Superstripes and the Excitation Spectrum of a Spin-Orbit-Coupled Bose-Einstein Condensate

Yun Li,[1] Giovanni I. Martone,[1] Lev P. Pitaevskii,[1,2] and Sandro Stringari[1]

[1]*INO-CNR BEC Center and Dipartimento di Fisica, Università di Trento, 38123 Povo, Italy*
[2]*Kapitza Institute for Physical Problems, RAS, Kosygina 2, 119334 Moscow, Russia*

Using Bogoliubov theory we calculate the excitation spectrum of a spinor Bose-Einstein condensed gas with an equal Rashba and Dresselhaus spin-orbit coupling in the stripe phase. The emergence of a double gapless band structure is pointed out as a key signature of Bose-Einstein condensation and of the spontaneous breaking of translational invariance symmetry. In the long wavelength limit the lower and upper branches exhibit, respectively, a clear spin and density nature. For wave vectors close to the first Brillouin zone, the lower branch acquires an important density character responsible for the divergent behavior of the structure factor and of the static response function, reflecting the occurrence of crystalline order. The sound velocities are calculated as functions of the Raman coupling for excitations propagating orthogonal and parallel to the stripes. Our predictions provide new perspectives for the identification of supersolid phenomena in ultracold atomic gases.

The search for supersolidity represents a challenging topic of research in different areas of condensed matter and atomic physics (for a recent review see, for example, [1]). Supersolidity was first predicted in the pioneering works by Andreev and Lifschitz [2], Leggett [3], and Chester [4]. It is characterized by the coexistence of two spontaneously broken symmetries. The breaking of gauge symmetry gives rise to off-diagonal long-range order yielding superfluidity, while the breaking of translational invariance yields diagonal long-range order characterizing the crystalline structure. The first experimental efforts toward the search of supersolidity were carried out in solid helium [5]. The strongly interacting nature of this system makes, however, the effects due to Bose-Einstein condensation (BEC) extremely small and no conclusive proof of supersolidity is still available in such a system [6]. More recently, systematic attempts to predict the occurrence of a supersolid phase have been carried out in atomic gases with dipolar [7–9] and soft core, finite range interactions [10–15]. However, these configurations have not yet been experimentally realized in the quantum degenerate phase required to observe the new effects.

The recent realization of spinor BECs with spin-orbit coupling [17–20] is opening new perspectives in the field. In systems with equal Rashba and Dresselhaus couplings and for small values of the Raman coupling, theory in fact predicts the occurrence of a stripe phase where translational invariance is spontaneously broken [21–23]. Actually these systems are periodic only in one direction and can be considered as superfluid nematic liquid crystals. Experiments are already available in the relevant range of parameters, but no direct evidence of the density modulations is still available, due to the smallness of the contrast and the microscopic distance separating consecutive stripes. A phase transition has been nevertheless detected

[20] at values of the Raman coupling below which theory predicts the occurrence of the stripe phase.

The purpose of this work is to show that the excitation spectrum of the gas in the stripe phase exhibits typical supersolid features, like the occurrence of two gapless bands and the divergent behavior of the static structure factor for wave vectors approaching the boundary of the Brillouin zone. The excitation spectrum is measurable in Bragg spectroscopy experiments, so the experimental characterization of the new phase should not represent a major difficulty.

Spin-orbit-coupled BECs can be described using the mean-field Gross-Pitaevskii picture. The interaction is zero ranged and is characterized by the values of the scattering lengths associated with the two hyperfine states involved in the Raman process (we limit here the discussion to spinor Bose gases). This differs from the case of other systems, like dipolar gases, where the origin of the supersolid phase is associated with the finite range of the force [15]. The validity of the Gross-Pitaevskii approach can be tested *a posteriori* by evaluating the quantum depletion of the condensate.

We use the single-particle Hamiltonian ($\hbar = m = 1$)

$$h_0 = \frac{1}{2}[(p_x - k_0\sigma_z)^2 + p_\perp^2] + \frac{\Omega}{2}\sigma_x, \qquad (1)$$

accounting for the effect of two counterpropagating and polarized laser fields, where k_0 is fixed by the momentum transfer of the two lasers, while Ω is the Raman coupling, accounting for the intensity of the laser beams causing the transition between the two spin states. The occurrence of the term $\delta\sigma_z/2$ has been ignored in h_0, since we will consider situations where the effective magnetic field δ is zero (experimentally this can be achieved with a proper detuning of the two laser fields). Hamiltonian (1) can be

formally derived by applying a unitary transformation to the Hamiltonian in the laboratory frame describing the system in the presence of two detuned, spin-polarized laser fields [26]. The unitary transformation consists of a local rotation in spin space around the z axis, causing the appearance of the spin-orbit term proportional to $p_x \sigma_z$.

Remarkable properties of this Hamiltonian are its translational invariance and, for $\Omega < 2k_0^2$, the occurrence of a double-minimum structure in the single-particle energy at momenta $p_x = \pm k_1$ with $k_1 = k_0\sqrt{1 - \Omega^2/4k_0^4}$, capable to host a BEC. This structure is at the origin of new intriguing features, like the existence of a spin-polarized plane-wave phase and of an unpolarized stripe phase [21,22] at even smaller values of Ω, resulting from the spontaneous breaking of translational symmetry. From general arguments one expects that the spontaneous breaking of this continuous symmetry is at the origin of a new gapless Goldstone mode.

The stripe phase arises due to the competition between the density and spin-density interaction terms in the mean-field Hamiltonian

$$H_{\text{int}} = \frac{1}{4} \int d^3r [(g + g_{\uparrow\downarrow})n(\mathbf{r})^2 + (g - g_{\uparrow\downarrow})s(\mathbf{r})^2], \quad (2)$$

where $n(\mathbf{r}) = n_\uparrow(\mathbf{r}) + n_\downarrow(\mathbf{r})$ and $s(\mathbf{r}) = n_\uparrow(\mathbf{r}) - n_\downarrow(\mathbf{r})$ correspond to the total and spin densities. In Eq. (2) we have assumed equal intraspecies interactions $g_{\uparrow\uparrow} = g_{\downarrow\downarrow} \equiv g$ [27] with $g_{\alpha\beta}$ ($\alpha, \beta = \uparrow, \downarrow$) being the coupling constants in the different spin channels. The stripe phase emerges only for $g_{\uparrow\downarrow} < g$, a condition yielding an unpolarized uniform ground state in the absence of the Raman coupling Ω. It is associated with the macroscopic occupation of a single-particle spinor state of the form

$$\begin{pmatrix} \psi_{0\uparrow} \\ \psi_{0\downarrow} \end{pmatrix} = \sum_{\bar{K}} \begin{pmatrix} a_{-k_1+\bar{K}} \\ -b_{-k_1+\bar{K}} \end{pmatrix} e^{i(\bar{K}-k_1)x}, \quad (3)$$

where $k_1 = \pi/d$ is related to the period d of the stripes, $\bar{K} = 2nk_1$, with $n = 0, \pm 1, \ldots$, are the reciprocal lattice vectors while $a_{-k_1+\bar{K}}$ and $b_{-k_1+\bar{K}}$ are expansion coefficients to be determined, together with the value of k_1, by a procedure of energy minimization, including the single-particle (1) and the interaction (2) terms in the Hamiltonian. In the stripe phase, energy minimization gives rise to the presence of terms with opposite phase ($e^{\pm ik_1x}, e^{\pm 3ik_1x}, \ldots$), responsible for the density modulations and characterized by the symmetry condition $a_{-k_1+\bar{K}} = b_{k_1-\bar{K}}^*$, causing the vanishing of the spin polarization. The stripe phase is favored at small values of the Raman coupling. In the limit of weak interactions, defined by the condition $G_1, G_2 \ll k_0^2$ where $G_1 = \bar{n}(g + g_{\uparrow\downarrow})/4$ and $G_2 = \bar{n}(g - g_{\uparrow\downarrow})/4$ with $\bar{n}$ being the average density, the critical value for the Raman frequency is given by $\Omega_{\text{cr}} = 2k_0^2\sqrt{2\gamma/(1 + 2\gamma)}$ with $\gamma = G_2/G_1$ independent

of the density, and the stripe phase is ensured for values of Ω smaller than Ω_{cr}. In the available experiments with [87]Rb atoms [20,28], the value of G_2 (and hence Ω_{cr}) is very small. To enlarge the range of values of Ω compatible with the stripe phase, it is useful to increase G_2 as much as possible. This is crucial to produce a significant contrast in the density profile which is proportional to Ω/k_0^2 [22]. In the following we will use the values $G_1/k_0^2 = 0.3$ and $G_2/k_0^2 = 0.08$ yielding $\Omega_{\text{cr}}/k_0^2 \simeq 1.3$. In Fig. 1 we show the ground state density profile calculated at $\Omega/k_0^2 = 1.0$. The other quantum phases predicted by theory at larger values of Ω are the plane-wave and the zero momentum phases. In these phases the sum (3) contains only the term e^{ik_1x} (or e^{-ik_1x}) with $k_1 \neq 0$ in the plane-wave phase and $k_1 = 0$ in the zero momentum one.

To evaluate the elementary excitations we apply Bogoliubov theory by writing the deviations of the order parameter with respect to equilibrium as

$$\begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = e^{-i\mu t}\left[\begin{pmatrix} \psi_{0\uparrow} \\ \psi_{0\downarrow} \end{pmatrix} + \begin{pmatrix} u_\uparrow(\mathbf{r}) \\ u_\downarrow(\mathbf{r}) \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} v_\uparrow^*(\mathbf{r}) \\ v_\downarrow^*(\mathbf{r}) \end{pmatrix} e^{i\omega t}\right]$$

(4)

and solving the corresponding linearized time-dependent Gross-Pitaevskii equations. The equations are conveniently solved by expanding $u_{\uparrow,\downarrow}(\mathbf{r})$ and $v_{\uparrow,\downarrow}(\mathbf{r})$ in the Bloch form in terms of the reciprocal lattice vectors:

$$u_{\mathbf{q}\uparrow,\downarrow}(\mathbf{r}) = e^{-ik_1x}\sum_{\bar{K}} U_{\mathbf{q}\uparrow,\downarrow\bar{K}}e^{i\mathbf{q}\cdot\mathbf{r}+i\bar{K}x}, \quad (5)$$

$$v_{\mathbf{q}\uparrow,\downarrow}(\mathbf{r}) = e^{ik_1x}\sum_{\bar{K}} V_{\mathbf{q}\uparrow,\downarrow\bar{K}}e^{i\mathbf{q}\cdot\mathbf{r}-i\bar{K}x}, \quad (6)$$

where $\mathbf{q}$ is the wave vector of the excitation. The same ansatz can be used to calculate the density and spin-density dynamic response function, by adding to the Hamiltonian a perturbation proportional to $e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)+\eta t}$ and $\sigma_z e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)+\eta t}$ with $\eta \to 0^+$, respectively.

The excitation spectrum predicted by Hamiltonians (1) and (2) has been already calculated in both the plane-wave phase and the zero momentum phases [26] where, despite
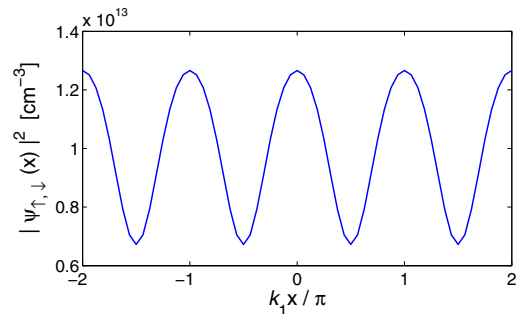

FIG. 1 (color online). Density profile along the x direction. The parameters are $\Omega/k_0^2 = 1.0$, $G_1/k_0^2 = 0.3$, and $G_2/k_0^2 = 0.08$.
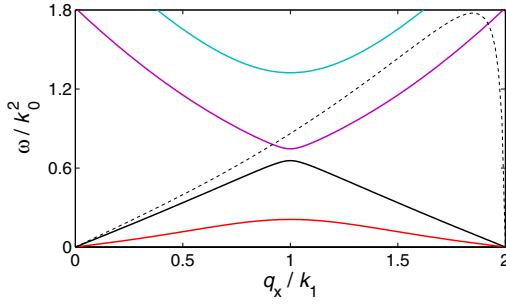
FIG. 2 (color online).   Lowest four excitation bands along the x direction. The parameters are the same as in Fig. 1. The thin dotted line corresponds to the Feynman relation $\omega = q_x^2/2S(q_x)$.

FIG. 3 (color online).   Density (a) and spin-density (b) static structure factor as a function of q_x (blue solid line). The contributions of the first (red dashed line) and second (black dash-dotted line) bands are also shown. The parameters are the same as in Fig. 1.

the spinor nature of the system, only one gapless branch is predicted as a consequence of the presence of the Raman coupling Ω. A peculiar feature exhibited by the plane-wave phase is the emergence of a rotonic structure whose gap becomes smaller and smaller as one decreases the value of Ω, providing the onset of the transition to the stripe phase.

The results for the dispersion law of the elementary excitations in the stripe phase are reported in Fig. 2 for the same parameters used in Fig. 1. We have considered excitations propagating in the x direction orthogonal to the stripes and labeled with the wave vector q_x. A peculiar feature, distinguishing the stripe phase from the other uniform phases, is the occurrence of two gapless bands. At small q_x we find that the lower branch is basically a spin excitation, while the upper branch is a density mode, as clearly revealed by Fig. 3(a) where we show the contributions of the two gapless branches to the static structure factor $S(q_x) = N^{-1}\sum_\ell |\langle 0|\rho_{q_x}|\ell\rangle|^2$ where ℓ is the band index and $\rho_{q_x} = \sum_i e^{iq_x x_i}$ is the density operator. The density nature of the upper branch, at small q_x, is further confirmed by the comparison with the Feynman relation $\omega = q_x^2/2S(q_x)$ (see Fig. 2). A two-photon Bragg scattering experiment with laser frequencies far from resonance, being sensitive to the density response, will consequently excite only the upper branch at small q_x. Bragg scattering experiments actually measure the imaginary part of the response function, a quantity which, at enough low temperature, can be identified with the $T = 0$ value of the dynamic structure factor $S(q_x, \omega) = \sum_\ell |\langle 0|\rho_{q_x}|\ell\rangle|^2 \delta(\omega - \omega_{\ell 0})$, where $\omega_{\ell 0}$ is the excitation frequency of the ℓth state [29]. The spin nature of the lower branch is clearly revealed by Fig. 3(b) where we report the contributions arising from the two gapless branches to the spin static structure factor $S_\sigma(q_x) = N^{-1}\sum_\ell |\langle 0|s_{q_x}|\ell\rangle|^2$, where $s_{q_x} = \sum_i \sigma_{zi} e^{iq_x x_i}$ is the spin-density operator. Notice that, differently from $S(q_x)$, the total spin structure factor does not vanish as $q_x \to 0$, being affected by the higher energy bands as a consequence of the Raman term in Hamiltonian (1). The lower branch exhibits a hybrid nature and, when approaching the Brillouin wave vector $q_B = 2k_1$, it is responsible
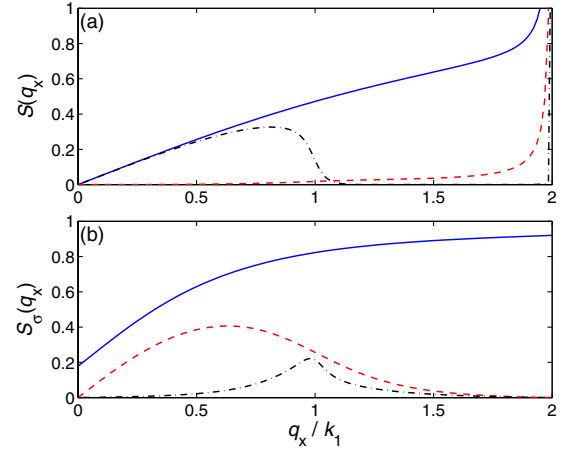
for the divergent behavior of the density static structure factor [see Fig. 3(a)], a typical feature exhibited by crystals.

It is worth pointing out that the occurrence of two gapless excitations is not by itself a signature of super-solidity and is exhibited also by uniform mixtures of BECs without spin-orbit and Raman couplings [30] as well as by the plane-wave phase of the Rashba Hamiltonian with $SU(2)$ invariant interactions ($G_2 = 0$) [31]. Only the occurrence of a band structure, characterized by the vanishing of the excitation energy and by the divergent behavior of the structure factor at the Brillouin wave vector, can be considered an unambiguous evidence of the density modulations characterizing the stripe phase. The divergent behavior near the Brillouin zone is even more pronounced (see Fig. 4) if one investigates the static response function $\chi(q_x) = 2N^{-1}\sum_\ell |\langle 0|\rho_{q_x}|\ell\rangle|^2/\omega_{\ell 0}$, proportional to the inverse energy-weighted moment of the dynamic structure factor. The divergent behaviors of $S(q_x)$ and $\chi(q_x)$ can be
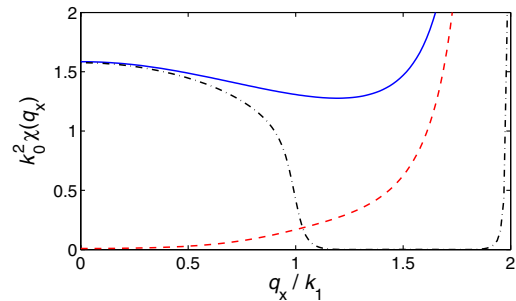

FIG. 4 (color online).   Static response as a function of q_x (blue solid line). The contributions of the first (red dashed line) and second (black dash-dotted line) bands are also shown. The parameters are the same as in Fig. 1.
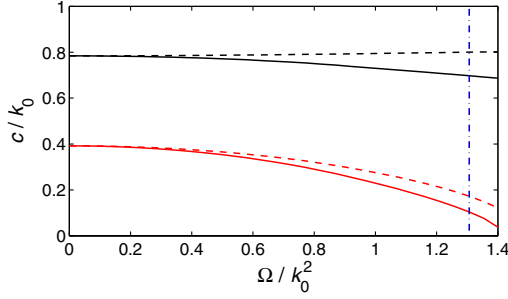
FIG. 5 (color online).   Sound velocities in the first (red) and second (black) bands along the x (c_x, solid lines) and transverse ($c_\perp$, dashed lines) directions as a function of Ω. The blue dash-dotted line represents the transition from the stripe phase to the plane-wave phase. The values of the parameters G_1/k_0^2 and G_2/k_0^2 are the same as in Fig. 1.

rigorously proven using the Bogoliubov [32] and the uncertainty principle [33] inequalities applied to systems with spontaneously broken continuous symmetries. These inequalities are based, respectively, on the relationships $m_{-1}(F)m_1(G) \geq |\langle[F, G]\rangle|^2$ and $m_0(F)m_0(G) \geq |\langle[F, G]\rangle|^2$ involving the pth moments $m_p(\mathcal{O}) = \sum_\ell (|\langle 0|\mathcal{O}|\ell\rangle|^2 + |\langle 0|\mathcal{O}^\dagger|\ell\rangle|^2)\omega_{\ell 0}^p$ of the ℓth strengths of the operators $F = \sum_j e^{iq_x x_j}$ and $G = \sum_j (p_{xj}e^{-i(q_x-q_B)x_j} + \text{H.c.})/2$. The commutator $\langle[F, G]\rangle = q_x N \langle e^{iq_B x}\rangle$, entering the right-hand side of the inequalities, coincides with the relevant crystalline order parameter and is proportional to the density modulations of the stripes [34]. The moments $m_{-1}(F)$ and $m_0(F)$ are instead proportional to the static response $\chi(q_x)$ and to the static structure factor $S(q_x)$, respectively. It is not difficult to show that the moments $m_1(G)$ and $m_0(G)$ are proportional, respectively, to $(q_x - q_B)^2$ and to $|q_x - q_B|$ as $q_x \rightarrow q_B$ due to the translational invariance of the Hamiltonian. This causes the divergent behaviors $S(q_x) \propto 1/|q_x - q_B|$ and $\chi(q_x) \propto 1/(q_x - q_B)^2$ with a weight factor proportional to the square of the order parameter [34].

In addition to the excitations propagating along x (longitudinal modes), another class of Bogoliubov modes is predicted in the transverse directions, parallel to the stripes. These modes are excited by the density and spin-density operators $\sum_i e^{iq_y y_i}$ and $\sum_i \sigma_{zi} e^{iq_y y_i}$. Remarkably, also in the transverse channel, Bogoliubov theory predicts the occurrence of two gapless spectra. Similarly to the longitudinal channel, at small q_y the lowest and upper branches have a spin and density character, respectively.

In Fig. 5 we compare the sound velocities of the two gapless branches in the longitudinal (c_x) and transverse ($c_\perp$) directions. We find that c_x is always smaller than $c_\perp$, reflecting the inertia of the flow caused by the presence of the stripes. The value of $c_\perp$ in the second band (second sound) is well reproduced by the Bogoliubov expression $\sqrt{2G_1}$ (equal to $0.78k_0$ in our case) for the sound velocity. Notice that the spin sound velocity becomes lower and lower as the Raman frequency increases, approaching the

transition to the plane-wave phase. The Bogoliubov solutions in the stripe phase exist also for values of Ω larger than the critical value $\Omega_{cr} = 1.3k_0^2$, due to the first-order nature of the transition.

We have finally checked that the quantum depletion of the condensate, due to the fluctuations associated with the Bogoliubov solutions, is always small, thereby confirming the validity of the mean-field approach.

In conclusion we have shown that the excitation spectrum in the stripe phase of a spin-orbit-coupled BEC exhibits a double gapless band structure, typical of super-solids. We predict that at small wave vectors the lower and upper branches have, respectively, a spin and density nature. The lower branch, whose gapless nature is due to the breaking of translational symmetry, is responsible for the divergent behavior of the static structure factor as the wave vector approaches the border of the Brillouin zone. The experimental verification of the new dynamic features predicted in this Letter is expected to provide a significant advance in our understanding of systems exhibiting simultaneously off-diagonal and diagonal long-range order.

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