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Supersymmetric Quantum Mechanics in Three-Dimensional Space. I

—— *One-Particle System with Spin-Orbit Potential* ——

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As a direct generalization of the model of supersymmetric quantum mechanics by Witten, which describes the motion of a spin one-half particle in the one-dimensional space, we construct a model of the supersymmetric quantum mechanics in the three-dimensional space, which describes the motion of a spin one-half particle in central and spin-orbit potentials in the context of the nonrelativistic quantum mechanics. With the simplest choice of the (super) potential, this model is shown to reduce to the model of the harmonic oscillator plus constant spin-orbit potential of unit strength of both positive and negative signs, which was studied in detail in our recent paper in connection with "accidental degeneracy" as well as the "graded groups". This simplest model is discussed in some detail as an example of the three-dimensional supersymmetric quantum mechanical system, where the supersymmetry is an exact symmetry of the system. More general choice of a polynomial superpotential is also discussed. It is shown that the supersymmetry cannot be spontaneously broken for any polynomial superpotential in our three-dimensional model; this result is contrasted to the corresponding one in the one-dimensional model.

§ 1. Introduction

Supersymmetry (SUSY) has recently attracted renewed interest in high energy physics. In his study of the dynamical breaking of the supersymmetry, Witten[1] has proposed "Supersymmetric Quantum Mechanics" and constructed a simple, but non-trivial model of supersymmetric quantum mechanics which describes the motion of a spin one-half particle in a one-dimensional space. An example of SUSY quantum mechanics in a two-dimensional space has been given in our recent paper.[2] In a series of papers, we shall present several realistic examples of the SUSY quantum mechanics in a three-dimensional space. The present paper, which is the first of the series, deals with a system of a spin one-half particle in central and spin-orbit potentials within the context of non-relativistic quantum mechanics. A more realistic case of a two-particle system will be treated in the second paper.

After brief introduction to SUSY quantum mechanics in the next section, we construct in §3 a three-dimension model of SUSY quantum mechanics, which describes the motion of a spin one-half particle in central and spin-orbit potentials, both of which are determined by a single (super) potential $V(r)$. With the simplest choice of the superpotential, this model is shown to reduce to the model of the harmonic oscillator plus constant spin-orbit potential of unit strength of both positive and negative signs, which was studied in detail in our recent paper[3] in connection with "accidental degeneracy" as well as "graded groups". This simplest model is discussed in §4 as an example of the three-dimensional SUSY quantum mechanical system, in which the supersymmetry is an exact symmetry of the system. A more general case of a polynomial superpotential is treated in the final section. It is shown that the supersymmetry is not broken for any polynomial superpotential. This result is contrasted to the corresponding result of the one-dimensional model.

§ 2. Supersymmetric quantum mechanics

In this section, we briefly summarize the supersymmetric quantum mechanics proposed by Witten[1] in a form well-adapted to our purposes. Let $Q_i (i=1, 2, \cdots, N)$ be the hermitian supercharges. Following Witten, we call the supersymmetric quantum mechanics if the Hamiltonian H of a system and the supercharges Q_i are governed by the following set of relations:

$$\{Q_i, Q_j\}=\delta_{ij}H \quad \text{and} \quad [H, Q_i]=0. \qquad (i=1, 2, \cdots, N) \tag{1}$$

Needless to say, this set of relations is one of the well-known definitions of "supercharge" in high-energy physics[4] when H is replaced by the energy-momentum operator P_μ.

In the case $N=2$, Witten has constructed a simple, but non-trivial model of the supersymmetric quantum mechanics, which describes the motion of spin one-half particle on a line (say, the x-axis). Q_1 and Q_2 are defined by

$$Q_1=(1/2)[\sigma_1 p_x+\sigma_2 W(x)\} \quad \text{and} \quad Q_2=(1/2)\{\sigma_2 p_x-\sigma_1 W(x)\}, \tag{2}$$

where $p_x=(\hbar/i)(d/dx)$, $W(x)$ a function of x and $\sigma_i (i=1, 2, 3)$ the Pauli spin matrix. The Hamiltonian of the system, whose wave function is a two-component spinor, is given by

$$H=\frac{1}{2}\left[p_x{}^2+\{W(x)\}^2+\hbar\sigma_3\frac{dW(x)}{dx}\right]. \tag{3}$$

It is easy to check that H, Q_1 and Q_2 satisfy (1) for $N=2$. Hence, the model constitutes a $(N=2)$ supersymmetric quantum mechanics. To examine the structure of the theory, let us introduce the non-hermitian supercharge $Q=(1/\sqrt{2})(-Q_2+iQ_1)$ and its hermitian conjugate, $Q^+=(1/\sqrt{2})(-Q_2-iQ_1)$, which — transcribed in two-by-two matrices — are represented as

$$Q^+=\begin{pmatrix}0 & D\\ 0 & 0\end{pmatrix}=\sigma_+ D \quad \text{and} \quad Q=\begin{pmatrix}0 & 0\\ D^+ & 0\end{pmatrix}=\sigma_- D^+, \tag{4}$$

where $D=(1/\sqrt{2})\{W(x)+\hbar d/dx\}$ and $D^+=(1/\sqrt{2})\{W(x)-\hbar d/dx\}$. In terms of these D and D^+, H can be written as

$$H=\{Q, Q^+\}=\begin{pmatrix}DD^+ & 0\\ 0 & D^+D\end{pmatrix}=(1/2)\{D, D^+\}+(1/2)\sigma_3[D, D^+]. \tag{5}$$

The defining set of Eq. (1) of the $(N=2)$ supersymmetric quantum mechanics reads

$$Q^2=\{Q^+\}^2=0, \tag{6a}$$

$$H=\{Q, Q^+\}, \tag{6b}$$

$$[H, Q]=[H, Q^+]=0. \tag{6c}$$

It is now trivial to see that (4) and (5) satisfy set (6), *independently* of the explicit forms of D and D^+. From this, we observe that a direct generalization of Witten's one-dimensional model to three-dimensional problems will be possible by appropriate choices of the operators D and D^+. In subsequent sections, we shall take a specific form of these operators which can be represented in two-by-two matrices. Therefore, Q and Q^+ in this case are four-by-four nilpotent matrices.

Finally, we note that the three relations in (6) are not mutually independent. Namely, $[H, Q]=[H, Q^+]=0$ which states the conservation of supercharges is a mere consequence of the nilpotent forms of Q and Q^+ in (6a) and the assumed form of H in (6b).

§ 3. A three-dimensional model of SUSY quantum mechanics

In this section, we shall construct a realistic model of a SUSY quantum mechanics in the three-dimensional space, which describes the motion of a spin one-half particle in central and spin-orbit potentials.

First, we introduce the supercharges through

$$Q=\rho_-(\sigma_x D_x{}^+ + \sigma_y D_y{}^+ + \sigma_z D_z{}^+)=\rho_- \boldsymbol{\sigma}\cdot\boldsymbol{D}^+ , \tag{7a}$$

$$Q^+=\rho_+(\sigma_x D_x + \sigma_y D_y + \sigma_z D_z)=\rho_+ \boldsymbol{\sigma}\cdot\boldsymbol{D} , \tag{7b}$$

where the operators $\boldsymbol{D}$ and $\boldsymbol{D}^+$ will be specified shortly. Here, $\boldsymbol{\sigma}$ is the spin of particle under consideration and $\rho_\pm=\frac{1}{2}(\rho_1 \pm i\rho_2)$, $\rho_i(i=1, 2, 3)$ being another set of the Pauli matrices. In the matrix form, Q and Q^+ are represented as

$$Q=\begin{pmatrix} 0 & 0 \\ \boldsymbol{\sigma}\cdot\boldsymbol{D}^+ & 0 \end{pmatrix} \quad \text{and} \quad Q^+=\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{D} \\ 0 & 0 \end{pmatrix} \tag{7c}$$

which should be compared with (4). The supersymmetric Hamiltonian H is then given by

$$\begin{aligned} H=\{Q, Q^+\}&=\begin{pmatrix} (\boldsymbol{\sigma}\cdot\boldsymbol{D})(\boldsymbol{\sigma}\cdot\boldsymbol{D}^+) & 0 \\ 0 & (\boldsymbol{\sigma}\cdot\boldsymbol{D}^+)(\boldsymbol{\sigma}\cdot\boldsymbol{D}) \end{pmatrix} \\ &=(1/2)\{(\boldsymbol{\sigma}\cdot\boldsymbol{D}), (\boldsymbol{\sigma}\cdot\boldsymbol{D}^+)\}1+(1/2)\rho_3[(\boldsymbol{\sigma}\cdot\boldsymbol{D}), (\boldsymbol{\sigma}\cdot\boldsymbol{D}^+)]. \end{aligned} \tag{8}$$

For the spin-independent D and D^+, we obtain

$$\{(\boldsymbol{\sigma}\cdot\boldsymbol{D}), (\boldsymbol{\sigma}\cdot\boldsymbol{D}^+)\}=\sum_{k=x,y,z}\{D_k, D_k{}^+\}+\sum_{\text{cyclic}} i\sigma_z([D_x, D_y{}^+]-[D_y, D_x{}^+]), \tag{9a}$$

$$[(\boldsymbol{\sigma}\cdot\boldsymbol{D}), (\boldsymbol{\sigma}\cdot\boldsymbol{D}^+)]=\sum_{k=x,y,z}[D_k, D_k{}^+]+\sum_{\text{cyclic}} i\sigma_z(\{D_x, D_y{}^+\}-\{D_y, D_x{}^+\}). \tag{9b}$$

As a natural generalization of Witten's construction, we now take the following forms of D and D^+:

$$\begin{aligned} D_x&=(1/\sqrt{2})(\partial V/\partial x+\hbar\partial/\partial x); & D_x{}^+&=(1/\sqrt{2})(\partial V/\partial x-\hbar\partial/\partial x), \\ D_y&=(1/\sqrt{2})(\partial V/\partial y+\hbar\partial/\partial y); & D_y{}^+&=(1/\sqrt{2})(\partial V/\partial y-\hbar\partial/\partial y), \\ D_z&=(1/\sqrt{2})(\partial V/\partial z+\hbar\partial/\partial z); & D_z{}^+&=(1/\sqrt{2})(\partial V/\partial z-\hbar\partial/\partial z). \end{aligned} \tag{10}$$

Here, the (super) potential V is assumed to be a function of r alone, so that we can write $\partial V(r)/\partial x=(x/r)(\partial V/\partial r)$ and so on. With these choices of D and D^+, the commutators and anticommutators appearing on the right-hand side of (9) are quickly calculated to obtain

$$[D_x, D_x^+]=\hbar\partial^2 V/\partial x^2=\hbar[(1/r)(\partial V/\partial r)+(x^2/r)\partial\{(1/r)(\partial V/\partial r)\}/\partial r],$$
$$[D_x, D_y^+]=[D_y, D_x^+]=\hbar\partial^2 V/\partial x\partial y\,,$$
$$\{D_x, D_x^+\}=(\partial V/\partial x)^2-\hbar^2\partial^2/\partial x^2\,,$$
$$\{D_x, D_y^+\}-\{D_y, D_x^+\}=2\hbar\{(\partial V/\partial y)\partial/\partial x-(\partial V/\partial x)\partial/\partial y)\}$$
$$=2\hbar(1/r)(\partial V/\partial r)(y\partial/\partial x-x\partial/\partial y)=-2i(1/r)(\partial V/\partial r)L_z\,.$$

Hence, we have

$$\{(\boldsymbol{\sigma}\cdot\boldsymbol{D}), (\boldsymbol{\sigma}\cdot\boldsymbol{D}^+)\}=-\hbar^2\nabla^2+(\nabla V\cdot\nabla V),$$
$$[(\boldsymbol{\sigma}\cdot\boldsymbol{D}), (\boldsymbol{\sigma}\cdot\boldsymbol{D}^+)]=\hbar^2\nabla^2 V+2(1/r)(\partial V/\partial r)(\boldsymbol{\sigma}\cdot\boldsymbol{L}) \tag{11}$$

from which we finally determine the explicit form of the supersymmetric Hamiltonian, H, in (8):

$$H=\frac{1}{2}\left\{-\hbar^2\nabla^2+\left(\frac{\partial V}{\partial r}\right)^2\right\}\mathbf{1}+\rho_3\left[\frac{\hbar}{2}\left\{\frac{3}{r}\frac{\partial V}{\partial r}+r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial V}{\partial r}\right)\right\}+\frac{1}{r}\frac{\partial V}{\partial r}(\boldsymbol{\sigma}\cdot\boldsymbol{L})\right], \tag{12}$$

where

$$\mathbf{1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \text{ and } \rho_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

It is now clear that our model Hamiltonian describes the motion of a spin one-half particle in the central and spin-orbit potentials, both of which are defined through the single (super) potential $V(r)$. Hence, our model may be regarded as a natural generalization of Witten's model in one-dimension to the three-dimensional space.

In summary, we have defined the supercharges, Q and Q^+, through (7a) and (7b). The relation, $\{Q\}^2=\{Q^+\}^2=0$, is evident from (7c). The explicit form of the supersymmetric Hamiltonian $H=\{Q, Q^+\}$ has been given by (12). As noted at the end of the preceding section, the relation, $[H, Q]=[H, Q^+]=0$, is trivially satisfied.

Finally, we present the spherical forms of $(\boldsymbol{\sigma}\cdot\boldsymbol{D})$ and $(\boldsymbol{\sigma}\cdot\boldsymbol{D}^+)$: By making use of (10), we obtain

$$\sqrt{2}(\boldsymbol{\sigma}\cdot\boldsymbol{D})=(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})(\partial V/\partial r)+\hbar(\boldsymbol{\sigma}\cdot\nabla)$$
$$=(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\left\{\frac{\partial V}{\partial r}-\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{L})}{r}+\hbar\frac{\partial}{\partial r}\right\}$$
$$=\left\{\frac{\partial V}{\partial r}+\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{L})}{r}+\frac{3\hbar}{r}+\hbar r\frac{\partial}{\partial r}\frac{1}{r}\right\}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}), \tag{13a}$$

$$\sqrt{2}(\boldsymbol{\sigma}\cdot\boldsymbol{D}^+)=(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})(\partial V/\partial r)-\hbar(\boldsymbol{\sigma}\cdot\nabla)$$
$$=(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\left\{\frac{\partial V}{\partial r}+\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{L})}{r}-\hbar\frac{\partial}{\partial r}\right\}$$
$$=\left\{\frac{\partial V}{\partial r}-\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{L})}{r}-\frac{3\hbar}{r}-\hbar r\frac{\partial}{\partial r}\frac{1}{r}\right\}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}). \tag{13b}$$

Elementary manipulations yield the result in (12). The spherical form of the supercharges are simply obtained by inserting (13a) and (13b) into (7c). This form of the supercharges will be particularly suited for examining the normalizability of the zero-energy solution of the system, by which we can conclude whether the supersymmetry is broken or not, as will be seen in the next section.

§ 4. An example

We have constructed the model of the supersymmetric quantum mechanics which describes the motion of spin one-half particle in the central and spin-orbit potentials, both of which are determined by a single (super) potential $V(r)$.

As an example, we now take the simplest form of the (super) potential: $V(r) = (1/2)\omega r^2$. With this choice, the Schrödinger equation $H\chi = E\chi$ of the system reads

$$\begin{pmatrix} \frac{1}{2}(\boldsymbol{p}^2+\omega^2\boldsymbol{r}^2)+\frac{3}{2}\hbar\omega+\omega(\boldsymbol{\sigma}\cdot\boldsymbol{L}) & \\ & \frac{1}{2}(\boldsymbol{p}^2+\omega^2\boldsymbol{r}^2)-\frac{3}{2}\hbar\omega-\omega(\boldsymbol{\sigma}\cdot\boldsymbol{L}) \end{pmatrix}\begin{pmatrix}\chi_B\\ \chi_F\end{pmatrix} = E\begin{pmatrix}\chi_B\\ \chi_F\end{pmatrix}. \quad (14)$$

As usual, we shall call the upper-component χ_B the bosonic state and the lower-component χ_F the fermionic state. As seen from (14), the bosonic and fermionic sectors are reversed when we take the negative sign of the (super) potential; $V(r) = -(1/2)\omega r^2$. Hereafter, we shall use the unit $\hbar = 1$ and take $\omega = +1$.

It is amusing to note that the bosonic and fermionic sectors of this model have been separately studied in detail in our recent paper[3] in connection with the accidental degeneracy as well as the graded groups. The full details of the algebraic structure of each sector can be found there. We here discuss the model with respect to the supersymmetric quantum mechanics. For this purpose, let us summarize the general properties of the ($N=2$) SUSY quantum mechanics, which can be directly obtained from its defining relations; $\{Q\}^2 = \{Q^+\}^2 = 0$, $H = \{Q, Q^+\}$ and $[H, Q] = [H, Q^+] = 0$.

(A) All the eigenvalues of H are non-negative.

(B) For the positive-energy solutions, Q brings a bosonic state to a fermionic state of the same eigenvalue E, while Q^+ transforms a fermionic state into a bosonic state, i.e., $Q\chi_B = \sqrt{E}\,\chi_F$ and $Q^+\chi_F = \sqrt{E}\,\chi_B$. Thus, the supersymmetry pairs the bosonic and fermionic states of all positive energy solutions — boson-fermion degeneracy.

(C) For the supersymmetry to be a good symmetry, the supercharges, Q and Q^+, should annihilate the ground state χ_g; $Q\chi_g = Q^+\chi_g = 0$. It then follows that $H\chi_g = \{Q, Q^+\}\chi_g = 0$. Namely, for the supersymmetry to be a good symmetry, the ground state energy should be zero. It is also clear that, when the supersymmetry is spontaneously broken, the ground-state energy cannot be zero. Since all the states are paired except the zero-energy solution from (B), there exist unpaired (zero-energy) solutions only if the supersymmetry is an exact symmetry of the system — the famous Witten index theorem.

We present in Fig. 1 the level scheme of our model, where the right-hand side is the fermionic sector, the left-hand side the bosonic sector. From the figure, we observe that there is boson-fermion degeneracy in all the excited states and that there exist zero-energy solutions in the fermionic sector alone. This latter observation immediately leads to the conclusion that the supersymmetry is a good symmetry of our model.

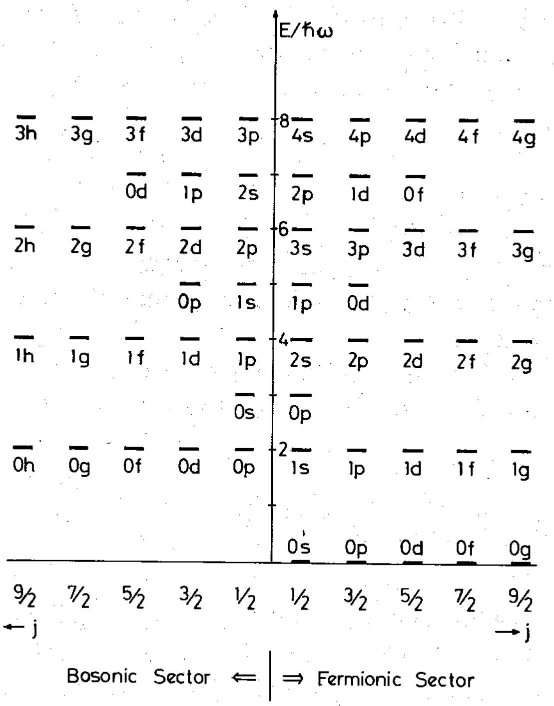

Fig. 1. The level scheme of the simplest model of the three-dimensional supersymmetric quantum mechanics in Eq. (14). The upper (bosonic) sector is displayed on the left-hand side, while the lower (fermionic) sector is shown on the right-hand side. The absissa of both sides denotes the total angular momentum j of the state and the ordinate the (absolute) energy in units of $\hbar\omega$. Note that the n-th excited states are infinitely degenerate when n=even (including the ground state), while it is finitely degenerate when n=odd. For detail of the algebraic structure of each sector, see our previous paper.[3)]

It will be instructive to derive here the zero-energy solutions of the model within the context of our SUSY quantum mechanics, because the same procedure is applicable to the more general choice of the (super) potential $V(r)$. As noted in (C), the zero-energy solutions can be obtained by $Q\chi_g = Q^+\chi_g = 0$. Denoting the bosonic and fermionic states in χ_g as $|B)$ and $|F)$, we have $Q|B)=0$ and $Q^+|F)=0$. Our problem is now to examine whether there are acceptable (normalizable, non-singular) solutions in these equations. We adopt the spherical forms of Q and Q^+ presented in (13a) and (13b). The radial parts of the equations, $Q|B)=0$ and $Q^+|F)=0$, are simply given by

$$\left(\frac{\partial V}{\partial r}+\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{L})}{r}-\frac{\partial}{\partial r}\right)|B)=0\,, \tag{15a}$$

$$\left(\frac{\partial V}{\partial r}-\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{L})}{r}+\frac{\partial}{\partial r}\right)|F)=0\,. \tag{15b}$$

These first-order differential equations can be easily integrated to obtain the solutions,

$$|B)=\text{const}\; e^{V(r)}r^{-(\boldsymbol{\sigma}\cdot\boldsymbol{L})}, \tag{16a}$$

$$|F)=\text{const}\; e^{-V(r)}r^{(\boldsymbol{\sigma}\cdot\boldsymbol{L})}. \tag{16b}$$

With our simplest choice of $V(r)=+r^2/2$, it is now clear from the behaviors of (16a) and (16b) at large r that there is no acceptable zero-energy solution in the bosonic sector, while the fermionic states in (16b) have the correct behavior $\exp(-r^2/2)$ at large r. To examine the behavior of $|F)$ at $r=0$, we note that the total angular momentum j is always the good quantum number of our model and that, for an eigenstate of the total angular momentum, $(\boldsymbol{\sigma}\cdot\boldsymbol{L})$ takes the value $+l$ or $-(l+1)$ according to whether $j=l+\frac{1}{2}$ or $j=l$

$-\frac{1}{2}$, l being the orbital angular momentum. From the requirement that the solutions should be non-singular at $r=0$, we exclude the states with $j=l-\frac{1}{2}$. Thus, we finally obtain the radial part of the normalizable, non-singular zero-energy solutions

$$\left|F{:}j=l+\frac{1}{2}\right)=\text{const } r^l e^{-r^2/2} \qquad \text{for all } l,$$

which constitute all the nodeless $j=l+\frac{1}{2}$ states, as displayed in Fig. 1. This procedure illustrates a method to examine the existence of the zero-energy solution for more general choice of (super) potential $V(r)$, from which we can easily conclude whether the supersymmetry is broken or not for this choice of V.

Finally, we present other, but equivalent forms of Q, Q^+ and H of our simplest model. Since $\partial V/\partial x=x$ for our specific choice $V=(1/2)r^2$, D_x and D_x^+ defined by (10) can be identified with the boson annihilation and creation operators, b_x and b_x^+ respectively. Hence, we have

$$Q=\rho_-(\sigma_x b_x^+ + \sigma_y b_y^+ + \sigma_z b_z^+)=\rho_-(\boldsymbol{\sigma}\cdot\boldsymbol{b}^+) \quad \text{and} \quad Q^+=\rho_+(\boldsymbol{\sigma}\cdot\boldsymbol{b})$$

and

$$H=\begin{pmatrix} \boldsymbol{b}\cdot\boldsymbol{b}^+ + \boldsymbol{\sigma}\cdot\boldsymbol{L} & 0 \\ 0 & \boldsymbol{b}^+\cdot\boldsymbol{b}-\boldsymbol{\sigma}\cdot\boldsymbol{L} \end{pmatrix}.$$

Since $(\boldsymbol{\sigma}\cdot\boldsymbol{b}^+)$ and $(\boldsymbol{\sigma}\cdot\boldsymbol{b})$ in the above forms of supercharge are pseudoscalar, the boson-fermion degeneracy in the positive energy solutions in Fig. 1 is now almost evident (for detail, see Ref. 3)). A more compact representation of the above H in terms of the paraboson operator, $(\boldsymbol{\sigma}\cdot\boldsymbol{b})$ and $(\boldsymbol{\sigma}\cdot\boldsymbol{b}^+)$, can be found in our previous paper.[3)]

§ 4. Discussion

In the preceding section, we have discussed a model of the three-dimensional SUSY quantum mechanics, in which the simplest form of the (super) potential $V(r)=(1/2)r^2$ is adopted. Here, we shall discuss the more general case of the polynomial (super) potential, $V(r)=\sum_{n=0}^N a_n r^n$, $(a_N>0)$. Although the excited states of this model cannot be solved analytically, its zero-energy solutions can be easily obtained in just the same way as that is the preceding section. Namely, by inserting this $V(r)$ into (16a) and (16b), we observe that there is no (acceptable) zero-energy solutions in the bosonic sector. Further, by applying the same arguments as given below Eq. (16b), we obtain the radial part of the zero-energy solutions in the fermionic sector

$$\left|F{:}\ j=l+\frac{1}{2}\right)=\text{const } r^l \exp\{\sum_{n=0}^N a_n r^n\}. \qquad (a_N>0)$$

Thus, we have again the *infinitely degenerate* (zero-energy) ground state in the fermionic sector. Needless to say, if we take $-V(r)$ instead of $+V(r)$, the fermionic and bosonic sectors are reversed.

From the existence of the zero-energy solutions, we conclude that the supersymmetry is a good symmetry in our three-dimensional model with any polynomial superpotential $V(r)$. In this connection, it is interesting to compare this result with that of the corresponding one-dimensional model,[1)] in which the one-dimensional superpotential $U(x)$,

$U(x)=dW(x)/dx$ in §2, is taken to be an N-th order polynomial of x: $U(x)=\sum_{n=0}^{N} a_n x^n$. In this model, the zero-energy solution can be obtained by $Q|B)=0$ and $Q^+|F)=0$ in (4), which in turn are written as

$$\left\{\frac{dU(x)}{dx}+\frac{d}{dx}\right\}|F)=0 \quad \text{and} \quad \left\{\frac{dU(x)}{dx}-\frac{d}{dx}\right\}|B)=0\,.$$

Hence, we have

$$|F)=\text{const}\ e^{-U(x)} \quad \text{and} \quad |B)=\text{const}\ e^{+U(x)}.$$

The normalizability of these formal solutions can be examined simply by observing their behaviours at $x=\pm\infty$. It is now clear that the zero-energy solution exists only when N is even — either in the fermionic sector or in the bosonic sector depending on whether a_N is positive or negative. When N is odd, there is no zero-energy solution. Namely, the supersymmetry is spontaneously broken for the N-th order polynomial superpotential when N=odd, while the supersymmetry is a good symmetry when N=even — the well-known result[1] of the one-dimensional model. On the contrary, we have shown that in our three-dimensional model, which is a natural generalization of the one-dimensional model by Witten, the supersymmetry cannot be spontaneously broken for any polynomial superpotential $V(r)$.

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