Supersymmetric Backgrounds From Generalized Calabi-Yau Manifolds

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Fluxes ⟷ Geometry

Type II sugra on $M_{10} = M_4 \times M_6$

Minimal supersymmetry

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M_6 is CY	Turn on fluxes ⟶	M_6 ~~CY~~ (at best conformal CY)
SU(3) holonomy	⟶	SU(3) structure
$\nabla_m \eta = 0$	⟶	$\nabla_m^{(T)} \eta = 0$ Torsion ~ flux
CY	⟶	**Generalized CY**

Hitchin '02

Outline

- Basic notions about SU(3) structure
- Torsion versus fluxes
- Generalized complex geometry / Generalized CY
- Generalized CY from supersymmetry equations
- Conclusions / open questions

Basic notions about SU(3) structure

$M_{10} = M_4 \times_w M_6$

- No flux) M_6 is CY → SU(3) holonomy

  - 9 SU(3) invt. spinor η

    $$\eta^\dagger \gamma_{mn} \gamma \eta = i\, w_{mn}$$
    $$\eta^\dagger \gamma_{mnp} (1+\gamma) \eta = i\, \Omega_{mnp}$$

    $$w \wedge \Omega = 0$$
    $$i\Omega \wedge \bar{\Omega} = w^3$$

  - η is cov. constant: r_m η =0

    $$dw = 0$$
    $$d\Omega = 0$$

    $$\epsilon_{10}^{1,2} = \theta^{1,2} \otimes \eta \;\; \Rightarrow \mathcal{N} = 2$$

- Turn on flux) back-reaction

  - $M_{10} = M_4 \, x_w \, M_6$ $\quad [ds_{10}^2 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + ds_6^2(y)]$
  - N =2 ! N =1 (relation between θ^1 and θ^2)
  - M_6 acquires torsion → SU(3) structure

    Gauntlett, Martelli, Pakis, Waldram ‘02

  - 9 SU(3) invt. spinor η

    $$\eta^\dagger \gamma_{mn} \gamma \eta = i\, w_{mn}$$
    $$\eta^\dagger \gamma_{mnp} (1+\gamma) \eta = i\, \Omega_{mnp}$$

  - $r_m^{(T)} \eta = 0$

    $r_m^{(T)}$ w = 0) dw ≠ 0 / T

    $r_m^{(T)} \Omega$ = 0) dΩ ≠ 0 / T

Torsion versus fluxes

Torsion: $dw = \mathrm{Im}\,(W_1\,\Omega) + W_4 \wedge w + W_3$
SU(3) reps $\quad 1 \oplus 1 \qquad 3 \oplus \bar{3} \qquad 6 \oplus \bar{6}$

$d\,\Omega = W_1\, w^2 + W_5 \wedge \Omega + W_2 \wedge w$
$\quad 1 \oplus 1 \qquad 3 \oplus \bar{3} \qquad 8 \oplus 8$

	$1 \oplus 1$	$3 \oplus \bar{3}$	$6 \oplus \bar{6}$	$8 \oplus 8$
Torsion	1 (W_1)	2 (W_4, W_5)	1 (W_3)	1 (W_2)
H_3	1	1	1	0
IIA: F_{2n}	2 (F_0, F_2, F_4)	2 (F_2, F_4)	0	1 (F_2, F_4)
IIB: F_{2n+1}	1 (F_3)	3 (F_1, F_3, F_5)	1 (F_3)	0

In IIB $W_2 = 0$ (integrability of complex structure)

In IIA $W_3 \sim H^{(6)}$ (symplectic geometry)

If also $W_1 = 0$ → IIB: $d\Omega = W_5 \wedge \Omega$ — M_6 is complex
(true in all susy vacua)

IIA: $d\,w = W_4 \wedge w + H^{(6)}$ — M_6 is "twisted symplectic"

Is there a mathematical construction that extends complex and symplectic geometry?

Generalized complex geometry

Hitchin
Gualtieri

Huybrechts
Kapustin
Lindström
Zabzine

- Usual differential geometry → T: tangent bundle (sections vector fields X) | T*: cotangent bundle (sections are 1-forms ζ)

- Want differential geometry on T © T* sections are X+ ζ

Natural metric I on T © T* : (X+ ζ, X+ζ) = i_X ζ I = $\begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}$

On T © T* define Generalized Almost Complex Structure (GACS) J: T© T* ! T© T*

$J^2 = -1_{2d}$
$J^t I J = I$

	T	T © T*
GACS	J s.t. $J^2 = -1_d$	J s.t. $J^2 = - 1_{2d}$
Projectors	$\pi_§ = (1_d$ § i J)	$\Pi_§ = (1_{2d}$ § i J)
Integrability	$\pi_{\cdot\cdot}[\pi_§ X, \pi_§ Y]=0$	$\Pi_{\cdot\cdot}[\Pi_§ (X+\zeta), \Pi_§ (Y+\zeta)]_C=0$

project down

T © T*	→	T
X + ζ	→	X
[,]$_C$	→	[,]
J	→	J ?

$[X+\zeta, Y + \eta]_C=[X,Y] + L_X \eta - L_Y \zeta -$ _ $d (i_X \eta - i_Y \zeta)$

when ζ = η = 0) [,]$_C$ = [,]

How is complex geometry embedded in generalized complex geometry?

Aside: GCG naturally incorporates B-field:

$e^B (X+\zeta) = X+\zeta + i_X B$ → $[e^B (X+\zeta), e^B (Y+\eta)]_C = e^B [X+\zeta,Y+\eta]_C$ → automorphisms of [,]$_C$

_ A GACS has the form

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & K \end{pmatrix}$$

J: TM ! TM	$J^m{}_n$
P: T*M ! TM	P^{mn}
L: TM ! T*M	L_{mn}
K: T*M ! T*M	$K_m{}^n$

Demanding $\mathcal{J}^2 = -1_{2d}$, $\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I}$

Get conditions

$$K = -J^t$$
$$P^t = -P$$
$$L^t = -L$$

$\rightarrow$ $$\mathcal{J} = \begin{pmatrix} J & P \\ L & -J^t \end{pmatrix}$$

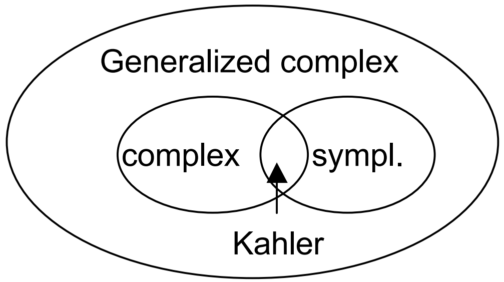
_ It is easy to guess how ACS _ GACS :

$$\mathcal{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}$$ Integrability of GACS $\mathcal{J}_1$) is J ıtegrable (it is a CS in T)) M is complex

_ But GACS have more…

Consider $$\mathcal{J}_2 = \begin{pmatrix} 0 & -w^{-1} \\ w & 0 \end{pmatrix}$$ Integrability of $\mathcal{J}_2$) dw=0. If w is non-degenerate) M is symplectic

Complex manifolds Symplectic manifolds	→ admit GCS



Complex: locally equivalent to $C^{d/2}$

Symplectic: locally equivalent to (R^d, w); $w = dx^1 Æ dx^2 + \ldots + dx^{d-1} Æ dx^d$

Generalized complex: locally equivalent to C^k _ (R^{d-2k}, w) k: rank. k=0 for symplectic
k=d/2 for complex

GACS ⟷ Pure spinors of Clifford (d,d)

	T	T © T*
	Clifford 6	Clifford (6,6)
Algebra	$\{\gamma^m,\gamma^n\}=g^{mn}$	$\{\gamma^m, \gamma^n\}=\{\gamma_m, \gamma_n\}=0$, $\{\gamma^m, \gamma_n\}= \delta^m{}_n$
Representation in terms of forms	$\gamma^i=dz^i$Æ and γ–= g–ʲ $\iota_{\partial j}$	$\gamma^m=dx^m$Æ and $\gamma_m = \iota_{\partial_m}$
Clifford vacuum	Ω	Ω
Pure spinor	$\gamma^i\ \Omega=0$	$\gamma^i\ \Omega=\gamma\_\ \Omega=0$
Basis	Ω : (3,0) γ– Ω : (2,0) γ–ʲ Ω : (1,0) γ–ʲᵏ Ω : (0,0)	Ω : (3,0) Ω : (3,0) γ– Ω : (3,1) $\gamma_i\Omega$: (2,0) γ–ʲ Ω : (3,2) $\gamma_{ij}\ \Omega$: (1,0) γ–ʲᵏ Ω : (3,3) $\gamma_{ijk}\ \Omega$: (0,0)
	Spinors $ (p,0) forms	Spinors $ (p,q) forms

$$e^{iw} = 1 + iw - \frac{1}{2}w^2 - i\frac{1}{3!}w^3$$

$(\gamma_m + i\, w_{mn}\, \gamma^n)\, e^{iw}=0$

- On a manifold with SU(3) structure in T ⟶ 2 pure spinors of Clifford(6,6) $\Omega\ ;\ e^{iw}$
- 1-1 correspondence between pure spinors and GACS J

Ω ↓ $J_1 = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}$

e^{iw} ↓ $J_2 = \begin{pmatrix} 0 & -w^{-1} \\ w & 0 \end{pmatrix}$

Integrability of J , 9 v and ξ s.t

$$d\varphi = (\iota_v + \xi\wedge)\varphi$$

has one GCS

⇑

Hitchin

Generalized Calabi-Yau ⟶ A manifold that has one closed pure spinor $(d\varphi = 0)$

Supersymmetry equations written in terms of pure spinors

$\delta\psi_m$:

$$IIA: D^H_m\eta_+ + e^\phi(\bar\Omega F^A)_m\eta_+ + e^\phi\left[(F^A_m e^{iw})_n + (F^A e^{iw})_0 g_{mn} + (F^A e^{iw})_{mn}\right]\gamma^n\eta_- = 0$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$IIB: D^H_m\eta_+ + e^\phi(e^{-iw}F^B)_m\eta_+ + e^\phi\left[(F^B_m\Omega)_n + (F^B\Omega)_0 g_{mn} + (F^B\Omega)_{mn}\right]\gamma^n\eta_- = 0$$

NSNS sector: $D^H_m\eta \equiv \left(\nabla_m + \frac{1}{8}H_{mnp}\gamma np\right)\eta$ We are not in CY) $0 \neq \nabla_m\eta \sim W\eta$

$$D^H_m\eta_+ = (W_4 + W_5 + iH^{(3)})_m\eta_+ + \left[(W_1 + iH^{(1)})g_{mn} + \left(W_3 + iH^{(6)} + W_2\right)_{mn}\right]\gamma^n\eta_-$$

RR fluxes: $F^A = F_0 + F_2 + F_4$ $\qquad F^B = F_1 + F_3 + F_5$

$$(F\Omega)_{mn} \equiv (\not{F}\ \not{\Omega})_{mn} = \left(F_{i_1\dots i_k}\Omega_{abc}\gamma^{i_1\dots i_k}\gamma^{abc}\right)_{mn}$$

Take coefficient of term with 2 γ's

Ex: $(F_1\Omega)_{mn} = \frac{1}{2}F^a\Omega_{amn}$

IIA $ IIB

F^A $ F^B

e^{iW} $ Ω

exchange of two pure spinors -- action of mirror symmetry

Fidanza, Minasian, Tomasiello '03

Using $\eta_+ \otimes \eta_+^\dagger = e^{i\omega}$, $\eta_+ \otimes \eta_-^\dagger = i\,\Omega$ we can go further and find equations **for** pure spinors

IIA

$$e^{-f} d(e^f e^{iw}) = H \bullet e^{iw}$$

$$e^{-g} d(e^g \Omega) = H \bullet \Omega + F_A e^{iw}$$

IIB

$$e^{-f} d(e^f e^{iw}) = H \bullet e^{iw} + F_B e^{iw}$$

$$e^{-g} d(e^g \Omega) = H \bullet \Omega$$

$$e^f \sim e^g \sim e^\phi$$

$e^f e^{iw}$ is "twisted" closed ← mirror symmetry → $e^g\,\Omega$ is "twisted" closed

M is twisted symplectic — M is complex (no (2,2) piece in $H \bullet \Omega$)

Sugra "twisting": $\left[d - H_{mnp}\left(dx^m \wedge dx^n \iota^p - \frac{1}{3}\iota^m \iota^n \iota^p\right)\right](e^f \varphi) = 0$ — $\varphi = e^{iw}$ in IIA, $\varphi = \Omega$ in IIB

Susy vacua are all twisted generalized Calabi-Yau's !

M twisted symplectic $H\bullet$

$d^H\,\Omega$ F^A

Caveat: Hitchin considered twisting by H

Hitchin 's twisting $[d + H\wedge]\varphi = 0$ ⟹ Associated GACS integrable with $[\Pi, \Pi]_C = 0$

IIA vacua

$$[X+\zeta, Y+\eta]_H = [X+\zeta, Y+\eta]_C + \iota_X \iota_Y H$$

F^B

IIB vacua $d^H e^{iW}$ M complex

Twisting by H in supergravity equations does not work à la Hitchin

Strominger '86
(Maldacena - Nuñez)

Conclusions / Open questions

- Type II supersymmetric vacua are twisted generalized Calabi-Yau's

- What is the meaning of supergravity twisting by H?

- How to compactify on these manifolds? (evade no-go theorems)

- Moduli spaces?

Generalized complex geometry is a nice tool for a systematic description of flux backgrounds.
But maybe strings see SO(d,d) structure of T©T* and we will learn more…