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Supersymmetric Nonlinear Sigma Models as Gauge Theories

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Supersymmetric nonlinear sigma models are obtained from linear sigma models by imposing supersymmetric constraints. If we introduce auxiliary chiral and vector superfields, these constraints can be expressed by D-terms and F-terms depending on the target manifolds. Auxiliary vector superfields appear as gauge fields without kinetic terms. If there are no D-term constraints, the target manifolds are always non-compact manifolds. When all the degrees of freedom in these non-compact directions are eliminated by gauge symmetries, the target manifold becomes compact. All supersymmetric nonlinear sigma models, whose target manifolds are hermitian symmetric spaces, are successfully formulated as gauge theories.

§1. Introduction

When the global symmetry G is spontaneously broken down to its subgroup H, there appear massless Nambu-Goldstone (NG) bosons corresponding to broken generators of the coset manifold G/H. At low energies, interactions among these massless particles are described by the so-called nonlinear sigma models, whose lagrangians are completely determined by the geometry of the target manifold G/H, parameterized by NG-bosons. [1)]

In supersymmetric theories, there appear massless fermions as supersymmetric partners of NG-bosons. [2)] These massless fermions together with NG-bosons are described by chiral superfields in four-dimensional theories with $N = 1$ supersymmetry. Since chiral superfields are complex, the supersymmetric nonlinear sigma models are closely related to the complex geometry; their target manifolds, where field variables take their values, must be Kähler manifolds. [3)] If the coset manifold G/H itself happens to be a Kähler manifold, both real and imaginary parts of the scalar components of chiral superfields are NG-bosons. If G/H is not a Kähler manifold, on the other hand, there is at least one chiral superfield whose real or imaginary part is not a NG-boson. This additional massless boson is called the quasi-Nambu-Goldstone (QNG) boson. [2),4)]

The general method to construct supersymmetric nonlinear sigma models has been discussed by Bando, Kuramoto, Maskawa and Uehara (BKMU). [5)] When QNG bosons are present, their effective lagrangians include arbitrary functions. This is always the case when the target manifold of the nonlinear sigma model is larger than the coset manifold G/H, where NG-bosons reside, since the geometry of the target manifold cannot be fixed by the metric of its subspace G/H. [6)-10)] The arbitrariness

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reflects the ambiguity of the metric in the direction of QNG bosons. When the coset manifold G/H is itself Kähler, the effective lagrangian is uniquely determined by the geometry of G/H, as has been shown in a beautiful paper by Itoh, Kugo and Kunitomo.[11] (See Appendix A for a review.) Kähler potentials in this case have been discussed by many authors[5),11)-18)] (See references in Ref. 18).), and have been used to construct the coset unification models, where fermionic partners of NG bosons are considered as quarks.[19]

Nonlinear sigma models are considered low energy effective theories for massless particles after integrating out the massive particles in the corresponding linear sigma models. In this context, Lerche and Shore have shown that nonlinear sigma models whose target manifolds are Kähler G/H manifolds cannot be obtained from linear sigma models.[6] (See also Ref. 7) and Appendix B for a review.) According to this theorem, there must exist at least one QNG bosons in effective field theories obtained from linear sigma models.

On the other hand, it is known that sigma models on some Kähler G/H manifolds, namely on $\boldsymbol{C}P^N$ or on the Grassmann manifold $G_{N,M}(\boldsymbol{C})$, are obtained by the introduction of gauge symmetry.[12),20)-22)] The implicit assumption of Lerche and Shore is the absence of gauge interactions in the linear sigma models. It seems possible to eliminate unnecessary QNG bosons if we introduce an appropriate gauge symmetry.

In this paper, we show that supersymmetric nonlinear sigma models on a certain class of Kähler G/H manifolds are obtained from linear sigma models with gauge symmetry. We define nonlinear sigma models by imposing supersymmetric constraints on linear sigma models. We introduce two kinds of constraints, D-term and F-term constraints. If we introduce auxiliary fields, these correspond to vector and chiral superfields. Vector auxiliary superfields appear as gauge fields. We successfully formulate nonlinear sigma models on (irreducible and compact) hermitian symmetric spaces[*] classified by Cartan as in Table I.[24),**]

This paper is organized as follows. In §2, we review simple cases without F-term constraints, namely the projective space $\boldsymbol{C}P^{N-1}$ and the Grassmann manifold $G_{N,M}(\boldsymbol{C})$. Although these cases are known, it is instructive to discuss them with emphasis on an interpretation in terms of NG and QNG bosons. In §3, we generalize to other hermitian symmetric spaces by introducing F-term constraints in addition to D-term constraints. Results in this section are new. As a by-product, we find explicit expressions of holomorphic constraints to embed G/H into $\boldsymbol{C}P^N$ or $G_{N,M}(\boldsymbol{C})$. Section 4 is devoted to conclusions and discussion. We discuss how the results can be generalized to an arbitrary Kähler G/H manifold. In Appendix A, we review the construction of the Kähler potentials for Kähler G/H using BKMU and IKK methods, in the case of hermitian symmetric spaces. In Appendix B, we review the theorem of Lerche and Shore. Appendices C, D and E are devoted to summaries of

[*] Symmetric spaces are homogeneous spaces G/H with an involutive automorphism. Since it can be shown that any G-invariant differential form ω in a symmetric space is closed, $d\omega = 0$, a fundamental two form of a hermitian symmetric space is also closed, and this is Kählerian. Hence, the expression "Kähler symmetric space" has the same meaning.

[**] We use '$\dim_{\boldsymbol{C}}$' for complex dimensions and 'dim' for real dimensions.

Table I. Hermitian symmetric spaces. The first three manifolds, $\boldsymbol{C}P^{N-1}$, $G_{N,M}(\boldsymbol{C})$ and $Q^{N-2}(\boldsymbol{C})$, are called a projective space, a Grassmann manifold and a quadratic surface, respectively. The projective space $\boldsymbol{C}P^{N-1}$ and the Grassmann manifold $G_{N,M}(\boldsymbol{C})$ are a set of complex lines and M-dimensional complex planes in $\boldsymbol{C}^N$, respectively. BI (DI) corresponds to odd (even) N. In the mathematical literature, EIII is written as $E_6/\mathrm{Spin}(10)\times U(1)$, since coset generators belong to the $SO(10)$ Weyl spinor.

Type	G/H	$\dim_{\boldsymbol{C}}(G/H)$
AIII_1	$\boldsymbol{C}P^{N-1} = SU(N)/S(U(N-1)\times U(1))$	$N-1$
AIII_2	$G_{N,M}(\boldsymbol{C}) = U(N)/U(N-M)\times U(M)$	$M(N-M)$
BDI	$Q^{N-2}(\boldsymbol{C}) = SO(N)/SO(N-2)\times U(1)$	$N-2$
CI	$Sp(N)/U(N)$	$\frac{1}{2}N(N+1)$
DIII	$SO(2N)/U(N)$	$\frac{1}{2}N(N-1)$
EIII	$E_6/SO(10)\times U(1)$	16
EVII	$E_7/E_6\times U(1)$	27

SO, E_6 and E_7 algebras.

In the rest of this section, we introduce the notation and terminology used in this paper.

The linear description of the nonlinear sigma model without a gauge symmetry is given by

$$\mathcal{L} = \int d^4\theta \phi^\dagger \phi + \left(\int d^2\theta \phi_0 g(\phi) + \text{c.c.}\right), \tag{1·1}$$

where the chiral superfield ϕ belongs to an irreducible representation of the global symmetry group G, and ϕ_0 is an auxiliary chiral superfield. The absence of kinetic term of ϕ_0 corresponds to the strong coupling limit of the Yukawa theory. Although the superpotential $W = \phi_0 g(\phi)$ is G-invariant, ϕ_0 and $g(\phi)$ need not be G-invariant separately. Instead, they may have indices transforming as a non-trivial representation of G, such as $W = \phi_{0i} g(\phi)^i$. If we integrate over the auxiliary field ϕ_0, we obtain F-term constraints, $g(\phi) = 0$, which are holomorphic functions. Therefore, the F-term constraints are invariant under the larger group $G^{\boldsymbol{C}}$, the complex extension of G.

Let the number of F-term constraints be N_{F}. If it is sufficiently large, the target manifold M' becomes a $G^{\boldsymbol{C}}$-orbit of the vacuum $v = \langle\phi\rangle$. Let the complex isotropy group of the vacuum be $\hat{H}$ ($\hat{H}v = v$). Then, the target manifold of the nonlinear sigma model is parameterized by the chiral superfields corresponding to complex broken generators in $\mathcal{G}^{\boldsymbol{C}} - \hat{\mathcal{H}}$.*) Therefore M' is a complex coset space, $M' \simeq G^{\boldsymbol{C}}/\hat{H}$, generated by these broken generators. As an example, let us consider a doublet $\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ of $G = SU(2)$ and suppose that they acquire the vacuum expectation values $v = \begin{pmatrix}1\\ 0\end{pmatrix}$. Since the raising operator $\tau_+ = \frac{1}{2}(\tau_1 + i\tau_2) = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ satisfies $\tau_+ v = 0$, it is the complex unbroken generator in $\hat{\mathcal{H}}$. On the other hand, τ_3 and the lowering operator $\tau_-(= {\tau_+}^\dagger)$ are the elements of the broken generators in $\mathcal{G}^{\boldsymbol{C}} - \hat{\mathcal{H}}$.

There are two kinds of broken generators: the hermitian broken generator X

*) We use the calligraphic font for a Lie algebra corresponding to a Lie group.

and the non-hermitian broken generator E.*) The superfields corresponding to non-hermitian and hermitian generators are called pure-type and mixed-type superfields, respectively.[5),6)] In the previous example, where the representative of $G^{\boldsymbol{C}}/\hat{H}$ is given by $\phi = \exp i(\varphi_3\tau_3 + \varphi\tau_-)\cdot v$, φ_3 is a mixed-type and φ is a pure-type superfield. The scalar components of the mixed-type multiplets consist of a QNG boson in addition to a NG boson, whereas those of the pure-type multiplets consist of two genuine NG bosons. Since the vacuum is invariant under $\hat{H}$, we can multiply the representative of the coset manifold by an arbitrary element of $\hat{H}$ from the right. In our previous example, we can rewrite it as $\exp i(\varphi_3\tau_3 + \Re\varphi\tau_1 + \Im\varphi\tau_2)\cdot v$ by multiplying an appropriate factor generated by τ_+ for sufficiently small $|\varphi_3|$ and $|\varphi|$. The NG-bosons parameterizing $S^3 \simeq SU(2)$ are $\Re\varphi_3, \Re\varphi$ and $\Im\varphi$, whereas $\Im\varphi_3$ is the QNG-boson parameterizing the radius of S^3. The number of chiral superfields parameterizing the target manifold is

$$N_\Phi = \dim_{\boldsymbol{C}} V - N_{\rm F} = N_{\rm M} + N_{\rm P}, \tag{1·2}$$

where V is the representation space. The numbers of the mixed-type and pure-type multiplets are denoted by $N_{\rm M}$ and $N_{\rm P}$, respectively.

The directions parameterized by QNG bosons are non-compact, whereas those of NG bosons are compact.**) From the theorem of Lerche and Shore (see Appendix B), there exists at least one mixed-type multiplet, and therefore the target manifold M' becomes non-compact. Since no two points in the non-compact direction can be connected by the compact isometry group G, M' is also non-homogeneous.

We rewrite the groups G and H defined above as G' and H', and therefore $M' \simeq G'^{\boldsymbol{C}}/\hat{H}'$. In order to eliminate the degree of freedom of QNG bosons, we elevate the subgroup of G' to a local gauge symmetry. We assume G' is the direct product of a global symmetry and the gauge symmetry $G_{\rm gauge}$; that is $G' = G \times G_{\rm gauge}$, where $G_{\rm gauge} = U(1)$ or $U(N)$. The gauged linear lagrangian can be written as

$$\mathcal{L} = \int d^4\theta \left(e^V \phi^\dagger \phi - cV\right) + \left(\int d^2\theta \phi_0 g(\phi) + \text{c.c.}\right), \tag{1·3}$$

where ϕ_0 and V are auxiliary chiral and vector superfields. The absence of the kinetic term of the gauge field corresponds to the strong coupling limit, where the gauge coupling constant tends to infinity. Here, for simplicity, the gauge group is assumed to be $U(1)$. (See §2.2 for the non-Abelian case.) Integration over ϕ_0 gives the F-term constraint to define the non-compact manifold M', as discussed above. The integration over V gives a D-term constraint that restricts M' to the compact manifold $M = M'/G^{\boldsymbol{C}}_{\rm gauge}$,[21)] whose dimension is

$$\dim_{\boldsymbol{C}} M = N_\Phi - \dim G_{\rm gauge}. \tag{1·4}$$

*) In general, $\hat{H}$ is larger than $H^{\boldsymbol{C}}$, due to the existence of non-hermitian generators $\bar{E}$. $\bar{E}$ is the hermitian conjugate of E. They constitute the so-called Borel subalgebra $\mathcal{B}$ in $\hat{\mathcal{H}}$.[5)]

**) We use the word "compactness" in the sense of topology. The kinetic terms of QNG bosons have the same sign as those of NG bosons.

Since we introduce gauge fields to absorb all mixed type multiplets,[*)] the dimension of the gauge group and the compact manifold M are

$$\dim G_{\mathrm{gauge}} = N_{\mathrm{M}}, \qquad \dim_{\boldsymbol{C}} M = N_{\mathrm{P}}. \tag{1·5}$$

The compact manifold M is parameterized by only pure-type multiplets.

§2. Nonlinear sigma models without the F-term constraint

Although examples in this section are well known,[21)] we describe them in detail, since the interpretation in terms of NG and QNG bosons is useful to find the nonlinear sigma models on other compact manifolds.

2.1. *Projective space*: $\boldsymbol{C}P^{N-1} = SU(N)/S(U(N-1)\times U(1))$

We consider the global symmetry $G' = U(N) = SU(N)\times U(1)_{\mathrm{D}} \stackrel{\mathrm{def}}{=} G\times U(1)_{\mathrm{D}}$. Below, the phase symmetry $U(1)_{\mathrm{D}}$ is gauged, while $G = SU(N)$ remains global. We prepare the fundamental fields $\vec{\phi} \in \boldsymbol{N}$, which acquire a vacuum expectation value. First of all, we consider the canonical Kähler potential

$$K(\vec{\phi}, \vec{\phi}^\dagger) = \vec{\phi}^\dagger \vec{\phi}. \tag{2·1}$$

For later purposes, we decompose $G = SU(N)$ under the subgroup $SU(N-1)\times U(1)$. A fundamental representation $\boldsymbol{N}$ is decomposed as $\boldsymbol{N} = (\boldsymbol{N-1}, 1) \oplus (\boldsymbol{1}, -N+1)$, where the second factors are $U(1)$ charges. Hence, we decompose the fields as $\vec{\phi} = \begin{pmatrix} x \\ y^i \end{pmatrix}$ $(i = 1, \cdots, N-1)$. Generators of $SU(N)$ can also be decomposed into the $SU(N-1)$ generators T_A $(A = 1, \cdots, N^2-2N)$, the $U(1)$ generator T, the $N-1$ raising operators E^i represented by upper triangle matrices, and the lowering operators represented by lower triangle matrices $\bar{E}_i = (E^i)^\dagger$. The transformation law of $\vec{\phi}$ under the complexified group $SU(N)^{\boldsymbol{C}}$ is

$$\begin{aligned}\delta\vec{\phi} &= \left(i\theta T + i\theta^A T_A + \bar{\epsilon}_i E^i + \epsilon^i \bar{E}_i\right)\vec{\phi} \\ &= \begin{pmatrix} i\sqrt{\frac{2(N-1)}{N}}\theta & \bar{\epsilon}_j \\ \epsilon^i & -i\theta^A \rho(T_A)^i{}_j - i\sqrt{\frac{2}{N(N-1)}}\theta\delta^i{}_j \end{pmatrix}\begin{pmatrix} x \\ y^j \end{pmatrix},\end{aligned} \tag{2·2}$$

where $\rho(T_A)$ is an $N-1$ by $N-1$ matrix for the fundamental representation of $SU(N-1)$. We normalized these generators as $\operatorname{tr} {T_A}^2 = \operatorname{tr} T^2 = \operatorname{tr} \bar{E}_i E^i = 2$ (no sum). When $\epsilon = \bar{\epsilon}$ and $\theta, \theta^A \in \boldsymbol{R}$, this transformation law reduces to that of the real group $SU(N)$. The $U(1)_{\mathrm{D}}$ transformation is generated by $T_{\mathrm{D}} = \boldsymbol{1}_N$.

When $\vec{\phi}$ acquires a vacuum expectation value, it can be transformed by $G'^{\boldsymbol{C}}$ to the standard form,

$$\vec{v} = \left\langle \vec{\phi} \right\rangle = \begin{pmatrix} 1 \\ \boldsymbol{0} \end{pmatrix}. \tag{2·3}$$

*) The supersymmetric Higgs mechanism acts as follows: A vector superfield absorbs one mixed-type multiplet to constitute a massive vector multiplet. If it absorbs a pure-type multiplet, one NG boson remains massless. They cannot constitute a massive vector multiplet, and the supersymemtry is spontaneously broken.[31)]

By this vacuum, the global symmetry is spontaneously broken down as $U(N) \to U(N-1) = SU(N-1) \times U(1)' \stackrel{\text{def}}{=} H'$. Here, $U(1)'$ is generated by $T' \sim \text{diag}(0, 1, \cdots, 1)$, which is a linear combination of $T_{\rm D}$ and T. The complex isotropy group $\hat{H}'$, which leaves $\vec{v}$ invariant, is larger than $H'^{\boldsymbol{C}}$, since upper triangle generators E^i annihilate the vacuum $\vec{v}$. Here, E^i generators constitute a Borel subalgebra $\mathcal{B}$ in $\hat{\mathcal{H}}'$. On the other hand, the complex broken generators are the lower triangle generators $\bar{E}_i$ and the diagonal generator $X = (1, 0, \cdots, 0)$, which is also a linear combination of T and $T_{\rm D}$. The non-hermitian generators $\bar{E}_i$ are pure-type generators, and the hermitian generator X is a mixed-type generator. The target manifold M' of the nonlinear sigma model is a complex coset manifold $M' \simeq G'^{\boldsymbol{C}}/\hat{H}'$ generated by these complex broken generators. Since, by using its representative $\xi' = \exp(\varphi^i \bar{E}_i + i\psi X)$, the fields can be written as $\vec{\phi} = \xi' \vec{v}$, its form near the vacuum is

$$\delta \vec{v} = (i\psi X + \varphi^i \bar{E}_i)\vec{v} = \begin{pmatrix} i\psi \\ \varphi^i \end{pmatrix}. \tag{2·4}$$

We thus find that ψ is a mixed-type chiral superfield, whose scalar components are NG and QNG bosons, while the φ^i are pure-type chiral superfields, whose scalar components are both NG bosons. Then the numbers of mixed-type and pure-type chiral superfields are $N_{\rm M} = 1$ and $N_{\rm P} = N - 1$, respectively. This Kähler manifold is non-compact and non-homogeneous due to the existence of the QNG boson.

To construct a compact homogeneous manifold, we wish to eliminate the QNG boson (the mixed-type multiplet). Hence, we gauge the $U(1)_{\rm D}$ symmetry by introducing a vector superfield V, which will absorb the mixed-type multiplet. The gauged Kähler potential is [31)]

$$K(\vec{\phi}, \vec{\phi}^\dagger, V) = e^V \vec{\phi}^\dagger \vec{\phi} - cV, \tag{2·5}$$

where cV is a Fayet-Iliopoulos (FI) D-term. [21),31)] Since the transformation law of V is

$$e^V \to e^{V'} = e^V e^{i(\theta^\dagger - \theta)}, \qquad e^{i\Re\theta} \in U(1)_{\rm D}, \tag{2·6}$$

where θ is a chiral superfield, the Kähler potential (2·5) is invariant under the complex extension of the gauge symmetry, $U(1)_{\rm D}{}^{\boldsymbol{C}}$. Note that the global symmetry $G = SU(N)$ cannot be complexified. The equation of motion of V is

$$\delta K/\delta V = e^V \vec{\phi}^\dagger \vec{\phi} - c = 0. \tag{2·7}$$

From this equation, V can be solved as

$$V(\vec{\phi}, \vec{\phi}^\dagger) = -\log\left(\frac{\vec{\phi}^\dagger \vec{\phi}}{c}\right). \tag{2·8}$$

To eliminate the gauge field, we substitute $V(\vec{\phi}, \vec{\phi}^\dagger)$ back into Eq. (2·5), obtaining

$$K(\vec{\phi}, \vec{\phi}^\dagger, V(\vec{\phi}, \vec{\phi}^\dagger)) = c\log(\vec{\phi}^\dagger \vec{\phi}), \tag{2·9}$$

where we have omitted constant terms.*) Since we have the gauge symmetry $U(1)_{\rm D}{}^{\boldsymbol{C}}$, we can fix the gauge as

$$\vec{\phi} = \begin{pmatrix} 1 \\ \varphi^i \end{pmatrix}. \tag{2·10}$$

By comparing Eqs. (2·4) and (2·10), we find that the mixed-type chiral superfield has been eliminated by this gauge fixing. The gauge fixed field (2·10) can be rewritten as

$$\vec{\phi} = \xi\vec{v}, \qquad \xi = e^{\varphi\cdot\bar{E}} = \begin{pmatrix} 1 & \mathbf{0} \\ \varphi^i & \mathbf{1}_{N-1} \end{pmatrix}, \tag{2·11}$$

where ξ can be considered as a representative of a complex coset manifold $G^{\boldsymbol{C}}/\hat{H} \simeq G/H = SU(N)/S(U(N-1)\times U(1))$. Since this is a compact homogeneous Kähler manifold, we have obtained the desired result. To obtain a compact manifold, gauge fields are necessary. By substituting Eq. (2·10) into Eq. (2·9), we obtain

$$K(\varphi,\varphi^\dagger,V(\varphi,\varphi^\dagger)) = c\log(1+|\varphi|^2). \tag{2·12}$$

This is the well-known Kähler potential of the Fubini-Study metric for $\boldsymbol{C}P^{N-1} = SU(N)/S(U(N-1)\times U(1))$.

2.2. *Grassmann manifold*: $G_{N,M}(\boldsymbol{C}) = U(N)/U(N-M)\times U(M)$

This subsection is a generalization of the last subsection. The picture of NG and QNG bosons is discussed in Ref. 22). We consider a global symmetry $G' = G_{\rm L}\times G_{\rm R} = U(N)_{\rm L}\times U(M)_{\rm R}\,(N>M)$. The basic fields are $\Phi \in (\boldsymbol{N},\overline{\boldsymbol{M}})$, which are $N\times M$ matrix-valued chiral superfields. The transformation law of Φ under $G'^{\boldsymbol{C}}$ is**)

$$\Phi \to \Phi' = g\cdot\Phi \overset{\rm def}{=} g_{\rm L}\Phi {g_{\rm R}}^{-1}, \qquad g=(g_{\rm L},g_{\rm R})\in G'^{\boldsymbol{C}}, \tag{2·13}$$

where $g_{\rm L}$ and $g_{\rm R}$ are $N\times N$ and $M\times M$ matrices, respectively.

The Kähler potential is canonical:

$$K(\Phi,\Phi^\dagger) = {\rm tr}\,(\Phi^\dagger\Phi). \tag{2·14}$$

Any vacuum can be transformed under $G'^{\boldsymbol{C}}$ to

$$V = \langle\Phi\rangle = \begin{pmatrix} \mathbf{1}_M \\ \mathbf{0} \end{pmatrix}, \tag{2·15}$$

where $\mathbf{1}_M$ is the $M\times M$ identity matrix and $\mathbf{0}$ is the $(N-M)\times M$ zero matrix. The global symmetry is spontaneously broken as $U(N)_{\rm L}\times U(M)_{\rm R}\to U(N-M)_{\rm L}\times U(M)_{\rm V}$. Here, $U(N-M)_{\rm L}$ is the group generated by $\left(\begin{pmatrix}\mathbf{0}_M & \mathbf{0}\\ \mathbf{0} & T\end{pmatrix},\mathbf{0}_M\right)\in(\mathcal{G}_{\rm L},\mathcal{G}_{\rm R})$,

*) Their contributions to the lagrangian vanish as a result of the $d^4\theta$ integration.

**) The conjugate representation $\vec{\phi}\in\overline{\boldsymbol{N}}$ is defined to transform as $\vec{\phi}\to(g^{-1})^T\vec{\phi}$, since the group is extended to its complexification and we must preserve the chirality.

where T are $(N-M)\times(N-M)$ matrices, and $U(M)_{\rm V}$ is generated by $\left(\begin{pmatrix} T & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{N-M}\end{pmatrix}, T\right) \in (\mathcal{G}_{\rm L}, \mathcal{G}_{\rm R})$, where T are $M\times M$ matrices. The complex isotropy $\hat{\mathcal{H}}'$ that leaves $\langle \Phi \rangle$ invariant is larger than $\mathcal{H}'^{\boldsymbol{C}}$ by $E \stackrel{\rm def}{=} \left(\begin{pmatrix} \mathbf{0}_M & T \\ \mathbf{0} & \mathbf{0}_{N-M}\end{pmatrix}, \mathbf{0}_M\right)$, where T are $M\times(N-M)$ matrices. Here, these E constitute a Borel subalgebra $\mathcal{B}$ of $\hat{\mathcal{H}}'$, and its dimension is $\dim_{\boldsymbol{C}} \mathcal{B} = M(N-M)$. On the other hand, the complex broken generators consist of non-hermitian generators, $\bar{E} \stackrel{\rm def}{=} \left(\begin{pmatrix} \mathbf{0}_M & 0 \\ T & \mathbf{0}_{N-M}\end{pmatrix}, \mathbf{0}_M\right)$, which are hermitian conjugates of E, and hermitian generators, $X \stackrel{\rm def}{=} \left(\begin{pmatrix} T & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{N-M}\end{pmatrix}, -T\right)$, which are elements of an axial symmetry $U(M)_{\rm A}$. The target manifold is a complex coset manifold $M' \simeq G'^{\boldsymbol{C}}/\hat{H}'$, and its representative is $\xi' = \exp(\varphi\cdot\bar{E} + i\psi\cdot X) \stackrel{\rm def}{=} ({\xi'}_{\rm L}, {\xi'}_{\rm R})$. The field can be written as $\vec{\phi} = \xi'\cdot V = {\xi'}_{\rm L} V {{\xi'}_{\rm R}}^{-1}$. Its form near the vacuum is

$$\delta V = \begin{pmatrix} 2i\psi \\ \varphi \end{pmatrix}. \qquad (2\cdot 16)$$

Here, ψ is an $M\times M$ matrix chiral superfield considered as mixed types and φ is an $(N-M)\times M$ matrix chiral superfield considered as pure types. Hence, the numbers of mixed-type and pure-type chiral superfields are $N_{\rm M} = M^2$ and $N_{\rm P} = M(N-M)(= \dim_{\boldsymbol{C}} \mathcal{B})$, respectively.

To absorb the M^2 mixed-type chiral superfields, we gauge $U(M)_{\rm R}$ by introducing M^2 vector superfields $V = V^A T_A$, where T_A represents generators of $U(M)_{\rm R}$. The gauged Kähler potential is

$$K(\Phi, \Phi^\dagger, V) = {\rm tr}\,(\Phi^\dagger \Phi e^V) - c\,{\rm tr}\,V, \qquad (2\cdot 17)$$

where $c\,{\rm tr}\,V$ is a Fayet-Iliopoulos D-term. Since the vector superfields are transformed as

$$e^V \to e^{V'} = g_{\rm R} e^V {g_{\rm R}}^\dagger, \qquad (2\cdot 18)$$

the gauged Kähler potential is invariant under the complexified gauge symmetry ${G_{\rm R}}^{\boldsymbol{C}}$. To eliminate vector superfields, we use the equation of motion of V,*)

$$\delta K/\delta V = \Phi^\dagger \Phi e^V - c\mathbf{1}_M = 0. \qquad (2\cdot 19)$$

Then V can be solved as

$$V(\Phi, \Phi^\dagger) = -\log\left(\frac{\Phi^\dagger \Phi}{c}\right). \qquad (2\cdot 20)$$

By substituting this into Eq. (2·17), we obtain

$$K(\Phi, \Phi^\dagger, V(\Phi, \Phi^\dagger)) = c\,{\rm tr}\,\log(\Phi^\dagger\Phi) = c\log\det(\Phi^\dagger\Phi), \qquad (2\cdot 21)$$

where we have omitted constant terms. We choose the gauge fixing as

$$\Phi = \begin{pmatrix} \mathbf{1}_M \\ \varphi \end{pmatrix}, \qquad (2\cdot 22)$$

*) We treat $e^{-V}\delta e^V$ as an infinitesimal parameter, since $\delta{\rm tr}\,(\Phi^\dagger\Phi e^V) = {\rm tr}\,(\Phi^\dagger\Phi e^V(e^{-V}\delta e^V))$. The second term is obtained from ${\rm tr}\,(\delta\log X) = {\rm tr}\,(X^{-1}\delta X)$, where $X = e^V$.

where φ is an $(N-M)\times M$ matrix-valued chiral superfield. By comparing Eq. (2·16) and Eq. (2·22), we find that all mixed-type multiplets ψ have disappeared by this gauge fixing condition. When ξ is a representative of $G^{\boldsymbol{C}}/\hat{H} = U(N)/U(N-M)\times U(M)$, $\varPhi$ can be rewritten as

$$\varPhi = \xi \cdot V = \xi_{\rm L} V {\xi_{\rm R}}^{-1}, \quad \xi = e^{\varphi\cdot\bar{E}} = \left(\begin{pmatrix} \mathbf{1}_M & 0 \\ \varphi & \mathbf{1}_{N-M} \end{pmatrix}, \ \mathbf{1}_M \right) = (\xi_{\rm L}, \xi_{\rm R}). \quad (2·23)$$

Since the target space M is parameterized solely by pure-type multiplets, it is a compact homogeneous Kähler manifold. By substituting Eq. (2·22) into Eq. (2·21), we obtain the Kähler potential of M:

$$K(\varphi, \varphi^\dagger, V(\varphi, \varphi^\dagger)) = c \log \det(\mathbf{1}_M + \varphi^\dagger \varphi). \quad (2·24)$$

This is the Kähler potential of the Grassmann manifold $G_{N,M} = U(N)/U(N-M)\times U(M)$.[3)]

§3. Nonlinear sigma models with F-term constraints

Only D-term constraints appeared in the last two examples. In this section we also introduce appropriate F-term constraints to define other Kählerian G/H manifolds.

3.1. $SO(N)/SO(N-2)\times U(1)$

We consider a global symmetry, $G' = SO(N)\times U(1)_{\rm D} = G\times U(1)_{\rm D}$. We will gauge $U(1)_{\rm D}$ symmetry later. The fields, which develop a vacuum expectation value, are $\vec{\phi}$ in the defining representation $\boldsymbol{N}$ of $SO(N)$. The $U(1)_{\rm D}$ charge of $\vec{\phi}$ is defined to be 1. The fundamental representation is decomposed under its subgroup $SO(N-2)\times U(1)$ as $\boldsymbol{N} = (\boldsymbol{N-2}, 0)\oplus(\mathbf{1}, 1)\oplus(\mathbf{1}, -1)$. Here, the second factor is the $U(1)$ charge. The fields can be written as

$$\vec{\phi} = \begin{pmatrix} x \\ y^i \\ z \end{pmatrix}, \quad (3·1)$$

where x, y^i $(i = 1, \cdots, N-2)$ and z are a scalar, a vector and a scalar of $SO(N-2)$, respectively. Their $U(1)$ charges are defined above. $SO(N)$ is defined as the group that leaves the quadratic form

$$I_2 \stackrel{\rm def}{=} \vec{\phi}^2 \stackrel{\rm def}{=} \vec{\phi}^T J \vec{\phi} = 2xz + y^2 \quad (3·2)$$

invariant, where we have written the invariant tensor of rank 2 in a rather unconventional way (see Appendix C):

$$J = \begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{1}_{N-2} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix}. \quad (3·3)$$

The generators of $SO(N)$ consist of the $SO(N-2)$ generator T_{ij} $(i, j = 1, \cdots, N-2)$, the $U(1)$ generator T, and the upper triangular matrices E^i $(i = 1, \cdots, N-2)$, which transform as $(\boldsymbol{N-2},1)$, and their complex conjugates $\bar{E}_i = (E^i)^\dagger$ in $(\boldsymbol{N-2},-1)$. $SO(N)^{\boldsymbol{C}}$ acts on the fundamental representation in our basis as

$$\begin{aligned} \delta\vec{\phi} &= \left(i\theta T + \frac{i}{2}\theta_{ij}T_{ij} + \bar{\epsilon}_i E^i + \epsilon^i \bar{E}_i\right)\vec{\phi} \\ &= \begin{pmatrix} i\theta & \bar{\epsilon}_j & 0 \\ \epsilon^i & \theta_{ij} & -\bar{\epsilon}_i \\ 0 & -\epsilon^i & -i\theta \end{pmatrix} \begin{pmatrix} x \\ y^j \\ z \end{pmatrix}, \end{aligned} \tag{3·4}$$

where $\frac{i}{2}\theta_{kl}(T_{kl})^i{}_j = \theta_{ij}$. Here, these coefficients are normalized so that $\operatorname{tr} T_{ij}^2 = \operatorname{tr} T^2 = \operatorname{tr} E^i \bar{E}_i = 2$ (no sum). All parameters are complex when we consider $SO(N)^{\boldsymbol{C}}$ and real when we consider $SO(N)$.

In order to impose the global symmetry $SO(N) \times U(1)_{\rm D}$, we introduce the superpotential

$$W(\phi_0, \vec{\phi}) = \phi_0 \vec{\phi}^2, \tag{3·5}$$

with the lagrange multiplier field ϕ_0. This is an $SO(N)$ singlet, and its $U(1)_{\rm D}$ charge is defined to be -2, so that W is invariant under G'. Since the superpotential is a holomorphic function of ϕ and ϕ_0, the symmetry is enhanced to its complexfication $G'^{\boldsymbol{C}} = SO(N)^{\boldsymbol{C}} \times U(1)_{\rm D}{}^{\boldsymbol{C}}$. We can eliminate the auxiliary field by using its equation of motion,*)

$$\partial W/\partial \phi_0 = I_2 = 2xz + y^2 = 0. \tag{3·6}$$

We thus obtain an F-term constraint ($N_F = 1$). This equation is immediately solved to give

$$z = -\frac{y^2}{2x}. \tag{3·7}$$

Then, the field $\vec{\phi}$ constrained by the F-term can be written as

$$\vec{\phi} = \begin{pmatrix} x \\ y^i \\ -\frac{y^2}{2x} \end{pmatrix}. \tag{3·8}$$

When this develops a vacuum expectation value, any vacuum can be transformed by $G'^{\boldsymbol{C}}$ to the standard form,

$$\vec{v} = \left\langle \vec{\phi} \right\rangle = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}. \tag{3·9}$$

By this vacuum expectation value, the global symmetry is spontaneously broken as $SO(N) \times U(1)_{\rm D} \to SO(N-2) \times U(1)'$, where the unbroken $U(1)'$ is generated

*) There is another way to obtain the F-term constraint. If we take $K = \lambda \phi_0{}^\dagger \phi_0 + \vec{\phi}^\dagger \vec{\phi}$ and $W = \phi_0 \vec{\phi}^2$, then the potential reads $V = \frac{1}{\lambda}|\vec{\phi}^2|^2 + |\phi_0|^2|\vec{\phi}|^2$. We obtain the F-term constraint in the limit $\lambda \to 0$.

by a linear combination of the $U(1)$ subgroup and $U(1)_{\rm D}$.*) The complex broken generators consist of X, which is hermitian and generates a mixed-type multiplet, and the E^i, which are non-hermitian and generate pure-type multiplets. Then, the number of the mixed- and pure-type multiplets are $N_{\rm M} = 1$ and $N_{\rm P} = N - 2$, respectively. The target manifold M' generated by these generators is non-compact and non-homogeneous due to the presence of the QNG boson. The field near $\vec{v}$ is

$$\delta\vec{v} = (i\psi X + \varphi^i \bar{E}_i)\vec{v} = \begin{pmatrix} i\psi \\ \varphi^i \\ 0 \end{pmatrix}, \qquad (3\cdot10)$$

where ψ is a mixed-type multiplet and φ^i are pure-type multiplets.

We elevate $U(1)_{\rm D}$ to a local gauge symmetry to obtain a compact manifold by eliminating the mixed-type multiplet, as in the case of $\boldsymbol{C}P^{N-1}$. The gauged Kähler potential is the same as Eq. (2·5). By integrating out the auxiliary superfields, we obtain Eq. (2·9), with the constraint $\vec{\phi}^2 = 0$. By using the gauge symmetry $U(1)_{\rm D}{}^{\boldsymbol{C}}$, we can choose the gauge fixing as $x = 1$:

$$\vec{\phi} = \begin{pmatrix} 1 \\ \varphi^i \\ -\frac{1}{2}\varphi^2 \end{pmatrix}. \qquad (3\cdot11)$$

Here we have rewritten y^i as φ^i. This $\vec{\phi}$ can be rewritten by using the representative ξ of the complex coset manifold $G^{\boldsymbol{C}}/\hat{H} = SO(N)/SO(N-2) \times U(1)$ as

$$\vec{\phi} = \xi\vec{v}, \qquad \xi = e^{\varphi\cdot\bar{E}} = \begin{pmatrix} 1 & \mathbf{0} & 0 \\ \varphi^i & \mathbf{1}_{N-2} & \mathbf{0} \\ -\frac{1}{2}\varphi^2 & -\varphi^i & 1 \end{pmatrix}. \qquad (3\cdot12)$$

We thus obtain a Kähler potential of $G^{\boldsymbol{C}}/\hat{H}$,

$$K(\varphi, \varphi^\dagger, V(\varphi, \varphi^\dagger)) = c \log\left(1 + |\varphi|^2 + \frac{1}{4}\varphi^{\dagger 2}\varphi^2\right). \qquad (3\cdot13)$$

This is exactly the Kähler potential of $SO(N)/SO(N-2) \times U(1)$.[13),23),10)]

In our derivation of the Kähler potential, we used the D-term constraint after imposing the F-term constraint first. Instead, we could impose the D-term constraint first. If we do so, we obtain the previous $\boldsymbol{C}P^{N-1}$ model. The F-term constraint is used as the holomorphic embedding condition of $Q^{N-2}(\boldsymbol{C}) = SO(N)/SO(N-2) \times U(1)$ to $\boldsymbol{C}P^{N-1}$. It is a well-known method to obtain $Q^{N-2}(\boldsymbol{C})$ in the mathematical literature.[24)] (See also p. 278 of Ref. 25).)

*) Note that the condition $I_2 = 0$ is essential to introduce the gauge symmetry. To impose $I_2 = f^2 \neq 0$, we have to use $W = g\phi_0(I_2 - f^2)$. In this case there is no $U(1)_{\rm D}$ symmetry, and there is a supersymmetric vacuum alignment.[9),10)] Thus the unbroken global symmetry H can depend on the choice of the vacuum expectation value: $H = SO(N-1)$ at the symmetric points, where $\phi^\dagger\phi = f^2$, and $H = SO(N-2)$ at the non-symmetric points, where $\phi^\dagger\phi > f^2$. Whereas $I_2 = 0$ corresponds to an open orbit, $I_2 \neq 0$ corresponds to closed orbits. In general, in closed orbits, there is a supersymmetric vacuum alignment. (See, e.g., §3.3 for the E_6 case.) In this paper, we do not discuss closed orbits, since we cannot gauge the $U(1)_{\rm D}$ symmetry.

3.2. $SO(2N)/U(N)$ and $Sp(N)/U(N)$

In this subsection, we consider the global symmetry $G' = G_{\rm L} \times G_{\rm R}$, where $G_{\rm L}$ is either $SO(2N)$ or $Sp(N)$ and $G_{\rm R} = U(N)_{\rm R}$, which will be gauged later. To embed $G_{\rm L}$ into a $2N \times 2N$ matrix of $U(2N)$, we write its elements by using four $N \times N$ matrices:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(2N). \tag{3·14}$$

This is an element of $SO(2N)$ or $Sp(N)$ if it satisfies

$$g^T J' g = J', \tag{3·15}$$

where J' is the invariant tensor of $SO(2N)$ or $Sp(N)$:

$$J' = \begin{pmatrix} \mathbf{0} & \mathbf{1}_N \\ \epsilon \mathbf{1}_N & \mathbf{0} \end{pmatrix}. \tag{3·16}$$

Here $\epsilon = +1$ corresponds to $SO(2N)$ and $\epsilon = -1$ to $Sp(N)$. Equation (3·15) can be written explicitly as

$$\begin{pmatrix} A^T C + \epsilon C^T A & A^T D + \epsilon C^T B \\ B^T C + \epsilon D^T A & B^T D + \epsilon D^T C \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{1}_N \\ \epsilon \mathbf{1}_N & \mathbf{0} \end{pmatrix}. \tag{3·17}$$

We consider the global symmetry as either $G' = SO(2N)_{\rm L} \times U(N)_{\rm R}$ for $\epsilon = +1$ or $G' = Sp(N)_{\rm L} \times U(N)_{\rm R}$ for $\epsilon = -1$. The field content is $\Phi \in (\mathbf{2N}, \overline{\mathbf{N}})$, which acquires a vacuum expectation value. Its transformation law under G' is

$$\Phi \to \Phi' = g \cdot \Phi = g_{\rm L} \Phi {g_{\rm R}}^{-1}, \qquad g = (g_{\rm L}, g_{\rm R}) \in G_{\rm L} \times G_{\rm R}. \tag{3·18}$$

The $G'^{\boldsymbol{C}}$ invariant superpotential is

$$W(\Phi_0, \Phi) = {\rm tr}\,(\Phi_0 \Phi^T J' \Phi), \tag{3·19}$$

where Φ_0 is an $N \times N$ auxiliary matrix chiral superfield, whose transformation law is

$$\Phi_0 \to g_{\rm R} \Phi_0 {g_{\rm R}}^T. \tag{3·20}$$

Since ${I_2}' \stackrel{\rm def}{=} \Phi^T J' \Phi$ is symmetric (anti-symmetric) for $\epsilon = 1$ ($\epsilon = -1$), Φ_0 satisfies

$${\Phi_0}^T = \epsilon \Phi_0. \tag{3·21}$$

Hence, Φ_0 belongs to a symmetric (anti-symmetric) rank-2 tensor representation of $SU(N)_{\rm R}$ for $\epsilon = 1$ ($\epsilon = -1$), and its $U(1)_{\rm D} (\in U(N)_{\rm R})$ charge is defined to be -2 to cancel with the Φ charge. Note that ${I_2}' = \Phi^T J' \Phi$ is invariant under $G_{\rm L}$, but not invariant under $G_{\rm R}$.

To eliminate the auxiliary field Φ_0, we solve its equation of motion

$$\delta W / \delta \Phi_0 = \Phi^T J' \Phi = 0. \tag{3·22}$$

We thereby obtain F-term constraints for the fields Φ. Their number is $N_F = \frac{1}{2}N(N+1)$ for $\epsilon = 1$ and $N_F = \frac{1}{2}N(N-1)$ for $\epsilon = -1$. Then the dimension of the resulting manifold M' constrained by the F-term is $N_\Phi = 2N^2 - \frac{1}{2}N(N+1) = \frac{3}{2}N^2 - \frac{1}{2}N$ for $\epsilon = 1$ and $N_\Phi = 2N^2 - \frac{1}{2}N(N-1) = \frac{3}{2}N^2 + \frac{1}{2}N$ for $\epsilon = -1$. When the field Φ acquires a vacuum expectation value, any vacuum can be transformed by $G'^{\boldsymbol{C}}$ to the standard form,

$$V = \langle \Phi \rangle = \begin{pmatrix} \mathbf{1}_N \\ \mathbf{0}_N \end{pmatrix}. \tag{3·23}$$

Hence, the F-term constrained manifold is a $G'^{\boldsymbol{C}}$-orbit of V. The breaking pattern of the global symmetry is either $SO(2N)_{\rm L} \times U(N)_{\rm R} \to U(N)_{\rm V}$ for $\epsilon = 1$ or $Sp(N)_{\rm L} \times U(N)_{\rm R} \to U(N)_{\rm V}$ for $\epsilon = -1$. Here, in both cases, the element of $U(N)_{\rm V}$ can be written as

$$\left(\begin{pmatrix} h & 0 \\ 0 & h^{-1T} \end{pmatrix},\ h \right) \in U(N)_{\rm V}, \tag{3·24}$$

where we have used Eq. (3·17). The complex isotropy group $\hat{H}'$ consists of complex extension of these elements and elements of the type

$$\left(\begin{pmatrix} \mathbf{1}_N & B \\ \mathbf{0}_N & \mathbf{1}_N \end{pmatrix},\ \mathbf{1}_N \right) \stackrel{\rm def}{=} e^E, \qquad E = \left(\begin{pmatrix} \mathbf{0}_N & B \\ \mathbf{0}_N & \mathbf{0}_N \end{pmatrix},\ \mathbf{0}_N \right), \tag{3·25}$$

with the constraints, from Eq. (3·17),

$$B + \epsilon B^T = 0. \tag{3·26}$$

These E constitute a Borel subalgebra $\mathcal{B}$ of $\hat{\mathcal{H}}'$. The dimensionality of $\mathcal{B}$ is $\dim_{\boldsymbol{C}} \mathcal{B} = \frac{1}{2}N(N-1)$ for $\epsilon = +1$ and $\dim_{\boldsymbol{C}} \mathcal{B} = \frac{1}{2}N(N+1)$ for $\epsilon = -1$. The pure-type broken generators are the complex conjugation of $E \in \mathcal{B}$: $\bar{E} = (E)^\dagger$.

To obtain a compact coset manifold, we gauge the $U(N)_{\rm R}$ symmetry by introducing vector superfields, as in the Grassmann manifold. The gauged Kähler potential is the same as Eq. (2·17), but with F-term constraints. Since the procedure of integrating out the gauge fields is also the same as for the Grassmann manifold, we obtain Eq. (2·21). We can choose the gauge fixing as

$$\Phi = \begin{pmatrix} \mathbf{1}_M \\ \varphi \end{pmatrix}, \tag{3·27}$$

where φ satisfies the F-term constraints Eq. (3·22):

$$\Phi^T J' \Phi = \varphi + \epsilon \varphi^T = 0. \tag{3·28}$$

The fields φ are all pure-type chiral superfields, since Φ is generated by the pure-type broken generators $\bar{E}$ from the vacuum V:

$$\Phi = \xi \cdot V, \qquad \xi = e^{\varphi \cdot \bar{E}} = \left(\begin{pmatrix} \mathbf{1}_N & \mathbf{0}_N \\ \varphi & \mathbf{1}_N \end{pmatrix},\ \mathbf{1}_N \right). \tag{3·29}$$

Here, from Eq. (3·17), φ satisfies $\varphi + \epsilon\varphi^T = 0$, which is consistent with (3·28). By substituting Eq. (3·27) into Eq. (2·21), we obtain the Kähler potential

$$K(\varphi, \varphi^\dagger, V(\varphi, \varphi^\dagger)) = c \log\det(\mathbf{1}_N + \varphi^\dagger\varphi), \qquad \varphi + \epsilon\varphi^T = 0. \quad (3·30)$$

The fields φ are anti-symmetric (symmetric) parts of the matrix chiral superfield of the Grassmann manifold $G_{2N,N}$ for $\epsilon = +1$ (-1). Their dimensions are $\dim_{\boldsymbol{C}} M = \frac{1}{2}N(N-1)$ for $\epsilon = +1$ and $\dim_{\boldsymbol{C}} M = \frac{1}{2}N(N+1)$ for $\epsilon = -1$. Again, it is well known that these manifolds are submanifolds of the Grassmann manifold $G_{2N,N}$ in the mathematical literature.[24)]

3.3. $E_6/SO(10) \times U(1)$

This and the next subsections are devoted to the gauge theory construction of exceptional-type hermitian symmetric spaces. The situation here is slightly different from the classical group cases. Namely, in the present case, an F-term constrained manifold M' is characterized by the derivative of a G-invariant ($\partial I = 0$), but not the G-invariant itself ($I = 0$), as in the case of classical types.

As in the $Q^{N-2}(\boldsymbol{C})$ case, we consider the global symmetry $G' = E_6 \times U(1)_{\rm D} = G \times U(1)_{\rm D}$. The field belongs to the fundamental representation of E_6: $\vec{\phi} \in \mathbf{27}$, which will acquire a vacuum expectation value. We decompose E_6 under its maximal subgroup $SO(10) \times U(1)$. Since the fundamental representation can be decomposed as $\mathbf{27} = (\mathbf{1}, 4) \oplus (\mathbf{16}, 1) \oplus (\mathbf{10}, -2)$,[27)] where the second entries are the $U(1)$ charges, the basic field $\vec{\phi}$ can be written as

$$\vec{\phi} = \begin{pmatrix} x \\ y_\alpha \\ z^A \end{pmatrix}. \quad (3·31)$$

Here, x, y_α $(\alpha = 1, \cdots, 16)$ and z^A $(A = 1, \cdots, 10)$ are an $SO(10)$ scalar, a Weyl spinor and a vector, respectively. The decomposition of the adjoint representation, $\mathbf{78} = (\mathbf{45}, 0) \oplus (\mathbf{1}, 0) \oplus (\mathbf{16}, 1) \oplus (\overline{\mathbf{16}}, -1)$,[27)] implies that the E_6 algebra can be constructed with the $SO(10)$ generators T_{AB} $(A, B = 1, \cdots, 10)$, the $U(1)$ generator T, upper half generators E_α, which belong to a Weyl spinor of $SO(10)$, and their conjugates $\bar{E}^\alpha$. (See Appendix D for details.)

The transformation law of $\vec{\phi}$ under the complex extension of E_6 is[15),28)]

$$\begin{aligned}\delta\vec{\phi} &= \left(i\theta T + \frac{i}{2}\theta_{AB}T_{AB} + \bar{\epsilon}^\alpha E_\alpha + \epsilon_\alpha \bar{E}^\alpha\right)\vec{\phi} \\ &= \begin{pmatrix} \frac{2i}{\sqrt{3}}\theta & \bar{\epsilon}^\beta & \mathbf{0} \\ \epsilon_\alpha & \frac{i}{2}\theta_{AB}(\sigma_{AB})_\alpha{}^\beta + \frac{i}{2\sqrt{3}}\theta\delta_\alpha^\beta & -\frac{1}{\sqrt{2}}(\bar{\epsilon}\sigma_B C)_\alpha \\ \mathbf{0} & -\frac{1}{\sqrt{2}}(C\sigma_A{}^\dagger\epsilon)^\beta & \theta_{AB} - \frac{i}{\sqrt{3}}\theta\delta_{AB} \end{pmatrix} \begin{pmatrix} x \\ y_\beta \\ z^B \end{pmatrix}, \end{aligned} \quad (3·32)$$

where $\frac{i}{2}\theta_{CD}\rho(T_{CD})^A{}_B = \theta_{AB}$, and $\rho(T_{AB})$ is the vector representation matrices of $SO(10)$. The 16×16 matrices σ_A, σ_{AB} and C are (off-diagonal blocks of) $SO(10)$ gamma matrices, spinor rotation matirices and the charge conjugation matrix, respectively. Normalizations are fixed by $\mathrm{tr}\, T^2 = \mathrm{tr}\, T_{AB}{}^2 = \mathrm{tr}\, E_\alpha \bar{E}^\alpha = 6$ (no sum).*)

The decomposition of the tensor product, $\mathbf{27} \otimes \mathbf{27} = \overline{\mathbf{27}}_{\mathrm{S}} \oplus \cdots$, implies that there exist a rank-3 symmetric invariant tensor Γ_{ijk} and its complex conjugate Γ^{ijk}.[28] By using this invariant tensor, a cubic invariant I_3 of E_6 is defined as

$$I_3 \stackrel{\mathrm{def}}{=} \Gamma_{ijk}\phi^i\phi^j\phi^k = xz^2 + \frac{1}{\sqrt{2}} z^A (yC\sigma_A{}^\dagger y). \tag{3·33}$$

Note that this is not invariant under $U(1)_{\mathrm{D}}$.

We construct the superpotential

$$W(\vec{\phi}_0, \vec{\phi}) = \Gamma_{ijk}\phi_0{}^i \phi^j \phi^k. \tag{3·34}$$

Here $\vec{\phi}_0$ represents auxiliary fields whose $U(1)_{\mathrm{D}}$ charges should be chosen so as to make the superpotential invariant. If we assign the $U(1)_{\mathrm{D}}$ charge 1 to $\vec{\phi}$, $\vec{\phi}_0$ must have charge -2, so that they belong to $(\mathbf{27}, -2)$. The equations of motion for the auxiliary fields $\phi_0{}^i$, $\delta W / \delta \phi_0 = \Gamma_{ijk}\phi^j\phi^k = 0$, are

$$\partial W / \partial z_0{}^A = \Gamma_{Ajk}\phi^j\phi^k = 2z_A x + \frac{1}{\sqrt{2}} y_\alpha (C\sigma_A{}^\dagger)^{\alpha\beta} y_\beta = 0, \tag{3·35}$$

$$\partial W / \partial y_{0\alpha} = \Gamma_{\alpha jk}\phi^j\phi^k = \sqrt{2}(C\sigma_A{}^\dagger)^{\alpha\beta} y_\beta z^A = 0, \tag{3·36}$$

$$\partial W / \partial x_0 = \Gamma_{0jk}\phi^j\phi^k = z^2 = 0. \tag{3·37}$$

In the second equation, we have used the fact that $(C\sigma_A{}^\dagger)^{\alpha\beta}$ is symmetric. Note that these equations can also be written as

$$\partial I_3 = 0, \tag{3·38}$$

where the differentiation is with respect to ϕ^i. In these 27 equations, only the first 10 equations are independent. The first equation can be solved to yield

$$z_A = -\frac{1}{2\sqrt{2}x} y (C\sigma_A{}^\dagger) y. \tag{3·39}$$

Then, the last two equations are not independent, since they are automatically satisfied as

$$\sqrt{2}(C\sigma_A{}^\dagger)^{\alpha\beta} y_\beta z^A = -\frac{1}{2x}(C\sigma_A{}^\dagger)^{\alpha\beta} y_\beta \left(y(C\sigma_A{}^\dagger)y \right) = 0, \tag{3·40}$$

$$z^2 = \frac{1}{8x^2}\left(y(C\sigma_A{}^\dagger)y \right)^2 = 0, \tag{3·41}$$

with the help of the identity

$$(\varepsilon C\sigma_A{}^\dagger \psi)(\psi C\sigma_A{}^\dagger \eta) = -\frac{1}{2}(\varepsilon C\sigma_A{}^\dagger \eta)(\psi C\sigma_A{}^\dagger \psi). \tag{3·42}$$

*) $\mathrm{tr}\, T_{AB}{}^2 = 6$ has been calculated from $\mathrm{tr}\, \rho(T_{AB})^2 = 2$, while $\mathrm{tr}\,(\sigma_{AB})^2 = 4$ and others have been fixed to this.

Hence, the number of F-term conditions is $N_F = 10$, and the dimension of M' is $N_\Phi = 27 - 10 = 17$. The manifold M' satisfying these F-term constraints can be written as

$$\vec{\phi} = \begin{pmatrix} x \\ y_\alpha \\ -\frac{1}{2\sqrt{2}x}(yC\sigma_A{}^\dagger y) \end{pmatrix}. \tag{3·43}$$

On M', the value of the E_6 invariant is

$$I_3 \sim (yC\sigma_A{}^\dagger y)^2 = 0, \tag{3·44}$$

by the identity (3·42). Note that I_3 must vanish, since it is not invariant under $U(1)_{\rm D}$.

When the fields $\vec{\phi}$ develop a vacuum expectation value, any vacuum can be transformed under $G'^{\boldsymbol{C}}$ to the standard form,

$$\vec{v} = \left\langle \vec{\phi} \right\rangle = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}. \tag{3·45}$$

The global symmetry is spontaneously broken as $E_6 \times U(1)_{\rm D} \to SO(10) \times U(1)' = H'$.*) The unbroken $U(1)'$ is generated by $T' = T - \frac{2}{\sqrt{3}}\mathbf{1}_{27} = {\rm diag}(0, -\frac{\sqrt{3}}{2}\delta_\alpha^\beta, -\sqrt{3}\delta_{AB})$, and $SO(10)$ is generated by T_{AB}. The complex isotropy $\hat{\mathcal{H}}'$ is larger than the complexification of $\mathcal{H}'$ due to the existence of the E_α. These 16 E_α constitute a Borel subalgebra $\mathcal{B}$ in $\hat{\mathcal{H}}'$. The complex broken generators are composed of pure-type generators $\bar{E}^\alpha$ and another combination of $U(1)$ generators of a mixed-type $X \sim (1, \cdots)$. Their numbers are $N_{\rm P} = 16$ and $N_{\rm M} = 1$, respectively. The target manifold M' generated by these broken generators has dimension $\dim M' = N_\Phi = 17$. Since this coincides with the dimension of the manifold constrained by the 10 independent F-term conditions, any vacuum that satisfies F-term constraints can be transformed to the form of Eq. (3·45) by a $G'^{\boldsymbol{C}}$ transformation.

To remove the mixed-type multiplet and to obtain a compact manifold, we gauge the $U(1)_{\rm D}$ symmetry as in the case of $\boldsymbol{C}P^{N-1}$. The gauged Kähler potential is the same as in Eq. (2·5). Since the procedure to eliminate the vector superfield is also the same as in the $\boldsymbol{C}P^{N-1}$ case, we obtain Eq. (2·9). We can choose a gauge fixing as

$$\vec{\phi} = \begin{pmatrix} 1 \\ \varphi_\alpha \\ -\frac{1}{2\sqrt{2}}(\varphi C\sigma_A{}^\dagger \varphi) \end{pmatrix}, \tag{3·46}$$

where we write φ_α for y_α. By using the representative ξ of the complex coset manifold $M = G^{\boldsymbol{C}}/\hat{H} \simeq E_6/SO(10) \times U(1)$, $\vec{\phi}$ can be rewritten as

$$\vec{\phi} = \xi\vec{v}, \qquad \xi = e^{\varphi\cdot\bar{E}} = \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \varphi_\alpha & \mathbf{1}_{16} & \mathbf{0} \\ -\frac{1}{2\sqrt{2}}(\varphi C\sigma_A{}^\dagger\varphi) & -\frac{1}{\sqrt{2}}(C\sigma_A{}^\dagger\varphi)^\beta & \mathbf{1}_{10} \end{pmatrix}. \tag{3·47}$$

*) As in the case of $SO(N)$ discussed in §3.1, there is no $U(1)_{\rm D}$ symmetry if $I_3 \neq 0$. In this case, the $E_6{}^{\boldsymbol{C}}$-orbit is closed, and, by a supersymmetric vacuum alignment, there exist two regions with different unbroken global symmetries,[9] symmetric points and non-symmetric points. The breaking patterns of E_6 are $E_6 \to F_4$ at the symmetric points and $E_6 \to SO(8)$ at generic points.[28]

By substituting Eq. (3·46) into Eq. (2·9), we obtain the Kähler potential

$$K(\varphi, \varphi^\dagger, V(\varphi, \varphi^\dagger)) = c \log \left(1 + |\varphi|^2 + \frac{1}{8}(\varphi^\dagger \sigma^A \varphi^\dagger)(\varphi \sigma_A{}^\dagger \varphi)\right), \qquad (3·48)$$

where we have used the basis in which $C = 1$.[15] This coincides with the Kähler potential of $E_6/SO(10) \times U(1)$ constructed in Refs. 24), 15) and 23). (It is also equivalent to Ref. 14).) Its dimension is $\dim_{\boldsymbol{C}} M = 27 - 10 - 1 = 16$. If we do not introduce the superpotential, the manifold is $\boldsymbol{C}P^{26}$. Hence, $E_6/SO(10) \times U(1)$ is embedded in $\boldsymbol{C}P^{26}$ by 10 F-term constraints, $\partial I_3 = 0$. In fact, Yasui constructed $E_6/SO(10) \times U(1)$ as a submanifold of $\boldsymbol{C}P^{26}$ by using the Jordan algebra.[16]

3.4. $E_7/E_6 \times U(1)$

In this subsection, we consider another exceptional group, E_7. The global symmetry in this case is $G' = E_7 \times U(1)_{\mathrm{D}} = G \times U(1)_{\mathrm{D}}$. The basic fields $\vec{\phi}$ belong to the fundamental representation **56**. Under a maximal subgroup $E_6 \times U(1)$, this representation can be decomposed as $\mathbf{56} = (\mathbf{27}, -\frac{1}{3}) \oplus (\overline{\mathbf{27}}, \frac{1}{3}) \oplus (\mathbf{1}, -1) \oplus (\mathbf{1}, 1)$.[27] Therefore, we write $\vec{\phi}$ as

$$\vec{\phi} = \begin{pmatrix} x \\ y^i \\ z_i \\ w \end{pmatrix}, \qquad (3·49)$$

where y^i and z_i are **27** and $\overline{\mathbf{27}}$, respectively, and x and w are scalars. By a decomposition of the adjoint representation under $E_6 \times U(1)$,[27] $\mathbf{133} = (\mathbf{78}, 0) \oplus (\mathbf{1}, 0) \oplus (\mathbf{27}, 1) \oplus (\overline{\mathbf{27}}, -1)$, we can construct the E_7 algebra from the E_6 algebra $T_A\,(A = 1, \cdots, 78)$, the $U(1)$ generator T, the upper half generators $E^i\,(i = 1, \cdots, 27)$, belonging to **27**, and their conjugates $\bar{E}_i = (E^i)^\dagger$, belonging to $\overline{\mathbf{27}}$. (Their commutation relations are discussed in Appendix E.) The action of the E_7 algebra on the fundamental representation is

$$\begin{aligned}\delta\vec{\phi} &= \left(i\theta T + i\theta_A T_A + \bar{\epsilon}_i E^i + \epsilon^i \bar{E}_i\right)\vec{\phi} \\ &= \begin{pmatrix} i\sqrt{\frac{3}{2}}\theta & \bar{\epsilon}_j & \mathbf{0} & 0 \\ \epsilon^i & i\theta_A \rho(T_A)^i{}_j + i\sqrt{\frac{1}{6}}\theta\delta^i{}_j & \Gamma^{ijk}\bar{\epsilon}_k & \mathbf{0} \\ \mathbf{0} & \Gamma_{ijk}\epsilon^k & -i\theta_A \rho(T_A)^T{}_i{}^j - i\sqrt{\frac{1}{6}}\theta\delta_i{}^j & \bar{\epsilon}_i \\ 0 & \mathbf{0} & \epsilon^j & -i\sqrt{\frac{3}{2}}\theta \end{pmatrix} \begin{pmatrix} x \\ y^j \\ z_j \\ w \end{pmatrix},\end{aligned} \qquad (3·50)$$

where $\rho(T_A)$ is the 27×27 representation matrix for the fundamental representation, Γ_{ijk} is the E_6 invariant tensor, defined in the last subsection, and Γ^{ijk} is its conjugate. Here normalizations have been determined by $\mathrm{tr}\, T^2 = \mathrm{tr}\, {T_A}^2 = \mathrm{tr}\, E^i \bar{E}_i = 12$ (no sum).[*]

[*] $\mathrm{tr}\, {T_A}^2 = 12$ has been calculated with the normalization $\mathrm{tr}\,(\rho(T_A)^2) = 6$ for the E_6 fundamental representation, as in the previous subsection. Other normalizations have been fixed relative to this. In the calculation of $\mathrm{tr}\, E^i \bar{E}_i = 12$, we have used the identity Eq. (D·5).

In the tensor products[27] $\mathbf{56}\otimes\mathbf{56}=\mathbf{1}_{\rm a}\oplus\cdots$ and $\mathbf{56}\otimes\mathbf{56}\otimes\mathbf{56}\otimes\mathbf{56}=\mathbf{1}_{\rm s}\oplus\cdots$, there exist the rank-2 anti-symmetric invariant tensor $f_{\alpha\beta}$ and the rank-4 symmetric invariant tensor $d_{\alpha\beta\gamma\delta}$, respectively. Their components are calculated in Appendix E. By using this invariant tensor, we can construct the quartic invariant of E_7 as

$$\begin{aligned} I_4 &\stackrel{\rm def}{=} d_{\alpha\beta\gamma\delta}\phi^\alpha\phi^\beta\phi^\gamma\phi^\delta \\ &= -\frac{1}{2}(xw-y^iz_i)^2-\frac{1}{3}w\Gamma_{ijk}y^iy^jy^k-\frac{1}{3}x\Gamma^{ijk}z_iz_jz_k \\ &\quad +\frac{1}{2}\Gamma^{ijk}\Gamma_{ilm}z_jz_ky^ly^m. \end{aligned} \tag{3·51}$$

Again, note that this is not invariant under $U(1)_{\rm D}$.

The superpotential invariant under $E_7\times U(1)_{\rm D}$ is

$$W(\vec{\phi}_0,\vec{\phi})=d_{\alpha\beta\gamma\delta}{\phi_0}^\alpha\phi^\beta\phi^\gamma\phi^\delta, \tag{3·52}$$

where the ${\phi_0}^\alpha$ are auxiliary fields belonging to $(\mathbf{56},-3)$. Here the second component is the $U(1)_{\rm D}$ charge assigned to cancel the $U(1)_{\rm D}$ charge of ϕ^α. (The term with rank-2 tensor $f_{\alpha\beta}$ is forbidden by $U(1)_{\rm D}$ symmetry.) To eliminate the auxiliary fields ϕ_0, we consider F-term constraints obtained from their equation of motions:

$$\begin{aligned} \partial W/\partial {y_0}^i &= w(xz_i-\Gamma_{ijk}y^jy^k)-z_iy^jz_j+\Gamma^{jkl}\Gamma_{jim}z_kz_ly^m=0, \\ \partial W/\partial w_0 &= xy^iz_i-wx^2-\frac{1}{3}\Gamma_{ijk}y^iy^jy^k=0, \\ \partial W/\partial z_{0i} &= x(wy^i-\Gamma^{ijk}z_jz_k)-y^iy^jz_j+\Gamma^{jik}\Gamma_{jlm}z_ky^ly^m=0, \\ \partial W/\partial x_0 &= wy^iz_i-xw^2-\frac{1}{3}\Gamma^{ijk}z_iz_jz_k=0. \end{aligned} \tag{3·53}$$

Note that these equations can be written as

$$\partial I_4=0, \tag{3·54}$$

where the differentiations are with respect to ϕ^α. We show that only half of these 58 equations are independent. To solve these equations, we put the ansatz

$$z_i=\frac{c}{x}\Gamma_{ijk}y^jy^k, \tag{3·55}$$

where c is a constant. By substituting this ansatz into the first and second equations, we obtain

$$w(c-1)\Gamma_{ijk}y^jy^k+\frac{c^2}{3x^2}\Gamma_{ijk}\Gamma_{lmn}y^jy^ky^ly^my^n=0, \tag{3·56}$$

$$w=\frac{c-\frac{1}{3}}{x^2}\Gamma_{ijk}y^iy^jy^k. \tag{3·57}$$

From these equations we obtain the equation

$$\frac{4(c-\frac{1}{2})^2}{3x^2}\Gamma_{ijk}y^jy^k\Gamma_{lmn}y^ly^my^n=0, \tag{3·58}$$

which gives $c = \frac{1}{2}$. By substituting $c = \frac{1}{2}$ back into Eqs. (3·55) and (3·57), we obtain the results,

$$z_i = \frac{1}{2x}\Gamma_{ijk}y^j y^k, \quad w = \frac{1}{6x^2}\Gamma_{ijk}y^i y^j y^k. \tag{3·59}$$

In the same way, the third and the fourth equations in Eq. (3·53) can be solved as

$$y^i = \frac{1}{2w}\Gamma^{ijk}z_j z_k, \quad x = \frac{1}{6w^2}\Gamma^{ijk}z_i z_j z_k. \tag{3·60}$$

We can show that these equations are not independent of Eq. (3·59) with the help of the Springer relation, Eq. (D·6). Then the number of F-term constraints is $N_F = 28$, and the dimension of M' is $\dim_{\boldsymbol{C}} M' = 56 - 28 = 28$. Thus, the F-term constraints can be solved as

$$\vec{\phi} = \begin{pmatrix} x \\ y^i \\ \frac{1}{2x}\Gamma_{ijk}y^j y^k \\ \frac{1}{6x^2}\Gamma_{ijk}y^i y^j y^k \end{pmatrix}. \tag{3·61}$$

On these points, the value of the E_7 invariant is

$$I_4 = 0, \tag{3·62}$$

where we have used the Springer relation (D·6). Note that $U(1)_{\mathrm{D}}$ invariance requires $I_4 = 0$.

By using $G'^{\boldsymbol{C}}$, any vacuum expectation value of $\vec{\phi}$ can be transformed to

$$\vec{v} = \left\langle \vec{\phi} \right\rangle = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}. \tag{3·63}$$

On this vacuum, global symmetry is spontaneously broken as $E_7 \times U(1)_{\mathrm{D}} \to E_6 \times U(1)' \stackrel{\mathrm{def}}{=} H'$. Here $U(1)'$ is generated by a linear combination of the $U(1)$ generator T and the $U(1)_{\mathrm{D}}$ generator $T_{\mathrm{D}} = \mathbf{1}_{56}$. From Eq. (3·50), we see that the complex isotropy $\hat{\mathcal{H}}'$ is larger than $\mathcal{H}'^{\boldsymbol{C}}$ due to the presence of the E^i, which constitute a Borel subalgebra. The complex broken generators constitute a hermitian generator X, which is a linear combination of T_{D} and T, and non-hermitian generators $\bar{E}_i$. Hence, the numbers of mixed- and pure-type multiplets are $N_{\mathrm{M}} = 1$ and $N_{\mathrm{P}} = 27$, respectively. The target manifold M' is generated by these broken generators, and its dimension is $\dim_{\boldsymbol{C}} M' = 28$, which coincides with the dimension of the manifold constrained by the F-term conditions in Eq. (3·61).

The target manifold M' obtained above is non-compact due to the QNG boson. We gauge the $U(1)_{\mathrm{D}}$ symmetry to remove the mixed-type multiplet and to obtain a compact manifold. Since the situation is the same as for the $\boldsymbol{C}P^{N-1}$, $Q_{N-2}(\boldsymbol{C})$ and $E_6/SO(10) \times U(1)$ cases, by integrating out the vector superfield, we obtain Eq. (2·9). We can choose the gauge fixing as

$$\vec{\phi} = \begin{pmatrix} 1 \\ \varphi^i \\ \frac{1}{2}\Gamma_{ijk}\varphi^j \varphi^k \\ \frac{1}{6}\Gamma_{ijk}\varphi^i \varphi^j \varphi^k \end{pmatrix}, \tag{3·64}$$

where we rewrite y^i as φ^i. As in the previous subsections, this can be written as

$$\vec{\phi} = \xi \vec{v}, \qquad \xi = e^{\varphi\cdot\bar{E}} = \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} & 0 \\ \varphi^i & \mathbf{1}_{27} & \mathbf{0}_{27} & \mathbf{0} \\ \frac{1}{2}\Gamma_{ijk}\varphi^j\varphi^k & \Gamma_{ijk}\varphi^j & \mathbf{1}_{27} & \mathbf{0} \\ \frac{1}{6}\Gamma_{ijk}\varphi^i\varphi^j\varphi^k & \frac{1}{2}\Gamma_{ijk}\varphi^j\varphi^k & \varphi^i & 1 \end{pmatrix}. \tag{3·65}$$

Hence the target manifold M, obtained by integrating out the vector superfield, is the coset manifold generated by $\bar{E}_i$, which is $M \simeq E_7/E_6 \times U(1)$. Then, by substituting (3·64) into Eq. (2·9), we obtain the Kähler potential

$$K(\varphi,\varphi^\dagger,V(\varphi,\varphi^\dagger)) = c\log\left(1+|\varphi|^2+\frac{1}{4}|\Gamma_{ijk}\varphi^j\varphi^k|^2+\frac{1}{36}|\Gamma_{ijk}\varphi^i\varphi^j\varphi^k|^2\right). \tag{3·66}$$

This form coincides with Ref. 13). Its dimension is $\dim_{\boldsymbol{C}} M = 56-28-1=27$. It can be embedded into $\boldsymbol{C}P^{55}$ by holomorphic constraints $\partial I_4 = 0$.

§4. Conclusions and discussion

We have obtained nonlinear sigma models whose target manifolds are the hermitian symmetric spaces G/H, which are compact and homogeneous, from linear models. For this purpose, we introduced appropriate superpotentials for $G = SO, SU, Sp, E_6$ and E_7 to impose F-term constraints. By solving these F-term constraint equations, we have obtained constrained manifolds M', which are non-compact and non-homogeneous due to the existence of QNG bosons. When there is no gauge symmetry, there must be at least one QNG boson, by the theorem of Lerche and Shore,[6] and the manifold inevitably becomes non-compact and non-homogeneous (see Appendix B). In order to get rid of these unwanted QNG-bosons, we further introduced suitable local gauge symmetry. By choosing suitable gauge conditions, we obtained the Kähler potentials of all the hermitian symmetric spaces, where decay constants (overall constants of Kähler potentials) originate from FI-terms of gauge fields.

The gauging procedures to eliminate QNG bosons can be summarized as follows:*)

$$\begin{aligned} \boldsymbol{R}^+ \times \frac{SU(N)\times U(1)_{\rm D}}{SU(N-1)\times U(1)'} &\overset{U(1)_{\rm D}}{\longrightarrow} \frac{SU(N)}{S(U(N-1)\times U(1))}, \\ (\boldsymbol{R}^+)^{M^2} \times \frac{U(N)_{\rm L}\times U(M)_{\rm R}}{U(N-M)_{\rm L}\times U(M)_{\rm V}} &\overset{U(M)_{\rm R}}{\longrightarrow} \frac{U(N)_{\rm L}}{U(N-M)_{\rm L}\times U(M)_{\rm L}}, \\ \boldsymbol{R}^+ \times \frac{SO(N)\times U(1)_{\rm D}}{SO(N-2)\times U(1)'} &\overset{U(1)_{\rm D}}{\longrightarrow} \frac{SO(N)}{SO(N-2)\times U(1)}, \end{aligned}$$

*) From the result in Ref. 9), in all cases considered in this paper, we know that there exists no supersymmetric vacuum alignment, since there is no non-singlet broken generators under the real unbroken subgroup H. Hence, the F-term constrained manifolds $M' \simeq G'^{\boldsymbol{C}}/\hat{H}'$ are topologically isomorphic to direct products of a QNG boson factor $\boldsymbol{R}^+ = \{\theta | \theta \in \boldsymbol{R}, \theta > 0\}$, which is non-compact, and a NG bosons factor G'/H', which is compact. For example, in the case of $\boldsymbol{C}^N$ without an F-term constraint, $M' \simeq G'^{\boldsymbol{C}}/\hat{H}' = \frac{(SU(N)\times U(1)_{\rm D})^{\boldsymbol{C}}}{(SU(N-1)\times U(1)')^{\boldsymbol{C}}\wedge\mathcal{B}} \simeq \boldsymbol{R}^+ \times \frac{SU(N)\times U(1)_{\rm D}}{SU(N-1)\times U(1)'} = \boldsymbol{R}^+ \times \frac{G'}{H'}$. Then, by gauging $U(1)$, we obtain $G^{\boldsymbol{C}}/\hat{H} \simeq G/H = \boldsymbol{C}P^{N-1}$.

$$(\boldsymbol{R}^+)^{N^2} \times \frac{SO(2N)_{\rm L} \times U(N)_{\rm R}}{U(N)_{\rm V}} \overset{U(N)_{\rm R}}{\longrightarrow} \frac{SO(2N)_{\rm L}}{U(N)_{\rm L}},$$
$$(\boldsymbol{R}^+)^{N^2} \times \frac{Sp(N)_{\rm L} \times U(N)_{\rm R}}{U(N)_{\rm V}} \overset{U(N)_{\rm R}}{\longrightarrow} \frac{Sp(N)_{\rm L}}{U(N)_{\rm L}},$$
$$\boldsymbol{R}^+ \times \frac{E_6 \times U(1)_{\rm D}}{SO(10) \times U(1)'} \overset{U(1)_{\rm D}}{\longrightarrow} \frac{E_6}{SO(10) \times U(1)},$$
$$\boldsymbol{R}^+ \times \frac{E_7 \times U(1)_{\rm D}}{E_6 \times U(1)'} \overset{U(1)_{\rm D}}{\longrightarrow} \frac{E_7}{E_6 \times U(1)}.$$

The left-hand sides denote the F-term constrained manifolds M' (if there is a superpotential). All M' are non-compact and non-homogeneous, due to the existence of QNG bosons represented by $\boldsymbol{R}^+$. This implies that they are scale factors. The arrows represent the gauging and the right-hand sides denote the manifold M obtained by integrating out the vector superfields. The relation between M and M' is a Kähler quotient, $M = M'/G^{\boldsymbol{C}}_{\rm gauge}$. All M are compact and homogeneous, since they are parameterized by only NG bosons. In the cases of $\boldsymbol{C}P^{N-1}$ and $G_{N,M}$, there are no F-term constraints. Other cases have $G^{\boldsymbol{C}}$-invariants, superpotentials and F-term constraints, as summarized in Table II.

The F-term constraints can be classified into two types:

- $G = SO,\ Sp$: $I = 0$. (They are $G^{\boldsymbol{C}}$-invariant.)
- $G = E_6,\ E_7$: $\partial I = 0$. (Although the ∂I are not $G^{\boldsymbol{C}}$-invariant, the constraints themselves are $G^{\boldsymbol{C}}$-invariant.)

In each case, the value of the $G^{\boldsymbol{C}}$-invariant vanishes on the constrained manifolds, since, even in the cases of the exceptional groups, the constraints $\partial I = 0$ lead to $I = 0$. This remarkable fact can be understood as follows: *Note that, in each case, the $G^{\boldsymbol{C}}$-invariant I is not invariant under a gauge group. Hence, it must vanish to be consistent with a gauge symmetry.* We call this the "consistency condition with a gauge symmetry".*)

If we forget the F-term constraints and impose only the D-term constraints, the manifolds become $\boldsymbol{C}P^{N-1}$ or $G_{2N,N}$. This means that all of the hermitian symmetric spaces are holomorphically embedded in $\boldsymbol{C}P^{N-1}$ or $G_{N,M}$ by F-term constraints, as is shown in the last column of Table II. Although some of the constraints are already known in the mathematical literature, the explicit forms of the constraints in the E_6 and E_7 cases are new results: $E_6/SO(10) \times U(1)$ is holomorphically embbedded in $\boldsymbol{C}P^{26}$ by 16 quadratic homogeneous constraints, and $E_7/E_6 \times U(1)$ is embedded in $\boldsymbol{C}P^{55}$ by 28 tripletic homogeneous constraints. The consistency condition with a gauge symmetry can be understood if we interpret the F-term constraints as the embedding conditions. Since $G_{N,M}$ can be embedded into $\boldsymbol{C}P^N$, all hermitian symmetric spaces are embedded in $\boldsymbol{C}P^N$. If we want to embed M into $\boldsymbol{C}P^N$, the constraint must be homogeneous, when it is written in terms of homogeneous

*) By combining the result in Ref. 10), this condition can be understood as the condition that the manifold before gauging must be an open orbit, not a closed orbit. In Ref. 10), it was shown that an open orbit includes a compact and homogeneous manifold as a submanifold. Contrastingly, a closed orbit does not have such a submanifold.

Table II. F-term constraints and embedding. Here, J, J', Γ and d are rank-2, rank-2, rank-3 and rank-4 invariant symmetric tensors of $SO(N)$, $SO(2N)$ or $Sp(N)$, E_6 and E_7, respectively, and I_2, ${I_2}'$, I_3 and I_4 are $G^{\boldsymbol{C}}$-invariants composed of them. Each superpotential gives an F-term constraint, which is $I = 0$ in the case of classical groups and $\partial I = 0$ in the case of exceptional groups. Only 10 equations of the 27 equations are independent in the E_6 case, and only 28 equations among 56 equations are independent in the E_7 case. The last column denotes the projective or Grassmann manifold, in which each hermitian symmetric space is embedded by the F-term constraint.

G/H	$G^{\boldsymbol{C}}$-invariants	superpotentials	constraints	embedding
$\frac{SO(N)}{SO(N-2)\times U(1)}$	$I_2 = \vec{\phi}^T J \vec{\phi}$	$\phi_0 I_2$	$I_2 = 0$	$\boldsymbol{C}P^{N-1}$
$\frac{SO(2N)}{U(N)}$, $\frac{Sp(N)}{U(N)}$	${I_2}' = \Phi^T J' \Phi$	$\mathrm{tr}\,(\Phi_0 {I_2}')$	${I_2}' = 0$	$G_{2N,N}$
$\frac{E_6}{SO(10)\times U(1)}$	$I_3 = \Gamma_{ijk}\phi^i\phi^j\phi^k$	$\Gamma_{ijk}{\phi_0}^i\phi^j\phi^k$	$\partial I_3 = 0$	$\boldsymbol{C}P^{26}$
$\frac{E_7}{E_6\times U(1)}$	$I_4 = d_{\alpha\beta\gamma\delta}\phi^\alpha\phi^\beta\phi^\gamma\phi^\delta$	$d_{\alpha\beta\gamma\delta}{\phi_0}^\alpha\phi^\beta\phi^\gamma\phi^\delta$	$\partial I_4 = 0$	$\boldsymbol{C}P^{55}$

coordinates.*)

In this paper, we have used the equation of motion for the vector auxiliary field. In the path integral formalism, this procedure corresponds to integrating over the vector field. In a separate paper,[26] we show that the path integration can be performed exactly.

Now we discuss possible generalizations of our results to a wider class of Kählerian G/H. In this paper, we treated hermitian symmetric spaces, which are a special class of homogeneous Kähler manifolds. We confined ourselves to the gauge groups of $U(1)$ or $U(N)$.

1. Even within this limitation, it is possible to generalize our construction to a wider class of homogeneous Kähler manifolds. Let us consider Kähler G/H, where H has only one $U(1)$ factor, $H = H_{\rm ss} \times U(1)$, with $H_{\rm ss}$ being a semisimple subgroup of H. To be specific, let us generalize $SO(2N)/U(N)$. By generalizing Φ to a $2N \times M$ matrix ($N \geq M$), transforming under $SO(2N) \times U(M)$ as $\Phi \to g_{\rm L}\Phi {g_{\rm R}}^{-1}$, with the same superpotential (3·19) (where J is the same as in Eq. (3·16)), we obtain

$$(\boldsymbol{R}^+)^{M^2} \times \frac{SO(2N)_{\rm L} \times U(M)_{\rm R}}{SO(2N-2M)_{\rm L} \times U(M)_{\rm V}} \xrightarrow{U(M)_{\rm R}} \frac{SO(2N)_{\rm L}}{SO(2N-2M)_{\rm L} \times U(M)_{\rm L}}.$$

   This reduces to $SO(2N)/SO(2N-2)\times U(1)$ when $M = 1$ and to $SO(2N)/U(N)$ when $N = M$. Similarly, $Sp(N)/U(N)$ can also be generalized. By generalizing Φ to a $2N \times M$ matrix ($N \geq M$), we obtain

$$(\boldsymbol{R}^+)^{M^2} \times \frac{Sp(N)_{\rm L} \times U(M)_{\rm R}}{Sp(N-M)_{\rm L} \times U(M)_{\rm V}} \xrightarrow{U(M)_{\rm R}} \frac{Sp(N)_{\rm L}}{Sp(N-M)_{\rm L} \times U(M)_{\rm L}}.$$

2. Now we consider generalization to the case of many $U(1)$ factors. Remember that the FI parameter c becomes a decay constant, which represents the size of G/H, after integrating out the vector superfield. Then, we can consider there

*) The manifold, which can be embedded into $\boldsymbol{C}P^N$, is a (projective) algebraic variety and can be understood as a Hodge manifold.

to be a one-to-one correspondence between the decay constants and the FI-parameters. Hence, to obtain G/H with $H = H_{\rm ss} \times U(1)^n$ we must prepare n FI-parameters. We thus consider a global symmetry, $G' = G \times G_1 \times \cdots G_n$, where each G_i includes a $U(1)$ factor. If we gauge all G_i, the gauged Kähler potential has n FI terms. After integrating out vector superfields, we obtain $G/H' \times G_1 \times \cdots G_n = G/H_{\rm ss} \times U(1)^n$, where H' is the remaining part after embedding all G_i into G. Here we have put $H_{\rm ss} = H' \times G_{1{\rm ss}} \times \cdots G_{n{\rm ss}}$. In the case of hermitian symmetric spaces, we have introduced an irreducible representation of G as the basic field. It seems that we have to introduce more irreducible representations in these generalizations. Then we must impose orthogonality relations on these fields with D-term or F-term constraints. At the moment, we are unable to find consistent constraints in these cases.

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Appendix A
—— *BKMU-IKK Construction of Kähler Potentials of Compact Homogeneous Kähler Manifolds* ——

Bando et al. (BKMU) gave the general method to construct the G invariant Kähler potential of $G^{\boldsymbol{C}}/\hat{H}$. [5] However, there remained an ambiguity in the choice of the projection operators η_i introduced below, Eq. (A·1). Itoh et al. (IKK) constructed these operators explicitly for the case that the target is compact, namely $G^{\boldsymbol{C}}/\hat{H} \simeq G/H$. [11] Note that their method does not ensure that such models can be obtained from linear models. In this appendix, we review their method to compare with our method, which, on the other hand, has a linear origin.

First of all, we need the projection matrices, which project a fundamental representation space onto a $\hat{H}$ invariant subspace. [5] They satisfy the projection conditions

$$\eta^\dagger = \eta, \qquad \eta \hat{H} \eta = \hat{H} \eta, \qquad \eta^2 = \eta. \tag{A·1}$$

In an arbitrary Kähler G/H, the number of projection matrices is equal to the number of $U(1)$ factors in H. Since there is only one $U(1)$ factor in the hermitian symmetric cases, there is one projection matrix. In each case, it can be written as [11]

$$\eta = \begin{pmatrix} 1 & \\ & \mathbf{0} \end{pmatrix}. \tag{A·2}$$

By using this, the Kähler potentials of compact Kähler manifolds can be written as [5]

$$K = c \log \det{}_\eta \xi^\dagger \xi, \tag{A·3}$$

where ξ is a representative of the complex coset $G^{\boldsymbol{C}}/\hat{H}$. Since the form of ξ can be calculated as Eqs. (2·11), (2·23), (3·12), (3·29), (3·47) and (3·65), they give the same Kähler potential obtained from linear models in this paper.

Appendix B

—— The Non-Compactness Theorem of Lerche and Shore ——

The nonlinear sigma model, whose target manifold is compact and homogeneous, has a unique Kähler potential, as discussed in the last appendix.[5),11)] Although these models include neither a QNG boson nor an arbitrariness in the Kähler potential and they are mathematically beautiful, they cannot be obtained from any linear model, at least when there is no gauge symmetry: It was shown that there exists at least one QNG boson, and therefore the target must be non-compact and non-homogeneous. In this appendix, we review the theorem obtained by Lerche and Shore[6)] (see also Ref. 7)).

The fact that the model has a linear origin implies that the target manifold can be obtained from some F-term conditions (if there is no gauge symmetry). Since they are holomorphic equations, the invariance under the global symmetry G enlarges to the complexification $G^{\boldsymbol{C}}$, and the manifold becomes a $G^{\boldsymbol{C}}$-orbit of the vacuum expectation value v.[*)] The pure-type multiplets require that the real broken generators are divided into complex unbroken and complex broken generators, E^i and $\bar{E}_i(= (E^i)^\dagger)$. Since $\bar{E}_i$ is broken, we obtain

$$0 \neq |\bar{E}_i v|^2 = v^\dagger \left[E^i, \bar{E}_i\right] v = \alpha(i)^a v^\dagger H_a v, \tag{B·1}$$

where $\alpha(i)^a$ is a root vector and H_a is a Cartan generator. Therefore, at least one Cartan generator, H_a, must be broken. Since this is hermitian, there exists at least one mixed-type generator, and therefore at least one QNG boson.

Appendix C

—— $SO(N)$ Algebra ——

Since the basis of $SO(N)$ used in §3.1 is not in the standard form, here we give its relation to the ordinary basis. The $SO(N)$ generators in the ordinary basis are

$$(T_{ij})^k{}_l = \frac{1}{i}(\delta_i^k \delta_{jl} - \delta_j^k \delta_{il}). \tag{C·1}$$

In the basis, the vacuum expectation value satisfying $\vec{v}^{\,2} = 0$ can be written as

$$\vec{v} = \begin{pmatrix} -\frac{i}{\sqrt{2}} \\ \mathbf{0} \\ \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{C·2}$$

*) If there are not enough F-term constraints, the manifold may become larger than a $G^{\boldsymbol{C}}$-orbit. However, the proof is valid also in such cases, since they include at least one $G^{\boldsymbol{C}}$-orbit.

The real unbroken generators, at the center of the matrix, generate $SO(N-2)$. The complex unbroken and broken generators are

$$E^i = \left(\begin{array}{c|ccc|c} & \cdots & -\frac{i}{\sqrt{2}} & \cdots & \\ \hline \vdots & & & & \vdots \\ \frac{i}{\sqrt{2}} & & \mathbf{0}_{N-2} & & -\frac{1}{\sqrt{2}} \\ \vdots & & & & \vdots \\ \hline & \cdots & \frac{1}{\sqrt{2}} & \cdots & \end{array}\right), \quad \bar{E}_i = \left(\begin{array}{c|ccc|c} & \cdots & -\frac{i}{\sqrt{2}} & \cdots & \\ \hline \vdots & & & & \vdots \\ \frac{i}{\sqrt{2}} & & \mathbf{0}_{N-2} & & \frac{1}{\sqrt{2}} \\ \vdots & & & & \vdots \\ \hline & \cdots & -\frac{1}{\sqrt{2}} & \cdots & \end{array}\right), \tag{C·3}$$

where $i = 1, \cdots, N-2$ and only the i-th components are nonzero. The broken $U(1)$ generator is

$$T = \left(\begin{array}{c|c|c} & & -i \\ \hline & \mathbf{0}_{N-1} & \\ \hline i & & \end{array}\right). \tag{C·4}$$

This generator will become unbroken after gauging $U(1)_{\mathrm{D}}$. Here, we change the basis by a unitary transformation with

$$U = \left(\begin{array}{c|c|c} \frac{i}{\sqrt{2}} & & \frac{1}{\sqrt{2}} \\ \hline & \mathbf{1}_{N-1} & \\ \hline -\frac{i}{\sqrt{2}} & & \frac{1}{\sqrt{2}} \end{array}\right). \tag{C·5}$$

Since U is a unitary matrix ($U^\dagger U = UU^\dagger = 1$), $\vec{\phi}^\dagger\vec{\phi}$ is invariant, and then $\log(\vec{\phi}^\dagger\vec{\phi})$ also is invariant. By the unitary transformation, the vacuum expectation value is transformed to the standard form,

$$U\vec{v} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}. \tag{C·6}$$

The $SO(N-2)$ generators are not transformed, and the other generators are transformed as

$$UE^iU^\dagger = \left(\begin{array}{c|ccc|c} & \cdots & 1 & \cdots & \\ \hline \vdots & & & & \vdots \\ 0 & & \mathbf{0}_{N-2} & & -1 \\ \vdots & & & & \vdots \\ \hline & \cdots & 0 & \cdots & \end{array}\right), \quad U\bar{E}_iU^\dagger = \left(\begin{array}{c|ccc|c} & \cdots & 0 & \cdots & \\ \hline \vdots & & & & \vdots \\ 1 & & \mathbf{0}_{N-2} & & 0 \\ \vdots & & & & \vdots \\ \hline & \cdots & -1 & \cdots & \end{array}\right),$$

$$UTU^\dagger = \left(\begin{array}{c|c|c} 1 & & \\ \hline & \mathbf{0}_{N-1} & \\ \hline & & -1 \end{array}\right). \tag{C·7}$$

We thus obtain the transformation law (3·4) used in §3.1. Moreover, the second rank invariant tensor is transformed as $\delta_{ij} \to (U\delta U^T)_{ij} = J_{ij}$, where J is defined in Eq. (3·3).

Appendix D
—— *E_6 Algebra* ——

In this appendix, we construct the E_6 algebra by referring to Refs. 15) and 28).

D.1. *Construction of E_6 algebra*

Since an adjoint representation is decomposed as $\mathbf{78} = (\mathbf{45}, 0) \oplus (\mathbf{1}, 0) \oplus (\mathbf{16}, 1) \oplus (\overline{\mathbf{16}}, -1)$,[27] we construct the E_6 algebra as $\mathcal{E}_6 = \mathcal{SO}(10) \oplus \mathcal{U}(1) \oplus \mathbf{16} \oplus \overline{\mathbf{16}}$: We prepare the $SO(10)$ generator T_{AB}, the $U(1)$ generator T, $\mathbf{16}$ as E_α, and $\overline{\mathbf{16}}$ as $\bar{E}^\alpha = (E_\alpha)^\dagger$. Then their commutation relations can be calculated as follows:[15),28)]

$$
\begin{aligned}
&[T_{AB}, T_{CD}] = -i(\delta_{BC} T_{AD} + \delta_{AD} T_{BC} - \delta_{AC} T_{BD} - \delta_{BD} T_{AC}), \quad [T, T_{AB}] = 0, \\
&[T_{AB}, E_\alpha] = -(\sigma_{AB})_\alpha{}^\beta E_\beta, \quad [T_{AB}, \bar{E}^\alpha] = (\sigma^*_{AB})^\alpha{}_\beta \bar{E}^\beta, \\
&[T, E_\alpha] = \frac{\sqrt{3}}{2} E_\alpha, \qquad [T, \bar{E}_\alpha] = -\frac{\sqrt{3}}{2} \bar{E}^\alpha, \\
&[E_\alpha, E_\beta] = [\bar{E}^\alpha, \bar{E}^\beta] = 0, \qquad [E_\alpha, \bar{E}^\beta] = -\frac{1}{2} (\sigma_{AB})_\alpha{}^\beta T_{AB} + \frac{\sqrt{3}}{2} \delta_\alpha{}^\beta T.
\end{aligned}
\tag{D·1}
$$

The $U(1)$ charge of E_α is determined by the difference between $U(1)$ charges of x and y or y and z in Eq. (3·32): $\frac{2}{\sqrt{3}} - \frac{1}{2\sqrt{3}} = \frac{1}{2\sqrt{3}} - (-\frac{1}{\sqrt{3}}) = \frac{\sqrt{3}}{2}$. The second coefficient of the last equation has the same value as the $U(1)$ charge of E_α, from the anti-symmetric property of the structure constants. The relative weight of the first and the second terms is determined by using the Jacobi identity, $[\bar{E}, [E, E]] + (\text{cyclic}) = 0$, and the nontrivial identity for the spinor generators,[15),28)]

$$
\Sigma_{AB} (\sigma_{AB})_\alpha{}^{[\beta} (\sigma_{AB})_\gamma{}^{\delta]} = \frac{3}{2} \delta_\alpha{}^{[\beta} \delta_\gamma{}^{\delta]}. \tag{D·2}
$$

D.2. *Invariant tensor of E_6*

From the tensor product[27] $\mathbf{27} \otimes \mathbf{27} = \overline{\mathbf{27}}_{\rm s} \oplus \cdots$, we know there exists a rank-3 symmetric tensor invariant under E_6. The components of Γ_{ijk} are[28]

$$
\Gamma_{ijk} = \begin{cases} \Gamma_{0AB} = \delta_{AB}, \\ \Gamma_{A\alpha\beta} = \frac{1}{\sqrt{2}} (C \sigma_A{}^\dagger)^{\alpha\beta}, \\ \text{otherwise } 0. \end{cases} \tag{D·3}
$$

These components can be calculated as follows. First, construct the $SO(10) \times U(1)$ invariant of order three:

$$
I_3 = Axz^2 + \frac{1}{\sqrt{2}} z^A y_\alpha (C \sigma_A{}^\dagger)^{\alpha\beta} y_\beta. \tag{D·4}
$$

By the requirement of the invariance of E or $\bar{E}$, we can conclude $A = 1$. (Here we have used the identity (3·42).) The components (D·3) can be read from this invariant.

It is known that there is an identity[29),*)]

$$
\Gamma_{ijk} \Gamma^{ijl} = 10 \delta_k^l. \tag{D·5}
$$

*) In the calculation of $\Gamma_{ijA} \Gamma^{ijB} = 10 \delta_{AB}$, we have used the identity $2^{-4} \mathrm{tr}\,(C \sigma_A{}^\dagger \sigma_B C) = \delta_{AB}$.[28]

Under the normalization in Eq. (D·5), there is the Springer relation[29)]

$$\Gamma_{ijk}(\Gamma^{jl\{m}\Gamma^{no\}k}) = \delta_i{}^{\{l}\Gamma^{mno\}}, \tag{D·6}$$

where we have used the notation $A^{\{ij\cdots\}} = A^{ij\cdots} + A^{ji\cdots} + \cdots$. These identities are used many times in the analysis of the E_7 algebra.

Appendix E
—— *E_7 Algebra* ——

In this appendix, we construct the E_7 algebra in the same way as in the last appendix.

E.1. *Construction of E_7 algebra*

The decomposition of the adjoint representation of E_7 under the maximal subgroup $E_6 \times U(1)$ is $\mathbf{133} = (\mathbf{78}, 0) \oplus (\mathbf{1}, 0) \oplus (\mathbf{27}, 1) \oplus (\overline{\mathbf{27}}, -1)$, where the second components are the $U(1)$ charges.[27)] Hence, we can construct the E_7 algebra by adding generators E^i and $\bar{E}_i (= (E^i)^\dagger)$ $(i = 1, \cdots, 27)$, which belong to the E_6 fundamental and anti-fundamental representations, respectively, to the $E_6 \times U(1)$ algebra, T_A $(A = 1, \cdots, 78)$ and T: $\mathcal{E}_7 = \mathcal{E}_6 \oplus \mathcal{U}(1) \oplus \mathbf{27} \oplus \overline{\mathbf{27}}$. In the same manner as we constructed the E_6 algebra in the last appendix, their commutation relations are obtained as follows:

$$\begin{aligned}
&[T_A, T_B] = i f_{AB}{}^C T_C, && [T, T_A] = 0, \\
&[T_A, E^i] = \rho(T_A)^i{}_j E^j, && [T_A, \bar{E}_i] = -\rho(T_A)^T{}_i{}^j \bar{E}_j, \\
&[T, E^i] = \sqrt{\frac{2}{3}} E^i, && [T, \bar{E}_i] = -\sqrt{\frac{2}{3}} \bar{E}_i, \\
&[E^i, E^j] = [\bar{E}_i, \bar{E}_j] = 0, && [E^i, \bar{E}_j] = \rho(T_A)^i{}_j T_A + \sqrt{\frac{2}{3}} \delta^i{}_j T.
\end{aligned} \tag{E·1}$$

Here $\rho(T_A)$ is a fundamental representation matrix, and the $f_{AB}{}^C$ are structure constants of E_6, whose explicit forms were obtained in the last section. The $U(1)$ charge of E^i is determined from the difference of x and y^i, etc., in Eq. (3·50), and $\bar{E}_i$ is its conjugate. In the last equation, the coefficient of the second term coincides with the $U(1)$ charge of E^i due to the anti-symmetricity of the structure constants of E_7. The first term is determined by the Jacobi identity $[\bar{E}, [E, E]] + \text{(cyclic)} = 0$ and the nontrivial identity for the E_6 fundamental representation,[30)]

$$\Sigma_A\, \rho(T_A)^{[i}{}_j \rho(T_A)^{k]}{}_l = -\frac{2}{3} \delta^{[i}{}_j \delta^{k]}{}_l. \tag{E·2}$$

This is satisfied when $\mathrm{tr}\, \rho(T_A)^2 = 6$.

E.2. *Invariant tensors of E_7*

From the tensor product of fundamental representations[27)], $\mathbf{56} \otimes \mathbf{56} = \mathbf{1}_{\mathrm{a}} \oplus \cdots$ and $\mathbf{56} \otimes \mathbf{56} \otimes \mathbf{56} \otimes \mathbf{56} = \mathbf{1}_{\mathrm{s}} \oplus \cdots$, there exist the rank-2 anti-symmetric tensor $f_{\alpha\beta}$ and the rank-4 symmetric tensor $d_{\alpha\beta\gamma\delta}$ as E_7 invariant tensors. To find their components, we construct a linear combination of $E_6 \times U(1)$ invariants of quartic order and require invariance under E or $\bar{E}$, as in the last appendix. The result is

$$\begin{aligned} I_4 &= d_{\alpha\beta\gamma\delta}\phi^\alpha\phi^\beta\phi^\gamma\phi^\delta \\ &= -\frac{1}{2}(xw - y^i z_i)^2 - \frac{1}{3}w\Gamma_{ijk}y^i y^j y^k - \frac{1}{3}x\Gamma^{ijk}z_i z_j z_k \\ &\quad + \frac{1}{2}\Gamma^{ijk}\Gamma_{ilm}z_j z_k y^l y^m. \end{aligned} \tag{E·3}$$

Here, I_4 is invariant due to the Springer relation for the E_6 invariant tensor, Eq. (D·6). The components can be read from this invariant. Since we do not use the anti-symmetric tensor $f_{\alpha\beta}$, we do not construct it here.

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