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Supersymmetric AdS_5 solutions of M-theory

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Abstract

We analyse the most general supersymmetric solutions of $D = 11$ supergravity consisting of a warped product of five-dimensional anti-de Sitter space with a six-dimensional Riemannian space M_6, with 4-form flux on M_6. We show that M_6 is partly specified by a one-parameter family of four-dimensional Kähler metrics. We find a large family of new explicit regular solutions where M_6 is a compact, complex manifold which is topologically a 2-sphere bundle over a four-dimensional base, where the latter is either (i) Kähler–Einstein with positive curvature, or (ii) a product of two constant-curvature Riemann surfaces. After dimensional reduction and T-duality, some solutions in the second class are related to a new family of Sasaki–Einstein spaces which includes $T^{1,1}/\mathbb{Z}_2$. Our general analysis also covers warped products of five-dimensional Minkowski space with a six-dimensional Riemannian space.

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1. Introduction

String or M-theory on a supersymmetric background that contains an AdS_5 factor is expected to be equivalent to a four-dimensional superconformal field theory [1]. The best-known example is the $AdS_5 \times S^5$ solution of type IIB string theory which is conjectured to be dual to $N = 4$ super-Yang–Mills theory. This geometry arises as the near-horizon limit of the supergravity solution describing D3-branes in flat space. It is a special case of a general class of supersymmetric solutions of the form $AdS_5 \times M_5$ where M_5 is a Sasaki–Einstein 5-manifold. These arise as the near-horizon limits of solutions describing D3-branes at the singularities of Calabi–Yau cones and are dual to $N = 1$ superconformal field theories [2–4].

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The purpose of this paper is primarily to study analogous solutions in M-theory. We first analyse the most general class of supersymmetric solutions of the form of a warped $\mathrm{AdS}_5 \times M_6$ product, deriving the conditions on the geometry of the six-dimensional manifold M_6. We then use our results to construct rich new families of explicit regular compact solutions.

To date, rather surprisingly, only a handful of supersymmetric AdS_5 solutions have been found in M-theory (There are also some examples of non-supersymmetric solutions [5, 6]). A rather trivial case is the maximally supersymmetric $\mathrm{AdS}_7 \times S^4$ solution: AdS_7 can be foliated by AdS_5 and S^1 (see, for example, [7]) and hence this solution can be viewed as a warped product of AdS_5 with a (non-compact) six-dimensional space. Another example was presented in [8] where it was interpreted as the near-horizon limit of two semi-localized M5-branes (the same solution also appears in [9]). It was subsequently shown in [7] that this solution can be obtained from the type IIB $\mathrm{AdS}_5 \times S^5$ solutions by a simple T-duality followed by an uplift to $D = 11$. It was noted in [8] that the six-dimensional manifold is singular indicating that the solution is not capturing all the degrees of freedom of the intersecting M5-brane system, and hence the $D = 11$ solution is of limited utility in analysing the dual field theory.

Two more interesting examples were found in [10]. These solutions describe the near-horizon limit of 5-branes wrapping holomorphic curves in Calabi–Yau 2- or 3-folds. In these solutions M_6 is an S^4 bundle over H^2/Γ where Γ is a discrete group of isometries of H^2, and hence H^2/Γ can be compact. They are dual to the $N = 2$ or $N = 1$ superconformal field theories arising on 5-branes wrapped on a holomorphic H^2/Γ cycle in a Calabi–Yau 2- or 3-fold, respectively. These solutions were first found in $D = 7$ gauged supergravity and then uplifted to obtain solutions of $D = 11$ supergravity. Regular and compact solutions where M_6 is an S^4 bundle over S^2 were found in [11] but these have yet to be connected with a dual field theory. Finally, AdS_5 solutions were also found in [12], and were argued to be related to $N = 2$ superconformal field theories arising on intersecting 5-branes. It would be interesting to know if these solutions are regular.

This small collection of solutions have all been found by guessing a suitable ansatz for the metric on M_6. Here we will systematically determine the general conditions placed on the geometry M_6, the 4-form field strength and the warp factor in order for the solution to have $N = 1$ supersymmetry. These conditions thus characterize the most general way in which four-dimensional superconformal field theories can arise in M-theory via the AdS/CFT correspondence. Furthermore, we use our results to obtain a rich family of new solutions in explicit form. It is quite simple to extend our analysis to characterize the most general warped products of five-dimensional Minkowski space with a six-dimensional manifold and this analysis is included in an appendix.

Our method follows that employed to analyse the most general kinds of supersymmetric solutions in supergravity theories, using the language of 'G-structures' and 'intrinsic torsion'. It provides a systematic way of translating the local supersymmetry constraints into differential conditions on a set of differential forms on M_6 defining the metric and flux. The utility of G-structures for analysing supersymmetric solutions was first advocated in [13]; for related developments in various string/M-theory settings see [18–29]. Essentially the same techniques have also proved very useful in determining the general form of supersymmetric solutions in various lower-dimensional supergravities [30–34]. In the present setting, we first concentrate on the local conditions on the geometry. We will see that the Killing spinor defines a preferred local $SU(2)$ structure on M_6, characterized by a number of tensor fields constructed as bilinears in the Killing spinor. The Killing spinor equation then implies a number of differential conditions on the tensors that determine the intrinsic torsion of the structure.

It is then straightforward to see what these conditions imply for the geometry of M_6. We will show that the metric on M_6 always admits a Killing vector. In terms of the AdS/CFT correspondence this corresponds to the R-symmetry of the $N = 1$ superconformal field theory. Locally the five-dimensional space orthogonal to the Killing vector is a warped product of a one-dimensional space with coordinate y and a four-dimensional complex space with a one-parameter family of Kähler metrics depending on y.

We then use these results to construct new explicit solutions. To do this, we impose a very natural geometric condition on the geometry. Specifically, we demand that the six-dimensional space is a complex manifold. Globally, the new regular compact solutions that we construct are all holomorphic 2-sphere bundles over a smooth four-dimensional Kähler base M_4. Using a recent mathematical result on Kähler manifolds [35], we are able to completely classify this class of solutions (assuming that the Goldberg conjecture is true). In particular, at fixed y the base is either (i) a Kähler–Einstein (KE) space or (ii) a non-Einstein space which is the product of two constant curvature Riemann surfaces.

In the KE class (i), we find regular solutions only if the curvature is positive. Such spaces have been classified by Tian and Yau [36, 37] and are topologically either $S^2 \times S^2$, $\mathbb{C}P^2$ or $\mathbb{C}P^2 \#_n \mathbb{C}P^2$ with $n = 3, \ldots, 8$. The KE metrics are of course explicitly known in the first two cases, but are not known explicitly for the other examples. In the second product class (ii), one finds regular compact solutions when the geometry is $S^2 \times S^2$, or $S^2 \times T^2$. There are also regular solutions of the form $S^2 \times H^2$, where H^2 is hyperbolic space, but these are not compact. Note, however, that we recover the $N = 1$ solution of [10], for which it is known that one can divide H^2 by a discrete group of isometries to obtain compact solutions while preserving supersymmetry. A number of additional singular solutions in both the first and second classes are also found in explicit form.

The $S^2 \times T^2$ class of solutions has the particularly interesting property that they can be reduced on an S^1 in T^2 to obtain regular supersymmetric type IIA supergravity solutions of the form $AdS_5 \times X'_5$. Then, via a T-duality on the other S^1, these give solutions of the form $AdS_5 \times X_5$ where X_5 is a new one-parameter family of Sasaki–Einstein spaces. Global properties of these will be studied in more detail in a separate paper [38], but we note that one special case gives $AdS_5 \times T^{1,1}/\mathbb{Z}_2$, where $T^{1,1}/\mathbb{Z}_2$ is a well-known Sasaki–Einstein manifold[4]; the superconformal field theory was identified in [4] (see also [2]). It will be clearly interesting to identify dual conformal field theories for our new solutions and determine under which conditions the M-theory, type IIA or type IIB supergravity solution is most useful.

The plan of the rest of the paper is as follows. Section 2 contains the analysis of the conditions imposed on the geometry of M_6 in order to get a supersymmetric solution. We have tried to minimize the details as much as possible, relegating most of the calculation to two appendices B and C. The conditions on the local form of the metric and flux are summarized at the end of section 2.3. Section 3 starts by analysing the additional local conditions on the geometry that arise by assuming that M_6 is a complex manifold. We then discuss a natural ansatz for the global topology of the solutions, requiring M_6 to be an S^2 fibration over a four-dimensional manifold M_4. Regular compact solutions of this type are shown to fall into two classes: one where the base is KE, the other where it is a product of constant curvature Riemann surfaces. In either case, we show that the supersymmetry conditions reduce to solving a single nonlinear ordinary differential equation. Section 4 discusses the explicit solutions in the first class, for positive, negative and zero curvature. Section 5 discusses the explicit solutions of the second type, again for each different possible sign of curvature. We also discuss the type IIA

[4] Note here, and throughout the paper, $T^{1,1}/\mathbb{Z}_2$ refers to the unique smooth quotient of $T^{1,1}$ by $\mathbb{Z}_2$ that preserves the Sasaki–Einstein structure.

and IIB duals to the $S^2 \times T^2$ solutions and how the solution of [10] is obtained in our formalism. Finally, we end with some conclusions. In the remaining appendices, we give a summary of our conventions (appendix A) and analyse the closely related case of supersymmetric warped products of Minkowski space with M_6 (appendix D).

2. The conditions for supersymmetry

We want to find the general structure of supersymmetric configurations of $D = 11$ supergravity that are warped products of an external AdS_5 space with an internal six-dimensional Riemannian manifold M_6:

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{e}^{2\lambda(v)}[\mathrm{d}s^2(\mathrm{AdS}_5) + \mathrm{d}s^2(M_6)] \\ G &= \frac{1}{4!} G_{\mu_1\mu_2\mu_3\mu_4} \mathrm{d}v^{\mu_1} \wedge \mathrm{d}v^{\mu_2} \wedge \mathrm{d}v^{\mu_3} \wedge \mathrm{d}v^{\mu_4} \end{aligned} \tag{2.1}$$

where $u^\alpha, \alpha = 0, 1, \ldots, 4$ are coordinates on AdS_5 and $v^\mu, \mu = 1, 2, \ldots, 6$ are coordinates on M_6. We assume that the AdS_5 space has radius squared given by m^{-2} so that

$$R_{\alpha\beta} = -4m^2 g_{\alpha\beta}. \tag{2.2}$$

Note that when $m = 0$ our ansatz reduces to warped products of five-dimensional Minkowski space with M_6. For completeness, this case is discussed separately in appendix D.

Our conventions for $D = 11$ supergravity are included in appendix A. We want solutions admitting at least one supersymmetry and satisfying the equations of motion. Supersymmetry implies we have a solution of the Killing spinor equation

$$\nabla_\mu \epsilon + \frac{1}{288}\left[\Gamma_\mu{}^{\nu_1\nu_2\nu_3\nu_4} - 8\delta_\mu^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}\right] G_{\nu_1\nu_2\nu_3\nu_4}\epsilon = 0 \tag{2.3}$$

where ϵ is a Majorana spinor in the representation where $\Gamma_{11} \equiv \Gamma_0\Gamma_1 \ldots \Gamma_{10} = 1$. To see how this reduces to a condition on M_6 we first decompose the $D = 11$ Clifford algebra $\mathrm{Cliff}(10, 1) \cong \mathrm{Cliff}(1, 4) \otimes \mathrm{Cliff}(6, 0)$. Explicitly, $\mathrm{Cliff}(6, 0) \cong H(4)$ and $\mathrm{Cliff}(1, 4) \cong H(2) \oplus H(2)$ and hence the tensor product gives $R(32) \oplus R(32) \cong \mathrm{Cliff}(10, 1)$. In other words, we can write the $D = 11$ gamma matrices as

$$\Gamma^a = \rho^a \otimes \gamma_7 \qquad \Gamma^m = 1 \otimes \gamma_m \tag{2.4}$$

where $a, b = 0, 1, \ldots, 4$ and $m, n = 1, 2, \ldots, 6$ are frame indices on AdS_5 and M_6 respectively, and we have

$$[\rho^a, \rho^b]_+ = -2\eta^{ab} \qquad [\gamma^m, \gamma^n]_+ = 2\delta^{mn} \tag{2.5}$$

with $\eta^{ab} = \mathrm{diag}(-1, 1, 1, 1, 1)$. Note that

$$\gamma_7 \equiv \gamma_1 \ldots \gamma_6 \tag{2.6}$$

so that $(\gamma_7)^2 = -1$. Given our conventions in $D = 11$, we then have $\rho_{01234} = -1$. The parameter ϵ decomposes as $\psi(u) \otimes \mathrm{e}^{\lambda/2}\xi(v)$, where the dependence on λ is added to simplify the resulting formulae. On the AdS_5 space, the Killing spinor satisfies

$$D_a\psi \equiv \left(\partial_a - \tfrac{1}{4}\omega_{abc}\rho^{bc}\right)\psi = \tfrac{1}{2}\mathrm{i}m\rho_a\psi. \tag{2.7}$$

The $D = 11$ Killing spinor equation then implies that the internal ξ must satisfy

$$\begin{aligned} &\left[\nabla_m + \frac{1}{2}\mathrm{i}m\gamma_m\gamma_7 - \frac{1}{24}\,\mathrm{e}^{-3\lambda}\gamma^{n_1n_2n_3}G_{mn_1n_2n_3}\right]\xi = 0 \\ &\left[\gamma^m\nabla_m\lambda + \frac{1}{144}\,\mathrm{e}^{-3\lambda}\gamma^{m_1m_2m_3m_4}G_{m_1m_2m_3m_4} - \mathrm{i}m\gamma_7\right]\xi = 0. \end{aligned} \tag{2.8}$$

For a true supergravity solution we must check that a background satisfying these equations is actually a solution of the equations of motion. By analysing the integrability conditions, and using the arguments presented in [20], we find that a geometry admitting solutions to (2.8) will satisfy the equations of motion if the Bianchi identity and the equation of motion for the 4-form G are imposed. Given our ansatz (2.1), these reduce to the following two conditions on M_6:

$$\mathrm{d}(\mathrm{e}^{3\lambda} *_6 G) = 0 \qquad \mathrm{d}G = 0. \tag{2.9}$$

The spinor ξ is a representation of the Clifford algebra Cliff(6, 0). Under $Spin(6, 0)$ it decomposes into chiral and anti-chiral pieces $\xi = \xi_+ + \xi_-$ where $-\mathrm{i}\gamma_7\xi_\pm = \pm\xi_\pm$. It is straightforward to analyse the Killing spinor equations (2.8) in the particular case that ξ is chiral. For either chirality, one finds that $m = 0$, $\lambda =$ constant, $G = 0$ and $\nabla\xi = 0$. This implies that solutions are, after possibly going to the covering space, simply the direct product $\mathbb{R}^{1,4} \times M_6$, with M_6 being a Calabi–Yau 3-fold, and with zero G-flux. In particular, there are no AdS_5 solutions, which is the case of primary interest in this paper[5]. Henceforth, we therefore assume that ξ is of indefinite chirality.

In summary, we want to analyse (2.8), combined with the Bianchi identity and equation of motion (2.9) for G. In particular, we will recast these conditions into equivalent conditions on the local geometry of M_6, the warp factor and the 4-form.

2.1. Spinor bilinears and local SU(2)-structure

We will derive the local form of the metric by first deriving a set of differential conditions on a set of spinor bilinears formed from ξ. The details of the calculation are described in appendices B and C. First note that generically the non-chiral ξ can be decomposed as

$$\xi \equiv \xi_+ + \xi_- = f_1\eta_1 + f_2[a\eta_1 + (1 - |a|^2)^{1/2}\eta_2]^* \tag{2.10}$$

where η_i are two orthogonal unit-norm chiral spinors, and f_i and a are real and complex functions, respectively. As we show in appendices B and C, the supersymmetry conditions (2.8) imply that ξ has constant norm and furthermore $\xi^{\mathrm{T}}\xi = 0$. This implies we can choose a normalization such that we have

$$\xi = \sqrt{2}(\cos\alpha\,\eta_1 + \sin\alpha\,\eta_2^*). \tag{2.11}$$

Since the stabilizer of (η_1, η_2) in $SO(6)$ is $SU(2)$ they define a particular privileged local $SU(2)$ structure on M_6. Equivalently, the structure can be specified by a set of bilinear forms constructed from η_i. It is worth emphasizing that since η_i are not globally defined in general (in particular, one or the other is not defined when $\sin\alpha = 0$ and $\cos\alpha = 0$), the $SU(2)$ structure on M_6 is not globally defined in general either. We will address the global G-structure in the case of a specific class of global solutions in section 3.

Nevertheless, in deriving the local form of the metric it is still very useful to derive differential conditions on the local $SU(2)$ structure. A basic set of bilinears specifying the local $SU(2)$ structure is given in equations (C.4) and (C.6) in appendix C. They consist of a fundamental (1, 1)-form J, a complex (2, 0)-form Ω and two 1-forms K^1 and K^2. The metric can be written as

$$\mathrm{d}s^2 = e^i e^i + (K^1)^2 + (K^2)^2 \tag{2.12}$$

[5] Note that this is somewhat analogous to M-theory compactifications on 8-manifolds, where it was shown in [39, 23] that a chiral internal spinor rules out most of the possible internal fluxes, and fixes the internal space to be conformal to a $Spin(7)$-holonomy manifold.

where $J = e^1 \wedge e^2 + e^3 \wedge e^4$ and $\Omega = (e^1 + \mathrm{i}e^2) \wedge (e^3 + \mathrm{i}e^4)$. The volume form is defined by $\mathrm{vol}_6 = \frac{1}{2} J \wedge J \wedge K^1 \wedge K^2$. These forms satisfy the following set of differential constraints, derived in appendices B and C:

$$\mathrm{e}^{-3\lambda}\, \mathrm{d}(\mathrm{e}^{3\lambda} \sin\zeta) = 2m K^1 \cos\zeta \tag{2.13}$$

$$\mathrm{e}^{-6\lambda}\, \mathrm{d}(\mathrm{e}^{6\lambda} \Omega \cos\zeta) = 3m\Omega \wedge (-K^1 \sin\zeta + \mathrm{i}K^2) \tag{2.14}$$

$$\mathrm{e}^{-6\lambda}\, \mathrm{d}(\mathrm{e}^{6\lambda} K^2 \cos\zeta) = \mathrm{e}^{-3\lambda} * G + 4m(J - K^1 \wedge K^2 \sin\zeta) \tag{2.15}$$

$$\mathrm{e}^{-6\lambda}\, \mathrm{d}(\mathrm{e}^{6\lambda} J \wedge K^2 \cos\zeta) = \mathrm{e}^{-3\lambda} G \sin\zeta + m(J \wedge J - 2J \wedge K^1 \wedge K^2 \sin\zeta). \tag{2.16}$$

In the last formula we have used $*J = J \wedge K^1 \wedge K^2$. Here we have defined $\zeta = \pi/2 - 2\alpha$, giving $\cos 2\alpha = \sin\zeta$, $\sin 2\alpha = \cos\zeta$. Note that multiplying each of the first three equations by a suitable power of e^{λ} and taking the exterior derivative so that the left-hand side vanishes leads to three more equations which should be separately imposed when $m = 0$. (This case is discussed separately in appendix D.)

We will argue in the following section that the above differential conditions are the set of necessary and sufficient local differential conditions for the geometry to admit a Killing spinor. In addition, we will show that they automatically imply that the equation of motion (2.9) and (for $m \neq 0$) the Bianchi identity for G are satisfied.

2.2. *Reduction from $d = 7$*

Before analysing what these conditions imply about the local form of the metric, we briefly pause to discuss how they can be obtained in a different way. If we write AdS_5 in Poincaré coordinates, the warped product of AdS_5 with M_6 that we are considering can be viewed as a special case of a warped product of Minkowski 4-space $\mathbb{R}^{1,3}$ with a seven-dimensional Riemannian manifold M_7. That is, we can rewrite (2.1) as

$$\mathrm{d}s^2 = \mathrm{e}^{2\lambda}\, \mathrm{e}^{-2mr}\, \mathrm{d}s^2(\mathbb{R}^{1,3}) + \mathrm{e}^{2\lambda}\big[\mathrm{d}r^2 + \mathrm{d}s_6^2\big] \tag{2.17}$$

with G being a 4-form on M_6 that is independent of r. General conditions on the geometry of M_7 imposed by supersymmetry have been analysed in [22] and then subsequently in [26, 28]. We now show how to obtain our conditions from these results.

It was shown in [22] that the $d = 7$ geometry is determined by an $SU(3)$ structure specified by a vector K', a 2-form J' and a 3-form Ω'. In particular, we write the $d = 7$ metric as

$$\mathrm{d}s_7^2 = e'^a e'^a + (K')^2 \tag{2.18}$$

with $J' = e'^1 \wedge e'^2 + e'^3 \wedge e'^4 + e'^5 \wedge e'^6$ and $\Omega' = (e'^1 + \mathrm{i}e'^2) \wedge (e'^3 + \mathrm{i}e'^4) \wedge (e'^5 + \mathrm{i}e'^6)$. The volume form is given by $\mathrm{vol}_7 = \frac{1}{3!} J' \wedge J' \wedge J' \wedge K'$. The differential conditions are given, in our conventions, by

$$\begin{aligned} &\mathrm{d}(\mathrm{e}^{2\Delta} K') = 0 \qquad && \mathrm{e}^{-4\Delta}\, \mathrm{d}(\mathrm{e}^{4\Delta} J') = *_7 G \\ &\mathrm{d}(\mathrm{e}^{3\Delta} \Omega') = 0 \qquad && \mathrm{e}^{-2\Delta}\, \mathrm{d}(\mathrm{e}^{2\Delta} J' \wedge J') = -2G \wedge K' \end{aligned} \tag{2.19}$$

with metric

$$\mathrm{d}s^2 = \mathrm{e}^{2\Delta}\, \mathrm{d}s^2(\mathbb{R}^{1,3}) + \mathrm{d}s_7^2. \tag{2.20}$$

The first three equations were derived in [22] while the last equation corrects that appearing in [22] by a factor. It was argued in [28] that these are sufficient conditions for a geometry to admit a Killing spinor. Furthermore, the second equation implies the G equation of motion and thus, given an integrability argument as in [20], only the Bianchi identity $\mathrm{d}G = 0$ need be imposed to give a solution to the full equations of motion.

It is easy to show that our set of equations (2.13)–(2.16) for the $d = 6$ geometry are precisely equivalent to the conditions (2.19) together with the AdS$_5$ metric ansatz (2.17). The correspondence is given by first identifying $e^{2\Delta} = e^{2\lambda} e^{-2mr}$. The forms (K', J', Ω') are then related to (J, Ω, K_1, K_2) by

$$\begin{aligned} K' &= e^{\lambda}(\cos\zeta K^1 - \sin\zeta\, dr) \\ J' &= e^{2\lambda}(J - (\sin\zeta K^1 + \cos\zeta\, dr) \wedge K^2) \\ \Omega' &= e^{3\lambda}\Omega \wedge (-\sin\zeta K^1 - \cos\zeta\, dr + iK^2). \end{aligned} \tag{2.21}$$

This structure arises as follows. In order to obtain AdS$_5$ we need to select a radial direction from the $d = 7$ geometry, which we denote by the unit 1-form $e^{\lambda} dr$. In general, this radial direction will point partly in the K' direction and partly in the $d = 6$ space orthogonal to K'. In other words we have

$$e^{\lambda} dr = -\sin\zeta K' - \cos\zeta W \qquad e^{\lambda} K^1 = \cos\zeta K' - \sin\zeta W \tag{2.22}$$

where W is a unit 1-form in M_6. We have also defined K^1 as the unit 1-form, orthogonal linear combination of K' and W. (The angle ζ is chosen to match the definition in the previous section). The almost complex structure determined by $J'^{\,b}_{a}$ in $d = 6$ pairs the 1-form W with another unit 1-form $e^{\lambda} K^2 \equiv J' \cdot W$. With these definitions, inverting the rotation (2.22) to write (K', W) in terms of (dr, K^1), one then gets the expressions in (2.21) for K', J' and Ω'.

As mentioned, the $d = 7$ conditions (2.19) are necessary and sufficient for a geometry which admits a Killing spinor. Given our metric ansatz they are equivalent to our conditions (2.13)–(2.16) on M_6. Hence, our conditions are also necessary and sufficient for supersymmetry.

To ensure we have a solution of the equations of motion, in general one also needs to impose the equation of motion and Bianchi identity (2.9) for G. The connection with the $d = 7$ results gives us a quick way of seeing that, in fact, provided $\sin\zeta$ is not identically zero, both conditions are a consequence of the supersymmetry constraints (2.13)–(2.16). As already noted, the equation of motion for G follows directly from the exterior derivative of the second equation in (2.19). For the Bianchi identity one notes that, given the ansatz for the $d = 7$ metric and G, the first and last equations in (2.19) imply in general that

$$\sin\zeta\, dG \wedge dr = 0 \tag{2.23}$$

since dG lies solely in M_6. This implies that $dG = 0$ provided $\sin\zeta$ is not identically zero–which can occur only when $m = 0$. Thus we see that, when $m \neq 0$, the constraints (2.13)–(2.16) are necessary and sufficient both for supersymmetry and for a solution of the equations of motion.

2.3. Local form of the metric

In this section, we use the differential conditions on the forms derived in section 2.1 to give the local form of the metric. We start by considering K^1 and K^2.

For K^1, we can immediately integrate condition (2.13) and introduce coordinates (w^M, y) with $M = 1, \ldots, 5$ and y defined by

$$2my = e^{3\lambda} \sin\zeta \tag{2.24}$$

so that

$$K^1 = e^{-3\lambda} \sec\zeta\, dy. \tag{2.25}$$

While we could eliminate either λ or ζ from the following formulae, for the moment it will be more convenient to keep both. The metric then has the form

$$ds^2 = g^5_{MN}(w, y)\, dw^M dw^N + e^{-6\lambda(w,y)} \sec^2\zeta(w, y)\, dy^2. \tag{2.26}$$

Turning to K^2, as discussed in appendix B, it is easy to show, starting from the Killing spinor equations, that $\tilde{K}^2_\mu = \frac{1}{2}\bar{\xi}\gamma_\mu\gamma_7\xi = \cos\zeta K^2_\mu$ satisfies $\nabla_{(\mu}\tilde{K}^2_{\nu)} = 0$ and hence, raising an index, we have the important condition that

$$\tilde{K}^2 = \cos\zeta K^2 \quad \text{defines a Killing vector.} \tag{2.27}$$

(Note that one could also show this from conditions (2.13)–(2.16) since they are necessary and sufficient—it is simply much easier to obtain this result directly from the Killing spinor equations). We also have $\mathcal{L}_{\tilde{K}^2}\lambda = 0$ (see (B.8)) and $\mathcal{L}_{\tilde{K}^2}\zeta = \mathcal{L}_{\tilde{K}^2}K^1 = 0$ where $\mathcal{L}_{\tilde{K}^2}$ is the Lie derivative with respect to $\tilde{K}^2$. This allows us to refine the local coordinates $w^M = (x^i, \psi)$ with $i = 1, \ldots, 4$, where as a vector $\tilde{K}^2 = 3m\partial/\partial\psi$, and the factor of $3m$ has been inserted for later convenience. Hence

$$K^2 = \frac{1}{3m}\cos\zeta(\mathrm{d}\psi + \rho) \tag{2.28}$$

with $\rho = \rho_i(x^j, y)\,\mathrm{d}x^i$. Locally the metric is then a product of a four-dimensional space M_4 and the K^1 and K^2 directions:

$$\mathrm{d}s^2 = g^4_{ij}(x, y)\,\mathrm{d}x^i\,\mathrm{d}x^j + \mathrm{e}^{-6\lambda(x,y)}\sec^2\zeta(x, y)\,\mathrm{d}y^2 + \frac{1}{9m^2}\cos^2\zeta(x, y)(\mathrm{d}\psi + \rho)^2 \tag{2.29}$$

with λ and ζ independent of ψ. Also note, from (2.15), that $i_{\tilde{K}^2} * G = -6\mathrm{e}^{3\lambda}\,\mathrm{d}\lambda$ and hence

$$\mathcal{L}_{\tilde{K}^2}G = 0. \tag{2.30}$$

In other words, the Killing vector generates a symmetry not only of the metric, and the AdS_5 warp factor λ, but also of the 4-form flux G.

Now let us turn to the metric g^4 on the four-dimensional part of the space M_4. The forms $J = \frac{1}{2}J_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j$ and $\Omega = \frac{1}{2}\Omega_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j$ define a local $SU(2)$ structure on M_4. Although the metric is independent of ψ, it does not necessarily follow that J and Ω are. They also explicitly depend on x^i and y. Let us write

$$\mathrm{d} = \mathrm{d}_4 + \mathrm{d}y \wedge \partial_y + \mathrm{d}\psi \wedge \partial_\psi. \tag{2.31}$$

It is also useful to define a rescaled structure and corresponding metric

$$J = \mathrm{e}^{-6\lambda}\hat{J} \qquad \Omega = \mathrm{e}^{-6\lambda}\hat{\Omega} \qquad g^4_{ij} = \mathrm{e}^{-6\lambda}\hat{g}_{ij} \tag{2.32}$$

so that the $d = 6$ metric becomes

$$\mathrm{d}s^2 = \mathrm{e}^{-6\lambda(x,y)}[\hat{g}_{ij}(x, y)\,\mathrm{d}x^i\,\mathrm{d}x^j + \sec^2\zeta(x, y)\,\mathrm{d}y^2] + \frac{1}{9m^2}\cos^2\zeta(x, y)(\mathrm{d}\psi + \rho)^2. \tag{2.33}$$

Considering first J, the supersymmetry condition (2.15), together with the 4-form equation of motion (2.9), implies

$$\begin{aligned}\mathrm{e}^{-6\lambda}\,\mathrm{d}(\mathrm{e}^{6\lambda}J) &= K^1 \wedge (\mathrm{d}\log\cos\zeta \wedge K^2 - \mathrm{d}K^2)\sin\zeta \\ &= -\frac{1}{3m}\,\mathrm{d}\rho \wedge K^1\cos\zeta\,\sin\zeta.\end{aligned} \tag{2.34}$$

Decomposing, we find

$$\mathrm{d}_4\hat{J} = 0 \tag{2.35}$$

$$\partial_y\hat{J} = -\tfrac{2}{3}y\,\mathrm{d}_4\rho \tag{2.36}$$

$$\partial_\psi\hat{J} = 0. \tag{2.37}$$

For Ω we can work directly from condition (2.14). We find

$$d_4\hat{\Omega} = (i\rho - d_4 \log\cos\zeta) \wedge \hat{\Omega} \tag{2.38}$$

$$\partial_y \hat{\Omega} = \left(-\tfrac{3}{2} y^{-1} \tan^2\zeta - \partial_y \log\cos\zeta\right)\hat{\Omega} \tag{2.39}$$

$$\partial_\psi \hat{\Omega} = i\hat{\Omega}. \tag{2.40}$$

We note in particular that $d\hat{\Omega} = A \wedge \hat{\Omega}$, for a suitable 1-form A. This implies that the almost complex structure defined by $\hat{\Omega}$, or equivalently $\hat{J}^i{}_j$, is independent of ψ and y and is integrable on M_4. In other words, M_4 is a complex manifold. Furthermore, from (2.35), we see that $d_4\hat{J} = 0$, and thus we have

$$\hat{g} \text{ is locally a family of Kähler metrics on } M_4 \text{ parametrized by } y. \tag{2.41}$$

The corresponding complex structure is independent of y and ψ, while from (2.37) we see the Kähler form $\hat{J}$ is independent of ψ.

Given the family of Kähler metrics $\hat{g}$, we have the general expression

$$d_4\hat{\Omega} = i\hat{P} \wedge \hat{\Omega} \tag{2.42}$$

where $\hat{P}$ is the canonical Ricci-form connection defined by the Kähler metric (at fixed y). That is, if the Ricci-form $\hat{\mathfrak{R}}$ is defined by $\hat{\mathfrak{R}}_{ij} = \frac{1}{2}\hat{R}^4_{ijkl}\hat{J}^{kl}$ then

$$\hat{\mathfrak{R}} = d_4\hat{P}. \tag{2.43}$$

Thus, the content of equation (2.38) is to fix the 1-form ρ in terms of $\hat{P}$ and $d_4\zeta$

$$\rho = \hat{P} + \hat{J} \cdot d_4 \log\cos\zeta. \tag{2.44}$$

The $\partial_\psi \hat{\Omega}$ condition is easily solved by redefining the phase of $\hat{\Omega}$

$$\hat{\Omega}(x, y, \psi) = e^{i\psi}\hat{\Omega}_0(x, y). \tag{2.45}$$

The remaining content of the $\partial_y \hat{\Omega}$ equation (2.39) is to fix the y-variation of the volume of $\hat{g}^4$. Recalling that $\hat{\Omega} \wedge \bar{\hat{\Omega}} = 4\widehat{\mathrm{vol}}_4$, we find

$$\partial_y \log\sqrt{\hat{g}} = -3y^{-1}\tan^2\zeta - 2\partial_y \log\cos\zeta. \tag{2.46}$$

Note that compatibility of the last equation with equation (2.44) implies that[6]

$$e^{3\lambda}\cos^3\zeta\, \partial_y\rho = -6m\hat{J} \cdot d_4\zeta \tag{2.47}$$

or equivalently that $e^{3\lambda}\cos^3\zeta\,\partial_y\rho - 6m i\, d_4\zeta$ is a $(1,0)$-form on M_4.

For a general 2-form ω on M_4 we can define self-dual and anti-self-dual combinations by $\omega^\pm = \frac{1}{2}(\omega \pm \hat{*}_4\omega)$. We have the identity

$$(\partial_y\hat{J})^+ = \tfrac{1}{2}\partial_y \log\sqrt{\hat{g}}\,\hat{J} \tag{2.48}$$

valid when the complex structure, $J_i{}^j$, is independent of y. Given the relation (2.36), we then see that the condition on the volume (2.46) can be written in the form

$$(d_4\rho)^+ = 3m^2 e^{-6\lambda}\sec^2\zeta(1 + 6y\partial_y\lambda)\hat{J}. \tag{2.49}$$

These conditions are in fact sufficient to ensure all the relations (2.13)–(2.16) are satisfied. The only remaining point is that one finds the flux G is given by

$$\begin{aligned} G = {} & -(\partial_y e^{-6\lambda})\widehat{\mathrm{vol}}_4 - e^{-9\lambda}\sec\zeta(\hat{*}_4 d_4 e^{6\lambda}) \wedge K^1 - \frac{1}{3m}\cos^3\zeta(\hat{*}_4\partial_y\rho) \wedge K^2 \\ & + e^{3\lambda}\left[\frac{1}{3m}\cos^2\zeta\,\hat{*}_4 d_4\rho - 4m e^{-6\lambda}\hat{J}\right] \wedge K^1 \wedge K^2. \end{aligned} \tag{2.50}$$

[6] Recall that when $\hat{g}$ is written in terms of four-dimensional complex coordinates, defined by the y-independent complex structure $\hat{J}_i{}^j$, locally we have $\hat{P} = \frac{i}{2}(\partial - \bar{\partial})\log\sqrt{\hat{g}}$.

In summary, we have shown that the necessary and sufficient local conditions for M_6 to be supersymmetric and solve the equations of motion are that the metric has the form

$$\mathrm{d}s^2 = \mathrm{e}^{-6\lambda}(\hat{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j + \sec^2\zeta\,\mathrm{d}y^2) + \frac{1}{9m^2}\cos^2\zeta(\mathrm{d}\psi + \rho)^2 \qquad (2.51)$$

where $\hat{g}$, λ and ζ and $\rho = \rho_i\,\mathrm{d}x^i$ are all functions of x^i and y. We also have

(a) $\partial/\partial\psi$ is a Killing vector (2.52)

(b) $\hat{g}$ is a family of Kähler metrics on M_4 parametrized by$y$ (2.53)

(c) the corresponding complex structure $\hat{J}_i{}^j$ is independent of y and ψ. (2.54)

In addition

(d) $2my = \mathrm{e}^{3\lambda}\sin\zeta$ (2.55)

(e) $\rho = \hat{P} + \hat{J}\cdot\mathrm{d}_4\log\cos\zeta$ (2.56)

where, in complex coordinates, $\hat{P} = \frac{1}{2}\hat{J}\cdot\mathrm{d}\log\sqrt{\hat{g}}$ is the Ricci-form connection defined by the Kähler metric, satisfying $\hat{\mathfrak{R}} = \mathrm{d}\hat{P}$. Finally, we have the conditions

(f) $\partial_y\hat{J} = -\frac{2}{3}y\,\mathrm{d}_4\rho$ (2.57)

(g) $\partial_y\log\sqrt{\hat{g}} = -3y^{-1}\tan^2\zeta - 2\partial_y\log\cos\zeta$ (2.58)

where, using (2.57), the last expression can also be written as

$$(\mathrm{d}_4\rho)^+ = 3m^2\,\mathrm{e}^{-6\lambda}\sec^2\zeta(1 + 6y\partial_y\lambda)\hat{J}. \qquad (2.59)$$

The 4-form flux G is given by (2.50) and is independent of ψ—that is, $\mathcal{L}_{\partial/\partial\psi}G = 0$. As shown in the previous subsection, the equations of motion for G and the Bianchi identity (2.9) are implied by expressions (2.52)–(2.58).

3. Complex M_6 and explicit solutions

In this section, we consider how the conditions on the metric specialize for solutions where the six-dimensional space M_6 is a complex manifold. As we will see this additional condition is equivalent to

$$\mathrm{d}_4\zeta = 0 \qquad \mathrm{d}_4\lambda = 0 \qquad \partial_y\rho = 0 \qquad (3.1)$$

and from the condition (2.56), we see that ρ then coincides with the canonical connection on the Kähler manifold M_4:

$$\rho = \hat{P}. \qquad (3.2)$$

Crucially, the supersymmetry conditions simplify considerably and we are able to find many solutions in closed form. In particular, we find many new regular and compact solutions that are topologically 2-sphere bundles over the Kähler base. These fall into two general classes and are discussed in detail in sections 4 and 5. In this section, we first analyse what the assumption that we have a complex structure implies about the geometry of M_6 and then describe the global topology of the class of solutions we consider.

3.1. Conditions on local geometry

We would like to specialize to the case where

$$\mathrm{d}s^2(M_6) \text{ is a Hermitian metric on a complex manifold } M_6.$$

There is a natural almost complex structure compatible with $\mathrm{d}s^2(M_6)$ and the local $SU(2)$-structure given by complex 3-form $\Omega_{(3)} = \Omega \wedge (K^1 + \mathrm{i}K^2)$ (Note one could equally well consider the 3-form $\Omega \wedge (K^1 - \mathrm{i}K^2)$ and get the same results). We find that $\mathrm{d}\Omega_{(3)}$ has the form

$$\mathrm{d}\Omega_{(3)} = A \wedge \Omega_{(3)} + v \wedge \Omega \wedge (K^1 - \mathrm{i}K^2) \tag{3.3}$$

where v is a 1-form given by

$$v = -\left(\tan\zeta + \tfrac{1}{2}\cot\zeta\right)\mathrm{d}_4\zeta + \mathrm{i}\tfrac{1}{6m}\,\mathrm{e}^{3\lambda}\cos^2\zeta\,\partial_y\rho. \tag{3.4}$$

In deriving (3.3) we have used the fact that $\mathrm{d}_4\rho \wedge \Omega = 0$, which is a consequence of (2.56). For $\Omega_{(3)}$ to define an integrable complex structure, the second term in (3.3) must vanish. In general, this implies $v \wedge \Omega = 0$ or equivalently v is a (0, 1)-form on M_4. This means that

$$\mathrm{e}^{3\lambda}\cos^2\zeta\,\partial_y\rho = -6m\left(\tan\zeta + \tfrac{1}{2}\cot\zeta\right)\hat{J}\cdot\mathrm{d}_4\zeta. \tag{3.5}$$

However, we also have the integrability condition (2.47). The only way both equations can hold is if $\mathrm{d}_4\zeta = \partial_y\rho = 0$ and, in fact, $v = 0$. From (2.55) we immediately have $\mathrm{d}_4\lambda = 0$ and from (2.56) $\rho = \hat{P}$. We thus get the results (3.1) and (3.2) stated above.

We will next derive conditions on the eigenvalues of the Ricci tensor for the metric on the four-dimensional space M_4. By definition, $\hat{\mathfrak{R}} = \mathrm{d}_4\hat{P}$ so conditions (2.57) and (2.59) now read

$$\hat{\mathfrak{R}} = -\frac{3}{2y}\partial_y\hat{J} \qquad \hat{\mathfrak{R}}^+ = 3m^2\,\mathrm{e}^{-6\lambda}\sec^2\zeta(1 + 6y\partial_y\lambda)\hat{J}. \tag{3.6}$$

Note that $\hat{\mathfrak{R}}^+$ is necessarily pointwise-proportional to $\hat{J}$ since it is a self-dual (1, 1)-form. However, here we see that the proportionality factor is independent of x^i. This implies that the Ricci scalar $\hat{R} \equiv \hat{J}^{ij}\hat{\mathfrak{R}}_{ij} = \hat{J}^{ij}\hat{\mathfrak{R}}^+_{ij}$ is constant:

$$\mathrm{d}_4\hat{R} = 0. \tag{3.7}$$

Recall that on a Kähler manifold, the Ricci tensor $\hat{R}_{ij}$ is related to the Ricci form by $\hat{R}_{ij} = -\hat{J}_i{}^k\hat{\mathfrak{R}}_{kj}$. From the third equation in (3.1) and the fact that the complex structure is independent of y, we have $\partial_y\hat{R}_{ij} = 0$. Also note that we can rewrite the first equation of (3.6) as

$$\hat{R}_{ij} = -\frac{3}{2y}\partial_y\hat{g}_{ij}. \tag{3.8}$$

Given $\partial_y\hat{g}^{ij} = -\hat{g}^{ik}\hat{g}^{jl}\partial_y\hat{g}_{kl}$ one finds

$$\hat{R}_{ij}\hat{R}^{ij} = \frac{3}{2y}\partial_y\hat{R} \tag{3.9}$$

and hence

$$\mathrm{d}_4(\hat{R}_{ij}\hat{R}^{ij}) = 0. \tag{3.10}$$

For a Kähler metric, the eigenvalues of the Ricci tensor come in pairs, since it is invariant under the action of the complex structure. Thus, in dimension four, there are *a priori* two distinct eigenvalues which each have a degeneracy of 2. From (3.7), we know that the sum of the eigenvalues is a constant (on M_4). Moreover, from (3.10) we see that the sum of the squares of the eigenvalues is also constant. Thus we find the useful condition that at fixed y, the Ricci tensor on M_4 has two pairs of constant eigenvalues.

Finally, we note that expression (2.50) for the flux G simplifies to

$$G = -(\partial_y\,\mathrm{e}^{-6\lambda})\widehat{\mathrm{vol}}_4 + \mathrm{e}^{3\lambda}\left(\frac{1}{3m}\cos^2\zeta\,\hat{*}_4\,\mathrm{d}_4\rho - 4m\,\mathrm{e}^{-6\lambda}\hat{J}\right)\wedge K^1 \wedge K^2. \tag{3.11}$$

3.2. Global structure of solutions

We would like to find global regular solutions for the complex manifold M_6. Our basic construction, which will cover almost all regular compact solutions that we find, will be as follows. We require that ψ and y describe a holomorphic $\mathbb{C}P^1$ bundle over a smooth Kähler base M_4

$$\begin{array}{ccc} \mathbb{C}P^1_{y,\psi} & \longrightarrow & M_6 \\ & & \downarrow \\ & & M_4 \end{array} . \tag{3.12}$$

Recall that, given $\mathrm{d}_4\zeta = 0$, the metric has the form

$$\mathrm{d}s^2 = \mathrm{e}^{-6\lambda(y)}\hat{g}_{ij}(x,y)\,\mathrm{d}x^i\,\mathrm{d}x^j + \mathrm{e}^{-6\lambda(y)}\sec^2\zeta(y)\,\mathrm{d}y^2 + \frac{1}{9m^2}\cos^2\zeta(y)[\mathrm{d}\psi + \hat{P}(x)]^2. \tag{3.13}$$

For the (y, ψ) coordinates to describe a smooth S^2 we take the Killing vector $\partial/\partial\psi$ to have compact orbits so that ψ defines an azimuthal angle. The coordinate y is taken to lie in the range $[y_1, y_2]$ with the $U(1)$ fibre, defined by ψ, shrinking to zero size at the two poles $y = y_i$—that is, we demand $\cos\zeta(y_i) = 0$.

It is necessary to check that the fibre is smooth at the poles $y = y_i$. As we shall see, in all cases in the above construction, a smooth S^2 is obtained by choosing the period of ψ to be 2π. (Note that this is consistent with the integrated expression (2.45) for Ω). By definition, $\hat{P}$ is a connection on the canonical bundle $\mathcal{L}$ of the base M_4. Thus at fixed generic y, so that $y_1 < y < y_2$, the resulting 5-manifold is the total space of a $U(1)$ bundle over M_4, which is in fact just the $U(1)$ bundle associated with the canonical line bundle $\mathcal{L}$ of M_4. For the full $\mathbb{C}P^1$ fibration we can think of each fibre $\mathbb{C}P^1$ as a projectivization of $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$. The transition functions of the $\mathbb{C}P^1$ bundle act on the relative phase of the two factors of $\mathbb{C}$. Thus we can take one to be the trivial bundle $\mathcal{O}$. The other is then $\mathcal{L}$. Projectivizing the bundles, we then have that, as a complex manifold, M_6 is the total space of the fibration

$$M_6 = P(\mathcal{O} \oplus \mathcal{L}). \tag{3.14}$$

We note that M_6 can also be viewed as the total space of the bundle of unit self-dual[7] 2-forms over M_4. Here we think of each S^2 fibre as being a unit sphere in $\mathbb{R}^3 = \mathbb{R} \oplus \mathbb{C}$ with the factor of $\mathbb{R}$ being the polar direction on the S^2. Our S^2 bundle may then be viewed as the unit sphere bundle in an $\mathbb{R}^3$ bundle, with the transition functions acting only in the $\mathbb{R}^2 = \mathbb{C}$ part of the fibre. The rank 3 real bundle thus splits into a direct sum $\mathcal{O} \oplus \mathcal{L}_{\mathbb{R}}$ of a trivial real line bundle $\mathcal{O}$, and (the realization of) the complex canonical line bundle $\mathcal{L}$. Consider now the 2-forms on M_4. These decompose into self-dual or anti-self-dual 2-forms:

$$\Lambda^2 M_4 \cong \Lambda^+ M_4 \oplus \Lambda^- M_4. \tag{3.15}$$

However, since M_4 is Kähler, the structure group of the tangent bundle is in fact $U(2) \subset SO(4)$. Thus, one can now further decompose the space of real 2-forms as follows:

$$\Lambda^+ M_4 \cong \mathbb{R}[\hat{J}] \oplus \mathcal{L}_{\mathbb{R}} \qquad \Lambda^- M_4 \cong \Lambda^{1,1}_0 M_4. \tag{3.16}$$

Here $\Lambda^{1,1}_0 M_4$ denotes the bundle of primitive (1, 1)-forms, i.e., 2-forms which are orthogonal to the Kähler form $\hat{J}$, and are invariant under the action of the complex structure. Thus we see that the bundle of self-dual 2-forms splits as $\Lambda^+ M_4 \cong \mathcal{O} \oplus \mathcal{L}_{\mathbb{R}}$ where $\mathcal{O}$ is a trivial real line bundle generated by the Kähler form on M_4. It is now clear that the $\mathbb{R}^3$ bundle over M_4

[7] This is not to be confused with the bundle of unit *anti*-self-dual 2-forms, which is the twistor space of M_4.

associated with our metrics is in fact the bundle of self-dual 2-forms. Note that the 'polar direction' is identified with the direction corresponding to the Kähler form.

Let us now return to considering the base manifold M_4. We showed in the previous subsection that, at fixed y, the Ricci tensor on M_4 has two pairs of constant eigenvalues. We may now invoke recent mathematical results from the literature on Kähler manifolds. Theorem 2 of [35] states that, if the Goldberg conjecture[8] is true, then a compact Kähler 4-manifold whose Ricci tensor has two distinct pairs of constant eigenvalues is locally the product of two Riemann surfaces of (distinct) constant curvature. If the eigenvalues are the same the manifold is by definition Kähler–Einstein. The compactness in the theorem is essential, since there exist non-compact counterexamples. However, for AdS/CFT purposes, we are most interested in the compact case (for example, the central charge of the dual CFT is inversely proportional to the volume).

Thus we naturally have two possibilities. First, the base M_4 is Kähler–Einstein. The cosmological constant can depend on y, so by definition we can write

$$\text{case 1:}\quad \hat{\mathfrak{R}} = \frac{k}{F(y)}\hat{J} \tag{3.17}$$

where $k \in \{0, \pm 1\}$ and $F(y) > 0$.

Second, at fixed y, the base is a product of two constant curvature Riemann surfaces. Thus the metric splits as

$$\mathrm{d}\hat{s}^2(M_4) = \mathrm{d}\hat{s}_1^2 + \mathrm{d}\hat{s}_2^2. \tag{3.18}$$

Let $\hat{J}_i$ be the corresponding Kähler forms. Again the cosmological constant on each Riemann surface can depend on y. By definition we can write

$$\hat{\mathfrak{R}}_1 = \frac{k_1}{F_1(y)}\hat{J}_1 \qquad \hat{\mathfrak{R}}_2 = \frac{k_2}{F_2(y)}\hat{J}_2 \tag{3.19}$$

where $k_i \in \{0, \pm 1\}$ and $F_i(y) > 0$. Thus on M_4 we have $\hat{J} = \hat{J}_1 + \hat{J}_2$ and

$$\text{case 2:}\quad \hat{\mathfrak{R}} = \hat{\mathfrak{R}}_1 + \hat{\mathfrak{R}}_2 = \frac{k_1}{F_1(y)}\hat{J}_1 + \frac{k_2}{F_2(y)}\hat{J}_2. \tag{3.20}$$

Clearly when $k_1 = k_2$ and $F_1 = F_2$ we get special examples of case 1 above.

Since $\partial_y \hat{P} = \partial_y \rho = 0$, we have $\partial_y \hat{\mathfrak{R}} = 0$. Thus, we immediately see that the rescaled Kähler forms given by

$$\tilde{J} = \frac{1}{F}\hat{J} \qquad \tilde{J}_i = \frac{1}{F_i}\hat{J}_i \tag{3.21}$$

are independent of y. The corresponding rescaled metrics $\mathrm{d}\tilde{s}^2 = (1/F)\,\mathrm{d}\hat{s}^2$ and $\mathrm{d}\tilde{s}_i^2 = (1/F_i)\,\mathrm{d}\hat{s}_i^2$, are independent of y and have fixed cosmological constants k and k_i, respectively

$$\tilde{\mathfrak{R}} = k\tilde{J} \qquad \tilde{\mathfrak{R}}_i = k_i\tilde{J}_i. \tag{3.22}$$

For the Riemann surfaces of case 2, we write the metrics as $\mathrm{d}\tilde{s}^2(C_{k_i})$. Depending on the value of k_i, they are the standard metrics on the torus $C_0 \equiv T^2$, the sphere $C_1 \equiv S^2$ and hyperbolic space[9] $C_{-1} \equiv H^2$. From the first equation in (3.6), we can also solve for the functions F and F_i. We find

$$F(y) = \tfrac{1}{3}(b - ky^2) \qquad F_i(y) = \tfrac{1}{3}(a_i - k_i y^2) \tag{3.23}$$

[8] The Goldberg conjecture says that any compact Einstein almost Kähler manifold is Kähler–Einstein, i.e., the complex structure is integrable.

[9] In this last case, hyperbolic space H^2 is of course non-compact. One might also consider compact H^2/Γ, where Γ is a discrete group of isometries of H^2, but in general we expect these to break supersymmetry. Exceptions do exist such as, for example, the solution discussed in section 5.4.

where b and a_i are constants. We then satisfy all the constraints of supersymmetry except for the $\partial_y \log \sqrt{g}$ condition (2.58).

Thus we have found that the metric on M_4 is given by

$$\begin{aligned} &\text{case 1:} \quad \mathrm{d}s^2(M_4) = \tfrac{1}{3}\,\mathrm{e}^{-6\lambda}(b - ky^2)\,\mathrm{d}\tilde{s}^2(M_4) \\ &\text{case 2:} \quad \mathrm{d}s^2(M_4) = \tfrac{1}{3}\,\mathrm{e}^{-6\lambda}(a_1 - k_1 y^2)\,\mathrm{d}\tilde{s}^2\big(C_{k_1}\big) + \tfrac{1}{3}\,\mathrm{e}^{-6\lambda}(a_2 - k_2 y^2)\,\mathrm{d}\tilde{s}^2\big(C_{k_2}\big) \end{aligned} \tag{3.24}$$

where $\mathrm{e}^{3\lambda(y)} \sin\zeta(y) = 2my$ and the Kähler metrics $\mathrm{d}\tilde{s}^2(M_4)$ and $\mathrm{d}\tilde{s}^2(C_k)$ each satisfy

$$\tilde{\mathfrak{R}} = k\tilde{J} \tag{3.25}$$

and are independent of y. The remaining equation (2.58), or equivalently the second equation in (3.6), implies

$$\begin{aligned} &\text{case 1:} \quad 3m^2 F(1 + 6y\partial_y\lambda) = k(\mathrm{e}^{6\lambda} - 4m^2y^2) \\ &\text{case 2:} \quad 6m^2 F_1F_2(1 + 6y\partial_y\lambda) = (k_1F_2 + k_2F_1)(\mathrm{e}^{6\lambda} - 4m^2y^2). \end{aligned} \tag{3.26}$$

In summary, our basic construction is to solve these final equations and look for solutions with a smooth base M_4 and a smooth $\mathbb{C}P^1$ fibre with the topology given in equation (3.14). We will consider both compact and non-compact examples, although the theorem leading to a consideration of these two classes of base manifold only applies to the compact case. Note it is also possible to find smooth metrics on M_6 when both the base and fibre metrics degenerate. One example of this more general class is described in section 5. In this case, the topology of the $U(1)$ fibration given by ψ also changes.

4. Case 1: Kähler–Einstein base

We start by considering the case where the base is Kähler–Einstein (KE). Recall that the metric has the form

$$\mathrm{d}s^2 = \tfrac{1}{3}\,\mathrm{e}^{-6\lambda}(b - ky^2)\,\mathrm{d}\tilde{s}^2(M_4) + \mathrm{e}^{-6\lambda}\sec^2\zeta\,\mathrm{d}y^2 + \frac{1}{9m^2}\cos^2\zeta(\mathrm{d}\psi + \tilde{P})^2 \tag{4.1}$$

where $\mathrm{d}\tilde{s}^2(M_4)$ is a y-independent KE metric on the four-dimensional base satisfying $\tilde{\mathfrak{R}} = \mathrm{d}_4\tilde{P} = k\tilde{J}$. The remaining supersymmetry condition (3.26) can be integrated explicitly. One finds, for $k = \pm 1$

$$\begin{aligned} \mathrm{e}^{6\lambda} &= \frac{2m^2(kb - y^2)^2}{cy + 2kb + 2y^2} \\ \cos^2\zeta &= \frac{-3y^4 - 2cy^3 - 6kby^2 + b^2}{(kb - y^2)^2}. \end{aligned} \tag{4.2}$$

The solution is fully specified by giving the 4-form flux which reads

$$G = \frac{(4y^3 + 3cy^2 + 12kby + kbc)}{18m^2(y^2 - kb)}\tilde{\mathrm{vol}}_4 + \frac{k(y^4 - 6kby^2 - 2kbcy - 3b^2)}{9m^2(y^2 - kb)^2}\tilde{J} \wedge \mathrm{d}y \wedge (\mathrm{d}\psi + \tilde{P}). \tag{4.3}$$

For $k = 0$, without loss of generality we can set $b = 3$, and we have simply

$$\mathrm{e}^{6\lambda} = \frac{1}{cy} \qquad \cos^2\zeta = 1 - 4m^2cy^3. \tag{4.4}$$

The 4-form flux for $k = 0$ will be given shortly.

By construction these give locally supersymmetric solutions. We next investigate the solutions in more detail, in particular, analysing when they are regular. We will find that there are regular solutions only for $k = 1$. An interesting aspect of the $k = 0$ case, when the hyper-Kähler manifold is taken to be a 4-torus, is that the solutions can be reduced to type IIA solutions and thence, on performing a T-duality, to (singular) Sasaki–Einstein type IIB solutions.

4.1. KE bases with positive curvature: $k = 1$

We consider the case where $k = 1$. From the form of the metric (4.1), we see that we require $b > 0$. Without loss of generality, by an appropriate rescaling of y we can set $b = 1$. In addition, by flipping the sign of y if necessary, we can take c to be positive. The warp factor $\mathrm{e}^{6\lambda}$ becomes zero at $y^2 = 1$ and the metric on M_4 also develops a singularity, and thus we require

$$y^2 < 1. \tag{4.5}$$

First we consider $0 \leqslant c < 4$, in which case $\cos^2\zeta$ has two zeros, at y_1, y_2, which are the two real roots of the quartic:

$$3y^4 + 2cy^3 + 6y^2 - 1 = 0. \tag{4.6}$$

We take $y_1 \leqslant y \leqslant y_2$ and note the remarkable fact that, within this range, the expression for $\cos^2\zeta$ is consistent with ζ running monotonically from $-\pi/2$ to $\pi/2$ (for example $\cos^2\zeta \leqslant 1$ and is equal to 1 just at $y = 0$). We also find that $y^2 < 1$ and so the base metric is well defined.

Now, if ψ is periodic, the (y, ψ) part of the metric is a metric on a non-round 2-sphere provided that a single choice for the period of ψ ensures the metric is smooth at both the poles, located at $y = y_1, y_2$. Let Σ_1 be the value of the derivative of $\cos^2\zeta$ at y_1. Assuming that Σ_1 is non-vanishing, which will be the typical case, then near y_1 the (y, ψ) part of the metric becomes

$$\frac{\mathrm{e}^{-6\lambda(y_1)}}{\Sigma_1(y - y_1)}\,\mathrm{d}y^2 + \frac{1}{9m^2}\Sigma_1(y - y_1)(\mathrm{d}\psi + \tilde{P})^2. \tag{4.7}$$

If we now introduce the coordinate $\epsilon = 2(y - y_1)^{1/2}$, this can be written as

$$\frac{\mathrm{e}^{-6\lambda(y_1)}}{\Sigma_1}[\mathrm{d}\epsilon^2 + \alpha^2\epsilon^2(\mathrm{d}\psi + \tilde{P})^2] \tag{4.8}$$

where

$$\alpha^2 = \frac{1}{36m^2}\,\mathrm{e}^{6\lambda(y_1)}\Sigma_1^2. \tag{4.9}$$

This is regular provided that ψ has period $2\pi\alpha$. Remarkably, for *both* y_1 and y_2 (in fact for all the roots of the quartic (4.6)) we find $\alpha^2 = 1$. Thus choosing the period of ψ to be 2π implies that we do indeed have a smooth 2-sphere. This is compatible with the fact that $\tilde{P}$ is the Kähler connection on the canonical bundle $\mathcal{L}$, and implies that at fixed $y \neq y_i$ the five-dimensional manifold is the total space of the associated $U(1) = S^1$ bundle to $\mathcal{L}$. In addition, it is compatible with solution (2.45) for the 2-form Ω, implying that Ω has charge 1 under the action of the Killing vector $\partial/\partial\psi$. In other words, for $0 \leqslant c < 4$ we have a one-parameter family of completely regular solutions which are S^2 fibrations over a smooth KE base.

Next consider $c = 4$. This case needs a special analysis since the quartic develops a triple root at $y = -1$. Another root is located at $y = 1/3$. As before, choosing the period of ψ to be 2π leads to a smooth metric near $y = 1/3$. On the other hand, this period leads to a conical singularity at $y = -1$. This is in fact a curvature singularity in general, the exception being when the base is $\mathbb{C}P^2$, in which case one has an orbifold singularity at this point. In fact the solution is in this case the weighted projective space $W\mathbb{C}P^3_{[1,1,1,3]}$.

Finally, when $c > 4$, the quartic only has a single real positive root. Once again, choosing the period of ψ to be 2π leads to a smooth metric at y equal to this root. On the other hand, the space includes the point $y = -1$ where the warp factor $\mathrm{e}^{6\lambda}$ is zero and the metric is singular.

In summary, we have found that:

For $0 \leqslant c < 4$, we have a one-parameter family of completely regular, compact, complex solutions with the topology of a $\mathbb{C}P^1$ fibration over a positive curvature KE space.

Since four-dimensional compact Kähler–Einstein spaces with positive curvature have been classified [36, 37], we have a classification for the above solutions. In particular, the base space is either $S^2 \times S^2$ or $\mathbb{C}P^2$, or $\mathbb{C}P^2\#_n\mathbb{C}P^2$ with $n = 3, \ldots, 8$. For the first two examples, the KE metrics are of course explicitly known and this gives explicit solutions of M-theory when fed into the above solutions. The remaining metrics, although proved to exist, are not explicitly known, and so the same applies to the corresponding M-theory solutions.

4.2. KE bases with negative curvature: $k = -1$

We now consider cases with $k = -1$. First consider the case $b \neq 0$. Arguing as above, by redefining y we can set $b = 1$ and we may also take c to be positive without loss of generality. To ensure that $e^{6\lambda}$ is positive we need to take

$$y \geqslant y_+ \equiv \tfrac{1}{4}\left(-c + \sqrt{c^2 + 16}\right), \qquad \text{or} \qquad y \leqslant y_- \equiv \tfrac{1}{4}\left(-c - \sqrt{c^2 + 16}\right). \tag{4.10}$$

Now, for all values of c we find that $\cos^2\zeta$ has two real roots, y_1 and y_2. The largest one, y_2, is always greater than y_+, while y_1 is always smaller than y_-. To ensure that $\cos^2\zeta$ is positive, we must take the range of y to be either: $y_+ \leqslant y \leqslant y_2$ or $y_1 \leqslant y \leqslant y_-$. When $y = y_{1,2}$, in each of the two solutions, by carrying out an analysis as above we find that the metric is regular provided that the period of ψ is once again taken to be 2π. However, when $y = y_\pm$, the warp factor $e^{6\lambda}$ goes to infinity and we have a singularity.

When $b = 0$ we can again take $c \geqslant 0$, without loss of generality. For $e^{6\lambda}$ to be positive and finite, we require $y < -\frac{1}{2}c$ or $y > 0$. The roots of $\cos^2\zeta$ are now $y = 0$ and $y = -\frac{2}{3}c$. Thus the possible ranges of y are either $y \geqslant 0$, in which case the solution is singular at $y = 0$, or $-\frac{2}{3}c \leqslant y < -\frac{1}{2}c$ with c strictly positive, $c > 0$. In the latter case, while $y = -\frac{2}{3}c$ is regular if the period of ψ is taken to be 2π, at $y = -\frac{1}{2}c$ again the solution is singular.

4.3. Hyper-Kähler bases: $k = 0$

Now we consider the case $k = 0$. This means the base is a Ricci-flat Kähler manifold, or in other words the y-independent four-dimensional metric $\tilde{g}$ is hyper-Kähler. Locally we can, and will, choose $\tilde{P} = 0$. As discussed above the solution has warp factor $e^{6\lambda} = 1/cy$ and hence $\cos^2\zeta = 1 - 4m^2cy^3$. Without loss of generality, we can assume $c \geqslant 0$ and hence $y > 0$ for the warp factor to be positive.

After rescaling $y \to y/c^{1/3}$ and $\tilde{g} \to c^{-2/3}\tilde{g}$, the full 11-metric takes the simple form

$$\begin{aligned} ds_{11}^2 = c^{-2/9}\bigg[&\frac{1}{y^{1/3}}\, ds^2(\mathrm{AdS}_5) + y^{2/3}\, d\tilde{s}^2(M_4) \\ &+ \frac{y^{2/3}}{1 - 4m^2y^3}\, dy^2 + \frac{1}{9m^2y^{1/3}}(1 - 4m^2y^3)\, d\psi^2\bigg] \end{aligned} \tag{4.11}$$

where, if the hyper-Kähler 4-metric $d\tilde{s}^2$ is to be compact, the base M_4 is either T^4 or $K3$ (modulo a quotient by a freely acting finite group). The flux takes the form

$$G = c^{-1/3}\left[-\widetilde{\mathrm{vol}}_4 - \tfrac{4}{3}y\, dy \wedge d\psi \wedge \tilde{J}\right] \tag{4.12}$$

where $\tilde{J}$ and $\widetilde{\text{vol}}_4$ denote the Kähler form and volume form on the hyper-Kähler space, respectively. We clearly see that the constant c is just an overall scaling and is not a genuine modulus, and so henceforth we set $c = 1$.

If we introduce the new variable:

$$y = \frac{1}{(4m^2)^{1/3}} \cos^{2/3}\theta \tag{4.13}$$

the $D = 11$ solution becomes

$$\begin{aligned} \mathrm{d}s_{11}^2 &= y^{-1/3}\left[\mathrm{d}s^2(\mathrm{AdS}_5) + \frac{1}{9m^2}(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\psi^2)\right] + y^{2/3}\,\mathrm{d}\tilde{s}^2(M_4) \\ G &= -\widetilde{\text{vol}}_4 + \frac{4}{9m} y^{1/2}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\psi\wedge\tilde{J}. \end{aligned} \tag{4.14}$$

If the period of ψ is 2π, then the solution is regular at $\theta = 0$. On the other hand, $\theta = \pi/2$ is clearly singular.

In the special case that the HK space is flat, we can obtain supersymmetric solutions of type IIA and type IIB supergravity by dimensional reduction and T-duality, respectively. Note that these solutions will be supersymmetric provided that the Killing spinor is invariant under the action of the Killing vector that one is reducing on or T-dualizing on. This will be the case for the flat torus directions but will not in general be the case if we reduce on ψ. If we write the flat HK metric as $\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2$ with $\tilde{J} = \mathrm{d}x_1\wedge\mathrm{d}x_2 + \mathrm{d}x_3\wedge\mathrm{d}x_4$ and then dimensionally reduce on the x_4 direction we obtain the type IIA supersymmetric solution

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}s^2(\mathrm{AdS}_5) + \frac{1}{9m^2}(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\psi^2) + y\left(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^3\right) \\ \mathrm{e}^{2\Phi} &= y \\ F^{\mathrm{RR}} &= \frac{4}{9m}y^{1/2}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\psi\wedge\mathrm{d}x_1\wedge\mathrm{d}x_2 \\ B^{\mathrm{NS}} &= -\frac{1}{2}x_1\,\mathrm{d}x_2\wedge\mathrm{d}x_3 + \frac{1}{2}x_2\,\mathrm{d}x_1\wedge\mathrm{d}x_3 - \frac{2}{3}y^2\,\mathrm{d}\psi\wedge\mathrm{d}x_3 \end{aligned} \tag{4.15}$$

where $\mathrm{d}s^2$ is the type IIA string metric, Φ is the dilaton, F^{RR} is the Ramond–Ramond 4-form field strength and B^{NS} is the Neveu–Schwarz 2-form potential. Here we are using the conventions of, for example, [40].

If we T-dualize now on the x^3 direction, using the formulae in [41], we obtain a supersymmetric type IIB solution whose metric is the direct product of AdS$_5$ with a 5-manifold X_5. The solution has constant dilaton and the only other non-vanishing field is the self-dual 5-form. The metric on X_5 is given by

$$\begin{aligned} \mathrm{d}s^2(X_5) = y\left(\mathrm{d}x_1^2 + \mathrm{d}x_2^2\right) &+ \frac{\sin^2\theta}{4y}\,(2\mathrm{d}x_3 + x_2\,\mathrm{d}x_1 - x_1\,\mathrm{d}x_2)^2 \\ &+ \frac{1}{9m^2}\,\mathrm{d}\theta^2 + \frac{1}{9m^2}[\mathrm{d}\psi - 3m^2 y(2\mathrm{d}x_3 + x_2\,\mathrm{d}x_1 - x_1\,\mathrm{d}x_2)]^2. \end{aligned} \tag{4.16}$$

For this to be a supersymmetric solution of type IIB string theory we know that the metric should be Sasaki–Einstein. It is satisfying to check that the Ricci tensor is given by $4m^2$ times the metric. A calculation of the square of the Riemann tensor is given by $8m^4(5 + 48\cos^{-4}\theta)$, which implies that the solution is singular at $\theta = \pi/2$.

5. Case 2: product base

Now we consider the case where the base is a product of two constant curvature Riemann surfaces. Recall that the metric has the form

$$\begin{aligned} ds^2 = \frac{1}{3}\, e^{-6\lambda}(a_1 - k_1 y^2)\, d\tilde{s}^2(C_1) + \frac{1}{3}\, e^{-6\lambda}(a_2 - k_2 y^2)\, d\tilde{s}^2(C_2) \\ + e^{-6\lambda} \sec^2\zeta\, dy^2 + \frac{1}{9m^2}\cos^2\zeta\,(d\psi + \tilde{P})^2 \end{aligned} \tag{5.1}$$

where the y-independent metrics $d\tilde{s}^2(C_i)$ describe constant curvature Riemann surfaces with curvature $k_i \in \{0, \pm 1\}$. In other words the metrics on C_i are the standard ones on either tori T^2, spheres S^2 or hyperbolic spaces H^2.

We must solve the remaining supersymmetry condition (3.26) to find λ and ζ as functions of y. Substituting for F_i, the condition can be written as

$$\begin{aligned} 2m^2 y \partial_y\, e^{6\lambda}(a_1 - k_1 y^2)(a_2 - k_2 y^2) = e^{12\lambda}(k_1 a_2 + k_2 a_1 - 2k_1 k_2 y^2) \\ + 2m^2\, e^{6\lambda}(3k_1 k_2 y^4 - (k_1 a_2 + k_2 a_1) y^2 - a_1 a_2). \end{aligned} \tag{5.2}$$

Remarkably, this can again be solved in closed form for general k_i. In the following, we will treat two cases separately. First we note that $k_1 = k_2 = 0$ is just a 4-torus and so this case was considered in the previous section. We can then distinguish between the cases where one of the k_i is zero and where neither is zero. We first consider the former case, when one of the Riemann surfaces is a flat 2-torus. For this case we will also show that, by dimensionally reducing on one of the circle directions, we can obtain additional type IIA solutions, and that a further T-duality on the other circle direction leads to type IIB solutions. In particular, in this way we obtain new Sasaki–Einstein spaces.

5.1. $S^2 \times T^2$ and $H^2 \times T^2$ base: $k_2 = 0$

Let us first suppose that one of the Riemann surfaces is flat. Then without loss of generality, we may set $k_1 = k = \pm 1$, $k_2 = 0$ and also write $a_1 = a$. The general solution to (5.2) is then

$$e^{6\lambda} = \frac{2m^2(a - ky^2)}{k - cy} \qquad \cos^2\zeta = \frac{a - 3ky^2 + 2cy^3}{a - ky^2} \tag{5.3}$$

where c is an integration constant. Without loss of generality we can set $a_2 = 3$. The full supersymmetric solution can then be written as

$$ds^2 = e^{-6\lambda}\, d\tilde{s}^2(T^2) + \frac{k - cy}{6m^2}\, d\tilde{s}^2(C_k) + e^{-6\lambda}\sec^2\zeta\, dy^2 + \frac{1}{9m^2}\cos^2\zeta\,(d\psi + \tilde{P})^2 \tag{5.4}$$

where $\tilde{P}$ denotes the canonical connection for the metric $d\tilde{s}^2(C_k)$. The constants a and c are arbitrary. We find that the flux is given by

$$\begin{aligned} G = \frac{-2y + ky^2 c + ca}{6m^2(a - ky^2)} \mathrm{vol}(C_k) \wedge \mathrm{vol}(T^2) - \frac{2(k - cy)}{9m^2}\, dy \wedge (d\psi + \tilde{P}) \wedge \mathrm{vol}(C_k) \\ - \frac{ak + y^2 - 2acy}{3m^2(a - ky^2)^2}\, dy \wedge (d\psi + \tilde{P}) \wedge \mathrm{vol}(T^2) \end{aligned} \tag{5.5}$$

where $\mathrm{vol}(C_k)$ is the standard volume form on $C_k \in \{S^2, H^2\}$ for $k = \pm 1$.

We are looking for solutions which are S^2 fibrations over $C_k \times T^2$. For compact fibres, as before, we consider $y_1 \leqslant y \leqslant y_2$ where y_i are two roots of $\cos^2\zeta = 0$ giving the poles of the S^2. Moreover, we also require that $k - cy$ and $e^{-6\lambda}$ should both be positive in this range so that the metric on $C_k \times T^2$ does not have singularities. Finally we must check that the poles of the fibre S^2 are free of conical singularities for a suitable choice of the period of ψ.

To investigate whether such solutions exist, first consider $c \neq 0$. In this case we can set $c = 1$ without loss of generality. For $k = 1$, we have

$$\mathrm{d}\tilde{s}^2(S^2) = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 \qquad \tilde{P} = -\cos\theta\,\mathrm{d}\phi. \tag{5.6}$$

In this case, for the range $0 < a < 1$ the cubic in the numerator of $\cos^2\zeta$ has three real roots. If we choose y to lie within the range between the two smallest roots we find that all the conditions mentioned above are satisfied, if we choose the period of ψ to be 2π. In other words, for $0 < a < 1$ we have a two-parameter family of completely regular, compact, complex solutions that are S^2 bundles over $S^2 \times T^2$. For $k = -1$ we find that the analogous solutions with $-1 < a < 0$ are singular.

For $k = 1$, still with $c = 1$, if one sets $a = 0$ it is straightforward to see that the resulting space has curvature singularities. The solution with $a = 1$ is also singular, but in a milder and more interesting way. It is convenient to perform a change of variables, defining

$$y = 1 - \tfrac{3}{2}\sin^2\sigma \tag{5.7}$$

so that the $a = 1$ metric becomes

$$\begin{aligned} m^2\,\mathrm{d}s^2 = \mathrm{d}\sigma^2 &+ \frac{1}{4}\sin^2\sigma\,(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \\ &+ \frac{\sin^2\sigma\cos^2\sigma}{1+3\cos^2\sigma}(\mathrm{d}\psi - \cos\theta\,\mathrm{d}\phi)^2 + \frac{1}{1+3\cos^2\sigma}\,\mathrm{d}\tilde{s}^2(T^2). \end{aligned} \tag{5.8}$$

The range of σ is $0 \leqslant \sigma \leqslant \pi/2$. It is easy to see that near $\sigma = \frac{1}{2}\pi$ one smoothly approaches a bolt $T^2 \times S^2$ of co-dimension 2 if and only if ψ has period 2π. On the other hand, near to $\sigma = 0$ the metric collapses to a co-dimension 4 bolt T^2. At constant σ, the collapsing fibre turns out to be $\mathbb{R}P^3 = S^3/Z_2$, rather than S^3, due to the periodicity already enforced on ψ. In fact, projecting out the T^2, the resulting 4-space is the weighted projective space $W\mathbb{C}P^2_{[1,1,2]}$. This is a complex orbifold. Note the close similarity of the metric we have to the standard Kähler–Einstein metric on $\mathbb{C}P^2$.

If one now starts with $W\mathbb{C}P^2_{[1,1,2]}$, one can consider blowing up the $\mathbb{Z}_2$ singularity, replacing it locally with T^*S^2. The resulting space is then clearly two copies of the cotangent bundle of S^2 glued back to back. This is precisely the topology of the non-singular spaces described above, when $0 < a < 1$. In fact, one can show that the resulting S^2 bundle over S^2 is topologically trivial. Thus, we can think of the limit $a \to 1$ as a limit in which we blow down an S^2 in the non-singular family to obtain an orbifold, and it follows that the parameter a measures the size of this cycle, with $a \to 1$ the zero-size limit.

The remaining case to consider is $c = 0$. Clearly for the metric to have the correct signature we must have $k = 1$. Without loss of generality, we can also set $a = 3$. With the change of coordinates $y = \cos\omega$ we find

$$6m^2\,\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 + \mathrm{d}\omega^2 + \frac{2\sin^2\omega}{2+\sin^2\omega}(\mathrm{d}\psi - \cos\theta\,\mathrm{d}\phi)^2 + \frac{3}{2+\sin^2\omega}\,\mathrm{d}\tilde{s}^2(T^2) \tag{5.9}$$

where $0 \leqslant \omega \leqslant \pi$. The total space is a regular S^2 bundle over $S^2 \times T^2$ provided that the period of ψ is taken to be 2π. In this case, projecting out the T^2, the resulting space is an S^2 bundle over S^2, which furthermore is topologically trivial (for more details, see the discussion in [38]).

In summary, we see that given $k = 1$ we have:

For $0 < a < 1$ and $c \neq 0$ we have a one-parameter family of completely regular, compact, complex solutions that are topologically trivial S^2 bundles over $S^2 \times T^2$. A single additional solution of this type is obtained when $c = 0$ and $a \neq 0$.

There are no non-singular solutions for $k = -1$.

5.2. New type IIA and type IIB solutions

This family of solutions also leads to solutions of type IIA and IIB supergravity. First, for any allowed value of the constants a and c, we can reduce the above solutions on one of the torus directions to obtain supersymmetric type IIA solutions. Concretely, writing $\mathrm{d}\tilde{s}^2(T^2) = \mathrm{d}x_3^2 + \mathrm{d}x_4^2$ and reducing along the x_4-direction we get

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}s^2(\mathrm{AdS}_5) + \frac{k-cy}{6m^2}\,\mathrm{d}\tilde{s}^2(C_k) + \mathrm{e}^{-6\lambda}\sec^2\zeta\,\mathrm{d}y^2 + \frac{1}{9m^2}\cos^2\zeta(\mathrm{d}\psi + \tilde{P})^2 + \mathrm{e}^{-6\lambda}\,\mathrm{d}x_3^2 \\
\mathrm{e}^{2\phi} &= \mathrm{e}^{-6\lambda} \\
G^{\mathrm{RR}} &= -\frac{2(k-cy)}{9m^2}\mathrm{d}y\wedge(\mathrm{d}\psi+\tilde{P})\wedge\mathrm{vol}(C_k) \\
B^{\mathrm{NS}} &= \frac{1}{6m^2k}\frac{-2y+ky^2c+ca}{a-ky^2}(\mathrm{d}\psi+\tilde{P})\wedge\mathrm{d}x_3.
\end{aligned} \tag{5.10}$$

For $k = 1$, the S^2 bundle is still smooth and in fact we have completely regular type IIA solutions for the values of a, c and the ranges of y and ψ where the $D = 11$ solutions are regular.

We can now T-dualize along the x_4-direction to get type IIB solutions. The metric is given by

$$\begin{aligned}
\mathrm{d}s^2 = {} & \mathrm{d}s^2(\mathrm{AdS}_5) + \frac{k-cy}{6m^2}\,\mathrm{d}\tilde{s}^2(C_k) + \mathrm{e}^{-6\lambda}\sec^2\zeta\,\mathrm{d}y^2 + \frac{1}{9m^2}\cos^2\zeta(\mathrm{d}\psi+\tilde{P})^2 \\
& + \mathrm{e}^{6\lambda}\left[\mathrm{d}x_3 + \frac{1}{6m^2k}\frac{-2y+ky^2c+ac}{a-ky^2}(\mathrm{d}\psi+\tilde{P})\right]^2.
\end{aligned} \tag{5.11}$$

We find that the dilaton is constant and the only other non-vanishing field is the self-dual 5-form. Thus the solutions should be the product of AdS_5 with a locally Sasaki–Einstein space. We have checked that it is Einstein with the appropriate factor of $4m^2$, and also locally Sasaki.

The global structure of these solutions for $k = 1$ will be discussed in detail in [38]. In particular, we find the regular M-theory solution with $c = 0$ leads to a regular type IIA solution and the resulting Sasaki–Einstein metric arising in the type IIB solution is simply the standard one on $T^{1,1}/\mathbb{Z}_2$. The four-dimensional superconformal field theory dual to this type IIB solution was identified in [4]. In [38], we will also show that for the solutions with $0 < a < 1$ and $c \neq 0$, there is a countably infinite number of values of a for which the type IIB solution gives a regular Sasaki–Einstein metric on $S^2 \times S^3$. These should correspond to new AdS_5 duals of $N = 1$ superconformal field theories and it will be interesting to see when the type IIB, type IIA or M-theory supergravity solutions are most useful. Interestingly, the mildly singular $D = 11$ limiting solution with $a = 1$ (and $c = 1$) is dual to $S^5/\mathbb{Z}_2$ in the type IIB theory.

5.3. $S^2 \times S^2$, $S^2 \times H^2$ and $H^2 \times H^2$ bases: $k_i = \pm 1$

We now consider $k_i = \pm 1$, corresponding to warped products of $S^2 \times S^2$, $H^2 \times H^2$ or $S^2 \times H^2$. Note that we should at least recover the KE result for the first two of these cases, when the warping is the same for each space. In fact, the warping can be different, corresponding to choosing the integration constants $a_1 \neq a_2$. The general solution to (5.2) is given by

$$
\begin{aligned}
e^{6\lambda} &= \frac{2m^2(y^2 - a_1k_1)(y^2 - a_2k_2)}{2y^2 + cy + k_1a_1 + k_2a_2} \\
\cos^2\zeta &= \frac{-3y^4 - 2cy^3 - 3(k_1a_1 + k_2a_2)y^2 + k_1k_2a_1a_2}{(y^2 - k_1a_1)(y^2 - k_2a_2)}
\end{aligned}
\tag{5.12}
$$

where c is an arbitrary integration constant. The 4-form flux is given by

$$
\begin{aligned}
G = {} & \frac{k_1k_2}{18m^2(y^2 - a_1k_1)(y^2 - a_2k_2)}\big[4y^5 + 3cy^4 + 4y^3(a_1k_1 + a_2k_2) \\
& - cy^2(a_1k_1 + a_2k_2) - 2y\big(a_1^2 + a_2^2 + 4a_1a_2k_1k_2\big) - ca_1a_2k_1k_2\big]\mathrm{vol}_1 \wedge \mathrm{vol}_2 \\
& + \Bigg\{\frac{k_2}{9m^2(y^2 - a_1k_1)^2}\big[y^4 - y^2(a_2k_2 + 5a_1k_1) - 2a_1k_1cy \\
& - a_1a_2k_1k_2 - 2a_1^2\big]\mathrm{vol}_2 + [1 \leftrightarrow 2]\mathrm{vol}_1\Bigg\} \wedge \mathrm{d}y \wedge (\mathrm{d}\psi + \tilde{P}).
\end{aligned}
\tag{5.13}
$$

We therefore have a three-parameter family of solutions. These reduce, on setting $a_1 = a_2 = \pm b$, to the KE solutions considered in section 4.1 with base $S^2 \times S^2$ or $H^2 \times H^2$ (for the cases $k_1 = k_2 = k = \pm 1$, respectively).

As usual we are particularly interested in regular solutions where the (y, ψ) part of the metric is a 2-sphere. This requires that y is bounded between two suitable roots y_1 and y_2 of the quartic appearing in the numerator of $\cos^2\zeta$. As before, we find the 2-sphere will be free of conical singularities if the period of ψ is taken to be 2π. A regular solution is obtained if, for values of y between the roots, the warp factor $e^{6\lambda}$ and the F_i are positive and $0 \leqslant \cos^2\zeta \leqslant 1$. We find:

> For various ranges of (a_1, a_2, c) there are regular solutions that are topologically S^2 bundles over $S^2 \times S^2$ and $S^2 \times H^2$.

In particular, for the $S^2 \times S^2$ case, there are solutions when a_1 is not equal to a_2 and hence this gives a broader class of solutions than in the Kähler–Einstein case considered above. The existence of regular solutions is rather easy to see if one sets $c = 0$. It is also not difficult to show that when $c = 0$ there are no regular solutions of the type being considered when the base is $H^2 \times H^2$, since the positivity conditions cannot be satisfied.

For the $S^2 \times H^2$ case there is a special degenerate limit which leads to a solution first found in [10], as we show in the next subsection. This solution is of a different global type from those considered thus far. First, the S^2 fibres are not smooth but have conical singularities at the poles. This is connected to the fact that the $U(1)$ bundle specified by ψ is globally $\mathcal{L}^{1/2}$ rather than $\mathcal{L}$. In addition, the volume of the S^2 of the $S^2 \times H^2$ base goes to zero at the poles of the (y, ψ) 2-sphere. Together these facts conspire to make the total space of S^2 and (y, ψ) a 4-sphere. In other words, the topology of these solutions is actually an S^4 bundle over H^2. Note that, for the class of regular solutions described in the last paragraph, the volume of the S^2 in the $S^2 \times H^2$ base is always finite and topologically we have an S^2 bundle over $S^2 \times H^2$. This combined with the fact that the ranges for the coordinate ψ are different indicate that they should probably not be viewed as deformations of the solution found in [10]. Finally, we comment that in the solution of [10] it was argued that the Killing spinors do not depend on the coordinates on H^2 and hence this space can be replaced with H^2/Γ where Γ is a discrete group of isometries. It would be interesting to know whether this is also possible for our more general solutions with base $S^2 \times H^2$. This could be established with an explicit expression for the Killing spinor.

5.4. Recovering the $\mathcal{N} = 1$ Maldacena–Núñez solution

Given our general formulation, it is a simple exercise to recover the regular solution with $\mathcal{N} = 1$ supersymmetry constructed by Maldacena and Núñez in [10]. This solution corresponds to setting $k_1 = -k_2 = 1$, so that the base 4-manifold is a warped product of $S^2 \times H^2$. Crucially though, the (y, ψ)-fibre is *not* a smooth S^2 and, in addition, the base metric is singular at certain values of y. Thus, the topology of this solution is different to that of the regular solutions we have considered so far.

For simplicity we set $m = 1$. One needs to make the following choice of integration constants:

$$a \equiv a_2 = 3a_1, \qquad c = 0. \tag{5.14}$$

In this case the expression for the quartic in the numerator of $\cos^2 \zeta$ factorizes, with one of the factors cancelling a quadratic in the denominator. The result is

$$\mathrm{e}^{6\lambda} = y^2 + a \qquad \cos^2 \zeta = \frac{a - 3y^2}{a + y^2}. \tag{5.15}$$

In fact, it is simple to see that the constant a is redundant; writing $y = \sqrt{a/3}\cos\alpha$ leads to the six-dimensional metric

$$\mathrm{d}s_6^2 = \frac{1}{3}\,\mathrm{d}s^2(H^2) + \frac{\sin^2\alpha}{3(3+\cos^2\alpha)}\,\mathrm{d}s^2(S^2) + \frac{1}{3}\,\mathrm{d}\alpha^2 + \frac{\sin^2\alpha}{3(3+\cos^2\alpha)}(\mathrm{d}\psi + \tilde{P})^2. \tag{5.16}$$

If we write the metrics on S^2 and H^2 as

$$\mathrm{d}s^2(S^2) = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\nu^2 \qquad \mathrm{d}s^2(H^2) = \frac{\mathrm{d}X^2 + \mathrm{d}Y^2}{Y^2} \tag{5.17}$$

then the connection $\tilde{P}$ is given by

$$\tilde{P} = -\cos\theta\,\mathrm{d}\nu - \frac{\mathrm{d}X}{Y}. \tag{5.18}$$

Note that, as usual, the (α, ψ)-fibre is a smooth S^2 only if we take the period of ψ to be 2π. However, the full space will then be singular, since the metric on the base S^2 is singular when $\sin\alpha = 0$. Remarkably, the full space can be made smooth by instead choosing the period of ψ to be 4π. The (θ, ν, ψ) part of the metric is then simply a round 3-sphere. Moreover, combined with the α part we get the metric on a squashed 4-sphere.

Thus, topologically this solution is different from our previous examples. The $U(1)$ ψ-fibration is now given by $\mathcal{L}^{1/2}$ instead of $\mathcal{L}$, where $\mathcal{L}$ is the canonical bundle on $S^2 \times H^2$ and the S^2 fibre has conical singularities at $\sin\alpha = 0$. As a check, one notes that this choice of period for ψ is still consistent with the integrated expression (2.45) for $\hat{\Omega}$.

To see the S^4 explicitly, we introduce the new coordinates

$$\psi = -(\phi_1 + \phi_2) \qquad \nu = \phi_1 - \phi_2 \tag{5.19}$$

as well as the constrained variables on a 2-sphere

$$\mu_0 = \cos\alpha \qquad \mu_1 = \sin\alpha\cos\frac{\theta}{2} \qquad \mu_2 = \sin\alpha\sin\frac{\theta}{2} \tag{5.20}$$

satisfying $\mu_0^2 + \mu_1^2 + \mu_2^2 = 1$. The full 11-dimensional solution then takes the form

$$\begin{aligned}\mathrm{d}s_{11}^2 = \Delta^{1/3}\,\mathrm{d}s_7^2 + \frac{1}{4}\Delta^{-2/3}\Bigg\{&\mathrm{e}^{-4\Phi}\,\mathrm{d}\mu_0^2 + \mathrm{e}^{\Phi}\Bigg[\mathrm{d}\mu_1^2 + \mathrm{d}\mu_2^2 + \mu_1^2\left(\mathrm{d}\phi_1 + \frac{1}{2Y}\,\mathrm{d}X\right)^2 \\ &+ \mu_2^2\left(\mathrm{d}\phi_2 + \frac{1}{2Y}\,\mathrm{d}X\right)^2\Bigg]\Bigg\}\end{aligned} \tag{5.21}$$

where

$$\Delta = e^{4\Phi}\mu_0^2 + e^{-\Phi}\left(\mu_1^2 + \mu_2^2\right). \tag{5.22}$$

The constant Φ is given by $e^{5\Phi} = 4/3$, and the 7-metric ds_7^2 is a warped product of AdS_5 and H^2, given by

$$ds_7^2 = e^{-8\Phi}\, ds^2(AdS_5)_{m=1} + e^{-3\Phi}\frac{dX^2 + dY^2}{4Y^2} \tag{5.23}$$

where $ds^2(AdS_5)_{m=1}$ is a unit-radius AdS_5-space. The metric (5.21) is precisely that of the solution, in the form they presented, found by Maldacena and Núñez.

6. Discussion

Let us summarize what we have found. First, we gave a geometric formulation of the general conditions that a six-dimensional manifold M_6 must satisfy in order to get a supersymmetric solution of M-theory of the form of a warped product $AdS_5 \times M_6$. These conditions are summarized in equations (2.52)–(2.58). The 4-form flux is completely determined from the geometry as in (2.50). We found that M_6 is locally the product of a complex four-dimensional space M_4 and a two-dimensional space spanned by a Killing vector $\partial/\partial\psi$ and an orthogonal direction with coordinate y. The complex structure on M_4 is independent of y and ψ. The six-dimensional metric is fixed by a one-parameter family of Kähler metrics on M_4 depending on y and a single angular function ζ, which also fixes the warp factor. The Bianchi identity and the equations of motion for the 4-form are both implied by the supersymmetry conditions.

We then specialized to the case where M_6 is a complex manifold and the metric is Hermitian, and found a number of new classes of AdS_5 solutions. The natural global structure to consider is where the (y, ψ) directions form a holomorphic S^2 bundle over the Kähler base M_4. For compact M_6 we found a complete classification of such geometries, assuming that the Goldberg conjecture is true. They fall into two classes, where the base M_4 is either (i) Kähler–Einstein (KE) or (ii) the product of two constant curvature Riemann surfaces. We found three families of regular compact solutions: those with positive curvature KE base (which are classified in [36, 37]) and those with $S^2 \times T^2$ or $S^2 \times S^2$ base. We also constructed several families of regular non-compact geometries, as well as singular geometries.

In the $S^2 \times T^2$ case discussed in 5.1 (or the KE case with base T^4), one can reduce on one torus direction to obtain a supersymmetric type IIA solution which is the direct product of AdS_5 with a 5-manifold. If one further T-dualizes on the second torus direction, one gets a supersymmetric type IIB Sasaki–Einstein solution. This family includes $T^{1,1}/\mathbb{Z}_2$ as a special case. A detailed discussion of the new Sasaki–Einstein metrics is presented in [38]. It is perhaps worth stressing that here the T-duality we implemented preserves supersymmetry, unlike the dualities on the canonical Sasaki–Einstein $U(1)$ Killing direction of S^5 considered in [42], or $T^{1,1}$ [43].

These new classes of AdS_5 solutions give an array of new candidates for the supergravity duals of $N = 1$ superconformal field theories. Of primary interest in this context are the regular compact solutions that we have constructed, but it is possible that the singular solutions can be resolved in physically interesting ways. Note that the general existence of the $\partial/\partial\psi$ Killing vector simply reflects the R-symmetry of the field theory.

A supergravity solution will serve as a good M/string-theory background only if the fluxes satisfy appropriate quantization conditions. This requirement puts additional constraints on the classical solutions we have found. This issue will be analysed in more detail for the solutions of section 5.1 that are S^2 bundles over $S^2 \times T^2$ in [38], and for this case flux quantization reduces the continuous family of solutions to a countably infinite subclass. In particular, quantization of the NS 3-form flux in the type IIA solution is T-dual to the classical requirement in type IIB that the curvature of a connection on a $U(1)$ bundle has integral periods (see for example [44]). On the other hand, lifting to M-theory, the quantization of the NS flux implies that the G-flux is also quantized. In general then, we expect that these quantization conditions put constraints on (some of) the parameters in our solutions. The continuous families of supergravity solutions that we have found would then only correspond to true string theory backgrounds at discrete values, and from the analysis in [38], these may be countably infinite in number.

It is natural to try to interpret our solutions in $D = 11$ in terms of wrapped or intersecting M5-branes. A very concrete way to make this connection would be to construct more general solutions with the property that the solutions presented here arise as near-horizon limits. For the families of solutions with $S^2 \times S^2 \times T^2$ topology discussed in section 5.1, and which have type II dual configurations, a standard interpretation would be as follows. Consider starting on the type IIB side, where the solutions arise as the near-horizon limit of N D3-branes placed at the tip of the Calabi–Yau cone over the Sasaki–Einstein manifold. Then T-duality along an appropriate transverse direction maps to a IIA solution which is expected to arise from a configuration of N D4-branes suspended between NS5-branes [43, 45]. Uplifting to $D = 11$, the M-theory solution is interpreted as the near-horizon limit of intersecting M5-branes. We expect that, depending on the choice of parameters, only one of the M-theory, type IIA or type IIB solutions would provide a good *supergravity* description of the underlying superconformal field theory. It would be interesting to see if in any of these limits one can extract the exact configuration of branes or the nature of the singularity and hence identify the corresponding conformal field theories.

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Appendix A. Conventions

We use the signature $(-, +, \ldots, +)$. In an orthonormal frame the gamma matrices satisfy

$$\{\Gamma_\alpha, \Gamma_\beta\} = 2\eta_{\alpha\beta} \tag{A.1}$$

and can be taken to be real in the Majorana representation. They satisfy, in our conventions, $\Gamma_{012345678910} = \epsilon_{012345678910} = 1$. Given a Majorana spinor ϵ its conjugate is given by $\bar{\epsilon} = \epsilon^T C$, where C is the charge conjugation matrix in $D = 11$ and satisfies $C^T = -C$. In the Majorana representation we can choose $C = \Gamma_0$.

The bosonic fields of $D = 11$ supergravity consist of a metric, g, and a 3-form potential C with 4-form field strength $G = \mathrm{d}C$. The action for the bosonic fields is given by

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^{11}x\sqrt{-g}R - \frac{1}{2}G \wedge *G - \frac{1}{6}C \wedge G \wedge G. \tag{A.2}$$

The Killing spinor equation is

$$\nabla_\mu\epsilon + \frac{1}{288}\left[\Gamma_\mu{}^{\nu_1\nu_2\nu_3\nu_4} - 8\delta_\mu^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}\right]G_{\nu_1\nu_2\nu_3\nu_4}\epsilon = 0 \tag{A.3}$$

where ϵ is a Majorana spinor. The equations of motion are given by

$$\begin{aligned} &R_{\mu\nu} - \tfrac{1}{12}\big(G_{\mu\sigma_1\sigma_2\sigma_3}G_\nu{}^{\sigma_1\sigma_2\sigma_3} - \tfrac{1}{12}g_{\mu\nu}G^2\big) = 0 \\ &\mathrm{d} * G + \tfrac{1}{2}G \wedge G = 0. \end{aligned} \tag{A.4}$$

Note that in M-theory, the field equation for the 4-form receives higher-order gravitational corrections—in our conventions they can be found in the appendix of [20].

The Hodge star of a p-form ω is defined by

$$*\omega_{\mu_1\ldots\mu_{11-p}} = \frac{\sqrt{-g}}{p!}\epsilon_{\mu_1\ldots\mu_{11-p}}{}^{\nu_1\ldots\nu_p}\omega_{\nu_1\ldots\nu_p}. \tag{A.5}$$

Appendix B. Differential conditions for spinor bilinears

We want to manipulate the Killing spinor equations (2.8) to define differential conditions on fermion bilinears on M_6. Following a similar calculation in [23], it is useful to write

$$\epsilon^+ = \tfrac{1}{\sqrt{2}}\xi \qquad \epsilon^- = -\tfrac{1}{\sqrt{2}}\mathrm{i}\gamma_7\xi. \tag{B.1}$$

We then get

$$\begin{aligned} &\nabla_m\epsilon^\pm \mp \frac{m}{2}\gamma_m\epsilon^\mp \mp \frac{1}{24}\gamma^{n_1n_2n_3}\,\mathrm{e}^{-3\lambda}G_{mn_1n_2n_3}\epsilon^\pm = 0 \\ &\gamma^m\nabla_m\lambda\epsilon^\pm \pm m\epsilon^\mp \pm \frac{1}{144}\gamma^{m_1m_2m_3m_4}\,\mathrm{e}^{-3\lambda}G_{m_1m_2m_3m_4}\epsilon^\pm = 0. \end{aligned} \tag{B.2}$$

Given that $\gamma_m^\dagger = \gamma_m$, one can derive some useful identities

$$\begin{aligned} &\frac{1}{144}\,\mathrm{e}^{-3\lambda}G_{pqrs}\bar\epsilon^\pm[\gamma^{pqrs}, A]_-\epsilon^\pm \pm \nabla_m\lambda\bar\epsilon^\pm[\gamma^m, A]_-\epsilon^\pm + m(\bar\epsilon^\mp A\epsilon^\pm - \bar\epsilon^\pm A\epsilon^\mp) = 0 \\ &\frac{1}{144}\,\mathrm{e}^{-3\lambda}G_{pqrs}\bar\epsilon^\pm[\gamma^{pqrs}, A]_+\epsilon^\pm \pm \nabla_m\lambda\bar\epsilon^\pm[\gamma^m, A]_+\epsilon^\pm + m(\bar\epsilon^\mp A\epsilon^\pm + \bar\epsilon^\pm A\epsilon^\mp) = 0 \\ &\frac{1}{144}\,\mathrm{e}^{-3\lambda}G_{pqrs}\bar\epsilon^\pm[\gamma^{pqrs}, A]_\pm\epsilon^- + \nabla_m\lambda\bar\epsilon^+[\gamma^m, A]_\mp\epsilon^- + m(\bar\epsilon^- A\epsilon^- \pm \bar\epsilon^+ A\epsilon^+) = 0 \end{aligned} \tag{B.3}$$

where $[\cdot,\cdot]_\pm$ refer to anticommutator or commutator, A is a general Clifford matrix and $\bar\epsilon \equiv \epsilon^\dagger$. There are similar formulae involving $\epsilon^{\pm\mathrm{T}}$.

Let us now consider the bilinears that can be constructed from $\epsilon^\pm$. We first analyse the scalars. By definition $\bar\epsilon^+\epsilon^+ = \bar\epsilon^-\epsilon^- = \frac{1}{2}\bar\xi\xi$. Using (B.2) we find $\nabla(\bar\epsilon^+\epsilon^+) = 0$. Thus we can normalize the spinor so that

$$\bar\epsilon^+\epsilon^+ = \bar\epsilon^-\epsilon^- = 1. \tag{B.4}$$

On the other hand $\nabla(\bar\epsilon^+\epsilon^-) \neq 0$, and we parametrize this non-trivial function, which takes values in the interval $[-1, 1]$, as

$$\bar\epsilon^+\epsilon^- \equiv \sin\zeta. \tag{B.5}$$

Of the other possible scalars, note that $\epsilon^{+\mathrm{T}}\epsilon^- = 0$ while $\epsilon^{+\mathrm{T}}\epsilon^+ = -\epsilon^{-\mathrm{T}}\epsilon^- \equiv f$ is *a priori* non-zero.

We can also construct the following tensor fields as bilinears:

$$\begin{aligned} &\tilde K^1_m = \bar\epsilon^+\gamma_m\epsilon^+ && \tilde K^2_m = \mathrm{i}\bar\epsilon^+\gamma_m\epsilon^- \\ &Y_{mn} = -\mathrm{i}\bar\epsilon^+\gamma_{mn}\epsilon^+ && Y'_{mn} = \mathrm{i}\bar\epsilon^+\gamma_{mn}\epsilon^- \\ &\tilde\Omega_{mn} = \epsilon^{+\mathrm{T}}\gamma_{mn}\epsilon^- && X_{mnp} = \epsilon^{+\mathrm{T}}\gamma_{mnp}\epsilon^+ \\ &V_{mnp} = \bar\epsilon^+\gamma_{mnp}\epsilon^-. \end{aligned} \tag{B.6}$$

Consideration of other bilinears turns out to be redundant and we will not include them in the following analysis.

Observe that the covariant derivative of $\tilde{K}^2$ is given by

$$\nabla_m \tilde{K}^2_n = -\tfrac{1}{4}\,\mathrm{e}^{-3\lambda} G_{mnpq} Y'^{pq} - m Y_{mn}. \tag{B.7}$$

This implies that $\nabla_{(m}\tilde{K}^2_{r)} = 0$ and therefore $(\tilde{K}^2)^m$ are the components of a Killing vector. Next, setting $A = 1$ in the last equation of (B.3) with the lower sign immediately gives the useful condition

$$\mathcal{L}_{\tilde{K}^2}\lambda = 0. \tag{B.8}$$

Conditions on the exterior derivatives of all the bilinears can be obtained, and we find

$$\mathrm{d}(\mathrm{e}^{3\lambda} f) = 0 \tag{B.9}$$

$$\mathrm{e}^{-3\lambda}\,\mathrm{d}(\mathrm{e}^{3\lambda}\sin\zeta) = 2m\tilde{K}^1 \tag{B.10}$$

$$\mathrm{d}(\mathrm{e}^{3\lambda}\tilde{K}^1) = 0 \tag{B.11}$$

$$\mathrm{e}^{-6\lambda}\,\mathrm{d}(\mathrm{e}^{6\lambda}\tilde{K}^2) = \mathrm{e}^{-3\lambda} * G + 4mY \tag{B.12}$$

$$\mathrm{d}(\mathrm{e}^{6\lambda} Y) = 0 \tag{B.13}$$

$$\mathrm{e}^{-6\lambda}\,\mathrm{d}(\mathrm{e}^{6\lambda}\tilde{\Omega}) = 3mX \tag{B.14}$$

$$\mathrm{e}^{-6\lambda}\,\mathrm{d}(\mathrm{e}^{6\lambda} X) = -f\,\mathrm{e}^{-3\lambda} G \tag{B.15}$$

$$\mathrm{e}^{-6\lambda}\,\mathrm{d}(\mathrm{e}^{6\lambda} V) = \mathrm{e}^{-3\lambda} G \sin\zeta + 2m * Y'. \tag{B.16}$$

Note that, when m is non-zero, which is the focus of this paper, (B.11) is implied by (B.10), and (B.13) is implied by (B.12) and the G equation of motion. Moreover, (B.14) together with (B.15) implies the important fact that

$$f \equiv \epsilon^{+\mathrm{T}}\epsilon^{+} = \tfrac{1}{2}\xi^{\mathrm{T}}\xi = 0. \tag{B.17}$$

Appendix C. $SU(3)$ and $SU(2)$ structures in d = 6

We now rewrite the conditions on the spinor bilinears derived in appendix B in terms of an explicit local $SU(2)$ structure. We start by defining $SU(3)$ and $SU(2)$ structures in $d = 6$. We work in a basis in which the gamma matrices are imaginary. A single unit-norm chiral spinor in $d = 6$, satisfying

$$\bar{\eta}_1\eta = 1 \qquad -\mathrm{i}\gamma_7\eta = \eta \tag{C.1}$$

defines an $SU(3)$ structure with 2-form j and (3, 0)-form ω given by

$$j = -\mathrm{i}\bar{\eta}_1\gamma_{(2)}\eta_1 \qquad \omega = \eta_1^T\gamma_{(3)}\eta_1 \tag{C.2}$$

where $\gamma_{(n)}$ is the n-form $\frac{1}{n!}\gamma_{i_1\ldots i_n} e^{i_1}\wedge\ldots\wedge e^{i_n}$. For further discussion, see for example [21].

Now consider two orthogonal unit-norm chiral spinors η_1, η_2 satisfying

$$\bar{\eta}_1\eta_1 = 1 \qquad \bar{\eta}_2\eta_2 = 1 \qquad \bar{\eta}_1\eta_2 = 0 \tag{C.3}$$

and $-\mathrm{i}\gamma_7\eta_a = \eta_a$. Together these define two canonical $SU(3)$ structures, specified by forms (j_1, ω_1) and (j_2, ω_2) defined as in (C.2). Equivalently, η_1, η_2 define a canonical $SU(2)$ structure in $d = 6$. Such a structure is specified by two 1-forms K^1, K^2 and three 2-forms J^m defined by

$$J^m = -\frac{\mathrm{i}}{2}\sigma_m^{\alpha\beta}\bar{\eta}_\alpha\gamma_{(2)}\eta_\beta \qquad K^1 - \mathrm{i}K^2 = -\frac{1}{2}\epsilon^{\alpha\beta}\eta_\alpha^T\gamma_{(1)}\eta_\beta \tag{C.4}$$

where σ_m are Pauli matrices. The $d = 6$ metric has the form

$$\mathrm{d}s^2 = e^i e^i + (K^1)^2 + (K^2)^2 \tag{C.5}$$

with $J^m = \frac{1}{2}J^m_{ij}e^i \wedge e^j$ satisfying the algebraic identities of a $d = 4$ $SU(2)$ structure. We also define

$$\Omega \equiv J^2 + \mathrm{i}J^1 \qquad J \equiv J^3 \tag{C.6}$$

so the $SU(2)$ structure in $d = 6$ is equivalently specified by (J, Ω, K^1, K^2). We naturally define $\mathrm{vol}_6 = \mathrm{vol}_4 \wedge K^1 \wedge K^2$ where $\mathrm{vol}_4 = \frac{1}{2}J \wedge J$. In addition we have that each J^i is self-dual with respect to vol_4 and also, after raising an index, we have $J^1 \cdot J^2 = J^3$.

For completeness, we note that the two $SU(3)$ structures defined by (C.2) can be written in terms of the $d = 6$ $SU(2)$ structure as

$$\begin{aligned} j_1 &= J - K^1 \wedge K^2 & \omega_1 &= -\Omega \wedge (K^1 - \mathrm{i}K^2) \\ j_2 &= -J - K^1 \wedge K^2 & \omega_2 &= -\bar{\Omega} \wedge (K^1 - \mathrm{i}K^2). \end{aligned} \tag{C.7}$$

In doing calculations it is often easiest to specify the spinors in terms of some concrete convenient projections. In particular, this provides a simple derivation of the above formulae. For example, for the spinor η_1 we take

$$\gamma_{12}\eta_1 = \gamma_{34}\eta_1 = -\gamma_{56}\eta_1 = \mathrm{i}\eta_1 \qquad \gamma_{135}\eta_1 = -\eta_1^*. \tag{C.8}$$

For the second spinor η_2 we take

$$-\gamma_{12}\eta_2 = -\gamma_{34}\eta_2 = -\gamma_{56}\eta_2 = \mathrm{i}\eta_2 \qquad \gamma_{135}\eta_2 = -\eta_2^*. \tag{C.9}$$

In addition, the two spinors are related by

$$\gamma_5\eta_2^* = \eta_1. \tag{C.10}$$

This then leads to $J^1 = e^{14} + e^{23}$, $J^2 = e^{13} - e^{24}$, $J^3 = e^{12} + e^{34}$, $K^1 = e^5$ and $K^2 = e^6$. Note that the J^m are indeed self-dual and that $J^1 \cdot J^2 = J^3$.

We would now like to relate the non-chiral spinor ξ entering the Killing spinor equations (2.8) to a canonical $d = 6$ $SU(2)$ structure, as defined above. We first observe that we can always take two non-orthogonal unit-norm chiral spinors to be η_1 and $a\eta_1 + b\eta_2$ with $|a|^2 + |b|^2 = 1$. Therefore, a general non-chiral spinor ξ can be decomposed as

$$\xi \equiv \xi_+ + \xi_- = f_1\eta_1 + f_2(a\eta_1 + b\eta_2)^* \tag{C.11}$$

where η_i are uniquely defined up to phases. We fix this phase freedom[10] by taking f_i real and $b = (1 - |a|^2)^{1/2}$. Now, since our spinor satisfies (B.4), we set $f_1 = \sqrt{2}\cos\alpha$ and $f_2 = \sqrt{2}\sin\alpha$. Moreover, since we found that f was zero (B.17), we conclude that we can choose $a = 0$ and hence $b = 1$. Hence we have

$$\xi_+ = \sqrt{2}\cos\alpha\,\eta_1 \qquad \xi_- = \sqrt{2}\sin\alpha\,\eta_2^* \tag{C.12}$$

[10] To see this, note that two chiral spinors ξ_i, $i = 1, 2$, define four real scalars: $\bar{\xi}_i\xi_i \equiv f_i^2$ and $\bar{\xi}_1\xi_2 \equiv f_1 f_2 a$. One can then define $\eta_1 = \xi_1/f_1$ and $\eta_2 = (1 - |a|^2)^{-1/2}(\xi_2/f_2 - a\xi_1/f_1)$.

and consistency with (B.5) implies $\cos 2\alpha = \sin\zeta$. Note that this implies $\cos\zeta = \sin 2\alpha$ where we have fixed an arbitrary sign.

Having established how our spinor ξ is related to a canonically defined $SU(2)$ structure defined by η_1, η_2, we can now translate the bilinears defined in (B.6) into those naturally defined by the $SU(2)$ structure. We find

$$\begin{aligned} &\tilde{K}^1 = K^1\cos\zeta && \tilde{K}^2 = K^2\cos\zeta \\ &Y = J - K^1\wedge K^2\sin\zeta && Y' = -J\sin\zeta + K^1\wedge K^2 \\ &\tilde{\Omega} = \Omega\cos\zeta && X = \Omega\wedge(-K_1\sin\zeta + \mathrm{i}K^2) \\ &V = J\wedge K^2\cos\zeta \end{aligned} \tag{C.13}$$

together with

$$\mathrm{vol}_6 = \tfrac{1}{2}J\wedge J\wedge K^1\wedge K^2. \tag{C.14}$$

We then find that (B.9)–(B.16) are equivalent to

$$\begin{aligned} &\mathrm{e}^{-3\lambda}\,\mathrm{d}(\mathrm{e}^{3\lambda}\sin\zeta) = 2mK^1\cos\zeta \\ &\mathrm{e}^{-6\lambda}\,\mathrm{d}(\mathrm{e}^{6\lambda}\Omega\cos\zeta) = 3m\Omega\wedge(-K^1\sin\zeta + \mathrm{i}K^2) \\ &\mathrm{e}^{-6\lambda}\,\mathrm{d}(\mathrm{e}^{6\lambda}K^2\cos\zeta) = \mathrm{e}^{-3\lambda} * G + 4m(J - K^1\wedge K^2\sin\zeta) \\ &\mathrm{e}^{-6\lambda}\,\mathrm{d}(\mathrm{e}^{6\lambda}J\wedge K^2\cos\zeta) = \mathrm{e}^{-3\lambda}G\sin\zeta + m(J\wedge J - 2J\wedge K^1\wedge K^2\sin\zeta) \end{aligned} \tag{C.15}$$

where in the last formula we have used $*J = J\wedge K^1\wedge K^2$.

Appendix D. Minkowski$_5$ solutions

Here we analyse the supersymmetry constraints on the geometry in the case of vanishing five-dimensional cosmological constant, $m = 0$. This corresponds to the most general supersymmetric solutions of $D = 11$ supergravity that are warped products of Minkowski$_5$-space with a six-dimensional manifold. It has been argued in [46], using an argument based on an effective superpotential, that it is not possible to have such supersymmetric M-theory vacua with non-trivial fluxes. Our results show that these are possible, if we consider a suitably general spinor ansatz. The reason for this apparent discrepancy is that the argument of [46] implicitly assumed that the internal manifold is Calabi–Yau, which indeed turns out to be incompatible with flux.

Let us now turn to the detailed analysis of the supersymmetry constraints. The basic conditions are given by equations (B.9)–(B.16), upon setting $m = 0$. From (B.11) we deduce that we have an integrable almost-product structure. Next, from (B.7) (which must in fact follow from (B.9)–(B.16)), we see that $\tilde{K}^2 = \cos\zeta K^2$ is a Killing vector. Moreover, we also have $\mathcal{L}_{\tilde{K}^2}\lambda = \mathcal{L}_{\tilde{K}^2}\zeta = \mathcal{L}_{\tilde{K}^2}K^1 = 0$. Therefore, similar to the $m \neq 0$ case, the $d = 6$ metric can be locally written as

$$\mathrm{d}s^2 = g^4_{ij}(x, y)\,\mathrm{d}x^i\,\mathrm{d}x^j + \mathrm{e}^{-6\lambda}\sec^2\zeta\,\mathrm{d}y^2 + \cos^2\zeta(\mathrm{d}\psi + \rho)^2 \tag{D.1}$$

with

$$K^1 = \mathrm{e}^{-3\lambda}\sec\zeta\,\mathrm{d}y \qquad K^2 = \cos\zeta(\mathrm{d}\psi + \rho) \tag{D.2}$$

and $\partial_\psi\lambda = \partial_\psi\zeta = 0$. Using equation (B.12), we get the following expression for the flux:

$$\begin{aligned} G = {} & \partial_y(\mathrm{e}^{6\lambda}\cos^2\zeta)\mathrm{vol}_4 - \mathrm{e}^{-3\lambda}\sec\zeta *_4 \mathrm{d}_4(\mathrm{e}^{6\lambda}\cos^2\zeta)\wedge K^1 - \cos^3\zeta\,\mathrm{e}^{6\lambda} *_4 \partial_y\rho\wedge K^2 \\ & + \mathrm{e}^{3\lambda}\cos^2\zeta *_4 \mathrm{d}_4\rho\wedge K^1\wedge K^2. \end{aligned} \tag{D.3}$$

Note this expression is slightly different from that one would naively obtain by setting $m = 0$ in (2.50).

Now from (B.10) we see that there are two cases to consider separately, depending on whether $\sin\zeta$ vanishes or not (note that $\sin\zeta = 0$ is not possible when $m \neq 0$).

D.1. $\sin\zeta = 0$

For this case we have $Y' = K^1 \wedge K^2$, so taking the antisymmetric part of (B.7) we get

$$\mathrm{d}\rho = -\mathrm{e}^{-3\lambda}(K^1 \wedge K^2) \lrcorner G. \tag{D.4}$$

Substituting (D.3) into (D.4) we immediately conclude that $\partial_y \rho = 0$ and

$$(\mathrm{d}_4\rho)^+ = 0. \tag{D.5}$$

Note that in the present case the chiral spinors $\xi_\pm$ have constant norm, and hence they define a *global* six-dimensional $SU(2)$-structure. Moreover, we find

$$\mathrm{d}(\mathrm{e}^{6\lambda} J) = 0 \qquad \mathrm{d}(\mathrm{e}^{6\lambda}\Omega) = 0 \tag{D.6}$$

implying that the four-dimensional slices, at fixed y, are conformal to y-independent hyper-Kähler manifolds. Therefore the general form of the metric is

$$\mathrm{d}s^2 = (\mathrm{d}\psi + \rho)^2 + \mathrm{e}^{-6\lambda}\left[\mathrm{d}\hat{s}_4^2(\mathrm{HK}) + \mathrm{d}y^2\right]. \tag{D.7}$$

It is a simple matter to check that (B.16) is trivially satisfied, while (B.15) forces $f = 0$. The G-flux reads

$$G = -\partial_y(\mathrm{e}^{-6\lambda})\hat{\mathrm{vol}}_4 + \hat{*}_4\, \mathrm{d}_4(\mathrm{e}^{-6\lambda}) \wedge \mathrm{d}y - \mathrm{d}_4\rho \wedge \mathrm{d}y \wedge (\mathrm{d}\psi + \rho) \tag{D.8}$$

and the (source-less) Bianchi identity $\mathrm{d}G = 0$ implies

$$\hat{\ }\mathrm{e}^{-6\lambda} + \partial_y^2\, \mathrm{e}^{-6\lambda} = -\tfrac{1}{2}\|\hat{\mathrm{d}_4\rho}\|^2 \tag{D.9}$$

which must be solved in order to complete the solution.

It is interesting to note that the geometries we have found arise when a 5-brane is transverse to a hyper-Kähler manifold times a flat direction, with a single flat longitudinal direction of the 5-brane fibred over the hyper-Kähler manifold. When $\partial/\partial y$ is a Killing vector, we can reduce to type IIA theory and recover the results of [21], where the corresponding geometry is related to wrapped NS5-branes. Examples of these solutions appeared in [47], where the 4-space was taken to be Eguchi–Hanson[11].

Alternatively, we can also make a Kaluza–Klein reduction along the ψ-direction. We thus obtain supersymmetric IIA solutions, with constant dilaton and non-trivial RR 2-form and 4-form, plus NS B-field $B^{\mathrm{NS}} = -y\, \mathrm{d}_4\rho$. Finally, we note that simply setting $\rho = 0$, we recover the conditions on the local geometry obtained by Witten [48] for M-theory compactifications to six-dimensional Minkowski space.

D.2. $\sin\zeta \neq 0$

We must now have $\mathrm{e}^{3\lambda}\sin\zeta = c$ for some constant c. Hence, from (B.16), we see that G is exact:

$$cG = \mathrm{d}(\mathrm{e}^{6\lambda}\cos\zeta\, J \wedge K^2) \tag{D.10}$$

[11] Note that the construction of [47] has been generalized in [21] to an arbitrary number of twisted directions. For instance, twisted T^2 bundles over a hyper-Kähler 4-manifold yield potentially interesting flux compactifications to four dimensions.

so that the Bianchi identity is automatically satisfied (as expected from the argument in section 2.2). The conditions on the base geometry can be conveniently expressed in terms of rescaled quantities, $\hat{g}_{ij} = \mathrm{e}^{6\lambda} g_{ij}$, $\hat{J} = \mathrm{e}^{6\lambda} J$ and $\hat{\Omega} = \mathrm{e}^{6\lambda}\Omega$. We find

$$\mathrm{d}_4\hat{J} = 0 \qquad \mathrm{d}_4(\cos\zeta\,\hat{\Omega}) = 0 \tag{D.11}$$

which state that the base manifold, at fixed y, is almost Calabi–Yau. In addition, we have

$$\partial_y\hat{J} = -c\,\mathrm{d}_4\rho \qquad \partial_y(\cos\zeta\,\hat{\Omega}) = 0. \tag{D.12}$$

Comparing the expressions for the flux computed from (D.3) and (B.16), we obtain an additional constraint:

$$c *_4 \partial_y\rho = J \wedge \mathrm{d}_4 \sec^2\zeta. \tag{D.13}$$

Note that here we can derive an expression for J that is similar to that in the $m \neq 0$ case, which reads

$$(\mathrm{d}_4\rho)^+ = \frac{-c}{\sin\zeta\,\cos\zeta}\partial_y\zeta\,J. \tag{D.14}$$

It is now a simple matter to deal with the remaining conditions (B.15) and (B.9). Using equations (D.11), (D.12), we find that (B.15) reduces to

$$f\,\mathrm{e}^{-3\lambda}G = -\cos\zeta\,\mathrm{d}y\wedge\Omega\wedge\left[\mathrm{i}\partial_y\rho + \frac{1}{c}\,\mathrm{d}_4\sec^2\zeta\right]. \tag{D.15}$$

However, upon using (D.13), one shows that the right-hand side vanishes. Hence $f = 0$ in the present case also. Observe that an integrability condition between the first equation in (D.12) and (D.13) yields the following second-order equation

$$\partial_y^2(\mathrm{e}^{6\lambda}J) - 2c\mathrm{i}\partial\bar{\partial}\sec^2\zeta = 0. \tag{D.16}$$

We conclude by presenting a simple example of a (singular) solution in this class. Consider the ansatz for the four-dimensional base metric

$$\mathrm{d}\hat{s}^2 = \Lambda(y)\,\mathrm{d}\hat{s}_0^2 \tag{D.17}$$

where the re-scaled metric is y-independent and hyper-Kähler, with associated hyper-Kähler structure $\hat{J}_0, \hat{\Omega}_0$. Setting $c = 1$ for simplicity, it is straightforward to check that all the conditions are satisfied, provided

$$\Lambda = \sec\zeta = 1 + y \qquad \mathrm{d}_4\rho = \hat{J}_0 \qquad \partial_y\rho = 0. \tag{D.18}$$

The corresponding flux reads

$$G = \mathrm{d}\left[\frac{1}{(1+y)}\hat{J}_0\wedge(\mathrm{d}\psi + \rho)\right] \tag{D.19}$$

and the six-dimensional metric is

$$\mathrm{d}s_6^2 = \frac{y(2+y)}{1+y}\left[\mathrm{d}\hat{s}_0^2 + (1+y)\,\mathrm{d}y^2\right] + \frac{1}{(1+y)^2}(\mathrm{d}\psi + \rho). \tag{D.20}$$

It is interesting to note that in the limit $y \to \infty$ ($\cos\zeta \to 0$) the flux vanishes and, correspondingly, the 11-dimensional metric asymptotes to

$$\mathrm{d}s^2 = \mathrm{d}s^2(\mathrm{Mink}_5) + y\,\mathrm{d}\hat{s}_0^2 + y^2\,\mathrm{d}y^2 + \frac{1}{y^2}(\mathrm{d}\psi + \rho)^2. \tag{D.21}$$

It is straightforward to check that the internal metric is indeed Calabi–Yau. This behaviour, in fact, is not accidental. In general, when $\cos\zeta \to 0$, the warp factor will become asymptotically constant[12], and in this case, from the Einstein equation, it follows that the flux must vanish (when $m = 0$). Therefore, this interpolating feature is likely to be quite generic. Unfortunately, as $y \to 0$ the metric develops a singularity.

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