the most popular matter multiplets and describe several famous models for them. Practically all results will be presented in conformal superspace. They can be recast in terms of the U(1) or GWZ superspace geometries by making use of the degauging formalism described in section 4.

# 6.1 Scalar multiplet

The minimal scalar multiplet was introduced by Wess and Zumino in their first paper on supersymmetry [50]. In conformal superspace, minimal scalar multiplets are described in terms of covariantly chiral primary scalar superfields. Such a superfield  $\phi$  obeys the constraints  $K^B \phi = 0$  and  $\bar{\nabla}^{\dot{\alpha}} \phi = 0$ . In general, every covariantly chiral primary superfield  $\phi$  of definite dimension  $\Delta$  satisfies equation (3.21). If we do not assume  $\phi$  to be an eigenvector of  $\mathbb{D}$  then it must hold that

$$K^B \phi = 0$$
,  $\bar{\nabla}^{\dot{\beta}} \phi = 0 \implies \Im \phi = -\frac{2}{3} \mathbb{D} \phi$ . (6.1)

The superfield  $\phi$  contains three independent component fields which can be chosen as follows:  $\varphi := \phi |, \eta_{\alpha} := \nabla_{\alpha} \phi |$  and  $F := -\frac{1}{4} \nabla^2 \phi |$ . In theories with at most two derivatives at the component level, the complex scalar *F* is an auxiliary field.

As a simple example of a supergravity-matter system, we consider a curved superspace extension of the massless Wess-Zumino model [50]. It corresponds to choosing a canonical dimension for the chiral scalar,  $\mathbb{D}\phi = \phi$ . The action is

$$S_{\rm WZ} = \int d^{4|4} z E \,\bar{\phi} \phi + \left\{ \frac{\lambda}{3} \int d^4 x d^2 \theta \,\mathscr{E} \,\phi^3 + {\rm c.c.} \right\} \,, \tag{6.2}$$

with  $\lambda$  a complex coupling constant.

## 6.2 Superconformal sigma model

In Minkowski superspace  $\mathbb{M}^{4|4}$ , general  $\mathcal{N} = 1$  supersymmetric two-derivative theories of scalar multiplets are nonlinear  $\sigma$ -models which are described by chiral scalar superfields  $\phi^{I}$  and their conjugates  $\bar{\phi}^{\bar{I}}$  taking their values in an arbitrary Kähler manifold  $\mathcal{M}$  [51]. In supergravity, however,  $\sigma$ -model couplings turn out to be more restrictive, and the target space  $\mathcal{M}$  must be a Kähler-Hodge manifold [52]. Within the locally superconformal setting, this means that we have to consider a superconformal sigma model on a Kähler cone (see, e,.g., [28, 53] for a more detailed discussion). Here, our goal is to show how these restrictions emerge.

Let  $N(\phi, \bar{\phi})$  be the Kähler potential of  $\mathcal{M}$ , and  $g_{I\bar{J}} = \partial_I \partial_{\bar{J}} N \equiv N_{I\bar{J}}$  its Kähler metric. We start with a naive curved-superspace extension of the  $\mathcal{N} = 1$  supersymmetric  $\sigma$ -model action<sup>7</sup>

$$S = -\int d^{4|4} z E N(\phi, \bar{\phi}) , \qquad K^{B} \phi^{I} = 0 , \quad \bar{\nabla}^{\dot{\beta}} \phi^{I} = 0 .$$
 (6.3)

Here the dynamical variables  $\phi^I$  are postulated to be covariantly chiral primary scalar superfields. Since  $\phi^I$  are local holomorphic coordinates,  $\mathbb{D}\phi^I$  and  $\mathbb{Y}\phi^I$  must be holomorphic vector fields on  $\mathcal{M}$ ,

$$\mathbb{D}\phi^{I} = \chi^{I}(\phi) \quad \Longleftrightarrow \quad \mathbb{Y}\phi^{I} = -\frac{2}{3}\chi^{I}(\phi) , \qquad (6.4)$$

where we have used (6.1). The action (6.3) must be locally superconformal. Then, in accordance with (3.29), N must be neutral under the U(1)<sub>R</sub> group and have dimension +2, and therefore

$$\chi^{I}(\phi)\partial_{I}N(\phi,\bar{\phi}) = \bar{\chi}^{I}(\bar{\phi})\bar{\partial}_{\bar{I}}N(\phi,\bar{\phi}) , \qquad (6.5a)$$

$$\chi^{I}(\phi)\partial_{I}N(\phi,\bar{\phi}) = \bar{\chi}^{I}(\bar{\phi})\bar{\partial}_{\bar{I}}N(\phi,\bar{\phi}) = N(\phi,\bar{\phi}) .$$
(6.5b)

Differentiating the condition  $\chi^I \partial_I N = N$  with respect to  $\bar{\partial}_{\bar{I}}$  gives

$$\chi^{I}(\phi)g_{I\bar{J}}(\phi,\bar{\phi}) = \bar{\partial}_{\bar{J}}N(\phi,\bar{\phi}) = \bar{\chi}_{\bar{J}}(\phi,\bar{\phi}) \implies \chi^{I}(\phi) = g^{I\bar{J}}\bar{\partial}_{\bar{J}}N.$$
(6.6)

The obtained relations have several nontrivial implications. First of all, the equations (6.5b) and (6.6) imply that N is a globally defined function on  $\mathcal{M}$ ,

$$N = g_{I\bar{J}} \chi^I \bar{\chi}^J . \tag{6.7}$$

Therefore, the Kähler two-form,  $\Omega = 2i g_{I\bar{J}} d\phi^I \wedge d\bar{\phi}^{\bar{J}}$ , is exact, hence  $\mathscr{M}$  is necessarily non-compact. Secondly, it follows that  $\chi^I$  is a homothetic conformal Killing vector field

$$\nabla_I \chi^J = \delta_I^J , \qquad \bar{\nabla}_{\bar{I}} \chi^J = \bar{\partial}_{\bar{I}} \chi^J = 0 .$$
(6.8)

The sigma model (6.3) can be generalised to include a superpotential

$$S = -\int \mathrm{d}^{4|4} z E N(\phi, \bar{\phi}) + \left\{ \int \mathrm{d}^4 x \mathrm{d}^2 \theta \,\mathscr{E} W(\phi) + \mathrm{c.c.} \right\} \,. \tag{6.9}$$

Here  $W(\phi)$  is a holomorphic scalar field on the target space. It should obey  $\mathbb{D}W(\phi) = 3W(\phi)$  and  $\mathbb{Y}W(\phi) = -2W(\phi)$ , which imply the homogeneity condition

$$\chi^{I}(\phi)\partial_{I}W(\phi) = 3W(\phi) . \qquad (6.10)$$

<sup>&</sup>lt;sup>7</sup> An overall minus sign is inserted in (6.3) in order to give the correct sign for the Einstein-Hilbert term at the component level, if (6.3) is viewed as the supergravity-matter action, see [28] for more details.

It should be mentioned that local complex coordinates in  $\mathcal{M}$  can be chosen in such a way that  $\chi^{I}(\phi) = \phi^{I}$ . Then the Kähler potential  $N(\phi, \bar{\phi})$  obeys the following homogeneity condition:

$$\phi^{I} \partial_{I} N(\phi, \bar{\phi}) = N(\phi, \bar{\phi}) . \tag{6.11}$$

# 6.3 Superconformal higher-derivative sigma model

Here we discuss a higher-derivative superconformal  $\sigma$ -model which was originally introduced in the GWZ superspace as an induced action [54]. It appears that its uplift to conformal superspace cannot be given solely in terms of the conformally covariant derivatives  $\nabla_A$  and should explicitly involve connection superfields.

Let  $K(\phi^I, \bar{\phi}^{\bar{J}})$  be the Kähler potential of an *arbitrary* Kähler manifold  $\mathcal{M}$ . We introduce a higher-derivative locally supersymmetric theory described in terms of covariantly chiral scalar superfields  $\phi^I, \bar{\mathscr{D}}^{\dot{\alpha}} \phi^I = 0$ , which are neutral under the super-Weyl transformations,  $\delta_{\sigma} \phi^I = 0$ . The higher-derivative action proposed in [54] is

$$S = \frac{1}{16} \int d^{4|4} z E g_{I\bar{J}}(\phi, \bar{\phi}) \left\{ \mathfrak{D}^{2} \phi^{I} \mathfrak{\bar{D}}^{2} \bar{\phi}^{\bar{J}} - 8G_{\alpha \dot{\alpha}} \mathscr{D}^{\alpha} \phi^{I} \mathfrak{\bar{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{J}} \right\} + \frac{1}{16} \int d^{4|4} z E \left\{ \alpha R_{I\bar{J}K\bar{L}}(\phi, \bar{\phi}) + \beta g_{I\bar{J}}(\phi, \bar{\phi}) g_{K\bar{L}}(\phi, \bar{\phi}) \right\} \times \mathscr{D}^{\alpha} \phi^{I} \mathscr{D}_{\alpha} \phi^{K} \mathfrak{\bar{D}}_{\dot{\alpha}} \bar{\phi}^{\bar{J}} \mathfrak{\bar{D}}^{\dot{\alpha}} \bar{\phi}^{\bar{L}} , \qquad (6.12)$$

where  $g_{I\bar{J}} = \partial_I \partial_{\bar{J}} K$  is the Kähler metric,  $R_{I\bar{J}K\bar{L}}$  the Riemann curvature of the Kähler manifold, and  $\mathfrak{D}^2 \phi^I$  is defined as follows<sup>8</sup>

$$\mathfrak{D}^2 \phi^I := \mathscr{D}^2 \phi^I + \Gamma^I_{KL} \mathscr{D}^\alpha \phi^K \mathscr{D}_\alpha \phi^L .$$
(6.13)

We recall that the Christoffel symbols  $\Gamma_{KL}^{I}$  and the curvature  $R_{IJ\bar{K}L}$  are given by the expressions

$$\Gamma_{JK}^{I} = g^{I\bar{L}}\partial_{J}\partial_{K}\partial_{\bar{L}}K , \quad R_{I\bar{J}K\bar{L}} = \partial_{I}\partial_{K}\partial_{\bar{J}}\partial_{\bar{L}}K - g^{M\bar{N}}\partial_{I}\partial_{K}\partial_{\bar{N}}K\partial_{\bar{J}}\partial_{\bar{L}}\partial_{M}K .$$
(6.14)

It is an instructive exercise to show that the action (6.12) is super-Weyl invariant. This action is manifestly invariant under Kähler transformations

$$K(\phi, \bar{\phi}) \to K(\phi, \bar{\phi}) + \Lambda(\phi) + \bar{\Lambda}(\bar{\phi}) , \qquad (6.15)$$

with  $\Lambda(\phi)$  being an arbitrary holomorphic function.

The super-Weyl invariance of (6.12) may be traced to the existence of a superconformal operator  $\Delta$  introduced in [55]. In conformal superspace, this operator is defined to act on a primary chiral weight-zero scalar  $\overline{\phi}$  as

<sup>&</sup>lt;sup>8</sup> The operator  $\mathfrak{D}^2$  in (6.13) should not be confused with  $\mathfrak{D}^{\alpha}\mathfrak{D}_{\alpha}$  in U(1) superspace.

$$\Delta \bar{\phi} = -\frac{1}{64} \bar{\nabla}^2 \nabla^2 \bar{\nabla}^2 \bar{\phi} \tag{6.16a}$$

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and the resulting weight-three chiral superfield is primary,

$$K^{B}\bar{\phi} = 0 , \quad \nabla^{\beta}\bar{\phi} = 0, \quad \mathbb{D}\bar{\phi} = 0 \quad \Longrightarrow \quad K^{B}\Delta\bar{\phi} = 0 , \quad \bar{\nabla}^{\dot{\beta}}\Delta\bar{\phi} = 0 . \quad (6.16b)$$

Degauging  $\Delta \bar{\phi}$  to the GWZ superspace gives

$$\Delta \bar{\phi} := -\frac{1}{64} (\bar{\mathscr{D}}^2 - 4R) \left\{ \mathscr{D}^2 \bar{\mathscr{D}}^2 \bar{\phi} + 8 \mathscr{D}^{\alpha} (G_{\alpha \dot{\alpha}} \bar{\mathscr{D}}^{\dot{\alpha}} \bar{\phi}) \right\}.$$
(6.17)

The super-Weyl transformation law of this superfield is  $\delta_{\sigma}\Delta\bar{\phi} = 3\sigma\Delta\bar{\phi}$ . For any covariantly chiral scalars  $\phi$  and  $\psi$ , it holds that

$$\int \mathrm{d}^4 x \, \mathrm{d}^2 \theta \, \mathscr{E} \, \psi \Delta \bar{\phi} = \frac{1}{16} \int \mathrm{d}^{4|4} z E \left\{ (\mathscr{D}^2 \psi) \bar{\mathscr{D}}^2 \bar{\phi} - 8(\mathscr{D}^\alpha \psi) G_{\alpha \dot{\alpha}} \bar{\mathscr{D}}^{\dot{\alpha}} \bar{\phi} \right\} \,. \tag{6.18}$$

If the chiral scalars  $\phi$  and  $\psi$  are inert under the super-Weyl transformations, this functional is super-Weyl invariant.

The operator (6.16) is a supersymmetric generalisation of the conformal fourthorder scalar operator in curved space

$$\Delta_0 = \Box \Box - \nabla^a \left( 2\mathscr{R}_{ab} \nabla^b - \frac{2}{3} \mathscr{R} \nabla_a \right) \,, \qquad \Box = \nabla^a \nabla_a \tag{6.19}$$

discovered by Fradkin and Tseytlin [56]. Here  $\nabla_a$  denotes the torsion-free Lorentz covariant derivative, with  $\mathscr{R}_{ab}$  and  $\mathscr{R}$  being its Ricci tensor and scalar curvature, respectively. The operator (6.16) was constructed for the first time in [55] using the conformal superspace approach, although there had been earlier attempts to construct such an operator, see the discussion in [57]. Making use of its degauged form, eq. (6.17), a new representation for the nonlocal action generating the super-Weyl anomalies was derived in [57].

#### 6.4 Tensor multipet

The massless tensor multiplet was introduced by Siegel [58] as a dual version of the minimal scalar multiplet. In conformal superspace, it is described by a primary covariantly chiral spinor superfield  $\eta_{\alpha}$  of dimension 3/2,

$$K^B \eta_{\alpha} = 0$$
,  $\bar{\nabla}^{\dot{\beta}} \eta_{\alpha} = 0$   $\mathbb{D} \eta_{\alpha} = \frac{3}{2} \eta_{\alpha}$ , (6.20)

which is defined modulo gauge transformations

$$\delta\eta_{\alpha} = -\frac{i}{4}\bar{\nabla}^2\nabla_{\alpha}U , \qquad \bar{U} = U , \qquad (6.21)$$

with the gauge parameter being a primary dimensionless real scalar. The descendant

$$\mathbb{G} = \frac{1}{2} \left( \nabla^{\alpha} \eta_{\alpha} + \bar{\nabla}_{\dot{\alpha}} \bar{\eta}^{\dot{\alpha}} \right) = \bar{\mathbb{G}}$$
(6.22)

is a gauge-invariant field strength. It has the following properties:

$$K^B \mathbb{G} = 0$$
,  $\overline{\nabla}^2 \mathbb{G} = 0$ ,  $\mathbb{D} \mathbb{G} = 2\mathbb{G}$ . (6.23)

These constraints define a *real linear multiplet*. Such superfields were originally introduced by Ferrara, Wess and Zumino [59] to describe flavour current multiplets.

The superconformal tensor multiplet is described by the action [60]

$$S = -\int d^{4|4} z E \mathbb{G} \ln \frac{\mathbb{G}}{\bar{\phi}\phi} , \qquad K^B \phi = 0 , \quad \bar{\nabla}^{\dot{\beta}}\phi = 0 , \quad \mathbb{D}\phi = \phi . \quad (6.24)$$

Both  $\mathbb{G}$  and  $\phi$  are assumed to be nowhere vanishing. Dependence of the action (6.24) on  $\phi$  and  $\overline{\phi}$  is fictitious, since the action remains unchanged under transformations  $\phi \rightarrow e^{\sigma}\phi$ , where  $\sigma$  is an arbitrary covariantly chiral weight-zero scalar. In the literature, (6.24) is referred to as the model for an improved tensor multiplet [60]. It is a unique superconformal representative in the family of tensor multiplet models introduced in [58].

# 6.5 Three-form multiplet

Let us consider the representation (3.23) for n = 0. The unconstrained prepotential  $\psi$  in (3.22) is necessarily complex for  $\Delta \neq 3$ . In the  $\Delta = 3$  case, however, one can impose the reality condition  $\bar{\psi} = \psi = P$ . This leads to the three-form multiplet<sup>9</sup> described by the primary covariantly chiral scalar

$$\Pi = -\frac{1}{4}\bar{\nabla}^2 P , \quad \bar{P} = P , \quad K^B P = 0 , \quad \mathbb{D}P = 2P .$$
 (6.25)

The main difference of the three-form multiplet from the minimal scalar multiplet is that the imaginary part of the auxiliary field  $F := -\frac{1}{4}\nabla^2 \Pi$  of  $\Pi$  is the field strength of a gauge three-form.

The prepotential *P* in (6.25) is defined modulo gauge transformations  $\delta P = \mathbb{G}$ , where the gauge parameter is a real linear superfield, eq. (6.22). The simplest superconformal and gauge-invariant action to describe the dynamics of this multiplet is given by

$$S = \int \mathrm{d}^{4|4} z E \left\{ \left( \bar{\Pi} \Pi \right)^{1/3} + 2\kappa P \right\}$$

<sup>&</sup>lt;sup>9</sup> In global supersymmetry, the three-form multiplet was originally proposed by Gates [61].

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$$= \int \mathrm{d}^{4|4} z E \left( \bar{\Pi} \Pi \right)^{1/3} + \left\{ \kappa \int \mathrm{d}^4 x \mathrm{d}^2 \theta \,\mathscr{E} \,\Pi + \mathrm{c.c.} \right\} \,, \tag{6.26}$$

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where  $\kappa$  is a real coupling constant.

#### 6.6 Non-minimal scalar multiplet

We next turn to a non-minimal scalar multiplet.<sup>10</sup> In conformal superspace it is described by a primary complex scalar superfield  $\Gamma$  satisfying the constraint

$$K^B \Gamma = 0$$
,  $\bar{\nabla}^2 \Gamma = 0 \implies \quad \Im \Gamma = \frac{2}{3} (2 - \mathbb{D}) \Gamma$ . (6.27a)

We choose to parametrise the dimension of  $\Gamma$  as

$$\mathbb{D}\Gamma = \frac{2}{3n+1}\Gamma \implies \mathbb{Y}\Gamma = \frac{4n}{3n+1}\Gamma , \qquad (6.27b)$$

following the notation introduced in [6]. For  $n \neq 0, -1/3$ , the constraint (6.27) defines a *complex linear superfield*. In the n = 0 case, the U(1)<sub>R</sub> charge of  $\Gamma$  is equal to zero, and  $\Gamma$  can be subject to the reality condition  $\overline{\Gamma} = \Gamma$ , which corresponds to the real linear multiplet (6.23). The general solution to the constraint (6.27) is

$$\Gamma = \bar{\nabla}_{\dot{\alpha}} \bar{\Psi}^{\dot{\alpha}} , \qquad (6.28)$$

where the unconstrained prepotential  $\bar{\Psi}^{\dot{\alpha}}$  may be chosen to be primary. It is defined modulo gauge transformations  $\delta \bar{\Psi}^{\dot{\alpha}} = \bar{\nabla}_{\dot{\beta}} \bar{\lambda}^{(\dot{\alpha}\beta)}$ , where the gauge parameter may be chosen to be primary.

A unique superconformal model, which is constructed solely in term of  $\Gamma$  and  $\overline{\Gamma}$  and involves at most two derivatives at the component level, is given by

$$S = -\frac{1}{n} \int d^{4|4} z E \left(\bar{\Gamma} \Gamma\right)^{(3n+1)/2}.$$
 (6.29)

In global supersymmetry, it was observed by Deo and Gates [63] that the complex linear constraint  $\overline{D}^2\Gamma = 0$  admits a deformation  $\overline{D}^2\Gamma = -4\Xi$  in the presence of a chiral scalar  $\Xi$ . This idea is compatible with local superconformal symmetry. Indeed, in conformal superspace the constraint (6.27) can be deformed to define the following *improved* linear constraint

$$-\frac{1}{4}\bar{\nabla}^{2}\Upsilon = \Xi , \qquad K^{B}\Xi = 0 , \quad \bar{\nabla}^{\dot{\beta}}\Xi = 0 , \quad \mathbb{D}\Xi = \frac{3(n+1)}{3n+1}\Xi . \tag{6.30}$$

<sup>&</sup>lt;sup>10</sup> In global supersymmetry, it was introduced by Gates and Siegel [62].

In general,  $\Xi$  may be a function of matter chiral scalars,  $\Xi = \Xi(\phi)$ , see [63, 64]. Such constraints naturally arise in the framework of the  $\mathcal{N} = 1$  superfield description of  $\mathcal{N} = 2$  supersymmetric sigma models [65].

It follows from (6.30) that the choice n = -1 is special in the sense that  $\Xi$  becomes dimensionless, and therefore one can impose the superconformal constraint

$$K^B \Upsilon = 0$$
,  $-\frac{1}{4} \bar{\nabla}^2 \Upsilon = \mu = \text{const} \implies \mathbb{D} \Upsilon = -\Upsilon$ . (6.31)

This multiplet originates as the compensator of the non-minimal AdS supergravity proposed in [46]. It is also used to describe the dynamics of a Goldstino [66].

#### 6.7 Vector multiplet

The Abelian vector multiplet was introduced by Wess and Zumino in their first paper on supersymmetry [50]. Its Yang-Mills extension was derived by Ferrara and Zumino [67] and, independently, by Salam and Strathdee [68]. Here we briefly review the conformal superspace formulation for the Abelian vector multiplet and related superconformal models.

The Abelian vector multiplet is described by a scalar dimension-zero prepotential *V* defined modulo gauge transformations of the form

$$\delta_{\Lambda} V = \Lambda + \bar{\Lambda} , \qquad \bar{\nabla}_{\dot{\alpha}} \Lambda = 0 .$$
 (6.32)

Both the prepotential V and the chiral gauge parameter may be chosen to be primary. Associated with V is the primary chiral spinor descendant

$$W_{\alpha} = -\frac{1}{4}\bar{\nabla}^2 \nabla_{\alpha} V , \qquad K^B W_{\alpha} = 0 , \quad \bar{\nabla}^{\dot{\beta}} W_{\alpha} = 0 , \quad \mathbb{D} W_{\alpha} = \frac{3}{2} W_{\alpha} , \quad (6.33)$$

which is gauge invariant,  $\delta_A W_\alpha = 0$ . The field strength  $W_\alpha$  is a reduced chiral superfield in the sense that it obeys the reality condition  $\nabla^\alpha W_\alpha = \bar{\nabla}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \equiv \nabla W$ . It should be pointed out that  $\nabla W$  is a primary dimension-2 superfield. Modulo purely gauge degrees of freedom, the independent components of *V* can be chosen as follows:  $\eta_\alpha = W_\alpha|, v_{\alpha\dot{\alpha}} = \frac{1}{2} [\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}] V|$  and  $D = -\frac{1}{2} \nabla W|$ .

Dynamics of the free vector multiplet is described by the action [8]

$$S = \frac{1}{4} \int d^4x d^2\theta \,\mathscr{E} W^2 + \text{c.c.}$$
(6.34)

For a single vector multiplet this is a unique locally superconformal action with at most two derivatives at the component level. In the case that  $(\nabla W)^{-1}$  exists, nonlinear superconformal actions exist of the form [69]

$$S = \frac{1}{4} \int d^4x d^2\theta \,\mathscr{E} W^2 + \text{c.c.} + \frac{1}{4} \int d^{4|4} z E \, \frac{W^2 \,\bar{W}^2}{(\nabla W)^2} \,\mathfrak{H}(u,\bar{u}) \,, \tag{6.35}$$

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where  $u := \frac{1}{8} \nabla^2 [W^2 (\nabla W)^{-2}]$  is a primary dimensionless antichiral superfield, and  $\mathfrak{H}(z, \overline{z})$  is a real function of a complex variable. This family includes a unique U(1) duality-invariant theory [70, 71]

$$S = \frac{1}{4}\cosh\gamma\int d^4x d^2\theta \,\mathscr{E}W^2 + \text{c.c.} + \frac{1}{4}\sinh\gamma\int d^{4|4}z E \,\frac{W^2 \bar{W}^2}{(\nabla W)^2 \sqrt{u\bar{u}}} , \quad (6.36)$$

where the coupling constant  $\gamma$  must be non-negative [70].<sup>11</sup> This nonlinear extension of the supersymmetric Maxwell action (6.34) is called the super ModMax theory.

Within the GWZ superspace formalism, the action (6.36) can be rewritten in a simpler, albeit not manifestly superconformal form, originally given in [70, 71]

$$S = \frac{1}{4}\cosh\gamma\int d^4x d^2\theta \,\mathscr{E}W^2 + \text{c.c.} + \frac{1}{4}\sinh\gamma\int d^{4|4}z E \,\frac{W^2 \bar{W}^2}{\sqrt{\mathbf{u}\bar{\mathbf{u}}}} \,, \quad (6.37)$$

where  $\mathbf{u} := \frac{1}{8} \mathscr{D}^2 W^2$ . In order to make direct contact with the U(1) duality-invariant formalism of [72], this action can be rewritten in the form

$$S = \frac{1}{4} \int d^4 x d^2 \theta \, \mathscr{E} \, W^2 + \text{c.c.} + \frac{1}{4} \int d^{4|4} z E \, W^2 \, \bar{W}^2 \Lambda \left( \mathbf{u}, \bar{\mathbf{u}} \right) \,, \qquad (6.38a)$$

$$\Lambda(\mathbf{u},\bar{\mathbf{u}}) = \frac{\sinh\gamma}{\sqrt{\mathbf{u}\bar{\mathbf{u}}}} + \frac{1}{2}(1-\cosh\gamma)\left(\frac{1}{\mathbf{u}}+\frac{1}{\bar{\mathbf{u}}}\right).$$
(6.38b)

A large class of other interesting, and not necessarily superconformal, models for supersymmetric nonlinear electrodynamics are based on deforming the super-Maxwell action (6.34) by a self-interaction  $\int d^{4|4}z E \mathscr{L}$ , where

$$\mathscr{L} = W^2 \bar{W}^2 \mathscr{H} \left( \omega, \bar{\omega}, \nabla W, \mathfrak{C} \right), \qquad \omega := \frac{1}{8} \nabla^2 \left[ W^2 \mathfrak{C}^{-2} \right], \qquad (6.39a)$$

$$K^B \mathfrak{C} = 0$$
,  $\mathbb{D}\mathfrak{C} = 2\mathfrak{C}$ ,  $\bar{\mathfrak{C}} = \mathfrak{C}$ . (6.39b)

Here  $\mathfrak{C}$  is a conformal compensator associated to an off-shell Poincaré supergravity (see the next section), while the composite  $\mathscr{H}$  is constrained to be a real primary superfield of dimension -4. Well-known theories of this type are, for instance, the supersymmetric Born-Infeld theory [73], and the generalised Fayet-Iliopoulos terms in supergravity without gauged *R*-symmetry, which were recently introduced in [74].

The supersymmetric Yang-Mills multiplet is well reviewed in the literature, see, e.g., [13, 17, 28, 35], and its description in conformal superspace does not bring in new features. We refer the interested reader to the literature, see [75].

<sup>&</sup>lt;sup>11</sup> The general formalism for U(1) duality-invariant  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$  supersymmetric theories was developed in [72].

### 7 Off-shell models for pure supergravity

As discussed in the introduction, there are several off-shell formulations for pure supergravity, including the old minimal [7, 8, 9, 10], new minimal [11, 12] and non-minimal [4, 5, 6] theories. Here we present their formulations in conformal superspace. Due to space limitations, a discussion of general supergravity-matter systems is beyond the scope of this review.

As discussed in section 3, conformal superspace can be identified with a pair  $(\mathcal{M}^{4|4}, \nabla)$ . In the case of Poincaré or AdS supergravity, the superspace geometric setup is a triple  $(\mathcal{M}^{4|4}, \nabla, \mathfrak{C})$ , where  $\mathfrak{C}$  is a compensator. The latter is a primary constrained scalar superfield such that (i)  $\mathfrak{C}$  is nowhere vanishing (more precisely,  $\mathfrak{C}^{-1}$  exists); and (ii) the dimension of  $\mathfrak{C}$  is non-zero. These conditions imply that the local scale and local  $U(1)_R$  gauge freedom can be used to impose the gauge condition  $\mathfrak{C} = 1$ . If the compensator is real,  $\overline{\mathfrak{C}} = \mathfrak{C}$ , the required gauge condition is achieved by applying a local scale transformation.

#### 7.1 Old minimal supergravity

In the old minimal formulation for supergravity, the compensator is a nowhere vanishing primary chiral scalar  $\phi$ , eq. (3.21), of non-zero dimension  $\Delta$ . Since  $\phi^{-1}$  exists, the primary chiral scalar  $\Phi = \phi^{1/\Delta}$  is also nowhere vanishing and its dimension is canonical,  $\mathbb{D}\Phi = \Phi$ . It is  $\Phi$  and its conjugate  $\overline{\Phi}$  which are chosen as the compensators in old minimal supergravity.

The action functional for pure old minimal supergravity is given by

$$S_{\text{old-minimal}} = -3 \int d^{4|4} z E \,\bar{\Phi} \Phi + \left\{ \mu \int d^4 x d^2 \theta \,\mathscr{E} \,\Phi^3 + \text{c.c.} \right\}.$$
(7.1)

The choice  $\mu = 0$  corresponds to Poincaré supergravity. For  $\mu \neq 0$  the action describes AdS supergravity. Let us analyse the equations of motion for this theory. The chirality constraint on  $\Phi$  and its equation of motion can be written as

$$\bar{\nabla}_{\dot{\alpha}}\Phi^3 = 0 , \qquad (7.2a)$$

$$-\frac{1}{4}\bar{\nabla}^2\left(\bar{\Phi}\Phi^{-2}\right) = \mu . \tag{7.2b}$$

The equation of motion corresponding to the gravitational superfield proves to be

$$\left[\nabla_{\alpha}, \bar{\nabla}_{\dot{\alpha}}\right] \left(\bar{\Phi}\Phi\right)^{-1/2} = 0.$$
(7.2c)

In general, given a primary real scalar *L* of dimension -1,  $\mathbb{D}L = -L$ , its real vector descendant  $[\nabla_{\alpha}, \bar{\nabla}_{\dot{\alpha}}]L$  is primary.

The equations (7.2) can be degauged to U(1) superspace, which results in

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Superspace approaches to  $\mathcal{N} = 1$  supergravity

$$\bar{\mathscr{D}}_{\dot{\alpha}}\Phi^3 = 0 , \qquad (7.3a)$$

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$$-\frac{1}{4}(\bar{\mathscr{D}}^2 - 4R)(\bar{\Phi}\Phi^{-2}) = \mu , \qquad (7.3b)$$

$$\left\{G_{\alpha\dot{\alpha}} + \left[\mathscr{D}_{\alpha}, \bar{\mathscr{D}}_{\dot{\alpha}}\right]\right\} \left(\bar{\Phi}\Phi\right)^{-1/2} = 0.$$
(7.3c)

Now, the super-Weyl and local  $U(1)_R$  gauge freedom can be used to impose the gauge condition  $\Phi = 1$ . This implies that the  $U(1)_R$  connection vanishes, and U(1) superspace geometry reduces to the GWZ geometry. The supergravity equations (7.3b) and (7.3c) turn into

$$R = \mu = \text{const}, \qquad G_{\alpha \dot{\alpha}} = 0, \qquad (7.4)$$

and all information about the dynamics of supergravity is encoded in the super-Weyl tensor  $W_{\alpha\beta\gamma}$ . By analogy with the terminology used in general relativity, the equations (7.4) define an Einstein superspace.

A unique maximally supersymmetric solution of (7.4) is characterised by the condition  $W_{\alpha\beta\gamma} = 0$ . It is called  $\mathcal{N} = 1$  AdS superspace, and can be identified with the homogeneous space OSp(1|4)/SO(3,1) introduced in [76, 77]. The superfield representations of AdS supersymmetry were classified by Ivanov and Sorin [78].

### 7.2 New minimal supergravity

In the new minimal formulation for Poincaré supergravity, the compensator is a tensor multiplet  $\mathbb{G}$  obeying the constraints (6.23). The supergravity action is given by the functional

$$S_{\text{new-minimal}} = 3 \int d^{4|4} z E \, \mathbb{G} \ln \frac{\mathbb{G}}{\bar{\phi}\phi} , \qquad (7.5)$$

which differs only by a negative overall factor from (6.24). The equation of motion for the chiral spinor prepotential  $\eta_{\alpha}$ , eq. (6.20), is

$$\bar{\nabla}^2 \nabla_\alpha \ln \frac{\mathbb{G}}{\bar{\phi}\phi} = 0 , \qquad (7.6)$$

and its general solution is given by

$$\mathbb{G} = \bar{\Phi}\Phi , \qquad K^{B}\Phi = 0 , \quad \bar{\nabla}^{\beta}\Phi = 0 , \quad \mathbb{D}\Phi = \Phi .$$
(7.7)

Here the chiral scalar  $\Phi$  is nowhere vanishing. Now, the constraint  $\bar{\nabla}^2 \mathbb{G} = 0$  leads to the equation on  $\bar{\Phi}$ 

- -

$$\bar{\nabla}^2 \bar{\Phi} = 0 , \qquad (7.8)$$

which is equivalent to the equation (7.2b) with  $\mu = 0$ . The equation of motion for the gravitational superfield can be shown to be equivalent to

$$\nabla_{\alpha} \ln \mathbb{G} \bar{\nabla}_{\dot{\alpha}} \ln \mathbb{G} - \left[ \nabla_{\alpha}, \bar{\nabla}_{\dot{\alpha}} \right] \ln \mathbb{G} = 0 , \qquad (7.9)$$

where the left-hand side proves to be a primary real vector superfield. This equation may be seen to be equivalent to (7.2c) if the representation (7.7) for  $\mathbb{G}$  in terms of  $\Phi$  is used.

The above results imply that new minimal supergravity is classically equivalent to old minimal supergravity without cosmological term.

# 7.3 Three-form supergravity

The only difference of three-form supergravity from the old minimal theory considered earlier is that the chiral compensator  $\Phi$  is realised in the former theory as  $\Phi = \Pi^{1/3}$ , where  $\Pi$  is the chiral field strength of the three-form multiplet, eq. (6.25). The supergravity action is

$$S_{\text{three-form}} = \int d^{4|4} z E \left\{ -3 \left( \bar{\Pi} \Pi \right)^{1/3} + 2mP \right\}, \qquad (7.10)$$

with *m* a real coupling constant. The equation of motion for *P* takes the simplest form in terms of  $\Phi = \Pi^{1/3}$  and its conjugate:

$$-\frac{1}{4}\bar{\nabla}^{2}\left(\bar{\Phi}\Phi^{-2}\right) - \frac{1}{4}\nabla^{2}\left(\Phi\bar{\Phi}^{-2}\right) = 2m.$$
 (7.11)

This is equivalent to the equation

$$-\frac{1}{4}\bar{\nabla}^2(\bar{\Phi}\Phi^{-2}) = \mu = \text{const}, \qquad \text{Re}\,\mu = m \tag{7.12}$$

which has the same form as (7.2b). The new feature of the supergravity theory (7.10) is that the imaginary part of  $\mu$  is now generated dynamically. It may be shown that the equation of motion for the gravitational superfield is equivalent to (7.3c). As a result, the supergravity theories (7.1) and (7.10) are classically equivalent.

The existence of three-form supergravity was first pointed out by Gates and Siegel [62]. Unlike the standard formulation of old minimal supergravity, the remarkable feature of three-form supergravity is that it allows a consistent coupling to the four-dimensional supermembrane [79] as demonstrated by Ovrut and Waldram [80] who built on the results of [43]. Within the GWZ formalism, the action (7.10) was presented in [81].

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### 7.4 Non-minimal supergravity

In the non-minimal formulation for Poincaré supergravity, the compensator is a complex linear superfield  $\Gamma$  constrained by (6.27). The supergravity action is described by the functional

$$S_{\text{non-minimal}} = \frac{1}{n} \int d^{4|4} z E \left(\bar{\Gamma} \Gamma\right)^{(3n+1)/2}, \qquad (7.13)$$

which differs from (6.29) by an overall sign. It may be shown that non-minimal supergravity is classically equivalent to old minimal supergravity without cosmological term, see [13, 35] for reviews. No supersymmetric cosmological term is allowed in non-minimal supergravity with the compensator  $\Gamma$  [13].

To describe non-minimal AdS supergravity [46], the compensator  $\Upsilon$  is chosen to obey the constraints (6.31), and the action is

$$S_{\text{non-minimal AdS}} = -\int d^{4|4} z E \left(\bar{\Upsilon}\Upsilon\right)^{-1}.$$
(7.14)

In order to derive the equation of motion for  $\Upsilon$ , we note that  $\delta\Upsilon$  is a complex linear superfield and hence  $\delta\Upsilon = \bar{\nabla}_{\dot{\alpha}} \delta\bar{\Psi}^{\dot{\alpha}}$ . Varying the action gives

$$\bar{\nabla}_{\dot{\alpha}} \left( \Upsilon^2 \bar{\Upsilon} \right)^{-1} = 0 \quad \Longrightarrow \quad \left( \Upsilon^2 \bar{\Upsilon} \right)^{-1} = \Phi^3 . \tag{7.15}$$

We see that the equation of motion for  $\Upsilon$  in the non-minimal theory (7.14) is equivalent to the off-shell constraint (7.2a) in old minimal supergravity. It follows that  $\Upsilon = \bar{\Phi}\Phi^{-2}$ , and the off-shell constraint (6.31) turns into the equation of motion (7.2b) in old minimal supergravity. Finally, it may be shown that, in the non-minimal theory (7.14), the equation of motion for the gravitational superfield is equivalent to (7.2c) once  $\Upsilon$  is expressed in terms of  $\Phi$  and its conjugate. We conclude that the minimal and non-minimal formulations for AdS supergravity, which are described by the actions (7.1) and (7.14), are classically equivalent.

## 7.5 Conformal supergravity

The conformal supergravity action [82, 18, 83], is

$$S_{\rm CSG} = -\frac{1}{4} \int d^4x d^2\theta \,\mathscr{E} \,W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} + {\rm c.c.}$$
(7.16)

The corresponding equation of motion is

$$B_{\alpha\dot{\alpha}} = \bar{B}_{\alpha\dot{\alpha}} = 0 , \qquad (7.17)$$

with  $B_{\alpha\dot{\alpha}}$  being the super-Bach tensor (3.25). This equation can be degauged to U(1) superspace to take the form

$$i\mathscr{D}_{\beta\dot{\alpha}}\mathscr{D}_{\gamma}W^{\alpha\beta\gamma} + \mathscr{D}_{\beta}(G_{\gamma\dot{\alpha}}W^{\alpha\beta\gamma}) = i\mathscr{D}_{\alpha\dot{\beta}}\bar{\mathscr{D}}_{\dot{\gamma}}\bar{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}} - \bar{\mathscr{D}}_{\dot{\beta}}(G_{\alpha\dot{\gamma}}\bar{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}) = 0.$$
(7.18)

It follows from the Bianchi identities (4.5c) and (4.10c) that every solution of the equations of motion for pure AdS supergravity (7.4) is also a solution of the equations of motion for conformal supergravity.

#### **Acknowledgements:**

We thank S. James Gates Jr. for his kind invitation to contribute this chapter to the *Handbook of Quantum Gravity*. We are grateful to Daniel Butter for useful discussions, and to Stefan Theisen for comments on the manuscript. The work of SK is supported in part by the Australian Research Council, project No. DP200101944. The work of ER is supported by the Hackett Postgraduate Scholarship UWA, under the Australian Government Research Training Program. The work of GT-M is supported by the Australian Research Council (ARC) Future Fellowship FT180100353, and by the Capacity Building Package of the University of Queensland.

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