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Supersymmetric Non-Linear Lagrangians of Kählerian Coset Spaces G/H: $G=E_6$, E_7 and E_8

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A simple and general procedure is given for constructing supersymmetric nonlinear σ model Lagrangian explicitly for any Kählerian coset space G/H. In particular, we derive explicit and full expressions of supersymmetric Lagrangians for the phenomenologically important manifolds $G/H=E_7/SU(5)\times SU(3)\times U(1)$ and $E_8/SO(10)\times SU(3)\times U(1)$, which are known to contain three generations of quarks and leptons as Nambu-Goldstone (NG) superfields. We discuss also (1) the arbitrariness of the choice of NG superfields when H contains more than one dimensional center, (2) how to gauge any subgroup of G and how to couple the system to supergravity, (3) a new anomaly of the supersymmetric nonlinear σ models induced by supergravity, etc.

§ 1. Introduction

One of the central problems in the present particle physics is certainly the problem of generations of quarks and leptons: Why are there the (at least) triplication of the quark / lepton generations in Nature? And further, how can we understand the mixings between them? Probably the composite model approach would be the most promising one to answer these questions. There is, however, a problem peculiar to composite models for quarks and leptons; namely, if they are composite, why their masses can be so small compared with the inverse of their (possible) typical size, say, >1 TeV? An interesting and natural idea answering this point is to regard quarks and leptons as quasi Nambu-Goldstone (NG) fermions appearing in supersymmetric theories, as was first proposed by Buchmüller et al.[1] Quasi NG fermions are the supersymmetric partners of the usual NG bosons which must exist if an internal symmetry G is spontaneously broken to a subgroup H. The number and quantum numbers of such quasi NG fermions, as well as their low energy effective Lagrangians, are unambiguously determined by the group structure of G and H.[2]

It was pointed out by Ong[3] that E_7 and E_8 were the only candidates for the group G that can accommodate three left-handed generations of quarks and leptons.*) It is indeed remarkable that the quasi NG fermions in the case $G/H=E_7/SU(5)\times SU(3)\times U(1)$ have precisely the required $SU(5)$ quantum number $3\times(\mathbf{5}^*+\mathbf{10})$ for three generations of quarks and leptons and $\mathbf{5}$ for Higgsino. This case was studied explicitly by Kugo and Yanagida[5] and the supersymmetric nonlinear Lagrangian was determined up to quartic order in the NG superfields. Recently the case $G/H=E_8/SO(10)\times SU(3)\times U(1)$ was also studied by Ong,[6] and also by Irié and Yasui,[7] independently, and was shown to predict a right-handed $\mathbf{16}$ multiplet of $SO(10)$ as the fourth generation in addition to the usual three left-handed generations $3\times\mathbf{16}$.

The last prediction of the right-handed forth generation in E_8 case is rather exciting.

*) Buchmüller, Peccei and Yanagida[4] also noticed (even earlier than Ong) that E_7 is sufficient to accommodate three generations. They, however, considered the model possessing mirror generations also.

Actually the group E_8 is a particular one since it is *maximal* simple group in the E series of exceptional groups. Hence it would not be quite unnatural if Nature selects E_8 and realizes the right-handed fourth generation. If so, a new heavy lepton with $V+A$ interaction may be found experimentally in the near future.

Of course, it is not clear in this stage whether the predicted $\mathbf{16}^*$ of $SO(10)$ in E_8 is actually realized as a right-handed new generation or unfortunately becomes an undetectably massive multiplet combined with a $\mathbf{16}$, hence leaving only two generations $2\times\mathbf{16}$. In order to answer this question as well as to make the other models realistic, it is necessary to study the mechanisms by which the quasi NG fermions and NG bosons acquire their masses. We have to introduce an explicit breaking of the global E_8 (or E_7) symmetry as well as a (spontaneous or explicit) supersymmetry breaking. Also necessary is to understand the origin of the gauge fields of GUT $SU(5)$ or $SO(10)$. If the GUT gauge interaction is introduced by gauging the subgroup $SU(5)$ (or $SO(10)$) by hand, it also works as a (natural) source of the explicit breaking of E_7 (or E_8) mentioned above. Another and more exciting possibility to obtain the GUT gauge interaction would be a dynamical realization; namely, since the $SU(5)$ or $SO(10)$ is the so-called "hidden local symmetry" of the nonlinear realization,[8] its gauge bosons may be generated dynamically as is realized in some 2- and 3-dimensional CP^{N-1} models.[9]

In order to discuss these problems, we need first of all the explicit forms of the supersymmetric Lagrangians for the nonlinear realizations $E_7/SU(5)\times SU(3)\times U(1)$, $E_8/SO(10)\times SU(3)\times U(1)$, etc. In the existing literature these are known only up to the quartic order in the NG superfields,[5),6)] or in a very abstract form.[7] So the purpose of the present paper is to give them in a complete and explicit form.

The construction of this paper is as follows: In § 2, we present a general procedure to construct supersymmetric Lagrangians, or equivalently, Kähler potentials explicitly for arbitrary Kählerian coset spaces G/H. Our method is based on the supersymmetric nonlinear realization theory by Bando, Kuramoto, Maskawa and Uehara (BKMU).[10] Since we have already discussed in our previous paper[11] various mathematical aspects concerning the BKMU construction of Kähler potentials, we explain here the general procedure for omitting the mathematical details but in such a manner to make the logical structure more transparent. Also we illustrate the procedure concretely by taking a simple example $G/H=SU(l+m+n)/S[U(l)\times U(m)\times U(n)]$.

The algebras of the exceptional groups E_6, E_7 and E_8 are given in § 3. We briefly explain a simple method to obtain the commutation relations of the generators based on a classical maximal subgroup. The decomposition of E_7 and E_8 generators with respect to the subgroups $SU(5)\times SU(3)\times U(1)$ and $SO(10)\times SU(3)\times U(1)$, respectively, is also performed there. We find three generations $3\times(\mathbf{5}^*+\mathbf{10})$ of NG superfields for $G/H=E_7/SU(5)\times SU(3)\times U(1)$ case and three left-handed and one right-handed generations $3\times\mathbf{16}+1\times\mathbf{16}^*$ for $E_8/SO(10)\times SU(3)\times U(1)$ case, as the previous authors did.[3)~7),12)] These contents of NG superfields are, however, no longer unique if the centers of H are relaxed to be more than one dimensional. By the help of the general method in § 2, we count all the possible choices of NG superfield sets for the cases E_7[resp. E_8]$/H$ with $H=SU(5)$ [resp. $SO(10)$]$\times SU(2)\times U(1)^2$ and $SU(5)$ [resp. $SO(10)$]$\times U(1)^3$. In particular we find it impossible to have a set of four left-handed NG superfields $4\times\mathbf{16}$ for $G=E_8$ case, unfortunately, although much freedom appears in general to replace the generations with their mirror ones.

The explicit construction of Kähler potentials is performed in § 4 for $G/H=E_6/SO(10)\times U(1)$, $E_7/SU(5)\times SU(3)\times U(1)$ and $E_8/SO(10)\times SU(3)\times U(1)$. The G-transformation laws of the Kähler potentials as well as of the NG superfields are also given.

Lastly in § 5, we discuss some problems in trying to make those nonlinear σ models realistic. We first remark a very simple formula which enables one to gauge an arbitrary subgroup of G. Contrary to the currently known formula,[21),23)] it is an off-shell formula which remains valid even if one adds matter superfields into the system freely. It has a straightforward extension to the local supersymmetry, i.e., coupling to supergravity, on which we comment also. Second we point out a new anomaly which necessarily appears when the supersymmetric nonlinear σ models are coupled to supergravity. This type of anomaly was first noticed by Ong[6)] and proposed to be a possible mechanism for the desired explicit breaking of G symmetry. But his observation is rather incomplete as for the origin of this anomaly and so we present a more complete discussion there.

The fundamental properties of $SO(2n)$ spinor representations, including our conventions for the gamma matrices, are summarized in the Appendix.

§ 2. Supersymmetric nonlinear realization for Kählerian G/H

The supersymmetric Lagrangian for the Nambu-Goldstone (NG) superfields $\phi^i(x,\theta)$ corresponding to G/H takes the form

$$\mathcal{L}=\int d^4\theta K(\phi,\bar{\phi})=g_{ij^*}\partial_\mu\varphi^i\partial^\mu\varphi^{j*}+(\text{fermion terms})\,, \tag{2·1}$$

$$g_{ij^*}(\varphi,\varphi^*)=\partial^2K(\varphi,\varphi^*)/\partial\varphi^i\partial\varphi^{j*}\,, \tag{2·2}$$

as is usual for the kinetic D-term for any supersymmetric theory, where $\varphi^i(x)$'s are the complex NG bosons standing for the first components of $\phi^i(x,\theta)$. The NG fields φ^i are the complex coordinates parametrizing the manifold G/H for which $g_{ij^*}(\varphi,\varphi^*)$ is the hermitian metric.[10),13),14)] From the particular form (2·2) of the metric, this manifold G/H is seen to be a special complex manifold called *Kählerian* and the function $K(\varphi,\varphi^*)$ is called Kähler potential.[15)] Hereafter we use the notation ϕ^i to denote the complex variables φ^i as well as the NG superfields ϕ^i, for simplicity. The G-invariance of the action $\int d^4x\mathcal{L}$ implies that the Kähler potential $K(\phi,\bar{\phi})$ transforms under the G-transformation as

$$K(\phi,\bar{\phi})\to K(\phi',\bar{\phi}')=K(\phi,\bar{\phi})+F(\phi)+F^*(\bar{\phi}) \tag{2·3}$$

with an arbitrary holomorphic function $F(\phi)$.

So the problem is how to find the Kähler potentials K for a given G/H. We now explain the general procedure in the following, and illustrate it concretely step by step by taking a simple example, a Grassmannian-like coset space $G/H=SU(l+m+n)/S[U(l)\times U(m)\times U(n)]$, when necessary. In order to avoid unnecessary complications we omit some mathematical proofs which are presented in our previous paper.[11)]

2.1. *A central charge Y of H and a general Kähler potential*

Before giving the details of the practical procedure, it will be helpful to know a general statement: *Each possible Kähler potential for G/H corresponds, one to one, to a central charge Y of H.* We explain this in this subsection first.

We assume that G is compact and semi-simple. The set of generators T_A of the Lie algebra $\mathcal{G}$ of G is first divided into two parts, generators $S_{\tilde{a}}$ of the unbroken subgroup H and the rest $X_{\hat{I}}$ orthogonal to $S_{\tilde{a}}$'s:

$$\{T_A\}=\{S_{\tilde{a}}\in\mathcal{H},\ X_{\hat{I}}\in\mathcal{G}-\mathcal{H}\}\,. \qquad (2\cdot4)$$

We understand that these generators are anti-hermitian matrices in a certain unitary representation of G.

Now we pick up from $\mathcal{H}$ all the independent *central charges* $(Y_\alpha)=(Y_1, Y_2, \cdots, Y_k)\equiv\boldsymbol{Y}$, i.e., the generators which are commutative with any elements of $\mathcal{H}$, so that the rest generators S_a of $\mathcal{H}$ span the semi-simple part $H_{\rm s.s.}$ of H. We assume that Y_α's are mutually orthogonal, for convenience, as $\mathrm{tr}\,(Y_\alpha Y_\beta)=N_\alpha\delta_{\alpha\beta}$. Here the number k $(1\leq k\leq\mathrm{rank}\ G)$ is the dimension of the center of H, which is known to be nonzero for the Kählerian G/H by a mathematical theorem.[16] This theorem further says that the generators $X_{\hat{I}}$ of $\mathcal{G}-\mathcal{H}$ are not commutable with one Y_α at least; i.e., the $\boldsymbol{Y}$-charge eigenvalues $\boldsymbol{y}_{\hat{I}}$ carried by $X_{\hat{I}}$'s are nonzero.

Let us recall the two basic objects in the nonlinear realization of Coleman, Wess and Zumino (CWZ);[17] namely, one is the element of the right coset space G/H

$$U(\phi,\bar{\phi})=e^{\pi^{\hat{I}}(\phi,\bar{\phi})X_{\hat{I}}}\in G/H \qquad (2\cdot5)$$

and the other is the Maurer-Cartan (Lie algebra valued) 1-form

$$\omega(\phi,\bar{\phi})=U^{-1}(\phi,\bar{\phi})\,dU(\phi,\bar{\phi})\,. \qquad (2\cdot6)$$

Here $\pi^{\hat{I}}$'s are real and U is a unitary matrix, ϕ^i's being the complex NG fields parametrizing the coset space G/H, half as many as $\pi^{\hat{I}}$'s. To give the Kähler metric $g_{ij^*}(\phi,\bar{\phi})$ in (2·2), it is convenient to introduce the fundamental 2-form defined by

$$\Omega(\phi,\bar{\phi})\equiv\frac{i}{2}g_{ij^*}(\phi,\bar{\phi})\,d\phi^i\wedge d\bar{\phi}^j\,, \qquad (2\cdot7)$$

which is closed, $\mathrm{d}\Omega=0$, when the manifold G/H is Kählerian.

We can now present a general statement proven in our previous paper: *For any possible Kähler metric g_{ij^*} for G/H, there exists a central charge $^\exists Y$ in H with which the fundamental 2-form Ω is given by*

$$\Omega=\frac{i}{2}\mathrm{tr}(-Y d\omega)\,. \qquad (2\cdot8)$$

Conversely, if we make a central charge

$$Y=\sum_{\alpha=1}^{k}v^\alpha Y_\alpha={}^t\boldsymbol{v}\cdot\boldsymbol{Y} \qquad (2\cdot9)$$

with arbitrary coefficients v^α, then the Ω of (2·8) gives a Kähler metric g_{ij^*} since (2·8) satisfies $d\Omega=0$ clearly. (Actually a trivial constraint must be imposed on $\boldsymbol{v}$ in order for the metric to be nondegenerate as will be seen below.) So the correspondence of the choices of central charge Y and Kähler potential is one to one, and hence the most general supersymmetric Lagrangian contains k arbitrary constants v^α (as many as independent

Y_a's) which represent the freedoms of coupling constants of nonlinear Lagrangian, like pion decay constant f_π.

We are now equipped with a general perspective, but how can we find the NG superfield contents ϕ^i and the Kähler potential function $K(\phi, \bar{\phi})$ explicitly? So we now turn to the more concrete procedure.

2.2. *The choice of a set NG superfields*

Once we fix the choice of a central charge $Y = {}^t\boldsymbol{v}\cdot\boldsymbol{Y}$ as in (2·9), then the corresponding set of NG superfields $\{\phi^i\}$ is determined as follows.

All the broken generators $X_{\hat{I}}$ of $\mathcal{G}-\mathcal{H}$ have nonvanishing $\boldsymbol{Y}$-charge eigenvalues $\boldsymbol{y}_{\hat{I}}$, as was mentioned above. So, if the coefficient vector $\boldsymbol{v}$ in the definition (2·9) of Y is chosen not orthogonal to any $\boldsymbol{y}_{\hat{I}}$, i.e.,

$$ {}^t\boldsymbol{v}\cdot\boldsymbol{y}_{\hat{I}} \neq 0 \qquad \text{for} \qquad {}^\forall y_{\hat{I}}\,, \tag{2·10} $$

then we can split the set of broken generators $X_{\hat{I}}$ into two parts, the generators X_I with positive Y-charge eigenvalues $y_I \equiv {}^t\boldsymbol{v}\cdot\boldsymbol{y}_I > 0$ and their anti-hermitian conjugates $\bar{X}_I \equiv (-X_I)^\dagger$ with negative Y-charge $-y_I$. This splitting defines a complex subgroup $\hat{H} \supset H^C$ (H^C: complex extention of H) spanned by the generators with positive or zero Y-charges:

$$ \hat{\mathcal{H}} = \{X_I, S_a, iY_a\}\,, \tag{2·11} $$

and a complex coset space $G^c/\hat{H}$ corresponding to the generators with negative Y-charges:

$$ \mathcal{G}^C - \hat{\mathcal{H}} = \{\bar{X}_I\} \tag{2·12} $$

We call these generators $\bar{X}_I \in \mathcal{G}^C - \hat{\mathcal{H}}$ *complex broken generators.*

The NG superfields Φ^I are introduced as the complex coordinate parametrizing the right coset space $G^C/\hat{H}$ as[10),13),14)]

$$ \xi(\phi) \equiv e^{\phi\cdot\bar{X}} \in G^C/H\,, \tag{2·13} $$

where $\phi\cdot\bar{X} \equiv \sum_I \phi^I \bar{X}_I$. This is the basic variable in the supersymmetric nonlinear realization theory by Bando, Kuramoto, Maskawa and Uehara (BKMU). So the quantum number contents of the NG superfields ϕ^I are the same as those of the complex broken generators $\bar{X}_I$.

By varying continuously the coefficient $\boldsymbol{v}$ in the definition (2·9) of Y, it happens that a Y-charge eigenvalue $Y_{I_0} = {}^t\boldsymbol{v}\cdot\boldsymbol{y}_{I_0}$ crosses the zero and changes its sign. Then a pair of the generators X_{I_0} and $\bar{X}_{I_0}$ exchange their roles with each other and the corresponding complex NG fields ϕ^{I_0} should be replaced by the conjugate representation field $\phi^{I_0 *}$ (still chiral as a superfield!), accordingly. In this way, for a given G/H, the choice of a set of NG superfields $\{\phi^I\}$, or equivalently the choice of $\hat{H}$ or $G^C/\hat{H}$, is not unique. Different choice corresponds to what is called different "invariant complex structure" in mathematics. We can find all the possible invariant complex structures $G^C/\hat{H}$ for G/H by varying the charge Y. We will present a simple method to do this task later.

It should be noted, as was shown in the previous paper, that the Kähler metric becomes degenerate at the crossing point ${}^t\boldsymbol{v}\cdot\boldsymbol{y}_{I_0} = 0$ from one complex structure to another.

Therefore if one keeps to use the old complex coordinates $\{\Phi^I\}$ even beyond that point, the Kähler metric becomes non-positive definite. So, for a fixed choice of the complex structure $G^C/\hat{H}$, the coefficient parameters v^a should be constrained in the region in which ${}^t\boldsymbol{v}\cdot\boldsymbol{y}_I>0$ holds for any eigenvalues $\boldsymbol{y}_I$ of the generators X_I in $\hat{\mathscr{H}}-\mathscr{H}^C$, for the positivity of the metric.

Let us illustrate the procedure up to here for the example of $G/H=SU(l+m+n)/S[U(l)\times U(m)\times U(n)]$. The center of H is two dimensional ($k=2$), whose two orthonormalized generators Y_1 and Y_2 can be chosen as, for instance,

$$Y_1=\begin{array}{cc} \begin{array}{ccc} l & m & n \end{array} & \\ \left[\begin{array}{ccc} n\mathbf{1}_l & & 0 \\ & 0 & \\ 0 & & -l\mathbf{1}_n \end{array}\right] & \begin{array}{c} l \\ m \\ n \end{array} \end{array}, \qquad Y_2=\left[\begin{array}{ccc} m\mathbf{1}_l & & 0 \\ & -(l+n)\mathbf{1}_m & \\ 0 & & m\mathbf{1}_n \end{array}\right] \tag{2·14}$$

in the fundamental representation of $G=SU(l+m+n)$. The other generators S_a of H are of course those of the semi-simple part $SU(l)\times SU(m)\times SU(n)$:

$$\mathscr{H}_{\text{s.s.}}=\{S_a\}=\left\{\left[\begin{array}{ccc} SU(l) & & \\ & SU(m) & \\ & & SU(n) \end{array}\right]\right\}. \tag{2·15}$$

The broken generators $X_{\bar{I}}$ of $\mathscr{G}-\mathscr{H}$ are given by the matrices, each of which takes non-vanishing value i (imaginary unit) at only one matrix element placed at the following off-diagonal parts

$$\begin{array}{cc} \begin{array}{ccc} l & m & n \end{array} & \\ \left[\begin{array}{ccc} 0 & A & B \\ \bar{A} & 0 & C \\ \bar{B} & \bar{C} & 0 \end{array}\right] & \begin{array}{c} l \\ m \\ n \end{array} \end{array}. \tag{2·16}$$

We use these names of the places $A, B, \cdots$ also to denote the generators $X_{\bar{I}}$ taking the value i there; $\{X_{\bar{I}}\}=\{A, B, C, \bar{A}, \bar{B}, \bar{C}\}$.

The $\boldsymbol{Y}$-charges of the basis vectors ψ in this fundamental representation are, of course,

$$(Y_1, Y_2)\text{ charge of }\begin{pmatrix}\psi_l\\ \psi_m\\ \psi_n\end{pmatrix}=\begin{pmatrix}(n, m)\\ (0, -l-n)\\ (-l, m)\end{pmatrix} \tag{2·17}$$

from (2·14), and hence the $\boldsymbol{Y}$-charges of broken generators are

$$\begin{array}{lll} & A=(n, l+m+n)\,, & \bar{A}=(-n, -l-m-n)\,, \\ (Y_1, Y_2)\text{ charge:} & B=(l+n, 0)\,, & \bar{B}=(-l-n, 0)\,, \\ & C=(l, -l-m-n)\,, & \bar{C}=(-l, l+m+n)\,. \end{array} \tag{2·18}$$

So if we choose Y_1 as the charge Y of (2·9), then the generators $\bar{X}_I$ of $\mathscr{G}^C-\mathscr{H}$, with negative Y-charges, are selected as

$$\mathcal{G}^c - \hat{\mathcal{H}} = \{\bar{A}, \bar{B}, \bar{C}\} \quad \text{when } Y = Y_1, \tag{2·19}$$

while if we choose $Y = Y_2$, for instance, then we have

$$\mathcal{G}^c - \hat{\mathcal{H}} = \{C, \bar{A}, \bar{B}\} \quad \text{when } \quad Y = Y_2. \tag{2·20}$$

For the choice $Y = Y_1$, the NG superfields are ϕ^A, ϕ^B, ϕ^C carrying the same quantum numbers of $H = S[U(l) \times U(m) \times U(n)]$ as A, B, C, respectively, and the BKMU's basic variable (2·13) now becomes

$$\xi(\phi) = e^{\phi^A \bar{A} + \phi^B \bar{B} + \phi^C \bar{C}} = \begin{bmatrix} 1 & 0 & 0 \\ \phi^A & 1 & 0 \\ \phi'^B & \phi^C & 1 \end{bmatrix} \quad \text{with} \quad \phi'^B = \phi^B + \phi^C \phi^A. \tag{2·21}$$

Notice that the explicit calculation of $\xi(\phi) = e^{\phi \cdot \bar{X}}$ is always simple thanks to the nilpotency of the matrices $\bar{X}_I$.

2.3. *Relation between nonlinear realization theories of CWZ's and BKMU's*

Since the supersymmetric nonlinear realization is just a special case of the usual nonlinear realization, the BKMU's variable $\xi(\phi) \in G^C/\hat{H}$ in (2·13) must correspond to a special complex parametrization of the coset element $U \in G/H$ in the CWZ's nonlinear realization. Indeed, as was shown in our previous paper, this relation is explicitly given by

$$G/H \ni U(\phi, \bar{\phi}) = \xi(\phi) e^{a(\phi,\bar{\phi}) \cdot X} e^{b(\phi,\bar{\phi}) \cdot S} e^{ic(\phi,\bar{\phi}) \cdot Y} \tag{2·22}$$

with abbreviations like $c \cdot Y = \sum_{\alpha=1}^{k} c_\alpha Y_\alpha$. The functions b and c are chosen purely imaginary since their real parts can be absorbed into an element of H (from the right). With a $\xi(\phi) = e^{\phi \cdot \bar{X}}$ given, the functions a, b and c are uniquely determined by the requirement that the real group element $U \in G/H$ must be unitary: $U^\dagger U = 1$, or equivalently,

$$\xi^\dagger(\bar{\phi}) \xi(\phi) = e^{a^* \cdot \bar{X}} e^{-2b \cdot S} e^{-2ic \cdot Y} e^{-a \cdot X}. \tag{2·23}$$

Corresponding to the usual CWZ's nonlinear transformation law

$$gU(\phi, \bar{\phi}) = U(\phi', \bar{\phi}') h(\phi, \bar{\phi}, g), \quad h \in H \tag{2·24}$$

under $g \in G$ transformation, BKMU required that the NG superfields ϕ^I transform as

$$g\xi(\phi) = \xi(\phi') \hat{h}(\phi, g), \qquad \hat{h} \in \hat{H}. \tag{2·25}$$

The comparison of these two transformation laws (2·24) and (2·25) leads to a remarkable transformation law of the functions $c_\alpha(\phi, \bar{\phi})$ appearing in the mapping equation (2·22) $\xi(\phi) \to U(\phi, \bar{\phi})$:[11)]

$$c_\alpha(\phi', \bar{\phi}') = c_\alpha(\phi, \bar{\phi}) + \frac{1}{2}(\gamma_\alpha(\phi, g) - \gamma_\alpha^*(\bar{\phi}, g)), \tag{2·26}$$

the same transformation law as the Kähler potential! Here the holomorphic functions $\gamma_\alpha(\phi, g)$ of ϕ are the ones contained in $\hat{h}(\phi, g)$ of (2·25) as $\hat{h} = e^{\alpha \cdot X} e^{\beta \cdot S} e^{i\gamma \cdot Y}$.

Furthermore, we have proved in Ref. 11) the equation

$$\mathrm{tr}(-Y_\alpha d\omega)=\frac{2}{i}N_\alpha\frac{\partial^2 c_\alpha(\phi,\bar\phi)}{\partial\phi^I\partial\bar\phi^J}d\phi^I\wedge d\bar\phi^J\,, \tag{2·27}$$

where N_α is the normalization factor of Y_α charge, $\mathrm{tr}\,Y_\alpha Y_\beta=N_\alpha\delta_{\alpha\beta}$. This implies that the Kähler metric g_{IJ^*} corresponding to the fundamental 2-form (2·8), $\Omega=(i/2)\mathrm{tr}(-Yd\omega)$, with a central charge $Y=\sum v^\alpha Y_\alpha$, is indeed given by the Kähler potential

$$K(\phi,\bar\phi)=\frac{2}{i}\sum_{\alpha=1}^{K}v^\alpha N_\alpha c_\alpha(\phi,\bar\phi)\,. \tag{2·28}$$

Since the fundamental 2-form (2·8) gave the most general Kähler metric, the functions $c_\alpha(\phi,\bar\phi)$ give a complete set of independent Kähler potential functions. This equation (2·28) gives the direct one-to-one connection between the Kähler potential and the central charge $Y=\sum v^\alpha Y_\alpha$.

To calculate the functions $c_\alpha(\phi,\bar\phi)$ in (2·22), the BKMU formula would be the simplest which we explain now.

2.4. *The BKMU formula and projection operators η*

BKMU assumed the existence of projection matrices η_i in the representation vector space V under consideration, each of which satisfies

$$\text{(i)}\qquad \eta_i=\eta_i{}^\dagger\,,\qquad (\eta_i)^2=\eta_i$$

$$\text{(ii)}\qquad \hat h\eta_i=\eta_i\hat h\eta_i\qquad\text{for}\quad {}^\forall\hat h\in\hat H\,. \tag{2·29}$$

Then, the candidates for the Kähler potentials are given by the following BKMU formula:

$$K_i(\phi,\bar\phi)=\mathrm{lndet}_{\eta_i}(\xi^\dagger(\bar\phi)\xi(\phi))\,, \tag{2·30}$$

where $\det_{\eta_i}$ denotes a determinant defined in the η_i-projected subspace $\eta_i V$. This formula in fact gives the quickest way to extract the above functions $c_\alpha(\phi,\bar\phi)$; indeed, as was noticed by BKMU themselves in their third paper,[10] the expression (2·23) gives

$$\mathrm{lndet}_{\eta_i}(\xi^\dagger\xi)=\mathrm{lndet}_{\eta_i}(e^{a^*\cdot\bar X})+\mathrm{lndet}_{\eta_i}(e^{-a\cdot X})$$
$$+\mathrm{lndet}_{\eta_i}(e^{-2b\cdot S})-2i\sum_\alpha c_\alpha(\phi,\bar\phi)\mathrm{tr}(\eta_i Y_\alpha)\,. \tag{2·31}$$

Since the projection matrix η_i satisfies $e^{-2b\cdot S}e^{-2ic\cdot Y}e^{-a\cdot X}\eta_i=\eta_i e^{-2b\cdot S}\eta_i.\eta_i e^{-2ic\cdot Y}\eta_i\cdot\eta_i e^{-a\cdot X}\eta_i$ by (2·29) and $\{S,iY,X\}\in\hat{\mathcal{H}}$. The two terms on the r.h.s. of (2·31) are zero because of the nilpotency of X_I and $\bar X_I$, and the third term also vanishes since S_a's are the generators of the semi-simple part $\mathcal{H}^C_{\text{s.s.}}$ of $\mathcal{H}^C$ and semi-simple Lie algebra is traceless in any representation. (Note that the η_i-projected subspace $\eta_i V$ still spans a representation space of $\mathcal{H}^C_{\text{s.s.}}$.) Thus (2·31) leaves us with the expression

$$K_i(\phi,\bar\phi)=\sum_\alpha x_\alpha\frac{1}{i}c_\alpha(\phi,\bar\phi)\,,\quad x_\alpha\equiv 2\mathrm{tr}(\eta_i Y_\alpha)\,. \tag{2·32}$$

The Kähler potential property of BKMU's function (2·30) is also seen more directly from the transformation law (2·25) of $\xi(\phi)$ under $g\in G$:[10] From (2·25), (2·29) and the unitarity of g, we find (omitting i of η_i)

$$\mathrm{lndet}_\eta(\xi^\dagger(\bar\phi')\xi(\phi'))=\mathrm{lndet}_\eta(\eta\hat h^{\dagger-1}\xi^\dagger(\bar\phi)\xi(\phi)\hat h^{-1}\eta)$$

$$=\mathrm{lndet}_\eta(\eta\hat{h}^{\dagger -1}\eta\cdot\eta\xi^\dagger(\bar{\phi})\xi(\phi)\eta\cdot\eta\hat{h}^{-1}\eta)$$
$$=\mathrm{lndet}_\eta(\xi^\dagger(\bar{\phi})\xi(\phi))+\mathrm{lndet}_\eta\hat{h}^{-1}+\mathrm{lndet}_\eta\hat{h}^{\dagger -1}. \qquad (2\cdot 33)$$

Since $\mathrm{lndet}_\eta(\hat{h}^{-1}(\phi, g))$ is a holomorphic functions of ϕ, this equation (2·33) indeed takes the desired form (2·3) of Kähler potential transformation law. (Incidentally, since $\mathrm{lndet}_{\eta_i}\hat{h}^{-1}=i\sum_a\gamma_a(\phi, g)\mathrm{tr}(\eta_i Y)$ for $\hat{h}=e^{\alpha\cdot X}e^{\beta\cdot S}e^{i\gamma\cdot Y}$ by the same reasoning as in (2·31), Eq. (2·33) again confirms the transformation law (2·26) of $c_a(\phi, \bar{\phi})$.)

In practice we need to find the projection operators η_i satisfying the property (2·29). The construction of η_i is extremely simple in our case owing to the existence of the central charge Y, and is well exemplified for the Grassmannian-like case $G/H=SU(l+m+n)/S[U(l)\times U(m)\times U(n)]$. i) First task is to arrange the H-irreducible blocks of the representation basis vector in the order of the Y-charge eigenvalues from top to bottom; for the Grassmannian example, the H-irreducible blocks of the basis vector in the fundamental representation are (ϕ_l, ϕ_m, ϕ_n) which carry the (Y_1, Y_2)-charges as shown in (2·17). We should take the following representation basis, for instance,

$$\begin{pmatrix}\phi_l[n]\\ \phi_m[0]\\ \phi_n[-l]\end{pmatrix} \text{ when } Y=Y_1, \quad \begin{pmatrix}\phi_m[l(l+n)]\\ \phi_n[0]\\ \phi_l[-m(l+n)]\end{pmatrix} \text{ when } Y=mY_1-lY_2. \qquad (2\cdot 34)$$

(Here we have shown the Y-charge eigenvalues in [ ] for the ease of understanding.) ii) Then, in this representation, projection operators η_i onto any upper blocks of the form $\begin{pmatrix}1&0\\0&0\end{pmatrix}$ satisfy the property (2·29), $\hat{H}\eta_i=\eta_i\hat{H}\eta_i$, since the generators $\{X_I, S_a, iY_a\}$ of $\hat{\mathscr{H}}$ were chosen to be those carrying positive or zero Y-charges and hence never lower the Y-charge of the basis vector. So, for the Grassmannian example, we have two projection operators

$$\eta_1=\begin{pmatrix}1&&\\&0&\\&&0\end{pmatrix}, \qquad \eta_2=\begin{pmatrix}1&&\\&1&\\&&0\end{pmatrix}, \qquad (2\cdot 35)$$

where 1's and 0's are block unit and null matrices of course.

The projection operators constructed this way are in fact shown to be sufficient for finding all the $c_a(\phi, \bar{\phi})$ functions via the BKMU formula (2·30) and (2·32). This is always true if we work in any representation of G.[11] We can also see this fact here by showing an interesting formula which expresses the general Kähler potential $K(\phi, \bar{\phi})$ of (2·28) corresponding to the charge $Y={}^t\boldsymbol{v}\cdot\boldsymbol{Y}$ directly in terms of the BKMU's functions (2·30). Consider any irreducible representation of G and decompose the representation basis vector into H-irreducible pieces ψ_i, whose dimension and Y-charges we denote by dim V_i and $\boldsymbol{y}_{(i)}$, respectively. We arrange them in the order of the Y-charge values ${}^t\boldsymbol{v}\cdot\boldsymbol{y}_{(i)}$:

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_N\end{pmatrix}, \qquad {}^t\boldsymbol{v}\cdot\boldsymbol{y}^{\psi}_{(1)}>{}^t\boldsymbol{v}\cdot\boldsymbol{y}^{\psi}_{(2)}>\cdots>{}^t\boldsymbol{v}\cdot\boldsymbol{y}^{\psi}_{(N)}. \qquad (2\cdot 36)$$

Let us define projection operators η_i such that $\eta_i{}^t\psi=({}^t\psi_1, {}^t\psi_2, \cdots, {}^t\psi_i, 0, 0, \cdots, 0)$ and construct Kähler potentials $K_i(\phi, \bar{\phi})$ by the BKMU formula (2·30) with those projection operators η_i. Since we have $K_i - K_{i-1} = (2/i)\boldsymbol{c}\cdot\boldsymbol{y}_{(i)} \times \dim V_i$ from (2·32) and $N_\alpha \delta_{\alpha\beta} = \mathrm{tr}(Y_\alpha Y_\beta) = \sum_{i=1}^N y_{\alpha(i)} y_{\beta(i)} \dim V_i$, we find that the Kähler potential $K(\phi, \bar{\phi})$ of (2·28) corresponding to the charge $Y = {}^t\boldsymbol{v}\cdot \boldsymbol{Y}$ becomes

$$K(\phi, \bar{\phi}) = \sum_{i=1}^N (K_i - K_{i-1})\, {}^t\boldsymbol{v}\cdot\boldsymbol{y}_{(i)}$$

$$= {}^t\boldsymbol{v}\cdot(\boldsymbol{y}_{(1)} - \boldsymbol{y}_{(2)}) K_1 + {}^t\boldsymbol{v}\cdot(\boldsymbol{y}_{(2)} - \boldsymbol{y}_{(3)}) K_2 + \cdots + {}^t\boldsymbol{v}\cdot(\boldsymbol{y}_{(N-1)} - \boldsymbol{y}_{(N)}) K_{N-1}\,. \quad (2·37)$$

Here $K_0 = K_N = 0$*) has been used. Although all of the K_i's are not necessarily mutually independent, this formula anyhow show the completeness of the BKMU functions K_i.

Equation (2·37) has another implication that the Kähler potential given by a linear combination of BKMU's functions K_i,

$$K(\phi, \bar{\phi}) = \sum_{i=1}^{N-1} w_i K_i(\phi, \bar{\phi}) = \sum_{i=1}^{N-1} w_i \ln\det{}_{\eta_i}[\xi^\dagger(\bar{\phi})\xi(\phi)] \quad (2·38)$$

is valid only when all the coefficients w_i are *positive*. The reason is as follows: As explained before, the positivity of metric requires ${}^t\boldsymbol{v}\cdot\boldsymbol{y}_I > 0$ for all the $\boldsymbol{Y}$-charges $\boldsymbol{y}_I$ of the generators $X_I \in \mathscr{H} - (\mathcal{G}^C - \mathscr{H}^C)$. The generators X_I are represented by upper triangular matrices on the basis vector (2·36) and so carry $\boldsymbol{Y}$-charge eigenvalues of the form $\boldsymbol{y}_I = \boldsymbol{y}_{(i)} - \boldsymbol{y}_{(j)}$ with $i > j$. Clearly the conditions ${}^t\boldsymbol{v}\cdot\boldsymbol{y}_I > 0$ are satisfied if (and only if) ${}^tv\cdot(\boldsymbol{y}_{(i)} - \boldsymbol{y}_{(i-1)}) > 0$ hold for $i = 1, 2, \cdots, N$, but they are just the coefficients of K_i in (2·37). q.e.d. It is interesting that somewhat complicated conditions on the parameters v^α are converted into simple ones $w_i > 0$ for the coefficients of K_i's (Of course, for the cases in which $K_1 \sim K_{N-1}$ are not mutually independent, K can be expressed in terms of suitably chosen independent K_i's and then the constraints on the coefficients of those K_i's remain no longer so simple.)

2.5. *Example* $G/H = SU(l+m+n)/S[U(l)\times U(m)\times U(n)]$

Let us give an explicit form of the Kähler potential for the above Grassmannian-like case. If we choose $Y = Y_1$ of (2·17), the H-irreducible blocks of the representation basis vector taken in § 2.2. are already in the correct order, and the BKMU variable $\xi(\phi)$ takes the form (2·1). (Notice that $\xi(\phi) = e^{\phi\cdot\bar{X}}$ always become the lower block triangular form like (2·21) in our representation convention, since $\bar{X}_I$'s carry negative Y-charges.) Then the BKMU formula (2·30) with the projection operators $\eta_{1,2}$ of (2·35) yields

$$K_1(\phi, \bar{\phi}) = \ln \det_{l\times l}(\xi^\dagger_{\eta_1}(\bar{\phi})\xi_{\eta_1}(\phi)) = \ln \det_{l\times l}(1 + \bar{\phi}_A \phi^A + \bar{\phi}_B{}' \phi'^B)\,,$$

$$K_2(\phi, \bar{\phi}) = \ln \det_{(l+m)\times(l+m)}(\xi^\dagger_{\eta_2}(\bar{\phi})\xi_{\eta_2}(\phi))\,, \quad (2·39)$$

where

*) $K_N = 0$ since $\eta_N = 1$ and $\mathrm{tr}\,\boldsymbol{Y} = \boldsymbol{0}$ owing to the semi-simpleness of G.

$$\xi_{\eta_1}(\phi)=\begin{array}{c} l \\ \left(\begin{array}{c} 1 \\ \phi^A \\ \phi'^B \end{array}\right) \end{array}\begin{array}{c} \\ l \\ m \\ n \end{array}, \qquad \xi_{\eta_2}(\phi)=\begin{array}{cc} l & m \\ \multicolumn{2}{c}{\left(\begin{array}{cc} 1 & 0 \\ \phi^A & 1 \\ \phi'^B & \phi^C \end{array}\right)} \end{array}\begin{array}{c} \\ l \\ m \\ n \end{array}. \tag{2·40}$$

Since $k=2$ in this example, the BKMU functions K_1 and K_2 are mutually independent and complete, and the general Kähler potential is given by

$$\begin{aligned} K(\phi, \bar{\phi}) &= w_1 K_1(\phi, \bar{\phi}) + w_2 K_2(\phi, \bar{\phi}) \\ &= [v^1 n + v^2(l+m+n)]K_1 + [v^1 l - v^2(l+m+n)]K_2, \end{aligned} \tag{2·41}$$

where use has been made of (2·37) and the $\boldsymbol{Y}$-charge eigenvalues $\boldsymbol{y}_{(i)}$ of the basis vector in (2·17). As we have seen generally in the above, this expression is valid only when both the coefficients w_1 and w_2 are positive. This can also be confirmed in this simple example directly by expanding the $K(\phi, \bar{\phi})$ around $\phi=0$:

$$K(\phi, \bar{\phi}) = (w_1+w_2)\mathrm{tr}(\bar{\phi}_A\phi^A + \bar{\phi}_B\phi^B) + w_2\mathrm{tr}(\bar{\phi}_C\phi^C) + O(\phi^3). \tag{2·42}$$

Since this implies the metric $g_{ij^*}=\mathrm{diag}[(w_1+w_2)\delta_{AA}, (w_1+w_2)\delta_{BB}, w_2\delta_{CC}]$ at the origin $\phi=0$, w_1 and w_2 clearly must be positive. The positive region of (w_1, w_2) corresponds to the region of the parameters v^1 and v^2 satisfying $-nv^1<(l+m+n)v^2<lv^1$ and the Kähler potential (2·41) is valid for the Y-charge $Y={}^t\boldsymbol{v}\cdot\boldsymbol{Y}$ in such a region.

2.6. *Invariant complex structures*

As promised we now explain a practical method how to find all the possible invariant complex structures $G^C/\hat{H}$ for a given G/H.

The problem is how to split the generators of $\mathcal{G}-\mathcal{H}$ into two sets, X_I's of $\hat{\mathcal{H}}-\mathcal{H}^C$ and $\bar{X}_I$'s of $\mathcal{G}^C-\hat{\mathcal{H}}$. So, first plot the $\boldsymbol{Y}$-charge vectors $\boldsymbol{y}_I$ of all the generators of $\mathcal{G}-\mathcal{H}$ in k-dimensional Euclidian space, and draw arbitrarily a $(k-1)$-dimensional plane containing the origin but none of the $\boldsymbol{y}_I$ vectors. Then we can choose as X_I's the generators with $\boldsymbol{y}_I$ vectors sitting in the one side of the plane and as $\bar{X}_I$'s those in the other side. A normal vector $\boldsymbol{v}=(v^1, \cdots, v^k)$ to the plane can be identified with the charge $Y={}^t\boldsymbol{v}\cdot\boldsymbol{Y}$ and the Y charge eigenvales ${}^t\boldsymbol{v}\cdot\boldsymbol{y}_I$ of X_I generators are ($\|v\|$ times of) the distances of the points $\boldsymbol{y}_I$ to the plane. As rotating the plane, for each time the plane crosses a vector $\boldsymbol{y}_{I_0}$, we find a new set of complex broken generators $\bar{X}_I$ corresponding to NG superfields ϕ^I. In this way we can count the number of possible invariant complex structure as well as the NG superfield content for each case. This task in easily carried out by hand for $k=1$, 2, 3 and would not be difficult also for higher k by the help of computer.

For the cases $E_7/SU(5)\times SU(3)\times U(1)$ and $E_8/SO(10)\times SU(3)\times U(1)$, for which we will calculate the Kähler potential explicitly in this paper, the central charge in H is unique, i.e., $k=1$. There are two invariant complex structures for such cases of $k=1$, which come from the sign change $Y\to -Y$ of the unique central charge, corresponding physically to replace all the chiral NG superfields by anti-chiral ones. So if there are no matter-superfields to which the chirality of the NG superfields can be referred, the supersymmetric nonlinear Lagrangian is actually unique.

For illustration of the above procedure, let us consider the case $G/H=SU(l+m+n)$

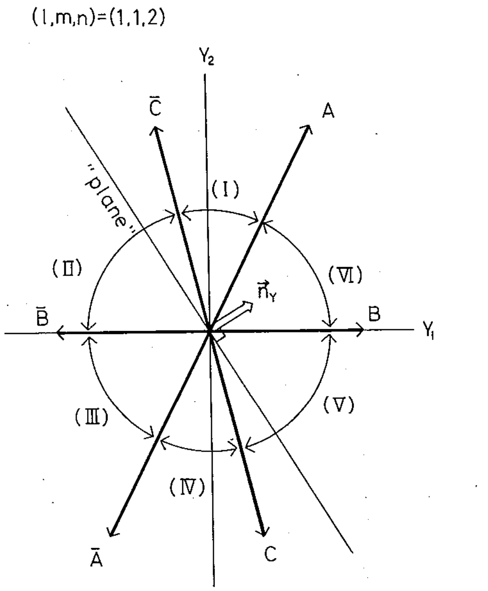

Fig. 1. Arrows show $\boldsymbol{Y}$-charge eigenvalue vectors $\boldsymbol{y}_I$ in (2·18) of the broken generators of $SU(l+m+n)/S[U(l)\times U(m)\times U(n)]$. This figure is drawn by taking the scale $(l, m, n)=(1, 1, 2)$.

$/S[U(l)\times U(m)\times U(n)]$ again. Broken generators in this case are $A, B, C, \bar{A}, \bar{B}, \bar{C}$ shown in (2·16) and carry $\boldsymbol{Y}$-charge eigenvalues $\boldsymbol{y}_I$ of (2·18). According to the above procedure, we have plotted these vectors $\boldsymbol{y}_I(I=A, B, C, \bar{A}, \bar{B}, \bar{C})$ by arrows in Fig. 1, and have drawn a typical "plane" (a line in this $k=2$ case) as well as its normal $\boldsymbol{n}_Y$, whose direction corresponds to the axis of the chosen Y-charge $Y\propto {}^t\boldsymbol{n}\cdot\boldsymbol{Y}$. For such a choice of "plane", the positive-Y-charge generators $X_I\in\hat{\mathcal{H}}-\mathcal{H}^c$ are $\bar{C}$, A and B evidently from the figure, and hence the corresponding NG superfields are $\phi_{\bar{C}}, \phi_A, \phi_B$ which transform under the $SU(l)\times SU(m)\times SU(n)$ as $(\mathbf{1}, \boldsymbol{m}^*, \boldsymbol{n})$, $(\boldsymbol{l}, \boldsymbol{m}^*, \mathbf{1})$, $(\boldsymbol{l}, \mathbf{1}, \boldsymbol{n}^*)$, respectively. We see clearly from Fig. 1 that there are six possibilities to draw an oriented "plane" leading to different complex structures; the selected set of X_I generators are (I) (A, B, C), (II) $(\bar{C}, A, B)$, (III) $(\bar{B}, \bar{C}, A)$, (IV) $(\bar{A}, \bar{B}, \bar{C})$, (V) $(C, \bar{A}, \bar{B})$ and (VI) $(B, C, \bar{A})$, respectively for each of the six cases. Notice that there exist impossible combinations like, for instance, $(A, B, \bar{A})$. It is this point that is interesting in building models. The possible combinations of H-quantum numbers of NG superfields are restricted and the restriction is more stringent as k becomes lower.

For $k=3$ cases, one needs to draw a 3-dimensional picture if he works similarly. We, however, notice that we needed in fact only the information of the *direction* of $\boldsymbol{y}$-charge eigenvalue vectors $\boldsymbol{y}_I$. So, by considering a cuboid with its center at the origin, for instance, we can replace the each vector $\boldsymbol{y}_I$ by a point on the surface of the cuboid at which a line in the direction of $\boldsymbol{y}_I$ from the origin crosses the surface. Similarly the plane containing the origin is replaced by a suitable line on the surface. Further if one changes suitably the length of the edges of the cuboid (and its direction also if necessary), all the points corresponding to $\boldsymbol{y}_I$'s can be gathered to appear only on a pair of (opposite) surfaces among six rectangular surfaces. Then the problem to find all the possible complex structures is reduced to drawing a line on the one surface of that pair.

§ 3. Exceptional Groups E_6, E_7 and E_8

3.1. *Lie algebras of E_6, E_7 and E_8*

The exceptional groups seem still not familiar to the usual physicists. So we present here the Lie algebras of $E_{6\sim 8}$ groups explicitly, briefly explaining how to obtain them. The easiest way to write down the Lie algebra for an exceptional group is to choose the generators referring to its convenient maximal subgroup.

3.1.1. E_8 algebra

Let us start with E_8. As a maximal subgroup of it we choose $SO(16)$ for our convenience to discuss $E_8/SO(10)\times SU(3)\times U(1)$ later. The dimensions of E_8 and $SO(16)$ are 248 and 120, respectively. The generators corresponding to the rest dimension $248-120=128$ must span a representation of the maximal subgroup $SO(16)$, and hence are easily guessed to be a $SO(16)$ Majorana-Weyl spinor possessing $2^{16/2-1}=128$ real components. So the generators of E_8 are given by $SO(16)$ generators, denoted by $T_{\hat{A}\hat{B}}(\hat{A}, \hat{B}=1\sim 16)$, and generators $E_{\hat{\alpha}}(\hat{\alpha}=1\sim 128)$ of a $SO(16)$ Weyl-spinor. Throughout this paper we adopt the convention to take the generators essentially anti-hermitian; so $T_{\hat{A}\hat{B}}$ is taken to satisfy $T_{\hat{A}\hat{B}}{}^{\dagger}=-T_{\hat{B}\hat{A}}=T_{\hat{A}\hat{B}}$, and $E_{\hat{\alpha}}$ is subject to an anti-Majorana condition:

$$(E_{\hat{\alpha}})^{\dagger}=-(C_{16})^{\hat{\alpha}\hat{\beta}}E_{\hat{\beta}}\,, \tag{3·1}$$

where C_{16} is the $SO(16)$ charge conjugation matrix. [See Appendix A for our conventions for the $SO(2n)$ Clifford algebra and spinor representations.]

The E_8 algebra is obtained as follows: First, $T_{\hat{A}\hat{B}}$'s of course satisfy the usual $SO(16)$ algebra,

$$[T_{\hat{A}\hat{B}},\ T_{\hat{C}\hat{D}}]=\delta_{\hat{B}\hat{C}}T_{\hat{A}\hat{D}}+\delta_{\hat{A}\hat{D}}T_{\hat{B}\hat{C}}-\delta_{\hat{A}\hat{C}}T_{\hat{B}\hat{D}}-\delta_{\hat{B}\hat{D}}T_{\hat{A}\hat{C}}\,. \tag{3·2a}$$

Second, since $E_{\hat{\alpha}}$ is an $SO(16)$ Weyl spinor, it obeys

$$[E_{\hat{\alpha}},\ T_{\hat{A}\hat{B}}]=(\sigma_{\hat{A}\hat{B}})_{\hat{\alpha}}{}^{\hat{\beta}}E_{\hat{\beta}} \tag{3·2b}$$

with a representation matrix $\sigma_{\hat{A}\hat{B}}$ of $T_{\hat{A}\hat{B}}$. [See Appendix A.] Finally, the commutator $[E_{\hat{\alpha}}, E_{\hat{\beta}}]$ is in general given by a linear combination of $T_{\hat{A}\hat{B}}$'s and $E_{\hat{\gamma}}$'s, but the $E_{\hat{\gamma}}$ terms are absent since a spinor-representation cannot be constructed from the product of two spinor-representations. Thus, from $SO(16)$ covariance, we obtain

$$[E_{\hat{\alpha}}, E_{\hat{\beta}}]=\frac{1}{2}(\sigma_{\hat{A}\hat{B}}C_{16}{}^{-1})_{\hat{\alpha}\hat{\beta}}T_{\hat{A}\hat{B}}\,. \tag{3·2c}$$

The factor $1/2$ in (3·2c) depends in fact on the normalization convention of $E_{\hat{\alpha}}$ generators. We have fixed it so that all the generators $T_I\equiv(T_{\hat{A}\hat{B}}, E_{\hat{\alpha}})$ have a common normalization in the sence of Killing form; namely, the adjoint representation matrices $(\mathrm{ad}\ T_I)^K{}_J\equiv f_{IJ}{}^K$, given by the structure constant $f_{IJ}{}^K$ through $[T_I,\ T_J]=f_{IJ}^K T_K$, satisfy

$$\mathrm{tr}(\mathrm{ad}(T_I{}^{\dagger})\cdot\mathrm{ad}\ T_J)=N\delta_{IJ} \tag{3·3}$$

with a common factor N. In the above case of E_8 algebra (3·2), $N=60$.

In our hermiticity convention for the generators $T_I=(T_{\hat{A}\hat{B}}, E_{\hat{\alpha}})$, the real-group E_8 elements are given by $\exp\theta^I T_I\equiv\exp(\sum_{\hat{A}>\hat{B}}\theta^{\hat{A}\hat{B}}T_{\hat{A}\hat{B}}+\epsilon^{\hat{\alpha}}E_{\hat{\alpha}})$ with parameters $\theta_I\equiv(\theta^{\hat{A}\hat{B}}, \epsilon^{\hat{\alpha}})$ satisfying hermiticy $(\theta^{\hat{A}\hat{B}})^*=\theta^{\hat{B}\hat{A}}$ and Majorana $\epsilon^{\hat{\alpha}}=(C_{16})^{\hat{\alpha}\hat{\beta}}(\epsilon^{\hat{\beta}})^*$ conditions. It should be noted that the corresponding matrix representations $\exp(\theta^I\mathrm{ad}\ T_I)$ are unitary *only when* the generators are commonly normalized like (3·3) since otherwise the structure constant does not satisfy $\mathrm{ad}(T_I{}^{\dagger})=[\mathrm{ad}\ T_I]^{\dagger}$.

3.1.2. E_7 algebra

Next is the 133 dimensional E_7 group. We take $SU(8)$ as a maximal subgroup and denote the 63 (traceless anti-hermitian) generators by $\hat{T}_I^J$ $(I, J=1\sim 8)$; $\hat{T}_I^I=0$, $(\hat{T}_I^J)^{\dagger}=-\hat{T}_J^I$. As is easily guessed also here, the remaining $133-63=70$ generators span a real representation $\boxminus$ or $SU(8)$, namely a totally antisymmetric tensor E_{IJKL} $(I, J, K, L=1\sim 8)$ subject to a reality constraint $(E_{IJKL})^{\dagger}=-(1/4!)\epsilon^{IJKLMNOP}E_{MNOP}$.

The E_7 algebra is obtained quite similarly to the above E_8 case, by taking into account the $SU(8)$ covariance:

$$[\hat{T}_I^J, \hat{T}_K^L] = i\delta_K^J \hat{T}_I^L - i\delta_I^L \hat{T}_K^J, \tag{3·4a}$$

$$[\hat{T}_I^J, E_{KLMN}] = i(\delta_K^J E_{ILMN} + \delta_L^J E_{KIMN} + \delta_M^J E_{KLIM} + \delta_N^J E_{KLMI} - \frac{1}{2}\delta_I^J E_{KLMN}), \tag{3·4b}$$

$$\begin{aligned}[E_{IJKL}, E_{MNOP}] = \frac{i}{2}(&\hat{T}_I^Q \epsilon_{QJKLMNOP} + \hat{T}_J^Q \epsilon_{IQKLMNOP} + \hat{T}_K^Q \epsilon_{IJQLMNOP} \\ &+ \hat{T}_L^Q \epsilon_{IJKQMNOP}) - \frac{i}{2}(\hat{T}_M^Q \epsilon_{IJKLQNOP} + \hat{T}_N^Q \epsilon_{IJKLMQOP} \\ &+ \hat{T}_O^Q \epsilon_{IJKLMNQP} + \hat{T}_P^Q \epsilon_{IJKLMNOQ}).\end{aligned} \tag{3·4c}$$

Here also the coefficient $1/2$ is fixed by the Killing form normalization condition (3·3) [with $N=494$ in this case]. The real group elements of E_7 are given by $\exp(\theta_I^J \hat{T}_J^I + (1/4!)\theta^{IJKL}E_{JIKL})$ with parameters θ satisfying reality conditions $(\theta_I^J)^* = \theta_J^I$ and $(\theta^{IJKL})^* = (1/4!)\epsilon_{IJKLMNOP}\theta^{MNOP}$.

3.1.3. E_6 algebra

A maximal subgroup of 78 dimensional E_6 is $SO(10)\times U(1)$ with 45+1 generators, T_{AB} ($A, B=1\sim 10$) and T. The rest $78-46=32$ generators just fall into a $SO(10)$ (anti-) Majorana spinor $(E_\alpha, \bar{E}^\alpha)$, where E_α ($\alpha=1\sim 16$) stand for its "upper" 16 components, a "right-handed" Weyl spinor, and $\bar{E}$ for their anti-hermitian conjugates $\bar{E}^\alpha \equiv -(E_\alpha)^\dagger$.

Considering $SO(10)$ covariance we obtain the E_6 algebra:

$$\begin{aligned}&[T_{AB}, T_{CD}] = \delta_{BC}T_{AD} + \delta_{AD}T_{BC} - \delta_{AC}T_{BD} - \delta_{BD}T_{AC}, \\ &[T_{AB}, T] = [T, T] = 0,\end{aligned} \tag{3·5a}$$

$$[E_\alpha, T_{AB}] = (\sigma_{AB})_\alpha{}^\beta E_\beta, \qquad \left[T, \begin{pmatrix} E_\alpha \\ \bar{E}^\alpha \end{pmatrix}\right] = \frac{\sqrt{3}}{2}i\begin{pmatrix} E_\alpha \\ -\bar{E}^\alpha \end{pmatrix}, \tag{3·5b}$$

$$\begin{aligned}&[\bar{E}^\alpha, T_{AB}] = (\bar{\sigma}_{AB})^\alpha{}_\beta \bar{E}^\beta, \\ &[E_\alpha, E_\beta] = [\bar{E}^\alpha, \bar{E}^\beta] = 0, \\ &[E_\alpha, \bar{E}^\beta] = -\frac{1}{2}(\sigma_{AB})_\alpha{}^\beta T_{AB} + \frac{\sqrt{3}}{2}i\delta_\alpha{}^\beta T.\end{aligned} \tag{3·5c}$$

Here the relative weight of T_{AB} and T terms appearing in the r.h.s. of the last equation can be determined by the Jacobi identity $[E_\alpha, [E_\beta, \bar{E}^\gamma]] + (\text{cyclic permutations}) = 0$. Other factors are fixed so that the normalization conditions (3·3) are satisfied with $N=24$. The E_6 real group elements $\exp(\theta T + \frac{1}{2}\theta_{AB}T_{AB} + \bar{\epsilon}^\alpha E_\alpha + \epsilon_\alpha \bar{E}^\alpha)$ are given in terms of "real" parameters satisfying $\theta^* = \theta$, $(\theta_{AB})^* = \theta_{BA}$ and $(\bar{\epsilon}^\alpha)^* = \epsilon_\alpha$.

Thus we have completed the presentation of the algebra E_8, E_7 and E_6 by Eqs. (3·2), (3·4) and (3·5), respectively. A comment may be useful on the same identities which are necessary in checking the Jacobi identities as a consistency of the above algebra. For the E_6 case, we used the Jacobi identity $[E_\alpha, [E_\beta, \bar{E}^\gamma]] + 2\text{-terms} = 0$ in the above, but there we had in fact needed a nontrivial identity

$$3\delta_\alpha{}^\gamma\delta_\beta{}^\delta-2(\sigma_{AB})_\alpha{}^\gamma(\sigma_{AB})_\beta{}^\delta=3\delta_\alpha{}^\delta\delta_\beta{}^\gamma-2(\sigma_{AB})_\alpha{}^\delta(\sigma_{AB})_\beta{}^\gamma\,. \tag{3·6}$$

Similar identities necessary for the E_7 and E_8 cases are:

$$\theta_{2ABEF}\bar{\theta}_1{}^{EFCD}-(1\leftrightarrow 2)=-\frac{2}{3}(\delta_{[A}{}^{[C}\theta_{2B]EFG}\bar{\theta}_1{}^{D]EFG}-(1\leftrightarrow 2))$$

with

$$\bar{\theta}^{ABCD}\equiv\frac{1}{4!}\epsilon^{ABCDEFGH}\theta_{EFGH}\,, \tag{3·7}$$

$$(\sigma_{\hat{A}\hat{B}})_{\hat{\alpha}}{}^{\hat{\delta}}(\sigma_{\hat{A}\hat{B}})_{\hat{\beta}}{}^{\hat{\varepsilon}}(C_{16}^{-1})_{\hat{\varepsilon}\hat{\gamma}}+(\text{cyclic in } \hat{\alpha},\,\hat{\beta} \text{ and } \hat{\gamma})=0\,. \tag{3·8}$$

3.2. *Decomposition of the generators into $\hat{\mathcal{H}}$ and $\mathcal{G}^c-\hat{\mathcal{H}}$*

For the purpose of constructing Kähler potentials for $G/H=E_7/SU(5)\times SU(3)\times U(1)$ and $E_8/SO(10)\times SU(3)\times U(1)$, it is necessary to further decompose the above generators into irreducible components with respect to the subgroups $SU(5)\times SU(3)\times U(1)$ and $SO(10)\times SU(3)\times U(1)$, respectively. For the $E_6/SO(10)\times U(1)$ case, the above construction of the algebra already gives the desired decomposition since $SO(10)\times U(1)$ was a maximal subgroup. However, including this E_6 case also, we need to specify the complex broken generators $\bar{X}_I$ corresponding to the coset space $G^C/\hat{H}$.

3.2.1. $\mathcal{G}^c-\mathcal{H}$ for $E_6/SO(10)\times U(1)$

The center of $SO(10)\times U(1)$ is the $U(1)$ itself. So the central charge Y discussed in §2 is unique in this case and given by T in (3·5). For convenience, however, we define Y by $Y\equiv -i(2T/\sqrt{3})$, so that Y becomes hermitian and has the simplest eigenvalues as follows:

$$Y\text{-charges of } (E_\alpha,\,T_{AB},\,T,\,\bar{E}^\alpha)=(1,0,0,-1)\,. \tag{3·9}$$

Hence, as the general procedure in §2 shows, the generators $\bar{X}_I$ carrying negative Y-charges are given by 16 generators $\bar{E}$:

$$\mathcal{G}^c-\mathcal{H}=\{\bar{X}_I\}=\{\bar{E}^\alpha\}\,. \tag{3·10}$$

We thus have an $SO(10)$ spinor NG superfield ϕ_α, namely one generation **16**, in this case.

3.2.2. $\mathcal{G}^c-\mathcal{H}$ for $E_7/SU(5)\times SU(3)\times U(1)$

The generators $\hat{T}_I{}^J$ and E_{IJKL} in the algebra (3·4) should first be decomposed into irreducible components with respect to $SU(5)\times SU(3)\times U(1)$. This is easily done by decomposing the $SU(8)$ indices $I, J, \cdots=1\sim 8$ into $SU(5)$ indices $a, b, \cdots$ and $SU(3)$ indices $i, j, \cdots$ running over $1\sim 5$ and $6\sim 8$, respectively. We thus find the following irreducible generators:

$$T_a{}^b\equiv\hat{T}_a{}^b-\frac{1}{2}\sqrt{\frac{3}{10}}T\,,\qquad T_i{}^j\equiv\hat{T}_i{}^j-\frac{1}{2}\sqrt{\frac{5}{6}}T\,,$$

$$T\equiv 2\sqrt{\frac{2}{15}}\sum_{I=1}^{5}\hat{T}_I{}^I=-2\sqrt{\frac{2}{15}}\sum_{I=6}^{8}\hat{T}_I{}^I\,,$$

$$T_a{}^i\equiv\hat{T}_a{}^i\,,\qquad T_i{}^a\equiv\hat{T}_i{}^a\,,$$

$$E^a \equiv \frac{1}{4!}\epsilon^{abcde}E_{bcde}\,, \qquad E_a \equiv \frac{1}{3!}\epsilon^{ijk}E_{aijk}\,,$$

$$E^i_{ab} \equiv \frac{1}{2!}\epsilon^{ijk}E_{abijk}\,, \qquad E_i{}^{ab} \equiv \frac{1}{3!}\epsilon^{abcde}E_{cdei}\,. \tag{3·11}$$

Here we still retain the common normalization condition (3·3) with $N=24$. The generators $T_a{}^b$, $T_i{}^j$ and T stand for the $SU(5)$, $SU(3)$ and $U(1)$ generators of the unbroken subgroup H, respectively, and the others for the broken generators.

The E_7 algebra (3·4) is immediately rewritten in terms of these generators of (3·12). We cite here only the commutation relations of broken generators since other ones trivial by $SU(5)$ $SU(3)$ covariance:

$$[T_a{}^i, T_j{}^b] = -i\left(\delta_a{}^b T_j{}^i - \delta_i{}^j T_a{}^b - 2\sqrt{\frac{2}{15}}\delta_j{}^i\delta_a{}^b T\right), \quad [T_a{}^i, E_b] = -iE^i_{ab}\,,$$

$$[T_a{}^i, E^j_{bc}] = -\frac{i}{2!}\epsilon^{ijk}\epsilon_{abcde}E_k{}^{de}\,, \quad [T_a{}^i, E_j{}^{bc}] = i\delta_j{}^i(\delta_a{}^b E^c - \delta_a{}^c E^b)\,,$$

$$[E^a, E_b] = i\left(\sqrt{\frac{6}{5}}\delta_b{}^a T - T_b{}^a\right), \quad [E^a, E^i_{bc}] = i(T_b{}^i\delta_c{}^a - T_c{}^i\delta_b{}^a)\,,$$

$$[E^i_{ab}, E_j{}^{cd}] = i\delta_j{}^i(T_a{}^c\delta_b{}^d + T_b{}^d\delta_a{}^c - T_a{}^d\delta_b{}^c - T_b{}^c\delta_a{}^d) - 2i\delta_a{}^{[c}\delta_b{}^{d]}\left(T_j{}^i + \sqrt{\frac{2}{15}}\delta_j{}^i T\right),$$

$$[E^i_{ab}, E^j_{ca}] = -i\epsilon^{ijk}\epsilon_{abcde}T_k{}^e\,, \quad [E^a, T_b{}^i] = [E^a, E^b] = [E^a, E_i{}^{bc}] = 0\,. \tag{3·12}$$

The unique central charge in H is T in this case, and we define the Y-charge by $Y \equiv -i\sqrt{15/2}\,T$. Then the Y-charge eigenvalues of the E_7 generators become:

$$Y\text{-charges of } (E^a, T_a{}^i, E_i{}^{ab}, T_a{}^b, T, T_i{}^j, \bar{E}^i_{ab}, \bar{T}_i{}^a, \bar{E}_a)$$
$$= (3, 2, 1, 0, 0, 0, -1, -2, -3)\,. \tag{3·13}$$

Therefore the complex broken generators $\bar{X}_I$ of $G^c/\hat{H}$, which carry negative Y-charges, are now given by

$$\mathcal{G}^c - \hat{\mathcal{H}} = \{\bar{E}^i_{ab}, \bar{T}_i{}^a, \bar{E}_a\}\,, \tag{3·14}$$

and hence the corresponding NG superfields are $(\phi_i{}^{ab}, \phi_a{}^i, \phi^a)$, which possess the $SU(5)\times SU(3)$ quantum numbers $\{(\mathbf{10}, \mathbf{3}^*), (\mathbf{5}^*, \mathbf{3}), (\mathbf{5}, \mathbf{1})\}$, namely three generations (of quarks / leptons plus one Higgs supermultiplet **5**).[3),5)]

3.2.3. $\mathcal{G}^c - \hat{\mathcal{H}}$ for $E_8/SO(10)\times SU(3)\times U(1)$

The E_8 generators $T_{\hat{A}\hat{B}}$ and $E_{\hat{\alpha}}$ in (3·1) are $SO(16)$ multiplets. The decomposition into $SO(10)\times SU(3)\times U(1)$ is accomplished in two steps; first $SO(16)\to SO(10)\times SO(6)$ and then $SO(6)\cong SU(4)\to SU(3)\times U(1)$.

Let us start with $T_{\hat{A}\hat{B}}$. The $SO(16)$ indices $\hat{A}, \hat{B}=1\sim 16$ are divided into the $SO(10)$ indices $A, B=1\sim 10$ and the $SO(6)$ indices $a, b=11\sim 16$. So $T_{\hat{A}\hat{B}}$ splits into three pieces T_{AB}, T_{Aa} and T_{ab}. The isomorphism $SO(6)\cong SU(4)$ implies that the $SO(6)$ generators T_{ab} and the $SO(6)$ vector T_{Aa} are equivalent to $SU(4)$ generators $T_{\hat{i}}{}^{\hat{j}}$ ($\hat{i}, \hat{j}=1\sim 4$) and $SU(4)$ **6** representation $T_{A[\hat{i},\hat{j}]}$ represented by $\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$, respectively. The conversion is performed by the matrices σ_{ab} and σ_a of $SO(6)$ Clifford algebra in Appendix A:

$$T_{A[\hat{i},\hat{j}]}=\frac{i}{2}(\sigma_a)_{\hat{i}\hat{j}}T_{Aa}\,,\qquad T_{\hat{i}}{}^{\hat{j}}=\frac{i}{2}(\sigma_{ab})_{\hat{i}}{}^{\hat{j}}T_{ab}\,. \tag{3·15}$$

These $SU(4)$ indices $\hat{i}$ are further decomposed into $SU(3)$'s one $i=1\sim3$ and a singlet one 4. Thus $T_{A[\hat{i},\hat{j}]}$ yields the following two irreducible generators:

$$T_{Ai}\equiv\sqrt{2}\,T_{A[i,4]}\,,\quad \bar{T}_A{}^i\equiv\frac{1}{2}\epsilon^{ijk}\sqrt{2}\,T_{A[j,k]}\,, \tag{3·16}$$

which are anti-hermitian conjugates of each other as can be shown by the help of Eq. (A·20). The $SU(4)$ generators $T_{\hat{i}}{}^{\hat{j}}$ yields $\mathbf{3}+\bar{\mathbf{3}}$ anti-hermitian conjugate pair

$$T^i\equiv T_4{}^{\hat{i}=i}\,,\qquad \bar{T}_i\equiv T_{\hat{i}=i}{}^4\,, \tag{3·17}$$

in addition to the $SU(3)$ and $U(1)$ generators defined by

$$T_i{}^j\equiv T_{\hat{i}=i}{}^{\hat{j}=j}-\frac{1}{3}\delta_i{}^j\sum_{\hat{k}=1}^{3}T_{\hat{k}}{}^{\hat{k}}\,,$$

$$T\equiv\sqrt{\frac{4}{3}}\,T_4{}^4=-\sqrt{\frac{4}{3}}\sum_{\hat{k}=1}^{3}T_{\hat{k}}{}^{\hat{k}}\,. \tag{3·18}$$

Next is the decomposition of the $SO(16)$ anti-Majorana Weyl spinor generator $E_{\tilde{\alpha}}$. Since the $SO(16)$ Γ-matrices can be constructed as a tensor product of the $SO(10)$ and $SO(6)$ Γ-matrices, the right-handed Weyl spinor $E_{\tilde{\alpha}}$ consists of a right×right spinor $E_{\alpha\hat{i}}$ and a left×left one $E^{\alpha\hat{i}}$, with α and $\hat{i}$ denoting the $SO(10)$ and $SO(6)\cong SU(4)$ spinor indices, respectively. The anti-Majorana property (3·1) of $E_{\tilde{\alpha}}$ implies $\bar{E}^{\alpha\hat{i}}=-(E_{\alpha\hat{i}})^\dagger$. [See Appendix A for these.] So the $SO(10)\times SU(3)\times U(1)$ decomposition yields

$$E_\alpha\equiv E_{\alpha4}\,,\qquad \bar{E}^\alpha\equiv E^{\alpha4}\,,$$

$$E^{\alpha i}\equiv E^{\alpha\hat{i}=i}\,,\qquad \bar{E}_{\alpha i}\equiv E_{\alpha\hat{i}=i}\,. \tag{3·19}$$

By summarizing the above procedure, the E_8 generators have decomposed into the unbroken $SO(10)\times SU(3)\times U(1)$ generators (T_{AB}, $T_i{}^j$, T) and the broken ones (T_{Ai}, $\bar{T}^{Ai}$, T^i, $\bar{T}_i$, E_α, $\bar{E}^\alpha$, $E^{\alpha i}$, $\bar{E}_{\alpha i}$). The various numerical factors in the above definitions of these generators were chosen so as to keep the common normalization condition (3·3). Now it is straightforward to rewrite the E_8 algebra in terms of these generators. We cite again only the broken generators' commutation relations here:

$$[T_{Ai},\,T_{Bj}]=-i\epsilon_{ijk}T^k\delta_{AB}\,,\quad [T_{Ai},\,\bar{T}_B{}^j]=-\delta_i{}^jT_{AB}+i\delta_{AB}\left(\frac{1}{\sqrt{3}}\delta_i{}^jT+T_i{}^j\right),$$

$$[T_{Ai},\,T_j]=-i\epsilon_{ijk}\bar{T}_A{}^k\,,\quad [T_{Ai},\,E_{\alpha j}]=-\frac{i}{\sqrt{2}}(\sigma_A)_{\alpha\beta}\epsilon_{ijk}E^{\beta k}\,,$$

$$[T_{Ai},\,E^{\alpha j}]=\frac{i}{\sqrt{2}}(\sigma_A{}^\dagger)^{\alpha\beta}E_\beta\delta_i{}^j\,,\quad [T_{Ai},\,E^\alpha]=-\frac{i}{\sqrt{2}}(\sigma_A{}^\dagger)^{\alpha\beta}E_{\beta i}\,,$$

$$[T_i,\,T^j]=i\left(T_i{}^j-\sqrt{\frac{4}{3}}\delta_i{}^jT\right),\quad [T_i,\,E_\alpha]=iE_{\alpha i}\,,$$

$$[T_i,\,E^{\alpha j}]=-i\delta_i{}^jE^\alpha\,,\quad [E_{\alpha i},\,E_{\beta j}]=-\frac{i}{\sqrt{2}}\epsilon_{ijk}(\sigma_A)_{\alpha\beta}T_A{}^k\,,$$

$$[E_{\alpha i}, E_\beta] = -\frac{i}{\sqrt{2}}(\sigma_A)_{\alpha\beta} T_{Ai} ,$$

$$[E_{\alpha i}, E^{\beta j}] = -\frac{1}{2}(\sigma_{AB})_\alpha{}^\beta \delta_i{}^j T_{AB} + i\left(T_i{}^j - \frac{1}{2\sqrt{3}}\delta_i{}^j T\right),$$

$$[E_{\alpha i}, E^\beta] = i\delta_\alpha{}^\beta T_i , \quad [E_\alpha, E^\beta] = -\frac{1}{2}(\sigma_{AB})_\alpha{}^\beta T_{AB} + i\sqrt{\frac{3}{4}}\delta_\alpha{}^\beta T ,$$

$$[T_i, T_A{}^j] = [T_i, T_j] = [T_i, E_{\alpha j}] = [T_i, E^\alpha] = [T_{Ai}, E_\alpha] = [E_\alpha, E_\beta] = 0 . \qquad (3\cdot20)$$

Again the unique central charge in H is T and the Y-charge is defined by $Y = -i2\sqrt{3}T$. The E_8 generators carry the following Y-charges:

$$Y\text{-charges of } (T^i, E_\alpha, T_{Ai}, E^{\alpha i}, T_{AB}, T, T_i{}^j, \bar{E}_{\alpha i}, \bar{T}_A{}^i, \bar{E}^\alpha, \bar{T}_i)$$
$$= (4, 3, 2, 1, 0, 0, 0, -1, -2, -3, -4) . \qquad (3\cdot21)$$

Hence the complex broken generators X_I of G^C/H are given by

$$\mathcal{G}^C - \hat{\mathcal{H}} = \{\bar{E}_{\alpha i}, \bar{T}_A{}^i, \bar{E}^\alpha, \bar{T}_i\} \qquad (3\cdot22)$$

in this $E_8/SO(10)\times SU(3)\times U(1)$ case. The appearing NG superfields $\{\phi^{\alpha i}, \phi_i{}^A, \phi_\alpha, \phi_i\}$ thus have $SO(10)\times SU(3)$ quantum number $\{(\mathbf{16}, \mathbf{3}), (\mathbf{10}, \mathbf{3}^*), (\mathbf{16}^*, \mathbf{1}), (\mathbf{1}, \mathbf{3}^*)\}$, containing three left-handed generations $3\times\mathbf{16}$ plus one right-handed generation $\mathbf{16}^*$ as announced in the Introduction.[6),7)]

3.3. *The cases of H with more than one central charges*

We have considered in the previous subsection only the cases of H having one dimensional center, for each of which the NG superfield content was unique. For the cases $G/H=E_7/SU(5)\times SU(3)\times U(1)$ and $E_8/SO(10)\times SU(3)\times U(1)$, however, we are free to reduce the unbroken subgroup H to smaller ones as far as keeping the grand unified group $SU(5)$ or $SO(10)$. All the possibilities not violating the Kähler property of G/H are to replace the $SU(3)\times U(1)$ in H[6)] by $SU(2)\times[U(1)]^2$ or $[U(1)]^3$. The invariant complex structure are no longer unique for these G/H cases as was explained in § 2. So it is an intriguing question whether the NG superfield contents can or cannot be changed in a phenomenologically interesting manner.

Let us first consider the case $G/H=E_7/SU(5)\times SU(2)\times[U(1)]^2$. The addittional central charge, say Y_2, is the so-called hypercharge $({}^1{}_{1}{}_{-2})$ of the $SU(3)$. The broken generators in the previous $E_7/SU(5)\times SU(3)\times U(1)$ case, E^a, $T_a{}^i$, $E_i{}^{ab}$ and their anti-hermitian conjugates, are now further decomposed into

$$E^a \to E^a(\mathbf{5}, \mathbf{1}; 3, 0) ,$$

$$T_a{}^i \to \begin{cases} T_a{}^i(\mathbf{5}^*, \mathbf{2}; 2, 1) , & (i=1, 2) \\ T_a{}^3(\mathbf{5}^*, 1; 2, -2) , \end{cases}$$

$$E_i{}^{ab} \to \begin{cases} E_i{}^{ab}(\mathbf{10}, \mathbf{2}; 1, -1) , \\ E_3{}^{ab}(\mathbf{10}, \mathbf{1}; 1, 2) \end{cases} \qquad (3\cdot23)$$

and their anti-hermitian conjugates, where the numbers in the brackets denote the multiplets under the $SU(5)\times SU(2)$ as well as Y_1 and Y_2 charges. Additional broken generators coming from previously unbroken $SU(3)$ part are

$$
T_3{}^i(\mathbf{1}, \mathbf{2}; 0, 3)\,, \quad T_i{}^3(\mathbf{1}, \mathbf{2}; 0, -3)\,. \tag{3·24}
$$

According to the procedure in § 2.6., we plot the $\boldsymbol{Y}=(Y_1, Y_2)$ charge eigenvalue vectors of those broken generators in Fig. 2. Clearly there are ten possibilities in drawing an oriented "plane" to choose a set of complex broken generators $\bar{X}_1 \in \mathcal{G}^c - \mathcal{H}$; by denoting $\bar{X}_I$'s by the $SU(5)\times SU(2)$ quantum numbers, we find

$$
\begin{array}{ll}
\text{I)} & [(\mathbf{5}^*, \mathbf{1}), (\mathbf{10}, \mathbf{2}), (\mathbf{5}, \mathbf{1}), (\mathbf{5}^*, \mathbf{2}), (\mathbf{10}, \mathbf{1}), (\mathbf{1}, \mathbf{2})]\,, \\
\text{II)} & [(\mathbf{5}, \mathbf{1}), (\mathbf{5}^*, \mathbf{2}), (\mathbf{10}, \mathbf{1}), (\mathbf{1}, \mathbf{2}), (\mathbf{5}, \mathbf{1}), (\mathbf{10}^*, \mathbf{2})]\,, \\
\text{III)} & [(\mathbf{5}^*, \mathbf{2}), (\mathbf{10}, \mathbf{1}), (\mathbf{1}, \mathbf{2}), (\mathbf{5}, \mathbf{1}), (\mathbf{10}^*, \mathbf{2}), (\mathbf{5}^*, \mathbf{1})]\,, \\
\text{IV)} & [(\mathbf{10}, \mathbf{1}), (\mathbf{1}, \mathbf{2}), (\mathbf{5}, \mathbf{1}), (\mathbf{10}^*, \mathbf{2}), (\mathbf{5}^*, \mathbf{1}), (\mathbf{5}, \mathbf{2})]\,, \\
\text{V)} & [(\mathbf{1}, \mathbf{2}), (\mathbf{5}, \mathbf{1}), (\mathbf{10}^*, \mathbf{2}), (\mathbf{5}^*, \mathbf{1}), (\mathbf{5}, \mathbf{2}), (\mathbf{10}^*, \mathbf{1})]
\end{array}
\tag{3·25}
$$

and their hermitian conjugate combinations. The first set (I) is the nearest one to the previous NG field set $[(\mathbf{5}^*, \mathbf{3}), (\mathbf{10}, \mathbf{3}^*), (\mathbf{5}, \mathbf{1})]$ in the case of $E_7/SU(5)\times SU(3)\times U(1)$. The other sets (II) $\sim$ (V) do not contain the NG field content which can be identified with the three generations of quarks and leptons $(\mathbf{5}^*+\mathbf{10})\times 3$. Similar result is obtained also for $E_7/SU(5)\times[U(1)]^3$.

Although not interesting phenomenologically, these simple examples tell us that increasing the central charges much loosen the constraints on the possible choices of NG superfield set. So it is interesting to see whether one can have four right-handed generations for E_8 case instead of the previous "three right-handed plus one left-handed generations."

The answer is "NO", unfortunately. We present, however, a discussion for the case $E_8/SO(10)\times[U(1)]^3$ to illustrate our procedure in particular for $k=3$, explicitly. The additional central charges Y_2 and Y_3 are the hypercharge $({}^{1}{}_{1}{}_{-2})$ and the third component of isospin $({}^{1}{}_{-1}{}_{0})$ of the $SU(3)$ of the previous $E_8/SO(10)\times SU(3)\times U(1)$. The broken generators there, T^i, E_α, T_{Ai}, $E^{\alpha i}$ and their anti-hermitian conjugates in (3·20), now decompose into following multiplets:

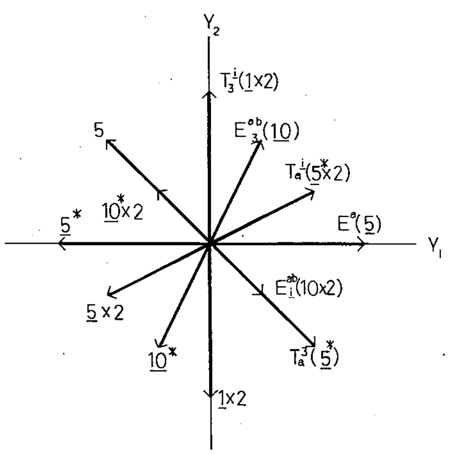

Fig. 2. $\boldsymbol{Y}=(Y_1, Y_2)$ charge eigenvalue vectors of the broken generators (3·23), (3·24) of $E_7/SU(5)\times SU(2)\times[U(1)]^2$.

$$
\begin{array}{llllll}
 & & SO(10); & Y_1, & Y_2, & Y_3, \\
T^i \to \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. \!\!\!\! & \begin{array}{l} T^1 \\ T^2 \\ T^3 \end{array} & \begin{array}{l} (\mathbf{1}; \\ (\mathbf{1}; \\ (\mathbf{1}; \end{array} & \begin{array}{r} 4, \\ 4, \\ 4, \end{array} & \begin{array}{r} 1, \\ 1, \\ -2, \end{array} & \begin{array}{r} 1), \\ -1), \\ 0), \end{array} \\
E_\alpha \to E_\alpha & & (\mathbf{16}^*; & 3, & 0, & 0)\,,
\end{array}
$$

$$T_{Ai}\to\left\{\begin{array}{llll} T_{A1} & (\mathbf{10};\ 2, & -1, & -1), \\ T_{A2} & (\mathbf{10};\ 2, & -1, & 1), \\ T_{A3} & (\mathbf{10};\ 2, & 2, & 0), \end{array}\right.$$

$$E^{\alpha i}\to\left\{\begin{array}{llll} E^{\alpha 1} & (\mathbf{16};\ 1, & 1, & 1), \\ E^{\alpha 2} & (\mathbf{16};\ 1, & 1, & -1), \\ E^{\alpha 3} & (\mathbf{16};\ 1, & -2, & 0) \end{array}\right. \tag{3·26}$$

and their anti-hermitian conjugates. Other than these there appear six broken generators $T_i^j(i\neq j)$ coming from the $SU(3)$, but we omit them for simplicity since they are phenomenologically uninteresting $SO(10)$ singlets.

According to the procedure explained in § 2.6., we consider the 3-dimensional $\boldsymbol{Y}=(Y_1, Y_2, Y_3)$-charge vector space and a cuboid with its center at the origin whose surfaces are taken to cross at $(\pm 1, 0, 0)$ $(0, \pm a, 0)$ and $(0, 0, \pm b)$ perpendicularly to each axis Y_1, Y_2 and Y_3, respectively. Then with a and b chosen suitably large, all the $\boldsymbol{Y}$-charge eigenvalue vectors $\boldsymbol{y}_I=(y_1, y_2, y_3)$ of the broken generators (3·26) with $y_1>0$, intersect the cuboid only on the "top surface" perpendicular to Y_1 axis with coordinate $Y_1=+1$, while their conjugate generators with $y_1<0$ on the "bottom surface" with $Y_1=-1$. We plot the cross points $(y_2/y_1, y_3/y_1)$ of $\boldsymbol{y}_I$ vectors of (3·26) with the top surface in Fig. 3. To draw a plane including the origin in the 3-dimensional $\boldsymbol{Y}$ space is equivalent to drawing an arbitrary line on the top surface, and the generators on the one side of the line plus the anti-hermitian conjugates of the generators on the other side are identified with the complex broken generators $\bar{X}_I$. We have drawn an example of such a line on the Fig. 3., for which the generators $\bar{X}_I$ are thus given by

$$\{\bar{X}_I\}=[E^{\alpha 1}(\mathbf{16}), E^{\alpha 2}(\mathbf{16}), T_{A3}, T^2, \bar{E}^{\alpha}(\mathbf{16}), \bar{E}_{\alpha 3}(\mathbf{16}^*), \bar{T}_1, \bar{T}_3, \bar{T}^{A1}, \bar{T}^{A2}]\,.$$

In this case also we have three **16** plus on **16***. We see from Fig. 3 that the point of the generator $E_\alpha(16^*)$ is placed inside a triangle made by the three points $E^{\alpha i}(\mathbf{16})$ $(i=1, 2, 3)$, and thus it is impossible to draw a line to have four **16** as the complex broken generators $\bar{X}_I$.

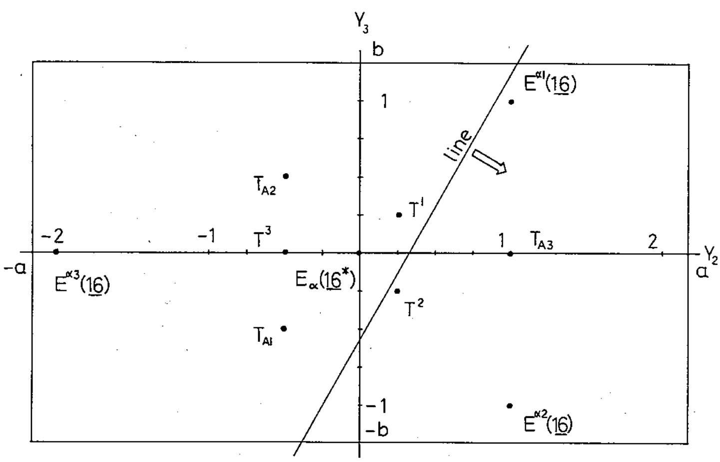

Fig. 3. Plot of the crossing points $(y_2/y_1, y_3/y_1)$ on the top surface $Y_1=1$ corresponding to $\boldsymbol{Y}$-charge eigenvalue vectors $\boldsymbol{y}=(y_1, y_2, y_3)$ of broken generators (3·26) of $E_8/SO(10)\times[U(1)]^3$.

§ 4. Kähler potentials

In this section we construct the Kähler potential $K(\phi, \bar{\phi})$ in a closed form for each of the physically interesting coset spaces $E_6/SO(10)\times U(1)$, $E_7/SU(5)\times SU(3)\times U(1)$ and $E_8/SO(10)\times SU(3)\times U(1)$. The Kähler potential is unique in these cases up to the possibility of chirality inversion of all the NG superfields. Hence, for simplicity, we calculate $K(\phi, \bar{\phi})$ based on the lowest dimensional representation of G for each case $G=E_{6\sim 8}$. Expanding the obtained $K(\phi, \bar{\phi})$ in a power series in ϕ and $\bar{\phi}$, we compare our results with the other authors' ones which were given up to quartic order for E_7 and E_8 cases.[5),6)]

We also give the explicit expression of the function $F(\phi)$ in the Kähler potential transformation law,

$$K(\phi', \bar{\phi}')=K(\phi, \bar{\phi})+F(\phi)+F^*(\bar{\phi}),$$

since $F(\phi)$ becomes important when the system is coupled to supergravity.

4.1. $E_6/SO(10)\times U(1)$

The lowest dimensional representation of E_6 is **27**, which is known to be decomposed, with respect to the subgroup $SO(10)\times U(1)$, as[18)]

$$\mathbf{27}=\left(\mathbf{1};\frac{4}{3}\right)+\left(\mathbf{16};\frac{1}{3}\right)+\left(\mathbf{10};-\frac{2}{3}\right). \tag{4·1}$$

Here the first numbers in the brackets denote $SO(10)$ multiplets and the second the eigenvalues with respect to the central charge $Y=-i(2/\sqrt{3})T$ defined previously in § 3.2.1. Corresponding to the $SO(10)\times U(1)$ decomposition (4·1), we can write the representation basis vector ϕ as

$$\phi=\begin{pmatrix} x \\ y_\alpha \\ z_A \end{pmatrix} \tag{4·2}$$

with $SO(10)$ spinor index $\alpha=1\sim 6$ and vector index $A=1\sim 10$ as before. [Notice that these H-irreducible pieces x, y, z are ordered according to our rule referring to the Y-charge values.]

The E_6 generators T_{AB}, T, E_α, $\bar{E}^\alpha$ in (3·5) are represented in this space as

$$\delta\phi\equiv\left(\theta T+\frac{1}{2}\theta_{AB}T_{AB}+\bar{\epsilon}^\alpha E_\alpha+\epsilon_\alpha\bar{E}^\alpha\right)\phi$$

$$=\begin{bmatrix} \frac{2}{\sqrt{3}}i\theta & \bar{\epsilon}^\beta & 0 \\ -\epsilon_\alpha & \frac{1}{2}\theta_{AB}(\sigma_{AB})_\alpha{}^\beta+\frac{i}{2\sqrt{3}}\theta\delta_\alpha{}^\beta & \frac{1}{\sqrt{2}}(\bar{\epsilon}\sigma_B)_\alpha \\ 0 & -\frac{1}{\sqrt{2}}(\sigma_A{}^\dagger\epsilon)^\beta & \theta_{AB}-\frac{i}{\sqrt{3}}\theta\delta_{AB} \end{bmatrix}\begin{bmatrix} x \\ y_\beta \\ z_B \end{bmatrix}, \tag{4·3}$$

where σ_A and σ_{AB} are the γ-matrices and rotation matrices of $SO(10)$ in spinor representation, respectively, defined in Appendix A. This expression (4·3) is easily obtainable by the help of the $SO(10)$ covariance and the E_6 algebra (3·5).

The complex broken generators $\bar{X}_I$ with negative Y-charge are $\bar{E}^\alpha$ alone as was seen in (3·10). Reading the representation of $\bar{E}^\alpha$ from (4·2), we can immediately obtain the following expression for the BKMU variable ξ:

$$\xi(\phi)=e^{\phi_\alpha\bar{E}^\alpha}=\begin{bmatrix} 1 & 0 & 0 \\ -\phi_\alpha & \delta_\alpha{}^\beta & 0 \\ \frac{1}{2\sqrt{2}}(\phi\sigma_A{}^\dagger\phi) & -\frac{1}{\sqrt{2}}(\phi\sigma_A{}^\dagger)^\beta & \delta_{AB} \end{bmatrix}, \tag{4·4}$$

Notice here that $e^{\phi\cdot\bar{E}}=1+\phi\bar{E}+(\phi\bar{E})^2/2$ because of the nilpotency $(\bar{E})^3=0$. As a projection operator η satisfying (2·29) necessary for the BKMU formula, we adopt the projection operator into the highest Y-charge subspace spanned by x:

$$\eta_x=\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}. \tag{4·5}$$

Then the BKMU formula (2·30) with (4·4) yields the desired Kähler potential:

$$\begin{aligned} K(\phi,\bar{\phi}) &= \ln\det{}_{\eta_x}(\xi^\dagger(\bar{\phi})\xi(\phi)) \\ &= \ln[1+\bar{\phi}\phi+\frac{1}{8}(\bar{\phi}\sigma_A\phi)(\bar{\phi}\sigma_A\phi)]. \end{aligned} \tag{4·6}$$

Here the determinant is trivial since the η_x-projected space is one dimensional.

We next determine the transformation law of the NG superfields ϕ_α under the infinitesimal E_6-transformation. By substituting $\xi(\phi)$ of (4·4) and $g=1+\delta g=1+(\theta T+\theta_{AB}T_{AB}/2+\bar{\epsilon}^\alpha E_\alpha+\epsilon_\alpha\bar{E}^\alpha)$ with matrix representation (4·3) into the BKMU transformation law (2·25)

$$g\xi(\phi)=\xi(\phi')\hat{h}(\phi,g), \tag{4·7}$$

we easily find the infinitesimal field change $\delta\phi=\phi'-\phi$ as well as $\hat{h}(\phi,1+\delta g)$: $\delta\phi$ is given by

Table I. Explicit matrix representation of Eq. (4·14) [E_7 generators].

$\delta\psi=$	$u_{k>l}$	$v_{c>d}$	w_{ck}
u_{ij}	$-4ig_{[i}^{[k}\delta_{j]}^{l]}+i\sqrt{\frac{5}{6}}\theta(\delta_i{}^k\delta_j{}^l-\delta_i{}^l\delta_j{}^k)$	$\epsilon_{ijm}\bar{h}^m_{cd}$	$-2i\bar{\Sigma}^c_{[i}\delta^k_{j]}$
v^{ab}	$-\epsilon^{klm}h_m{}^{ab}$	$4i\Lambda^{[a}_{[c}\delta^{b]}_{d]}+i\sqrt{\frac{3}{10}}\theta(\delta_c{}^a\delta_d{}^b-\delta_c{}^b\delta_d{}^a)$	$-\frac{1}{2}\epsilon^{abcef}\bar{h}^k_{ef}$
w_{ai}	$-2i\Sigma_a{}^{[k}\delta_i{}^{l]}$	$\frac{1}{2}\epsilon_{acdef}h_i{}^{ef}$	$-i(\Lambda_a{}^c\delta_i{}^k-g_i{}^k\delta_a{}^c)+\frac{i}{3}\theta\delta_a{}^c\delta_i{}^k$
x^{ai}	$-\epsilon^{ikl}\epsilon^a$	$-2i\Sigma^i_{[c}\delta^a_{d]}$	$\epsilon^{ikm}h_m{}^{ac}$
y_{ab}	0	$\epsilon_{abcde}\epsilon^e$	$2i\Sigma^k_{[a}\delta^c_{b]}$
z^{ij}	0	0	$-\epsilon^{ijk}\epsilon^c$

$$\delta\phi_\alpha = \epsilon_\alpha - \frac{i}{2\sqrt{3}}\theta\phi_\alpha - \frac{1}{2}\theta_{AB}(\sigma_{AB}\phi)_\alpha - \frac{1}{4}(\phi\sigma_A\phi)(\bar{\epsilon}\sigma_A)_\alpha\,, \tag{4·8}$$

and, as for the $\hat{h}(\phi, 1+\delta g)$, we only cite its (1,1) matrix element $\propto \eta_x \hat{h} \eta_x$,

$$\begin{aligned}\eta_x \hat{h}(\phi, 1+\delta g)\eta_x &= \eta_x g \xi(\phi)\eta_x \\ &= \left(1 + \frac{2}{\sqrt{3}} i\theta - \bar{\epsilon}\phi\right)\eta_x\,,\end{aligned} \tag{4·9}$$

where the first equality follows from (4·7) and the properties $\hat{h}\eta_x = \eta_x \hat{h}\eta_x$ and $\eta_x \xi(\phi')\eta_x = \eta_x$. Thus, since the change of the Kähler potential $\ln\det_\eta(\xi^\dagger\xi)$ is generally given by $\ln\det_\eta \hat{h}^{-1}(g, \phi) + \text{h.c.}$ as seen in (2·33), the infinitesimal change of the present Kähler potential (4·6) is now found to be

$$\delta K(\phi, \bar{\phi}) = \bar{\epsilon}\phi + \epsilon\bar{\phi}\,. \tag{4·10}$$

The purely imaginary term $2i\theta/\sqrt{3}$ has dropped here as it should be since the Kähler potential is truly invariant under the $H = SO(10) \times U(1)$ transformation which is linearly realized on ϕ as is indeed seen in (4·8).

The Kähler potential for this irreducible manifold $E_6/SO(10)\times U(1)$ was also obtained by Achiman, Aoyama and van Holten[19] as well as by Delduc and Valent,[20] in a heuristic way. Their results of $K(\phi, \bar{\phi})$ and the transformation law coincide exactly with ours (4·6), (4·8) and (4·10). It is however noted that the expression $K(\phi, \bar{\phi})$ by the former authors is apparently different:

$$\begin{aligned} K(\phi, \bar{\phi}) &= \frac{1}{48}\bar{\phi}[Q^{-1}\ln(1+Q)]\phi\,, \\ Q_\alpha{}^\beta &\equiv \left\{\frac{3}{2}\delta_\alpha{}^\beta\delta_\gamma{}^\delta - (\sigma_{AB})_\alpha{}^\beta(\sigma_{AB})_\gamma{}^\delta\right\}\bar{\phi}^\gamma\phi_\delta\,. \end{aligned} \tag{4·11}$$

The equivalence of this to ours (4·6) is rather difficult to show directly but can be seen from the fact that both (4·6) and (4·11) transform in the same way as (4·10) under the E_6 transformation and coincide with each other around $\phi = \bar{\phi} = 0$; that is, they satisfy the same set of first order differential equations and the same boundary condition at $\phi = \bar{\phi} = 0$, and hence must coincide with each other for all ϕ and $\bar{\phi}$ owing to the uniqueness of the solution of that set of differential equations.

x^{ck}	$y_{c>d}$	$z^{k>l}$
$\epsilon_{ijk}\bar{\epsilon}_c$	0	0
$-2i\bar{\Sigma}_k{}^{[a}\delta_c{}^{b]}$	$-\epsilon^{abcde}\bar{\epsilon}_e$	0
$-\epsilon_{ikm}\bar{h}^m_{ac}$	$2i\bar{\Sigma}_i{}^{[c}\delta_a{}^{d]}$	$\epsilon_{ikl}\bar{\epsilon}_a$
$i(\Lambda_c{}^a\delta_k{}^j - g_k{}^j\delta_c{}^a) - \frac{i}{3}\theta\delta_c{}^a\delta_k{}^j$	$-\frac{1}{2}\epsilon^{acdef}\bar{h}^i_{ef}$	$2i\bar{\Sigma}^a_{[k}\delta^i_{l]}$
$\frac{1}{2}\epsilon_{abcef}h_k{}^{ef}$	$-4i\Lambda^{[c}_{[a}\delta^{d]}_{b]} - i\sqrt{\frac{3}{10}}\theta(\delta_a{}^c\delta_b{}^d - \delta_a{}^d\delta_b{}^c)$	$\epsilon_{klm}\bar{h}^m_{ab}$
$2i\Sigma_c{}^{[i}\delta_k{}^{j]}$	$-\epsilon^{ijm}h_m{}^{cd}$	$4ig^{[i}_{[k}\delta^{j]}_{l]} - i\sqrt{\frac{5}{6}}\theta(\delta_k{}^i\delta_l{}^j - \delta_k{}^j\delta_l{}^i)$

4.2. $E_7/SU(5)\times SU(3)\times U(1)$

The lowest dimensional representation of E_7 is **56** and its $H=SU(5)\times SU(3)\times U(1)$ decomposition is[18)]

$$\mathbf{56}=\left(\mathbf{1},\mathbf{3};\frac{5}{2}\right)+\left(\mathbf{10},\mathbf{1};\frac{3}{2}\right)+\left(\mathbf{5}^*,\mathbf{3}^*;\frac{1}{2}\right)$$
$$+\left(\mathbf{5},\mathbf{3};-\frac{1}{2}\right)+\left(\mathbf{10},\mathbf{1};-\frac{3}{2}\right)+\left(\mathbf{1},\mathbf{3}^*;-\frac{5}{2}\right). \tag{4·12}$$

As before, here the H-irreducible components are ordered according to the $Y=-i\sqrt{15/2}\,T$ charge eigenvalues denoted by the last numbers in the brackets. The representation basis vector ψ is written accordingly as

$$\psi={}^t(u_{[i,j]},\ v^{[a,b]},\ w_{ai},\ x^{ai},\ y_{[a,b]},\ z^{[i,j]})\,, \tag{4·13}$$

where $a, b, \cdots$ and $i, j, \cdots$ denote the $SU(5)$ and $SU(3)$ indices, respectively, and the square bracket [ ] implies that the indices are anti-symmetric. For those quantities with anti-symmetric indices we treat hereafter only the independent quantities like $u_{i>j}$ (half as many as u_{ij}); therefore, for instance, $M^{[ij]}u_{i>j}=\sum_{i>j}M^{[ij]}u_{ij}=1/2\sum_{i,j}M^{[ij]}u_{ij}$.

The matrix representation of the E_6 generators of (3·11) in this space are also easily obtained by the help of the $SU(5)\times SU(3)$ covariance and the E_7 algebra (3·12). We show in Table I the explicit matrix expression of the infinitesimal E_7-transformation

$$\delta\psi=(\Lambda_b{}^aT_a{}^b+g_j{}^iT_i{}^j+\theta T+\bar{\Sigma}_i{}^aT_a{}^i+\Sigma_a{}^i\bar{T}_i{}^a$$
$$+\bar{\epsilon}_aE^a+\epsilon^a\bar{E}_a+\frac{1}{2}\bar{h}^i_{ab}E_i^{ab}+\frac{1}{2}h_i{}^{ab}\bar{E}^i_{ab})\psi\,, \tag{4·14}$$

where parameters are "hermitian"; $(\Lambda_b{}^a)^*=\Lambda_a{}^b$, $(g_j{}^i)^*=g_i{}^j$, $\theta^*=\theta$, $(\bar{\Sigma}_i{}^a)^*=\Sigma_a{}^i$, $(\bar{\epsilon}_a)^*=\epsilon^a$, $(\bar{h}^i_{ab})^*=h_i{}^{ab}$. The symbol [ ] in Table I denotes the anti-symmetrization with "weight 1"; e.g., $\Sigma^i_{[a}\delta^c_{b]}=1/2(\Sigma_a{}^i\delta_b{}^c-\Sigma_b{}^i\delta_a{}^c)$.

The complex broken generators $\bar{X}_I$ in this case are $\bar{E}^i_{ab}$, $\bar{T}_i{}^a$ and $\bar{E}_a$ as was shown in (3·14), and the corresponding NG superfields $\phi_i{}^{ab}$, $\phi_a{}^i$, ϕ^a are now denoted by using different letters for the ease of distinction as $\phi_i{}^{ab}$, $\psi_a{}^i$, χ^a. The exponent of the BKMU variable $\xi(\phi)$ is seen from Table I to have the following matrix representation:

$$\frac{1}{2}\phi_i{}^{ab}\bar{E}^i_{ab}+\psi_a{}^i\bar{T}^a{}_i+\chi^a\bar{E}_a$$

$$=\begin{array}{c}\\ u_{ij}\\ v^{ab}\\ w_{ai}\\ x^{ai}\\ y_{ab}\\ z^{ij}\end{array}\begin{array}{c}\begin{array}{cccccc} u_{k>l} & v_{c>d} & w_{ck} & x^{ck} & y_{c>d} & z^{k>l}\end{array}\\ \left[\begin{array}{cccccc} 0 & & & & & \\ -\epsilon^{klm}\phi_m{}^{ab} & 0 & & 0 & & \\ -2i\psi_a{}^{[k}\delta_i{}^{l]} & \frac{1}{2}\epsilon_{acdef}\phi_i{}^{ef} & 0 & & & \\ -\varepsilon^{ikl}\chi^a & -2i\psi^i_{[c}\delta^a_{d]} & \epsilon^{ikm}\phi_m{}^{ac} & 0 & & \\ 0 & \epsilon_{abcde}\chi^e & 2i\psi^k_{[c}\delta^c_{b]} & \frac{1}{2}\epsilon_{abcef}\phi_k{}^{ef} & 0 & \\ 0 & 0 & -\epsilon^{ijk}\chi^c & 2i\psi_c{}^{[i}\delta_k{}^{j]} & -\epsilon^{ijm}\phi_m{}^{cd} & 0 \end{array}\right]\end{array}. \tag{4·15}$$

Instead of the original BKMU parametrization $\xi=\exp(\phi^{ab}{}_i\bar{E}^i_{ab}/2+\psi_a{}^i\bar{T}_i{}^a+\chi^a\bar{E}_a)$, we adopt the parametrization

$$\xi(\phi)=\exp\left(\frac{1}{2}\phi_i{}^{ab}\bar{E}^i_{ab}\right)\cdot\exp(\psi_a{}^i\bar{T}_i{}^a)\cdot\exp(\chi^a\bar{E}_a)\,, \tag{4·16}$$

which is slightly easier to calculate and equivalent to the original one through the change of third variable $\chi^a\to\chi^a+c\phi_i{}^{ab}\psi_b{}^i$ with a certain constant c. Now we take the projection operator η_u into the $u_{i>j}$ sector (3-dimensional) with the highest Y-charge 5/2 and calculate the first column ($u_{i>j}$-column) $\xi_u(\phi)$ of the matrix $\xi(\phi)$ (4·16). This is because we need only the column "vector" $\xi_u(\phi)$ (56×3 matrix, more precisely) in applying the BKMU formula (2·30) with η_u:

$$K(\phi,\bar{\phi})=\ln\det{}_{\eta_u}(\xi^\dagger(\bar{\phi})\xi(\phi))=\ln\det_{3\times 3}(\xi_u{}^\dagger(\bar{\phi})\xi_u(\phi))\,. \tag{4·17}$$

The first column "vector" $\xi_u(\phi)$ is calculated straightforwardly and is given explicitly by

$$\xi_u(\phi)=\begin{array}{c}u_{k>l}\\ v^{cd}\\ \\ w_{ck}\\ x\\ y\\ \\ z\\ \\ \\ \end{array}\left(\begin{array}{l}
2\delta_k{}^{[i}\delta_l{}^{j]}\\
-\epsilon^{ijm}\phi_m{}^{cd}\\
\\
-2i\psi_c{}^{[i}\delta_k{}^{j]}-\dfrac{1}{4\times 2!}\epsilon^{ijm}\epsilon_{abcde}\phi_m{}^{ab}\phi_k{}^{de}\\
-\epsilon^{ijk}\chi^c+i\epsilon^{ijm}\phi_m{}^{ec}\phi_e{}^k-\dfrac{1}{4\times 3!}\epsilon^{ijl}\epsilon^{kmn}\epsilon_{abefg}\phi_l{}^{ab}\phi_m{}^{ec}\phi_n{}^{fg}\\
-2\psi^{[i}_{[c}\psi^{j]}_{d]}-\dfrac{1}{2}\epsilon^{ijm}\epsilon_{cdefg}\chi^e\phi_m{}^{fg}+\dfrac{i}{4}\epsilon^{ijm}\epsilon_{abef[c}\psi^n_{d]}\phi_m{}^{ab}\phi_n{}^{ef}\\
\quad-\dfrac{1}{8\times 4!}\epsilon^{ijm}\epsilon^{pno}\epsilon_{\alpha\beta\gamma\delta\varepsilon}\epsilon_{cdefg}\phi_m{}^{\alpha\beta}\phi_n{}^{\delta\varepsilon}\phi_p{}^{\gamma e}\phi_0{}^{fg}\\
-2i\epsilon^{kl[i}\psi_e{}^{j]}\chi^e+\dfrac{1}{8}\epsilon^{ijm}\epsilon^{kln}\epsilon_{abefg}\phi_m{}^{ab}\phi_n{}^{fg}\chi^e\\
\quad+\epsilon^{ijm}\phi_m{}^{ab}\psi_a{}^k\psi_b{}^l+\dfrac{i}{12}\epsilon^{ijm}\epsilon_{abefg}\epsilon^{on[k}\psi_\epsilon{}^{l]}\phi_m{}^{ab}\phi_0{}^{e\epsilon}\phi_n{}^{fg}\\
\quad+\dfrac{1}{16\times 5!}\epsilon^{ijm'}\epsilon^{l'n'n}\epsilon^{klm}\epsilon_{abe\alpha'\beta'}\epsilon_{\alpha\beta\gamma fg}\phi_{m'}{}^{ab}\phi_{n'}{}^{\alpha'\beta'}\phi_{l'}{}^{e\gamma}\phi_m{}^{\alpha\beta}\phi_n{}^{fg}
\end{array}\right). \tag{4·18}$$

Equation (4·17) with (4·18) gives the desired full order explicit expression of the Kähler potential for $E_7/SU(5)\times SU(3)\times U(1)$. Here for comparison with the previously obtained quartic order result,[5] we expand it up to quartic order in ϕ and $\bar{\phi}$:

$$\begin{aligned}\ln\det_{3\times 3}(\xi_u{}^\dagger(\bar{\phi})\xi_u(\phi))=\;&\phi_i{}^{ab}\bar{\phi}^i_{ab}+4\psi_a{}^i\bar{\psi}_i{}^a+6\bar{\chi}_a\chi^a\\
&+\frac{1}{8}\{(\phi_i{}^{ab}\bar{\phi}^i_{ab})^2-(\phi_i{}^{ab}\bar{\phi}^j_{ab})(\phi_j{}^{cd}\bar{\phi}^i_{cd})-4(\phi_i{}^{ac}\bar{\phi}^i_{bc})(\phi_j{}^{bd}\bar{\phi}^j_{ad})\}\\
&-2(\psi_a{}^i\bar{\psi}_i{}^b)(\psi_b{}^j\bar{\psi}_j{}^a)-3(\bar{\chi}_a\chi^a)^2-(\bar{\phi}^i_{ab}\phi_i{}^{ab})(\bar{\psi}_j{}^c\psi_c{}^j)+(\phi_i{}^{ab}\bar{\phi}^j_{ab})(\psi_c{}^i\bar{\psi}_j{}^c)\\
&+2(\phi_i{}^{ab}\bar{\phi}^i_{ac})(\psi_b{}^j\bar{\psi}_j{}^c)-\frac{2}{3}(\bar{\phi}^i_{ac}\phi_j{}^{ab})(\bar{\psi}_i{}^c\psi_b{}^j)-2\phi_i{}^{ac}\bar{\phi}^i_{bc}\chi^b\bar{\chi}_a\end{aligned}$$

Table II. Explicit matrix representation of Eq. (4·25) [E_8 generators].

$\delta\psi=$	p_k	q^β	$r_C{}^k$	$s_{\beta k}$	t_{CD}
p_i	$-ig_i{}^k + i\frac{2}{\sqrt{3}}\theta\delta_i{}^k$	$-i\bar{\omega}_{\beta i}$	$i\epsilon_{ijk}\bar{\Lambda}_C{}^j$	$i\bar{\epsilon}^\beta\delta_i{}^k$	0
q^α	$-i\omega^{\alpha k}$	$-\frac{1}{2}\Sigma^{IJ}(\sigma_{IJ})_\beta{}^\alpha + \frac{\sqrt{3}}{2}\theta\delta_\beta{}^\alpha$	$-\frac{i}{\sqrt{2}}\bar{\omega}_{\gamma k}(\sigma_C{}^\dagger)^{\gamma\alpha}$	$\frac{i}{\sqrt{2}}\bar{\Lambda}_C{}^k(\sigma_C{}^\dagger)^{\alpha\beta}$	$\bar{\epsilon}^\gamma(\sigma_{CD})_\gamma{}^\alpha$
$r_A{}^i$	$-i\epsilon^{ijk}\Lambda_{Aj}$	$-\frac{i}{\sqrt{2}}\omega^n(\sigma_A)_{\gamma\beta}$	$\Sigma^{AC}\delta_k{}^i + i\delta_{AC}g_k{}^i + \frac{i}{\sqrt{3}}\theta\delta_{AC}\delta_k{}^i$	$-\frac{i}{\sqrt{2}}\epsilon^{ijk}\bar{\omega}_{\gamma j}(\sigma_A{}^\dagger)^{\gamma\beta}$	$-2\delta^A_{[C}\bar{\Lambda}^i_{D]}$
$s_{\alpha i}$	$i\epsilon_\alpha\delta_\delta{}^k$	$\frac{i}{\sqrt{2}}\Lambda_{IJ}(\sigma_I)_{\alpha\beta}$	$\frac{i}{\sqrt{2}}\epsilon_{ijk}\omega^{\gamma j}(\sigma_C)_{\gamma\alpha}$	$\frac{1}{2}\Sigma^{IJ}(\sigma_{IJ})_\alpha{}^\beta\delta_i{}^k - ig_i{}^k\delta_\alpha{}^\beta + \frac{i}{2\sqrt{3}}\theta\delta_\alpha{}^\beta\delta_i{}^k$	$-\bar{\omega}_{\gamma i}(\sigma_{CD})_\alpha{}^\gamma$
t_{AB}	0	$\epsilon_\gamma(\sigma_{AB})_\beta{}^\gamma$	$2\delta^C_{[A}\Lambda_{D]k}$	$-\omega^{\gamma k}(\sigma_{AB})_\gamma{}^\beta$	$-4\delta^C_{[A}\Sigma_{B]D}$
u	$-i\frac{2}{\sqrt{3}}h^k$	$-i\frac{\sqrt{3}}{2}\epsilon_\beta$	$-i\frac{1}{\sqrt{3}}\Lambda_{Ck}$	$-i\frac{1}{2\sqrt{3}}\omega^{\beta k}$	0
$v_j{}^i$	$ih^i\delta_j{}^k - \frac{i}{3}\delta_j{}^i h^k$	0	$-i\Lambda_{Cj}\delta_k{}^i + \frac{i}{3}\delta_j{}^i\Lambda_{Ck}$	$i\delta_j{}^k\omega^{\beta i} - \frac{i}{3}\delta_j{}^i\omega^{\beta k}$	0
$w^{\alpha i}$	0	$ih^i\delta_\beta{}^\alpha$	$\frac{i}{\sqrt{2}}\epsilon_\gamma(\sigma_C{}^\dagger)^{\alpha\beta}\delta_k{}^i$	$-\frac{i}{\sqrt{2}}\epsilon^{ijk}\Lambda_{IJ}(\sigma_I{}^\dagger)^{\alpha\beta}$	$\omega^{\gamma i}(\sigma_{CD})_\gamma{}^\alpha$
x_{Ai}	0	0	$i\epsilon_{ijk}h^j\delta_{AC}$	$-\frac{i}{\sqrt{2}}\epsilon_\gamma(\sigma_A{}^\dagger)^{\gamma\beta}\delta_i{}^k$	$-2\delta^A_{[C}\Lambda_{D]i}$
y_α	0	0	0	$-ih^k\delta_\alpha{}^\beta$	$-\epsilon_\gamma(\sigma_{CD})_\alpha{}^\gamma$
z^i	0	0	0	0	0

$$+4\{-(\psi_d{}^i\bar{\psi}_i{}^d)(\chi^a\bar{\chi}_a)+(\bar{\psi}_i{}^a\psi_b{}^i)(\chi^b\bar{\chi}_a)\}$$

$$-\frac{1}{2}\epsilon_{ijk}\epsilon^{abcde}(\phi^i_{ab}\psi_c{}^j)(\bar{\chi}_d\psi_e{}^k)-\frac{1}{2}\epsilon^{ijk}\epsilon_{abcde}(\bar{\psi}_j{}^c\phi_i{}^{ab})(\bar{\psi}_k{}^e\chi^d)\,, \qquad (4\cdot19)$$

where χ^a denotes the redefined field $\tilde{\chi}^a\equiv\chi^a+\frac{i}{3}\phi_i{}^{ab}\psi_b{}^i$ with omission of $\sim$. This expression (4·19) indeed agrees with the known one in Ref. 5) aside from trivial scale transformations.

We can also obtain the E_7 transformation laws of the NG superfields $\phi_i{}^{ab}$, $\psi_a{}^i$, χ^a and of the Kähler potential following the same steps as for the previous E_6 case. The change of the Kähler potential (4·17) under the infinitesimal E_7 transformation with parameters defined in (4·14) is simply given by

u	$v_l{}^k$	$w^{\beta k}$	x_{Ck}	y_β	z^k
$-i\frac{2}{\sqrt{3}}\bar{h}_i$	$i\bar{h}_k\delta_i{}^l - \frac{i}{3}\delta_k{}^l\bar{h}_i$	0	0	0	0
$-i\frac{\sqrt{3}}{2}\bar{\epsilon}^\alpha$	0	$i\bar{h}_k\delta_\beta{}^\alpha$	0	0	0
$-i\frac{1}{\sqrt{3}}\bar{\Lambda}_A{}^i$	$-i\bar{\Lambda}_A{}^l\delta_k{}^i + \frac{i}{3}\delta_k{}^l\bar{\Lambda}_A{}^i$	$\frac{i}{\sqrt{2}}\bar{\epsilon}^\gamma(\sigma_A)_{\gamma\beta}\delta_k{}^i$	$-i\delta_{AC}\epsilon^{ijk}\bar{h}_j$	0	0
$-i\frac{1}{2\sqrt{3}}\bar{\omega}_{\alpha i}$	$i\bar{\omega}_{\alpha k}\delta_i{}^l - \frac{i}{3}\delta_k{}^l\bar{\omega}_{\alpha i}$	$\frac{i}{\sqrt{2}}\epsilon_{ijk}\bar{\Lambda}_I{}^j(\sigma_I)_{\alpha\beta}$	$-\frac{i}{\sqrt{2}}\bar{\epsilon}^\gamma(\sigma_C)_{\alpha\gamma}\delta_i{}^k$	$-i\bar{h}_i\delta_\alpha{}^\beta$	0
0	0	$\bar{\omega}_{\gamma k}(\sigma_{AB})_\beta{}^\gamma$	$2\delta^C_{[A}\bar{\Lambda}_{B]}{}^k$	$-\bar{\epsilon}^\gamma(\sigma_{AB})_\gamma{}^\beta$	0
0	0	$\frac{i}{2\sqrt{3}}\bar{\omega}_{\alpha i}$	$i\frac{1}{\sqrt{3}}\bar{\Lambda}_C{}^k$	$i\frac{\sqrt{3}}{2}\bar{\epsilon}^\beta$	$i\frac{2}{\sqrt{3}}\bar{h}_k$
0	$ig_k{}^i\delta_j{}^l - ig_j{}^l\delta_k{}^i$	$-i\omega_{\beta j}\delta_k{}^i + \frac{i}{3}\delta_j{}^i\omega_{\beta k}$	$i\bar{\Lambda}_C{}^i\delta_j{}^k - \frac{i}{3}\delta_j{}^i\bar{\Lambda}_C{}^k$	0	$-i\bar{h}_j\delta_k{}^i + \frac{i}{3}\delta_j{}^i\bar{h}_k$
$i\frac{1}{2\sqrt{3}}\omega^{\alpha i}$	$-i\omega^{\alpha l}\delta_k{}^i + \frac{i}{3}\delta_k{}^l\omega^{\alpha i}$	$-\frac{1}{2}\Sigma^{IJ}(\sigma_{IJ})_\beta{}^\alpha\delta_k{}^i + ig_k{}^i\delta_\beta{}^\alpha - \frac{i}{2\sqrt{3}}\theta\delta_\beta{}^\alpha\delta_k{}^i$	$-\frac{i}{2}\epsilon^{ijk}\bar{\omega}_{\gamma j}(\sigma_C{}^\dagger)^{\gamma\alpha}$	$-\frac{i}{\sqrt{2}}\bar{\Lambda}_I{}^j(\sigma_I{}^\dagger)^{\alpha\beta}$	$-i\bar{\epsilon}^\alpha\delta_k{}^i$
$i\frac{1}{\sqrt{3}}\Lambda_{Ai}$	$i\Lambda_{Ak}\delta_i{}^l - \frac{i}{3}\delta_k{}^l\Lambda_{Ai}$	$\frac{1}{\sqrt{2}}\epsilon_{ijk}\omega^{\gamma j}(\sigma_A)_{\gamma\beta}$	$\Sigma^{AC}\delta_i{}^k - ig_i{}^k\delta_{AC} - \frac{i}{\sqrt{3}}\theta\delta_{AC}\delta_i{}^k$	$\frac{i}{\sqrt{2}}\bar{\omega}_{\gamma j}(\sigma_A{}^\dagger)^{\gamma\beta}$	$i\epsilon_{ijk}\bar{\Lambda}_A{}^j$
$i\frac{\sqrt{3}}{2}\epsilon_\alpha$	0	$-\frac{i}{\sqrt{2}}\Lambda_{Ik}(\sigma_I)_{\alpha\beta}$	$\frac{i}{\sqrt{2}}\omega^{\gamma k}(\sigma_C)_{\gamma\alpha}$	$\frac{1}{2}\Sigma^{IJ}(\sigma_{IJ})_\alpha{}^\beta - \frac{\sqrt{3}}{2}i\theta\delta_\alpha{}^\beta$	$i\bar{\omega}_{\alpha k}$
$i\frac{2}{\sqrt{3}}h^i$	$-ih^l\delta_k{}^i + \frac{i}{3}\delta_k{}^l h^i$	$-i\epsilon_\beta\delta_k{}^i$	$-i\epsilon^{ijk}\Lambda_{Cj}$	$i\omega^{\beta i}$	$ig_k{}^i - \frac{2}{\sqrt{3}}\theta\delta_k{}^i$

$$\delta K(\phi, \bar{\phi}) = \frac{1}{2}\bar{h}^i_{ab}\phi_i{}^{ab} + 2(\bar{\Sigma}_i{}^a\psi_a{}^i) + 3(\bar{\epsilon}_a\chi^a) + \text{h.c.} \tag{4·20}$$

It requires more tedious calculation although straightforward to obtain the field transformation laws in a closed form. We omit them since they are lengthy and not illuminating.

4.3. $E_8/SO(10)\times SU(3)\times U(1)$

The lowest dimensional representation of E_8 is the adjoint representation **248** itself. We have decomposed the E_8 generators into $SO(10)\times SU(3)\times U(1)$ irreducible pieces in § 3.2 already. They are arranged in the order of Y-charge eigenvalue in (3·20) as

$$ {}^t\boldsymbol{T} = (\boldsymbol{T}_I) \equiv (T^i, E_\alpha, T_{Ai}, E^{\alpha i}, T_{AB}, T, T_i{}^j, \bar{E}_{\alpha i}, \bar{T}_A{}^i, \bar{E}^\alpha, \bar{T}_i)\,. \tag{4·21}$$

On this basis the basis vector in the adjoint representation is given by

$$\boldsymbol{\phi} = {}^t\boldsymbol{T} \cdot \psi \tag{4·22}$$

with the component vector ψ

$${}^t\psi = (p_i,\ q^\alpha,\ r_A{}^i,\ s_{\alpha i},\ t^{AB},\ u,\ v^i{}_j,\ w^{\alpha i},\ x_{Ai},\ y_\alpha,\ z^i)\ . \tag{4·23}$$

Notice that it is the component vector ψ that we mean by "representation basis vector", as is the usual convention among physicists. We are following this rule in this paper.

The matrix representation of the generators in the adjoint representation are given by the structure constant of course:

$$[\boldsymbol{T}_I,\ \boldsymbol{T}_J] = f_{IJ}{}^K \boldsymbol{T}_K \equiv \boldsymbol{T}_K (ad\boldsymbol{T}_I)^K{}_J, \quad \text{i.e.,} \quad (\mathrm{ad}\,\boldsymbol{T}_I)^K{}_J = f_{IJ}{}^K\ . \tag{4·24}$$

Thus, from (4·22), the basis vector ψ is transformed under the infinitesimal E_8 transformation as follows:

$$\delta\psi = \mathrm{ad}\Big(\frac{1}{2}\Sigma_{AB}T_{AB} + g_j{}^i T_i{}^j + \theta T + \bar{\Lambda}_A{}^i T_{Ai} + \Lambda_{Ai}\bar{T}_A{}^i$$
$$+ \bar{h}_i T^i + h^i \bar{T}_i + \bar{\epsilon}^\alpha E_\alpha + \epsilon_\alpha \bar{E}^\alpha + \bar{\omega}_{\alpha i}E^{\alpha i} + \omega^{\alpha i}\bar{E}_{\alpha i}\Big) \cdot \psi\ , \tag{4·25}$$

The structure constant is readable from the E_8 algebra (3·19) and we cite the explicit form of the adjoint matrix in (4·25) in Table II.

The complex broken generators $\bar{X}_I$ in this case are $\bar{E}_{\alpha i}$, $\bar{T}_A{}^i$, $\bar{E}^\alpha$ and $\bar{T}_i$ as was shown in (3·21) and we denote the corresponding NG superfields by $\phi^{\alpha i}$, ψ_{Ai}, χ_α and ζ^i, respectively. Then the exponent of the BKMU variable is represented by the following matrix as is seen from Table II:

$$\phi^{\alpha i}\bar{E}_{\alpha i} + \psi_{Ai}\bar{T}_A{}^i + \chi_\alpha \bar{E}^\alpha + \zeta^i \bar{T}_i =$$

$$\begin{array}{c|cccc}
 & p_k & q^\beta & r_C{}^k & s_{\beta k} \\
\hline
p_i & 0 & & & \\
q^\alpha & -i\phi^{\alpha k} & 0 & & \\
r_A{}^i & -i\epsilon^{ijk}\psi_{Aj} & -\frac{i}{\sqrt{2}}\phi^{\gamma i}(\sigma_A)_{\gamma\beta} & 0 & \\
s_{\alpha i} & i\chi_\alpha\delta_i{}^k & \frac{i}{\sqrt{2}}\psi_{Ai}(\sigma_A)_{\alpha\beta} & \frac{i}{\sqrt{2}}\epsilon_{ijk}\phi^{\gamma j}(\sigma_C)_{\gamma\alpha} & 0 \\
t_{AB} & 0 & \chi_\gamma(\sigma_{AB})_\beta{}^\gamma & 2\delta^C_{[A}\psi_{B]k} & -\phi^{\gamma k}(\sigma_{AB})_\gamma{}^\beta \\
u & -i\frac{2}{\sqrt{3}}\zeta^k & -i\frac{\sqrt{3}}{2}\chi_\beta & -\frac{i}{\sqrt{3}}\psi_{Ck} & -\frac{i}{2\sqrt{3}}\phi^{\beta k} \\
v^i{}_j & i\zeta^i\delta_j{}^k - \frac{i}{3}\delta_j{}^i\zeta^k & 0 & -i\psi_{Cj}\delta_k{}^i + \frac{i}{3}\delta_j{}^i\psi_{Ck} & i\phi^{\beta i}\delta_j{}^k - \frac{i}{3}\delta_j^i\phi^{\beta k} \\
w^{\alpha i} & 0 & i\zeta^i\delta_\beta{}^\alpha & \frac{i}{\sqrt{2}}\chi_\beta(\sigma_C{}^\dagger)^{\alpha\beta}\delta_k{}^i & -\frac{i}{\sqrt{2}}\epsilon^{ijk}\psi_{Aj}(\sigma_A{}^\dagger)^{\alpha\beta} \\
x_{Ai} & 0 & 0 & i\epsilon_{ijk}\zeta^j\delta_{AC} & -\frac{i}{\sqrt{2}}\chi_\gamma(\sigma_A{}^\dagger)^{\gamma\beta}\delta_i{}^k \\
y_\alpha & 0 & 0 & 0 & -i\zeta^k\delta_\alpha{}^\beta \\
z^i & 0 & 0 & 0 & 0
\end{array}$$

$$
\begin{array}{c|ccccccc}
 & t_C & u & v^k{}_l & w^{\beta k} & x_{Ck} & y_\beta & z^k \\
\hline
p\sim u & 0 & 0 & & & & & \\
v^i{}_j & 0 & 0 & 0 & & 0 & & \\
w^{\alpha i} & \phi^{\gamma i}(\sigma_{CD})_\gamma{}^\alpha & \frac{i}{2\sqrt{3}}\phi^{\alpha i} & -i\phi^{\alpha l}\delta_k{}^i+\frac{i}{3}\delta_k{}^l\phi^{\alpha i} & 0 & & & \\
x_{Ai} & -2\delta^A_{[C}\psi_{D]i} & \frac{i}{\sqrt{3}}\psi_{Ai} & i\psi_{Ak}\delta_i{}^l-\frac{i}{3}\delta_k{}^l\psi_{Ai} & \frac{i}{\sqrt{2}}\epsilon_{ijk}\phi^{\gamma j}(\sigma_A)_{\gamma\beta} & 0 & & \\
y_\alpha & -\chi_\gamma(\sigma_{CD})_\alpha{}^\gamma & i\frac{\sqrt{3}}{2}\chi_\alpha & 0 & -\frac{i}{\sqrt{2}}\psi_{Ak}(\sigma_A)_{\alpha\beta} & \frac{i}{\sqrt{2}}\phi^{\gamma k}(\sigma_C)_{\gamma\alpha} & 0 & \\
z^i & 0 & i\frac{2}{\sqrt{3}}\zeta^i & -i\zeta^l\delta_k{}^i+\frac{i}{3}\delta_k{}^l\zeta^i & -i\chi_\beta\delta_k{}^i & -i\epsilon^{ijk}\psi_{Cj} & i\phi^{\beta i} & 0
\end{array}
\tag{4·26}
$$

Similarly to the previous E_7 case, we adopt the parametrization

$$
\xi(\phi)=\exp(\phi^{\alpha i}\bar{E}_{\alpha i})\cdot\exp(\psi_{Ai}\bar{T}_A{}^i)\cdot\exp(\chi_\alpha\bar{E}^\alpha)\cdot\exp(\zeta^i\bar{T}_i)
\tag{4·27}
$$

for the BKMU variable $\xi(\phi)$. Taking the projection operator η_p into the p_i sector with highest Y-charge $+4$, we can calculate the BKMU Kähler potential straightforwardly (although somewhat tedious):

$$
K(\phi,\bar{\phi})=\ln\det{}_{\eta_p}\xi^\dagger(\bar{\phi})\xi(\phi)=\ln\det_{3\times 3}\xi_p{}^\dagger(\bar{\phi})\xi_p(\phi)\,,
\tag{4·28}
$$

where the first column "vector" $\xi_p(\phi)$ (in fact 248×3 matrix) is given explicitly by

$$
\xi_p(\phi)=
\begin{array}{c}
 \\
p_k \\ q^\beta \\ r_C{}^k \\ s_{\beta k} \\ t_{CD} \\ \\ u \\ v_l{}^k \\ \\ w^{\beta k} \\ \\
\end{array}
\begin{array}{l}
\qquad\qquad p_i \\
\left[\begin{array}{l}
\delta_k{}^i \\
-i\phi^{\beta i} \\
-i\epsilon^{imk}\psi_{Cm}-\frac{1}{2\sqrt{2}}(\sigma_C)_{\gamma\delta}\phi^{\gamma i}\phi^{\delta k} \\
i\delta_k{}^i\chi_\beta+\frac{1}{\sqrt{2}}(\sigma_E)_{\gamma\beta}\psi_{Ek}\phi^{\gamma i}-\frac{1}{12}\epsilon_{kmn}(\sigma_E)_{\gamma\delta}(\sigma_E)_{\alpha\beta}\phi^{\gamma i}\phi^{\delta m}\phi^{\alpha n} \\
-i\epsilon^{imn}\psi_{Cm}\psi_{Dn}-i(\sigma_{CD})_\gamma{}^\delta\phi^{\gamma i}\chi_\delta+\frac{i}{\sqrt{2}}(\sigma_{[C})_{\gamma\delta}\psi_{D]m}\phi^{\gamma i}\phi^{\delta m} \\
-\frac{1}{4\times 4!}\epsilon_{lmn}(\sigma_{CDE})_{\alpha\beta}(\sigma_E)_{\gamma\delta}\phi^{\alpha m}\phi^{\beta n}\phi^{\gamma l}\phi^{\delta i} \\
-i\frac{2}{\sqrt{3}}\zeta^i-\frac{\sqrt{3}}{2}\phi^{\gamma i}\chi_\gamma+\frac{i}{6}\sqrt{\frac{3}{2}}(\sigma_E)_{\gamma\delta}\phi^{\gamma i}\phi^{\delta j}\psi_{Ej} \\
i\delta_l{}^i\zeta^k-\frac{i}{3}\delta_l{}^k\zeta^i-\frac{1}{2}\epsilon^{ijk}\psi_{Aj}\psi_{Al}+\frac{i}{2\sqrt{2}}(\sigma_A)_{\gamma\delta}\phi^{\gamma i}\phi^{\delta k}\psi_{Al} \\
-\frac{i}{6\sqrt{2}}\delta_l{}^k(\sigma_A)_{\gamma\delta}\phi^{\gamma i}\phi^{\delta j}\psi_{Aj}+\frac{1}{48}\epsilon_{lmn}(\sigma_A)_{\gamma\delta}(\sigma_A)_{\alpha\beta}\phi^{\gamma i}\phi^{\delta m}\phi^{\alpha n}\phi^{\beta k} \\
-\frac{1}{\sqrt{2}}\epsilon^{ikm}(\sigma_E{}^\dagger)^{\delta\beta}\psi_{Em}\chi_\delta+\phi^{\beta i}\zeta^k-\frac{i}{2}\epsilon^{kmn}(\sigma_{EF})_\gamma{}^\beta\psi_{Em}\psi_{Fn}\phi^{\gamma i} \\
-\frac{i}{4}(\sigma_E)_{\gamma\delta}(\sigma_E{}^\dagger)^{\alpha\beta}\phi^{\gamma i}\phi^{\delta k}\chi_\alpha+\frac{\sqrt{2}}{12}(\sigma_E)_{\gamma\delta}\phi^{\gamma i}\phi^{\delta[k}\phi^{\beta m]}\psi_{Em}
\end{array}\right.
\end{array}
$$

$$+\frac{\sqrt{2}}{6}(\sigma_E)_{\gamma\delta}(\sigma_{EF})_\alpha{}^\beta\phi^{\gamma i}\phi^{\delta[k}\phi^{\alpha m]}\psi_{Fm}$$

$$-\frac{i}{2\times 5!}\epsilon_{lmn}(\sigma_E)_{\gamma\delta}(\sigma_E)_{\alpha\epsilon}\phi^{\gamma i}\phi^{\delta l}\phi^{\alpha m}\phi^{\beta n}\phi^{\epsilon k}$$

$$-\frac{1}{8\times 5!}\epsilon_{lmn}(\sigma_E)_{\gamma\delta}(\sigma_{EFG})_{\alpha\epsilon}(\sigma_{FG})_\kappa{}^\beta\phi^{\gamma i}\phi^{\delta n}\phi^{\alpha l}\phi^{\epsilon m}\phi^{\kappa k}$$

x_{Ck}

$$\frac{1}{2\sqrt{2}}\delta_k{}^i(\sigma_C{}^\dagger)^{\gamma\delta}\chi_\gamma\chi_\delta-\delta_k{}^i\psi_{Cm}\zeta^m+\psi_{Ck}\zeta^i+\frac{i}{2}\epsilon^{imn}\psi_{Cm}\psi_{En}\psi_{Ek}$$

$$-\frac{i}{2}\psi_{Ck}\chi_\gamma\phi^{\gamma i}+i(\sigma_{C\dot{E}})_\gamma{}^\delta\psi_{Ek}\chi_\delta\phi^{\gamma i}-\frac{i}{2\sqrt{2}}\epsilon_{kmn}(\sigma_C)_{\gamma\delta}\phi^{\gamma i}\phi^{\delta m}\zeta^n$$

$$+\frac{1}{4\sqrt{2}}(\sigma_C)_{\gamma\delta}\phi^{\gamma i}\phi^{\delta m}\psi_{Em}\psi_{Ek}-\frac{\sqrt{2}}{4}(\sigma_E)_{\gamma\delta}\phi^{\gamma i}\phi^{\delta m}\psi_{Ek}\psi_{Cm}$$

$$-\frac{\sqrt{2}}{24}\epsilon_{kmn}(\sigma_C)_{\gamma\delta}\phi^{\gamma i}\phi^{\delta m}\phi^{\alpha n}\chi_\alpha+\frac{i\sqrt{2}}{12}\epsilon_{kmn}(\sigma_{CE})_\alpha{}^\epsilon(\sigma_E)_{\gamma\delta}\phi^{\gamma i}\phi^{\delta m}\phi^{\alpha n}\chi_\epsilon$$

$$-\frac{i}{96}\epsilon_{lmn}(\sigma_{CEF})_{\alpha\epsilon}(\sigma_E)_{\gamma\delta}\phi^{\gamma i}\phi^{\delta n}\phi^{\alpha l}\phi^{\epsilon m}\psi_{Fk}$$

$$+\frac{i}{48}\epsilon_{kmn}(\sigma_E)_{\gamma\delta}(\sigma_E)_{\alpha\epsilon}\phi^{\gamma i}\phi^{\delta m}\phi^{\alpha n}\phi^{\epsilon j}\psi_{Cj}$$

$$+\frac{1}{16\times 6!\sqrt{2}}\epsilon_{lmn}\epsilon_{kpq}(\sigma_E)_{\gamma\delta}(\sigma_{EFG})_{\alpha\epsilon}(\sigma_{FGC})_{\lambda\kappa}\phi^{\gamma i}\phi^{\delta n}\phi^{\alpha l}\phi^{\epsilon m}\phi^{\lambda p}\phi^{\kappa q}$$

$$-\frac{1}{2\times 6!\sqrt{2}}\epsilon_{lmn}\epsilon_{kpq}(\sigma_E)_{\gamma\delta}(\sigma_E)_{\alpha\epsilon}(\sigma_C)_{\lambda\kappa}\phi^{\gamma i}\phi^{\delta n}\phi^{\alpha m}\phi^{\epsilon p}\phi^{\lambda l}\phi^{\kappa q}$$

y_β

$$\zeta^i\chi_\beta+\frac{i}{2}\epsilon^{imn}(\sigma_{EF})_\beta{}^\delta\chi_\delta\psi_{Em}\psi_{Fn}+\frac{i}{4}(\sigma_{EF})_\epsilon{}^\gamma(\sigma_{EF})_\beta{}^\delta\chi_\gamma\chi_\delta\phi^{\varepsilon i}$$

$$-\frac{i}{\sqrt{2}}(\sigma_E)_{\gamma\beta}\psi_{Em}\zeta^m\phi^{\gamma i}-\frac{3i}{8}\chi_\beta\chi_\gamma\phi^{\gamma i}-\frac{\sqrt{2}}{24}\epsilon^{lmn}(\sigma_{EFG})_{\gamma\beta}\psi_{El}\psi_{Fm}\psi_{Gn}\phi^{\gamma i}$$

$$+\frac{\sqrt{2}}{4}(\sigma_E)_{\gamma\delta}(\sigma_{EF})_\beta{}^\alpha\phi^{\gamma i}\phi^{\delta m}\psi_{Fm}\chi_\alpha-\frac{\sqrt{2}}{8}(\sigma_E)_{\gamma\delta}\phi^{\gamma i}\phi^{\delta m}\psi_{Em}\chi_\beta$$

$$-\frac{1}{12}\epsilon_{lmn}(\sigma_E)_{\gamma\delta}(\sigma_E)_{\alpha\beta}\phi^{\gamma i}\phi^{\delta m}\phi^{\alpha n}\zeta^l$$

$$+\frac{i}{24}(\sigma_E)_{\gamma\delta}(\sigma_{EFG})_{\alpha\beta}\phi^{\gamma i}\phi^{\delta[m}\phi^{\alpha n]}\psi_{F[m}\psi_{Gn]}$$

$$+\frac{i}{12}(\sigma_{[E})_{\gamma\delta}(\sigma_{F]})_{\alpha\beta}\phi^{\gamma i}\phi^{\delta[m}\phi^{\alpha n]}\psi_{Em}\psi_{Fn}$$

$$+\frac{1}{8\times 4!}\epsilon_{lmn}(\sigma_E)_{\gamma\delta}(\sigma_{EFG})_{\alpha\epsilon}(\sigma_{FG})_\beta{}^\lambda\phi^{\gamma i}\phi^{\delta n}\phi^{\alpha l}\phi^{\epsilon m}\chi_\lambda$$

$$-\frac{\sqrt{2}}{32\times 5!}\epsilon_{lmn}(\sigma_E)_{\gamma\delta}(\sigma_{EFG})_{\alpha\epsilon}(\sigma_{FGA})_{\lambda\beta}\phi^{\gamma i}\phi^{\delta n}\phi^{\alpha l}\phi^{\epsilon m}\phi^{\lambda j}\psi_{Aj}$$

$$-\frac{\sqrt{2}}{4\times 5!}\epsilon_{lmn}(\sigma_E)_{\gamma\delta}(\sigma_E)_{\alpha\epsilon}(\sigma_A)_{\lambda\beta}\phi^{\gamma i}\phi^{\delta n}\phi^{\alpha l}\phi^{\epsilon j}\phi^{\lambda m}\psi_{Aj}$$

$$-\frac{\sqrt{2}}{32\times 5!}\epsilon_{lmn}(\sigma_E)_{\gamma\delta}(\sigma_{EFG})_{\alpha\epsilon}(\sigma_F)_{\lambda\beta}\phi^{\gamma i}\phi^{\delta n}\phi^{\alpha l}\phi^{\epsilon m}\phi^{\lambda j}\psi_{Gj}$$

$$-\frac{1}{4\times 7!}\epsilon_{lmn}\epsilon_{pqr}(\sigma_E)_{\gamma\delta}(\sigma_A)_{\alpha\beta}\Bigg[\frac{1}{8}(\sigma_{EBC})_{\lambda\epsilon}(\sigma_{ABC})_{\eta\kappa}\phi^{\epsilon m}\phi^{\eta p}$$

$$+(\sigma_E)_{\lambda\epsilon}(\sigma_A)_{\eta\kappa}\phi^{\epsilon p}\phi^{\eta m}\Bigg]\phi^{\gamma i}\phi^{\delta n}\phi^{\lambda l}\phi^{\kappa q}\phi^{\alpha r}$$

$$
z^k \left[\begin{array}{l}
\zeta^i\zeta^k - \frac{i\sqrt{2}}{4}\epsilon^{imk}(\sigma_E{}^\dagger)^{\gamma\delta}\psi_{Em}\chi_\gamma\chi_\delta + \frac{i}{2}\epsilon^{imk}\psi_{Am}\psi_{An}\zeta^n \\
-\frac{1}{8}\epsilon^{imn}\epsilon^{kpq}\psi_{Em}\psi_{Fn}\psi_{Ep}\psi_{Fq} - i\zeta^k\chi_\gamma\phi^{\gamma i} + \frac{1}{2}\epsilon^{kmn}(\sigma_{EF})_\gamma{}^\delta\chi_\delta\psi_{Em}\psi_{Fn}\phi^{\gamma i} \\
-\frac{i}{8}(\sigma_E)_{\gamma\delta}(\sigma_E{}^\dagger)^{\epsilon\alpha}\phi^{\gamma i}\phi^{\delta k}\chi_\epsilon\chi_\alpha + \frac{\sqrt{2}}{4}(\sigma_E)_{\gamma\delta}\phi^{\gamma i}\psi_{Em}\left[\phi^{\delta k}\zeta^m - \frac{1}{2}\phi^{\delta m}\zeta^k\right] \\
+\frac{i\sqrt{2}}{8}\epsilon^{klm}(\sigma_E)_{\gamma\delta}\psi_{El}\psi_{Fm}\psi_{Fn}\phi^{\gamma i}\phi^{\delta n} \\
-\frac{i\sqrt{2}}{12}(\sigma_E)_{\gamma\delta}\phi^{\gamma i}\phi^{\delta[k}\phi^{\alpha l]}[\chi_\alpha\psi_{El} + 2(\sigma_{EF})_\alpha{}^\epsilon\chi_\epsilon\psi_{Fl}] \\
+\frac{1}{192}\epsilon_{lmn}(\sigma_E)_{\gamma\delta}\phi^{\gamma i}\phi^{\delta n}\phi^{\alpha l}[\epsilon^{kpq}(\sigma_{EFG})_{\epsilon\alpha}\phi^{\epsilon m}\psi_{Fp}\psi_{Gq} - 4i(\sigma_E)_{\alpha\epsilon}\phi^{\epsilon k}\zeta^m] \\
-\frac{1}{48}(\sigma_E)_{\gamma\delta}(\sigma_E)_{\alpha\epsilon}\phi^{\gamma i}\phi^{\delta[k}\phi^{\alpha l]}\phi^{\epsilon m}\psi_{Am}\psi_{Al} \\
-\frac{1}{8\times 5!}\epsilon_{lmn}(\sigma_E)_{\gamma\delta}\phi^{\gamma i}\phi^{\delta n}\phi^{\alpha l}\phi^{\epsilon k}\chi_\beta[(\sigma_{EFG})_{\alpha\eta}(\sigma_{FG})_\epsilon{}^\beta\phi^{\eta m} + 4(\sigma_E)_{\alpha\epsilon}\phi^{\beta m}] \\
+\frac{i\sqrt{2}}{32\times 5!}\epsilon_{lmn}(\sigma_E)_{\gamma\delta}[(\sigma_{EAB})_{\alpha\epsilon}(\sigma_{ABF})_{\lambda\eta} \\
+8(\sigma_E)_{\alpha\lambda}(\sigma_F)_{\epsilon\eta}]\phi^{\gamma i}\phi^{\delta n}\phi^{\alpha l}\phi^{\epsilon m}\phi^{\lambda[k}\phi^{\eta j]}\psi_{Fj} \\
-\frac{1}{32\times 8!}\epsilon_{lmn}\epsilon_{pqr}(\sigma_E)_{\gamma\delta}\phi^{\gamma i}\phi^{\delta n}\phi^{\lambda l}\phi^{\beta q}\phi^{\alpha r}\phi^{\tau k}[8(\sigma_E)_{\lambda\epsilon}(\sigma_A)_{\eta\beta}(\sigma_A)_{\alpha\tau}\phi^{\epsilon p}\phi^{\eta m} \\
+(\sigma_{EAB})_{\lambda\epsilon}(\sigma_{ABC})_{\eta\beta}(\sigma_C)_{\alpha\tau}\phi^{\epsilon m}\phi^{\eta p}]
\end{array}\right]
\tag{4·29}
$$

where $\sigma_{ABC}=\sigma_{[A}\sigma_B{}^\dagger\sigma_{C]}$, i.e., antisymmetrization in A, B and C with weight 1.

Equation (4·28) with (4·29) gives our final answer to the explicit closed form of the Kähler potential for $E_8/SO(10)\times SU(3)\times U(1)$. Here again we cite its expansion up to quartic order in ϕ and $\bar{\phi}$:

$$
\begin{aligned}
K(\phi,\bar{\phi}) = & (\bar{\phi}\phi)+2(\bar{\psi}\psi)+3(\bar{\chi}\chi)+4(\bar{\zeta}\zeta) \\
& -2(\bar{\zeta}\zeta)^2-2(\bar{\zeta}\zeta)(\bar{\psi}\psi)+2(\bar{\psi}_A\bar{\zeta})(\psi_A\zeta)-(\bar{\phi}_i\phi^j)(\bar{\zeta}_j\zeta^i)-3(\bar{\zeta}\zeta)(\bar{\chi}\chi) \\
& -\bar{\psi}_A{}^i\psi_{Aj}\bar{\psi}_B{}^j\psi_{Bi}+\frac{1}{2}\psi_{Ai}\psi_{Aj}\bar{\psi}_B{}^i\bar{\psi}_B{}^j+\frac{1}{3}\bar{\psi}_A{}^i\psi_{Bj}(\phi^j\sigma_{AB}\bar{\phi}_i) \\
& -(\bar{\psi}_A\psi_B)(\phi\sigma_{AB}\bar{\phi})+\frac{5}{6}(\bar{\phi}_i\phi^j)\bar{\psi}_A{}^i\psi_{Aj}-\frac{1}{2}(\bar{\phi}\phi)(\bar{\psi}\psi) \\
& +2(\bar{\psi}_A\psi_B)(\bar{\chi}\sigma_{AB}\chi)-(\bar{\chi}\chi)(\bar{\psi}\psi)-\frac{1}{2}(\bar{\phi}_i\phi^j)(\bar{\phi}_j\phi^i) \\
& +\frac{1}{8}(\phi^i\sigma_A\phi^j)(\bar{\phi}_i\sigma_A{}^\dagger\bar{\phi}_j)+\frac{1}{2}(\bar{\phi}_i\bar{\chi})(\phi^i\chi)-\frac{1}{2}(\bar{\chi}\sigma_{AB}\bar{\phi}_i)(\phi^i\sigma_{AB}\chi) \\
& -\frac{3}{2}(\bar{\chi}\chi)^2+\frac{3}{8}(\chi\sigma_A{}^\dagger\chi)(\bar{\chi}\sigma_A\bar{\chi})-\frac{1}{\sqrt{2}}\epsilon_{ijk}(\bar{\chi}\sigma_A\phi^i)\zeta^j\bar{\psi}_A{}^k \\
& -\frac{1}{\sqrt{2}}\epsilon^{ijk}(\chi\sigma_A{}^\dagger\bar{\phi}_i)\bar{\zeta}_j\psi_{AK}-(\bar{\chi}\chi)(\bar{\phi}\phi)-\frac{1}{\sqrt{2}}(\bar{\chi}\sigma_A\bar{\chi})(\zeta\psi_A)-\frac{1}{\sqrt{2}}(\chi\sigma_A{}^\dagger\chi)(\bar{\zeta}\bar{\psi}_A)
\end{aligned}
$$

$$-\frac{1}{2}\epsilon^{ijk}\psi_{Aj}\psi_{Bk}(\bar{\chi}\sigma_{AB}\bar{\phi}_i)+\frac{1}{2}\epsilon_{ijk}\bar{\psi}_A{}^j\bar{\psi}_B{}^k(\phi^i\sigma_{AB}\chi)\,. \tag{4·30}$$

Here in (4·30), and only here, the fields χ and ζ denote the following tilde fields

$$\tilde{\chi}_\alpha=\chi_\alpha-\frac{i}{3\sqrt{2}}(\sigma_A)_{\alpha\beta}\phi^{\beta i}\psi_{Ai}\,,$$

$$\tilde{\zeta}^i=\zeta^i-\frac{i}{4}\phi^{\gamma i}\chi_\gamma+\frac{1}{12\sqrt{2}}(\sigma_A)_{\alpha\beta}\phi^{\alpha i}\phi^{\beta j}\psi_{Aj}\,, \tag{4·31}$$

which was introduced to eliminate the $(\bar{\phi})^1(\phi)^2$ or $(\bar{\phi})^2(\phi)^1$ terms like $\bar{\chi}^\alpha(\sigma_A)_{\alpha\beta}\phi^{\beta i}\psi_{Ai}$ from the Kähler potential (4·30). One can see after a suitable Fierz transformation that this expression (4·30), *except for* the last four terms, agrees with Ong's results[6] which was obtained up to quartic order by a different method. The last four terms are multiplied by a factor 2 in Ong's results, but it is probably his error or simply a misprint. We actually cross-checked the correctness of our result (4·30) by confirming directly the E_8 invariance of (4·30) by substituting the field transformation law which we give now below.

As before, $g\xi(\phi)=\xi(\phi')\hat{h}(\phi,g)$ determines the transformation laws of the NG superfields $\phi=(\phi^{\alpha i},\psi_{Ai},\chi_\alpha,\zeta^i)$. We cite here them up to quadratic order in ϕ since they have appeared in no literature. Omitting the unbroken generators of $SO(10)\times SU(3)\times U(1)$ under which the transformations are trivial, we take the infinitesimal transformation parameters as

$$g-1=\delta g=(\bar{\Lambda}_A{}^iT_{Ai}+\bar{h}_iT^i+\bar{\epsilon}^\alpha E_\alpha+\bar{\omega}_{\alpha i}E^{\alpha i})-\text{h.c.}$$

as in (4·25). Then the explicit matrix forms of this given in Table II and of $\xi(\phi)$ of (4·27) lead to the following transformation laws:

$$\begin{aligned}\delta\phi^{\alpha i}=\;&\omega^{\alpha i}+\frac{1}{\sqrt{2}}\epsilon^{ijk}(\bar{\omega}_j\sigma_A{}^\dagger)^\alpha\psi_{Ak}-\frac{1}{4}(\bar{\omega}_j\sigma_A{}^\dagger)^\alpha(\phi^i\sigma_A\phi^j)+(\bar{\omega}_j\phi^i)\phi^{\alpha j}\\&-\frac{1}{\sqrt{2}}\bar{\Lambda}_A{}^i(\chi\sigma_A{}^\dagger)^\alpha+\frac{1}{2}(\bar{\Lambda}\psi)\phi^{\alpha i}+\bar{\Lambda}_A{}^j\psi_{Bj}(\phi^i\sigma_{AB})^\alpha-\bar{\Lambda}_A{}^i\psi_{Aj}\phi^{\alpha j}+(\bar{\epsilon}\chi)\phi^{\alpha i}\\&+\frac{1}{2}\epsilon^{ijk}(\bar{\epsilon}\sigma_{AB})^\alpha\psi_{Aj}\psi_{Bk}+\frac{1}{2}(\bar{\epsilon}\sigma_{AB})^\alpha(\phi^i\sigma_{AB}\chi)-i\bar{\epsilon}^\alpha\zeta^i-\frac{3}{4}\bar{\epsilon}^\alpha(\phi^i\chi)\\&+\frac{1}{\sqrt{2}}\epsilon^{ijk}\bar{h}_j\psi_{Ak}(\chi\sigma_A{}^\dagger)^\alpha+\bar{h}_j\phi^{\alpha j}\zeta^i\,,\end{aligned}$$

$$\begin{aligned}\delta\psi_{Ai}=\;&\Lambda_{Ai}+\bar{\Lambda}_B{}^k\psi_{Bi}\psi_{Ak}+\frac{1}{2}\epsilon_{ijk}(\phi^i\sigma_{AB}\chi)\bar{\Lambda}_B{}^k-\frac{1}{4}\epsilon_{ijk}\bar{\Lambda}_A{}^j(\phi^k\chi)-i\epsilon_{ijk}\bar{\Lambda}_A{}^j\zeta^k\\&-\frac{1}{2}\bar{\Lambda}_A{}^j\psi_{Bi}\psi_{Bj}+\frac{i}{2\sqrt{2}}\epsilon_{ijk}(\phi^j\sigma_A{}^\dagger\omega^k)+\frac{i}{\sqrt{2}}(\bar{\omega}_i\sigma_A{}^\dagger\chi)+\frac{1}{4}(\bar{\omega}\phi)\psi_{Ai}-\frac{1}{4}(\bar{\omega}_i\phi^j)\psi_{Aj}\\&+\frac{1}{2}(\phi^j\sigma_{AB}\bar{\omega}_j)\psi_{Bi}-\frac{1}{2}(\phi^j\sigma_{AB}\bar{\omega}^j)\psi_{Bj}+\frac{1}{2}(\bar{\epsilon}\chi)\psi_{Ai}-(\bar{\epsilon}\sigma_{AB}\chi)\psi_{Bi}\\&+\frac{1}{2\sqrt{2}}\epsilon_{ijk}(\bar{\epsilon}\sigma_A\phi^j)\zeta^k+\frac{1}{2\sqrt{2}}\bar{h}_i(\chi\sigma_A{}^\dagger\chi)-\bar{h}_i(\psi_A\zeta)+(\bar{h}\zeta)\psi_{Ai}\,,\end{aligned}$$

$$\delta\chi_\alpha=\epsilon_\alpha+\frac{1}{\sqrt{2}}(\bar{\epsilon}\sigma_A)_\alpha(\psi_A\zeta)-\frac{1}{4}(\bar{\epsilon}\sigma_A)(\chi\sigma_A{}^\dagger\chi)+(\bar{\epsilon}\chi)\chi_\alpha+\frac{1}{2}(\bar{\Lambda}\psi)\chi_\alpha-(\bar{\Lambda}_A\psi_B)(\sigma_{AB}\chi)_\alpha$$

$$+\frac{i}{\sqrt{2}}(\omega^i\sigma_A)_\alpha\psi_{Ai}+\frac{1}{12}\epsilon_{ijk}(\omega^i\sigma_A\phi^j)(\phi^k\sigma_A)_\alpha+i\bar{\omega}_{\alpha i}\zeta^i$$

$$+\frac{1}{2}\epsilon^{ijk}(\sigma_{AB}\bar{\omega}_i)_\alpha\psi_{Aj}\psi_{Bk}+(\bar{h}\zeta)\chi_\alpha\,,$$

$$\delta\zeta^i=h^i+(\bar{h}\zeta)\zeta^i+(\bar{\epsilon}\chi)\zeta^i+\frac{1}{2}\epsilon^{ijk}\Lambda_{Aj}\psi_{Ak}-\frac{1}{4\sqrt{2}}\Lambda_{Aj}(\phi^i\sigma_A\phi^j)+\frac{1}{2}(\bar{\Lambda}\psi)\zeta^i-\frac{1}{2}\bar{\Lambda}_A{}^i\psi_{Aj}\zeta^j$$

$$+\frac{1}{2\sqrt{2}}\bar{\Lambda}_A{}^i(\chi\sigma_A{}^\dagger\chi)+i(\omega^i\chi)-\frac{1}{2\sqrt{2}}(\omega^j\sigma_A\phi^i)\psi_{Aj}-\frac{1}{2\sqrt{2}}\epsilon^{ijk}(\bar{\omega}_j\sigma_A{}^\dagger\chi)\psi_{Ak}\,. \qquad (4\cdot32)$$

The Kähler potential transformation law can be derived much more simply and in a closed form as explained in the E_6 case, and is given in this case by

$$\delta K(\phi,\,\bar{\phi})=(\bar{\omega}_{\alpha i}\phi^{\alpha i}+2\bar{\Lambda}_A{}^i\psi_{Ai}+3\bar{\epsilon}^\alpha\chi_\alpha+4\bar{h}_i\zeta^i)+\text{h.c.} \qquad (4\cdot33)$$

Direct substitution of the field transformation laws (4·32) into the above quartic order expression (4·30) of $K(\phi,\,\bar{\phi})$ also confirms Eq. (4·33) as we have mentioned above.

§ 5. Summary and discussion

In this paper we have first clarified the general procedure how to construct the supersymmetric Lagrangians for any given Kählerian coset spaces G/H. Next we presented the explicit form of the Lie algebras of the exceptional grous E_6, E_7 and E_8 as well as the $SU(5)\times SU(3)\times U(1)$ and $SO(10)\times SU(3)\times U(1)$ decompositions of generators for the latter two groups respectively. Based on these we have explicitly constructed the Kähler potential (or equivalently, supersymmetric Lagrangians) in a closed form for the three phenomenologically important cases $G/H=E_6/SO(10)\times U(1)$, $E_7/SU(5)\times SU(3)\times U(1)$ and $E_8/SO(10)\times SU(3)\times U(1)$. A comment was also made on the point how the NG superfield contents can be changed if the center of H is relaxed to be more than one dimensional as is the case, e.g., $E_8/SO(10)\times SU(2)\times[U(1)]^2$ or $E_8/SO(10)\times[U(1)]^3$.

Our general theory for construction of supersymmetric Lagrangians will probably have wider applications. Our work in the latter part in this paper, however, just provides a basis for the further study of the models of the origin of generations based on E_7 and E_8 groups. Much more works have to be done in order to make these models actually realistic. As stated in the Introduction the problems are: i) how to introduce an explicit breaking of the global E_7 (or E_8) symmetry, ii) how to obtain the explicit or spontaneous breaking of supersymmetry, iii) how to understand the GUT gauge interaction, dynamically or elementary, and so on.

If these questions are solved in a natural and satisfactory manner, then those models indeed become exciting super-GUT's which can answer the origin of generations as well as mixings among generations. To those models, which may be named "nonlinear σ-model super-GUT's", we can have two alternative attitudes. One is the ordinary one of composite approach; the nonlinear σ-model is regarded as a "low energy effective Lagrangians" of a certain preon theory in which the global symmetry $G=E_7$ or E_8 is linearly realized and supposed to be spontaneously broken into H. The other is an unorthodox viewpoint to regard the nonlinear σ-model as already a fundamental Lagrangian. This viewpoint rather differs from that in conventional GUT approach in which the renormalizability of theory is one of the central principles. Nevertheless, we think it very interesting viewpoint since nobody knows whether it is actually meaningful to impose the usual renormalizability constraint to select the theories in such a high energy scale as large as Planck mass. In addition we know that the nonlinear Lagrangian indeed appears for instance even in supergravity theories which is currently believed to be most fundamental theory.

To conclude this paper we add comments on some points concerning the above quoted problems. First is about a technical problem in gauging a subgroup of or the full global group G. Such gauging is necessary when one wants to introduce the GUT $SU(5)$ or $SU(10)$ gauge interaction into the above E_7 or E_8 nonlinear models simply by hand. Further it is known that if the gauging is performed on a subgroup larger than $H[SU(5)$ $(SO(10))\times SU(3)\times U(1)$ in this case], then the supersymmetry is necessarily broken spontaneously,[21),23)] So it can serve as a possible natural mechanism for the required supersymmetry breaking. The gauging problem of the isometry group G (or its subgroup) was in fact solved by Bagger and Witten[21)] and by Ong[22)] independently, but their gauging method is based on the "trial and errors" Noether procedure and further is restricted to the so-called on-shell formalism. Therefore one cannot use their results and must repeat the tedious Noether procedure once any matter superfields are introduced into system. The gauging procedure in an off-shell (i.e., system independent) manner is indeed simple in our formalism based on the BKMU formula (2·30): For any subgroup S of G which one wants to gauge, introduce vector superfields V^a transforming under superspace gauge transformations as

$$e^{V'}=e^{i\Lambda^\dagger}e^{V}e^{-i\Lambda} \tag{5·1}$$

Here $\Lambda=\Lambda^aT^a$ (Λ^a; chiral superfield parameter) and $V=V^aT^a$ are Lie algebra valued and the matrices T^a are hermitian generators of S. The original global transformation (2·25), $g\xi(\phi)=\xi(\phi')\hat{h}(\phi, g)$, of the BKMU variable $\xi(\phi)$ under $g\in S\subseteq G$, is now replaced by the following superspace gauge transformation,

$$e^{i\Lambda}\xi(\phi)=\xi(\phi')\hat{h}(\phi, \Lambda), \quad \hat{h}(\phi, \Lambda)\in\hat{H}\,. \tag{5·2}$$

It is easy to see in the same way as for the global transformation case that the action $\int d^4xd^4\theta K(\phi, \bar{\phi})$ is supersymmetric and gauge invariant under $S\subseteq G$ if the BKMU Kähler potential $K(\phi, \bar{\phi})=\ln\det_\eta(\xi^\dagger(\bar{\phi})\xi(\phi))$ is replaced by*)

$$K(\phi, \bar{\phi};\, V)=\ln\det_\eta(\xi^\dagger(\bar{\phi})e^V\xi(\phi))\,. \tag{5·3}$$

This formula is clearly of off-shell since it is written in terms of superfields. One more advantage of this formula is that it is made trivial to couple to supergravity. This is simply achieved, for instance, in old minimal supergravity by the Lagrangian

$$\mathcal{L}=-\frac{1}{2}[\phi_0\bar{\phi}_0e^{-(1/3)K(\chi,\bar{\phi}\,;\,V)}]_D\,, \tag{5·4}$$

*) Actually the present authors were informed of this formula (5·3) by S. Uehara. The priority of this formula should be attributed to BKMU.

where ϕ_0 is the compensating chiral multiplet and $[\cdots]_D$ means the D-term action formula in local superconformal framework.[24),25)] Hence it is only necessary to apply the superconformal tensor calculus formula[26)] to the function $e^{-(1/3)K(\phi,\bar{\phi};V)}$, but not to repeat the Noether procedure (further tedious in supergravity!) as Bagger[27)] did actually.

Second comments are concerned with some problems occuring in coupling the nonlinear system to supergravity. Coupling to supergravity is an important issue in this context since the scale of "decay constant" in those nonlinear models should be already of the order of Planck mass and further it may also provide a possible source of spontaneous supersymmetry breaking (and even of explicit global G symmetry breaking as will be explained shortly). When the system is coupled to supergravity as in (5·4), the change of the Kähler potential $\delta K=F(\phi)+F^*(\bar{\phi})$ no longer vanishes as in global supersymmetry case (in which $\mathcal{L}=[K]_D$ and $\delta\mathcal{L}=[F(\phi)]_D+\text{h.c.}=0$), but yields a change on the compensating multiplet:

$$\phi_0 \to \phi_0 e^{-(1/3)F(\phi)}. \tag{5·5}$$

This change is equivalent to a combination of local superconformal transformations, dilatation, chiral and S-supersymmetry, on the scalar and fermion component fields of the compensating multiplet ϕ_0, after the auxiliary fields are eliminated as we assume henceforth. Since those local supercomformal transformations are the symmetry of the system (at least at the classical level), the above change (5·5) has no effect on the Lagrangian if the same superconformal transformations are perfomed simultaneously on the component fields of ϕ^I, scalar φ^I and fermions χ^I, as well as on the vierbein $e_\mu{}^m$ and Rarita-Schwinger field ψ_μ; these transformations are given as follows explicitly.*)

$$\begin{aligned}
&\chi^I \to \exp\left(-\frac{1}{6}\mathrm{Re}F(\varphi)+\frac{i}{2}\mathrm{Im}F(\varphi)\gamma_5\right)\cdot\chi^I\,,\\
&e_\mu{}^m \to \exp\left(\frac{1}{3}\mathrm{Re}F(\varphi)\right)e_\mu{}^m\,,\\
&\psi_\mu \to \exp\left(\frac{1}{6}\mathrm{Re}F(\varphi)-\frac{i}{2}\mathrm{Im}F(\varphi)\gamma_5\right)\cdot\psi_\mu-\gamma_\mu\zeta\,.
\end{aligned}\tag{5·6}$$

At the quantum level those superconformal transformations suffer from anomalies[28)] and are no longer symmetries of the system. Thus the G global symmetry of the nonlinear model, whose transformation induces a charge of the Kähler potential as $\delta K=F(\phi)+F^*(\bar{\phi})$, becomes broken *explicitly* by the supercomformal anomalies in the presence of supergravity. [The linearly realized subgroup H remains still unbroken since $F(\phi)$ vanishes for H transformation.] This may serve as a possible mechanism by which

*) Scalar fields φ^I remain intact since they have vanishing Weyl and chiral weights. ζ is the S-supersymmetry transformation parameter whose right-handed component is given by

$$\zeta_R=\frac{\chi_{0R}}{2\varphi_0}\left[1-\exp\left(-\frac{1}{6}\mathrm{Re}F(\varphi)+\frac{i}{2}\mathrm{Im}F(\varphi)\right)\right]-\frac{1}{6\varphi_0}\chi_R{}^I\frac{\partial F(\varphi)}{\partial\varphi^I}\,.$$

Here φ_0 and χ_{0R} are scalar and spinor components of chiral compensator ϕ_0 which are fixed in such a way that (5·4) yields the canonical Einstein and Rarita-Schwinger terms;[29)] explicitly,

$$\varphi_0=\sqrt{3}e^{(1/6)K(\varphi,\varphi^*)},\quad \chi_{0R}=\frac{1}{3}\varphi_0\chi_R{}^I\frac{\partial K(\varphi,\varphi^*)}{\partial\varphi^I}\,.$$

the global symmetry G (or $\mathcal{G}-\mathcal{H}$ part, more precisely) is broken explicitly. This mechanism was in fact proposed by Ong,[6] although he mentioned incompletely only to the conformal chiral ($=R$-symmetry) anomaly.

There may be, however, a problem in this idea, since those anomalies imply that the nonlinear σ model cannot be defined globally on the manifold G/H. On a topologically nontrivial manifold G/H, the coordinate system ϕ^I covers only a portion of G/H. That is, we must cover G/H by patches $\{\theta_\alpha\}$ and the Kähler potential is defined patchwise: On the overlap regions $\theta_\alpha \cap \theta_\beta$, the Kähler potential K_α of θ_α is not equal to K_β of θ_β in general but is related by a Kähler transformation $K_\alpha - K_\beta = F_{\alpha\beta}(\phi) + F_{\alpha\beta}(\bar{\phi})$. Owing to the above stated superconformal anomalies, however, the Lagrangian is not invariant under the Kähler transformation and therefore the nonlinear σ model coupled to supergravity is not consistently defined globally on G/H. If this global problem is crucial for the consistency of the model itself, one cannot couple the nonlinear σ models to supergravity unless the superconformal anomalies are cancelled. It may, however, be meaninguful to consider the nonlinear σ model only within a patch θ which is extended as far as $K(\phi, \bar{\phi})$ remains nonsingular with given coordinate ϕ^I. If so, the model can be coupled to supergravity and the superconformal anomalies can be used peacefully as a source of G symmetry breaking as stated above.

Witten and Bagger[30] discussed a similar global consistency problem of the same system, putting the superconformal anomaly problem aside. They found that the global consistency of the conformal chiral transformation phase (Im$F(\phi)$ part) of (5·6) associated with the Kähler transformations on the overlap regions require that the Kähler manifold G/H be of restricted type (Hodge manifold). This constraint is met[31] fortunately for the present exceptional type manifolds $E_6/SO(10)\times U(1)$, $E_7/SU(5)\times SU(3)\times U(1)$ and $E_8/SO(10)\times SU(3)\times U(1)$ as was noticed by Iriê and Yasui.[7]

Similar global obstruction problem occurs already in rigid supersymmetry case in the nonlinear σ models, and is recently discussed by many authors[32),33] under the name "nonlinear σ model anomaly." (This anomaly is associated with the field dependent (i.e., local H transformation induced by the global G transformation. The difference with the usual gauge theory anomaly is only that the H gauge fields here are not elementary vectors but are given by certain functions of the NG boson fields.) Here also, if this anomaly is present, not only the global G symmetry is broken explicitly but the nonlinear σ model itself becomes ill-defined globally on G/H. There has been found no simple method to judge which supersymmetric nonlinear σ models have this type of anomalies. Recently, however, Moriya and Yasui[33] reported that $E_6/SO(10)\times U(1)$ is free of this anomaly but $E_7/SU(5)\times SU(3)\times U(1)$ and $E_8/SO(10)\times SU(3)\times U(1)$ are not. If their conclusion is true, then one has to introduce some modifications to the E_7 and E_8 models, such as to enlarge the dimension of the center of H or to introduce matter superfields other than NG ones. The above-mentioned standpoint to consider the nonlinear σ model only within one patch may also be good.

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Appendix A
—— *Properties of Spinor Representation of SO(2n)* ——

We summarize here some properties of spinor representation of $SO(6)$, $SO(10)$ and $SO(16)$, which are used in §§ 3 and 4 in the text, omitting the proofs. [See Refs. 34) and 35) for details.]

A.1. $SO(2n)$

The $SO(2n)$ spinor ψ has 2^n components and the $2n$ gamma matrices $\Gamma^\mu(\mu=1, 2, \cdots, 2n)$ are $2^n\times 2^n$ matrices satisfying the Clifford algebra:

$$\{\Gamma^\mu, \Gamma^\nu\}=2\delta^{\mu\nu}, \qquad \Gamma^{\mu\dagger}=\Gamma^\mu . \tag{A·1}$$

The complete set of $2^n\times 2^n$ matrices is spanned by the $\{\Gamma^{(f)}\}$, $f=0, 1, \cdots, 2n$, with

$$\Gamma^{(f)}\equiv i^{[f/2]}\Gamma^{\mu_1\mu_2\cdots\mu_f}\equiv i^{[f/2]}\Gamma^{[\mu_1}\Gamma^{\mu_2}\cdots\Gamma^{\mu_f]} ,$$
$$\Gamma^{(f)\dagger}=\Gamma^{(f)} , \qquad (\Gamma^{(f)})^2=1 , \tag{A·2}$$

where $[f/2]$ is the largest integer $\leq f/2$ and $[\mu_1, \cdots\mu_f]$ indicates antisymmetrization with "strength one."

Since $\pm\Gamma^{\mu*}$ form an equivalent representation of the Clifford algebra, the charge conjugation matrix C exist such that

$$\Gamma^\mu=\eta C^{-1}\Gamma^{\mu*}C \tag{A·3}$$

for either choice of $\eta=\pm1$. Further it can be shown that[34)]

$$C^\dagger C=1, \qquad C^T=\epsilon C \text{ with } \epsilon=\cos\frac{\pi}{2}n+\eta\sin\frac{\pi}{2}n , \tag{A·4}$$

$$C\Gamma^{(f)}=\epsilon\eta^f(-)^{[f/2]}[C\Gamma^{(f)}]^T . \quad (f=0, 1, \cdots, 2n) \tag{A·5}$$

In Table III, we summarize and the symmetry properties of $C\Gamma^{(f)}$ in various dimensions $2n$ implied by these equations.

Majorana spinor can exist only when $\epsilon=+1$ and is defined by

$$\psi^*=C\psi . \qquad (\epsilon=+1) \tag{A·6}$$

The usual γ_5 matrix analogue, Γ^{2n+1}, is definable:

$$\Gamma^{2n+1}\equiv i^n\Gamma^1\Gamma^2\cdots\Gamma^{2n} , \qquad (\Gamma^{2n+1})^2=1 \tag{A·7}$$

$$C^{-1}\Gamma^{2n+1*}C=(-)^n\Gamma^{2n+1} . \tag{A·8}$$

Weyl spinors $\psi_\pm$ with chirality ±1 is defined by

$$\psi_\pm=P_\pm\psi=\frac{1}{2}(1\pm\Gamma^{2n+1})\psi . \tag{A·9}$$

Table III. Sign factor ϵ and symmetry properties of $C\Gamma^{(f)}$ of $SO(2n)$.

choice of η	n (mod 4)	ϵ	$C\Gamma^{(f)}$ symmetric	$C\Gamma^{(f)}$ antisymmetric
$\eta=1$	0, 1	$+1$	$f=0, 1$ (mod 4)	$f=2, 3$ (mod 4)
	2, 3	-1	$f=2, 3$ (mod 4)	$f=0, 1$ (mod 4)
$\eta=-1$	0, 3	$+1$	$f=0, 3$ (mod 4)	$f=1, 2$ (mod 4)
	1, 2	-1	$f=1, 2$ (mod 4)	$f=0, 3$ (mod 4)

Majorana-Weyl spinor is defined by the equation

$$(P_{\pm}\psi)^* = CP_{\pm}\psi\,, \tag{A·10}$$

which has nonzero solution only if both conditions n=even (by (A·8)) and $\epsilon=+1$ (by (A·6)) are satisfied. That is possible only when $n=0$ (mod 4), i.e., $2n=8, 16, \cdots$.

We cite here a very useful Fierz identity in Ref. 35) which is needed to prove some Jacobi identities in § 3 in the text:

$$(\psi_1{}^\dagger\Gamma^{(\lambda)}\psi_2)\cdot(\phi_1{}^\dagger\Gamma^{(\lambda)}\phi_2) = \sum_{f=0}^{2n} a_{\lambda f}(\psi_1{}^\dagger\Gamma^{(f)}\phi_2)\cdot(\phi_1{}^\dagger\Gamma^{(f)}\psi_2)\,,$$
$$a_{\lambda f} = 2^{-n}(-)^{f\lambda}\{\text{coefficient of } x^\lambda \text{ in } (1+x)^{2n-f}(1-x)^f\}\,, \tag{A·11}$$

where $\Gamma^{(f)}\cdot\Gamma^{(f)} \equiv \sum_{\mu_1>\mu_2>\cdots>\mu_f} \Gamma^{\mu_1\cdots\mu_f}\Gamma^{\mu_1\cdots\mu_f}$.

A.2. *SO(6), SO(10) and SO(16)*

$SO(16)$ is a maximal subgroup of E_8, which we need decompose into $SO(10)\times SO(6)$ in § 3. For these three groups we fix to choose the following sign factors,

$$\begin{aligned} SO(16):&\quad \eta_{16}=-1\,, \quad \epsilon_{16}=+1\,,\\ SO(10):&\quad \eta_{10}=-1\,, \quad \epsilon_{10}=-1\,,\\ SO(6):&\quad \eta_{6}=+1\,, \quad \epsilon_{6}=-1\,, \end{aligned} \tag{A·12}$$

so as to satisfy $\eta_{16}=\eta_{10}\cdot\eta_6$ since it is necessary when the gamma matrices of $SO(16)$ are constructed by a tensor product of those of $SO(10)$ and of $SO(6)$. The sign factors ϵ in (A·12) are read from Table III.

Consider $SO(6)$ and $SO(10)$ first. We take (chirality)Γ^{2n+1} diagonal representation $\Gamma^{2n+1}=\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, and write spinor ψ by the chirality ±1 components ${}^t(\xi_{\tilde{a}}, \eta^{\tilde{a}})$ with lower and upper spinor indices $\tilde{a}(=1\sim2^{n-1})$, respectively. The gamma matrices are block off-diagonal in this representation since they change chirality. Using different letters for indices of $SO(6)$ and $SO(10)$, we denote them by

$$SO(6):\quad \Gamma_6{}^a=\begin{pmatrix} 0 & (\sigma_a)_{\hat{i}\hat{j}} \\ (\sigma_a{}^\dagger)^{\hat{i}\hat{j}} & 0 \end{pmatrix}, \qquad \sigma_a{}^T=-\sigma_a\,, \quad a=1\sim6, \quad \hat{i},\hat{j}=1\sim4\,, \tag{A·13}$$

$$SO(10):\quad \Gamma_{10}^A=\begin{pmatrix} 0 & (\sigma_A)_{\alpha\beta} \\ (\sigma_A{}^\dagger)^{\alpha\beta} & 0 \end{pmatrix}, \qquad \sigma_A{}^T=\sigma_A\,, \quad A=1\sim10, \quad \alpha,\beta=1\sim16\,. \tag{A·14}$$

The charge conjugation matrix C also changes chirality when n is odd by (A·8), i.e., changes ξ into η and vice versa. Changing the definition of the base η relative to ξ in such cases, we can always bring C into the form $\begin{pmatrix} & 1 \\ \pm 1 & \end{pmatrix}$ depending on the sign $\epsilon = \pm 1$. In our case $\epsilon_6 = \epsilon_{10} = -1$, and so we have

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{for both } SO(6) \text{ and } SO(10)\,. \tag{A·15}$$

The $SO(16)$ gamma matrices $\Gamma_{16}^{\hat{A}}(\hat{A}=1\sim16)$ can be constructed by a tensor product of those of $SO(10)$, (A·14), and of $SO(6)$, (A·13):

$$\begin{aligned} &\Gamma_{16}^{\hat{1}\sim\hat{10}} = \Gamma_{10}^{1\sim10}\otimes 1\,, \\ &\Gamma_{16}^{\hat{11}\sim\hat{16}} = \Gamma_{10}^{11}\otimes\Gamma_6^{1\sim6}\,. \end{aligned} \tag{A·16}$$

From this the chirality multiplication rule follows: $\Gamma_{16}^{17} = \Gamma_{10}^{11}\otimes\Gamma_6^{7}$. So is the charge conjugation matrix, $C_{16} = C_{10}\otimes C_6$. Hence now the charge conjugation matrix C_{16} commutes with chirality matrix Γ_{16}^{17} and therefore is block diagonal; more explicitly, from (A·15),

$$C_{16} = \left(\begin{array}{cc:c} 0 & \delta_\alpha{}^\beta\delta_{\tilde{i}}{}^{\tilde{j}} & \\ \delta^\alpha_\beta\delta_{\tilde{j}}{}^{\tilde{i}} & 0 & \\ \hdashline & & * \end{array}\right), \tag{A·17}$$

where we have shown explicitly only the positive Γ^{17} chirality sector since we are interested only in the Majorana-Weyl spinor generators $E_{\tilde{\alpha}}$ of E_8 with $\Gamma^{17} = +1$ in the text. Notice that $\Gamma^{17} = +1$ Weyl spinor is given as $(\xi_\alpha\xi_{\tilde{i}},\ \eta^\alpha\eta^{\tilde{i}})$ in terms of $SO(10)$ and $SO(6)$ spinors.

The generators $\Sigma^{\hat{A}\hat{B}}$ of $SO(16)$ group are defined by $[\Gamma_{16}^{\hat{A}}, \Gamma_{16}^{\hat{B}}]/4$ and have the following form in terms of $SO(10)$ and $SO(6)$ gamma matrices:

$$\begin{aligned} &\Sigma_{16}^{AB} = \Sigma_{10}^{AB}\otimes 1\,, \qquad \Sigma_{16}^{ab} = 1\otimes\Sigma_6{}^{ab}\,, \\ &\Sigma_{16}^{Aa} = \frac{1}{2}\left(\begin{array}{cc:c} 0 & -(\sigma^A)_{\alpha\beta}(\sigma^a)_{\tilde{i}\tilde{j}} & 0 \\ (\sigma_A{}^\dagger)^{\alpha\beta}(\sigma_a{}^\dagger)^{\tilde{i}\tilde{j}} & 0 & \\ \hdashline & 0 & * \end{array}\right), \end{aligned} \tag{A·18}$$

where Σ_{10}^{AB} is the $SO(10)$ generators

$$\Sigma_{10}^{AB} = \begin{pmatrix} (\sigma^{AB})_\alpha{}^\beta & \\ & (\bar{\sigma}^{AB})^\alpha{}_\beta \end{pmatrix}, \qquad \begin{aligned} &\sigma^{AB} = \frac{1}{4}(\sigma^A\sigma^{B\dagger} - \sigma^B\sigma^{A\dagger})\,, \\ &\bar{\sigma}^{AB} = \frac{1}{4}(\sigma^{A\dagger}\sigma^B - \sigma^{B\dagger}\sigma^A)\,, \end{aligned} \tag{A·19}$$

and similar expressions for the $SO(6)$ generators $\Sigma_6{}^{ab}$.

Finally we note a relation

$$(\sigma_a)_{\tilde{i}\tilde{j}} = \frac{1}{2}\epsilon_{\tilde{i}\tilde{j}\tilde{k}\tilde{l}}(\sigma_a{}^\dagger)^{\tilde{k}\tilde{l}} \tag{A·20}$$

for the $SO(6)$ matrices σ_a, which implies that the representation **6** of $SO(6)$ is a self-dual representation $\boxminus$ of $SU(4)$ ($\simeq SO(6)$), and an identity of the $SO(10)$ matrices:

$$\Sigma_{\alpha\beta}^{\gamma\delta}=\Sigma_{\beta\alpha}^{\gamma\delta}=\Sigma_{\alpha\beta}^{\delta\gamma}$$

$$\text{for}\quad \Sigma_{\alpha\beta}^{\gamma\delta}\equiv(\sigma_{AB})_\alpha{}^\gamma(\sigma_{AB})_\beta{}^\delta-\frac{3}{2}\delta_\alpha{}^\gamma\delta_\beta{}^\delta\,. \tag{A$\cdot$21}$$

This identity is necessary to check the Jacobi identity consistency of E_6 algebra, and can be proven by using the Fierz identity (A·11) for the cases $\lambda=0$ and 2.

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