

Supersymmetric Solutions with Fluxes from Algebraic Killing Spinors

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Abstract

We give a general framework for constructing supersymmetric solutions in the presence of non-trivial fluxes of tensor gauge fields. This technique involves making a general Ansatz for the metric and then defining the Killing spinors in terms of very simple projectors on the spinor fields. These projectors and, through them, the spinors, are determined *algebraically* in terms of the metric Ansatz. The Killing spinor equations then fix the tensor gauge fields algebraically, and, with the Bianchi identities, provide a system of equations for all the metric functions. We illustrate this by constructing an infinite family of massive flows that preserve eight supersymmetries in M -theory. This family constitutes all the radially symmetric Coulomb branch flows of the softly broken, large N scalar-fermion theory on $M2$ -branes. We

reduce the problem to the solution of a single, non-linear partial differential equation in two variables. This equation governs the flow of the fermion mass, and the function that solves it then generates the entire M -theory solution algebraically in terms of the function and its first derivatives. While the governing equation is non-linear, it has a very simple perturbation theory from which one can see how the Coulomb branch is encoded.

1 Introduction

It has been an outstanding open problem in AdS/CFT to find a relatively simple characterization of *supersymmetric* backgrounds that involve non-trivial fluxes. Such solutions are particularly important in the study of holographic RG flows in supergravity: First, there is always the flux generated by the branes upon which the holographic field theory lies, and then one often wants to add further fluxes that are typically holographically dual to fermion mass terms on the brane. Such “multiple flux” solutions thus lie at the heart of the holographic study of softly broken supersymmetry.

There are two basic methods of generating such solutions: One can either work with the appropriate supergravity theory in ten or eleven dimensions, or one can work with the appropriate gauged supergravity in lower dimensions. While some beautiful results have been obtained using the latter approach, it is ultimately limited because a gauged supergravity theory truncates the fields of the higher-dimensional theory to some especially simple sub-sector of “lowest harmonics”. The resulting solutions thus involve very symmetric fluxes with very smooth brane distributions. The huge advantage of gauged supergravity is that it produces an extremely simple description of flows whose higher dimensional analogues, or “lifts” can be extremely complicated. One version of the problem that we will address in this paper is how to extract a general geometric principle that may be used to construct multi-flux solutions directly in ten or eleven dimensions, and yet embody the remarkable simplicity apparent in the gauged supergravity description.

The second method of generating supersymmetric flux solutions is to work directly in ten or eleven dimensions. There have been some beautiful results, like the “harmonic principle” for intersecting branes. In holographic field theory this leads to, amongst other things, a very nice picture of the Coulomb branch of a supersymmetric theory. More generally, one would like to study flows that not only involve Coulomb branch flows, but also involve softly broken supersymmetry. The difficulty lies in attempting to classify such solutions with “multiple fluxes” that lie in different directions

with respect to the underlying branes. Much important work has been done of work on this, particularly for solutions that involve wrapped NS 5-branes (see, for example, [1], [2], [3], [4], [5]). There was also some important early work on four-dimensional Einstein-Maxwell solutions [6], [7]. The general philosophy of the more recent papers on this subject has been to try to exploit the ideas of G -structures (see for example, [8], [4], [5]). This approach is very promising, but the concept of a G -structure is rather a general one, and thus far, the new results have been largely restricted to low dimensions or to the consideration of a single background flux, and most particularly to a flux that may be thought of as a torsion.

In this paper we will present an approach to generating whole new families of solutions with multiple fluxes. In particular, we believe that our approach will enable one to obtain the most general holographic analogues of the Donagi-Witten ($\mathcal{N} = 2^*$) flows to an arbitrary point on the Coulomb moduli space. While we do not solve this particular problem here, we do solve an M -theory analogue: We find an infinite family of M -theory flow solutions with eight supersymmetries. This new family is obtained from a solution to a second order PDE in two variables, and it represents a generalization of the flow of [9] to a spherically symmetric Coulomb distribution of $\mathcal{N} = 4^*$ flows of the $\mathcal{N} = 8$ scalar-fermion theory in $(2 + 1)$ -dimensions. The fact that we can find such a solution represents progress in what has been a technically complicated subject, however we think that the method by which it is obtained will admit many interesting variations and generalizations. Indeed, an analysis of a class of $\mathcal{N} = 2^*$ flows in IIB supergravity will appear in [10].

Our approach is very much in the spirit of the ideas of G -structures, however rather than constructing all the differential forms associated with the Killing spinor, we make a rather general Ansatz for the metric, and then we make an *algebraic* Ansatz for the Killing spinors. To be more precise, we will work in eleven dimensions, and thus a Killing spinor will mean a solution of the equation:

$$\delta\psi_\mu \equiv \nabla_\mu \epsilon + \frac{1}{144} \left(\Gamma_\mu^{\nu\rho\lambda\sigma} - 8\delta_\mu^\nu \Gamma^{\rho\lambda\sigma} \right) F_{\nu\rho\lambda\sigma} = 0. \quad (1)$$

We will require that the space of Killing spinors be defined by two very simple projection operators, Π_0 and Π_1 , on the spinor space. Each projector reduces the number of spinor components by half, and so the two projectors reduce the 32-component spinor to eight components. These remaining eight spinors, ϵ^j , $j = 1, \dots, 8$ are then required to be supersymmetries. The form of the projector, Π_1 , is elementary and it is naturally motivated in terms of the moduli space of brane probes. The other projector, Π_0 , is a non-trivial

deformation of the usual Dirichlet projector parallel to the $M2$ -brane. The key to understanding this projector lies in the fact that the spinor bilinears:

$$K_{(ij)}^\mu \equiv \bar{\epsilon}^i \Gamma^\mu \epsilon^j, \quad (2)$$

are necessarily Killing vectors in M -theory. For a harmonic $M2$ -brane distribution, equation 2 will only give Killing vectors, K^μ , parallel to the $M2$ -brane. The deformation is constructed so that 2 gives rise to another “internal” Killing vector, L^μ , transverse to the original $M2$ -brane distribution. This deformation involves an, *a priori*, arbitrary function. However, this function is fixed in terms of the metric coefficients by requiring that L^μ is indeed a Killing vector (and not a functional multiple of a Killing vector). Moreover, requiring that the vectors, K^μ , parallel to the branes are Killing vectors completely determines the normalization of the Killing spinors in terms of the metric. Thus the Killing spinor Ansatz is completely determined in terms of *algebraic* combinations of functions that appear in the metric Ansatz. Since the Killing spinor equation is a first order differential equation, the Killing spinors would normally involve first derivatives of metric coefficients. Our approach thus produces a rather special class of “algebraic Killing spinors”.

Our result generalizes the well known result about Killing spinors for harmonic distributions of branes:

$$\epsilon = H^{-\alpha} \epsilon_0, \quad (3)$$

in which ϵ_0 is a constant spinor, α is a rational number ($\frac{1}{6}$ for $M2$ -branes) and H is the harmonic function that appears in the metric. Such harmonic solutions have 16 supersymmetries in maximal supergravity. The formula 3 can be deduced from the form of the metric and requiring that 2 yields Killing vectors. Our Killing spinors will be characterized by two metric functions. The second function arises in the further reduction of the supersymmetry.

We believe that our method can easily be adapted to study softly broken theories in which there is a non-trivial space of moduli, such as a Coulomb branch. For example, there will presumably be whole classes of solutions where that involve another projector, Π_2 , that reduces the number of symmetries to four. The projector, Π_2 , could be obtained from the reduction of the supersymmetry on the space of moduli.

Having made a metric Ansatz and fixed the Killing spinors there remains the problem of reconstructing the background tensor gauge field. This turns out to be very straightforward: The Killing spinor equation 1 can be used to determine the Maxwell tensor algebraically in terms of the metric and

Killing spinor. The interesting differential equations will then emerge from the Bianchi identities.

Alternatively, one can use some of the G -structure to obtain the tensor gauge fields. That is, one defines forms from bilinears of Killing spinors:

$$\Omega_{\mu_1\mu_2\dots\mu_p}^{(ij)} \equiv \bar{\epsilon}^i \Gamma_{\mu_1\mu_2\dots\mu_p} \epsilon^j. \quad (4)$$

These forms satisfy first order differential equations that may be derived from 1. These equations have been catalogued in [11]. We can use one or two of the simpler equations to deduce almost all the components of the background 3-form potential, $A_{\mu\nu\rho}^{(3)}$. Specifically, we can show that if $K_{(ij)}^\rho$ is a Killing vector given by 2 then there is a gauge in which:

$$\Omega_{\mu\nu}^{(ij)} \sim A_{\mu\nu\rho}^{(3)} K_{(ij)}^\rho. \quad (5)$$

Thus we may obtain most of the components of the tensor gauge field. Indeed, in the example that we consider here, this gives us all but one component of $A^{(3)}$. Again notice that these components are algebraic combinations of the metric functions.

Either way, it is very straightforward to obtain an expression for $F_{\nu\rho\lambda\sigma}$ in terms of derivatives of metric functions. In the specific example considered in this paper, the Bianchi identities then reduce to a single second order, *non-linear* PDE:

$$\frac{1}{u^3} \frac{\partial}{\partial u} \left(u^3 \frac{\partial}{\partial u} \left(\frac{1}{u^2} g(u, v) \right) \right) + \frac{1}{v} \frac{\partial}{\partial v} \left(g(u, v) \frac{1}{v} \frac{\partial}{\partial v} \left(\frac{v^2}{u^2} g(u, v) \right) \right) = 0. \quad (6)$$

While we do not know if this PDE is explicitly solvable in general, it does have a very simple perturbation theory in which the solution at n^{th} order involves solving a simple linear PDE with a source that is a quadratic form in the lower-order solutions and their derivatives. The zeroth order solution is a constant, and the first order “seed” is *any* solution of the homogeneous linear equation:

$$\frac{1}{u} \frac{\partial}{\partial u} \left(u^3 \frac{\partial}{\partial u} \left(\frac{1}{u^2} g(u, v) \right) \right) + \frac{1}{v} \frac{\partial}{\partial v} \left(\frac{1}{v^3} \frac{\partial}{\partial v} (v^4 g(u, v)) \right) = 0. \quad (7)$$

Thus we see that there is an infinite family of solutions to the non-linear PDE generated by the family of solutions of the linearized PDE.

There is one potential danger in our procedure: Solving the Killing spinor equation does *not* necessarily guarantee a solution to the full set of equations

of motion of the theory. This seems to violate some basic preconceptions about supersymmetry, but harmonic distributions of branes illustrate this point. One can easily check that the Ansatz 3 satisfies 1 for the appropriate “harmonic” Ansatz for the metric and tensor gauge field. However the Killing spinor equation works for any arbitrary function H : The harmonic condition on H *only* comes from imposing the equations of motion¹. The supergravity preconception arises from the fact that the commutator of two supersymmetries generates equations of motion, and so it seems that one cannot solve 1 without solving the equations of motion. However, the commutator of two supersymmetries does not necessarily generate *all* equations of motion. For the harmonic brane configuration it generates a combination of the Einstein and Maxwell equations from which the Laplacian on H cancels. Thus there is a danger that solving the equations of motion might impose further conditions on our solution and perhaps render it trivial. This does not happen in our example: We have explicitly checked that the second order, non-linear PDE is both necessary and sufficient to solve all the equations of motion. While we verified every equation of motion, it turns out that the procedure can be significantly simplified: There is a very nice discussion of “sufficiency” in section 2 of [11], where it is shown, very generally, that once one has satisfied the supersymmetry variations, one only needs to check, at most, one of the Einstein equations and only a subset of the Maxwell equations. We also suspect that any remaining “insufficiency” of solving 1 is a pathology of the special structure of the “harmonic” brane Ansatz, and that for any sufficiently complicated solution (like ours), solving 1 will be sufficient to solve all the equations of motion.

In section 2 of this paper we will describe our basic approach to generating families of supersymmetric flux solutions, while in section 3 we will consider a detailed example that generalizes the flow of [9]. Section 4 contains some remarks about the geometry transverse to the branes, and section 5 contains some final comments.

2 The Ansatz: Generalities

Our purpose here is to outline the aspects of our Ansatz that should generalize readily to other settings. While our comments will be made primarily for M -theory, they should be readily applicable to any supergravity theory.

Our M -theory conventions are those of [12]. Our metric is “mostly plus,”

¹We are grateful to J. Gomis for pointing this out to us.

and we will take the gamma-matrices to be

$$\begin{aligned} \Gamma_1 &= -i \Sigma_2 \otimes \gamma_9, & \Gamma_2 &= \Sigma_1 \otimes \gamma_9, & \Gamma_3 &= \Sigma_3 \otimes \gamma_9, \\ \Gamma_{j+3} &= \mathbf{1}_{2 \times 2} \otimes \gamma_j, & j &= 1, \dots, 8, \end{aligned} \tag{8}$$

where the Σ_a are the Pauli spin matrices, $\mathbf{1}$ is the Identity matrix, and the γ_j are real, symmetric $SO(8)$ gamma matrices. As a result, the Γ_j are all real, with Γ_1 skew-symmetric and Γ_j symmetric for $j > 2$. One also has:

$$\Gamma^{1 \dots 11} = \mathbf{1},$$

where $\mathbf{1}$ will henceforth denote the 32×32 identity matrix. The gravitino variation will be as in 1. With these conventions, sign choices and normalizations, the equations of motion are:

$$\begin{aligned} R_{\mu\nu} + R g_{\mu\nu} &= \frac{1}{3} F_{\mu\rho\lambda\sigma} F_{\nu}{}^{\rho\lambda\sigma}, \\ \nabla_{\nu} F^{\mu\nu\rho\sigma} &= -\frac{1}{576} \varepsilon^{\nu\rho\sigma\lambda_1\lambda_2\lambda_3\lambda_4\tau_1\tau_2\tau_3\tau_4} F_{\lambda_1\lambda_2\lambda_3\lambda_4} F_{\tau_1\tau_2\tau_3\tau_4}. \end{aligned} \tag{9}$$

2.1 The harmonic brane solutions

It is first worth recalling the situation for a harmonic distribution of $M2$ branes. The metric is given by:

$$ds_{11}^2 = H^{-2/3} (-dt^2 + dx_1^2 + dx_2^2) + H^{1/3} \left(\sum_{j=1}^8 dy^j dy^j \right), \tag{10}$$

and the Killing spinors are given by

$$\epsilon = H^{-\frac{1}{6}} \epsilon_0, \tag{11}$$

where ϵ_0 is a constant spinor satisfying the projection condition

$$(\mathbf{1} - \Gamma^{123}) \epsilon_0 = 0. \tag{12}$$

Consider a Killing vector of the form:

$$K_{(ij)}^{\mu} \equiv \bar{\epsilon}^i \Gamma^{\mu} \epsilon^j, \tag{13}$$

Observe that Γ^{123} is hermitian (indeed, it is real and symmetric) and anti-commutes with Γ^{μ} for $\mu = 4, 5, \dots, 11$. Inserting Γ^{123} in front of ϵ^j in 13 still yields $K_{(ij)}^{\mu}$ because of the projection condition 12. Now commute it through Γ^{μ} , and use its action on $\bar{\epsilon}^i$ (remembering that the Dirac conjugate

contains Γ^1), and one gets $-K_{(ij)}^\mu$ for $\mu = 4, 5, \dots, 11$. We thus learn that $K_{(ij)}^\mu \equiv 0$ for $\mu = 4, 5, \dots, 11$, and hence any Killing vectors generated by 13 must be parallel to the brane.

Since we will use the foregoing kind of argument several times, will refer to it as “the standard projector argument.”

2.2 The generalized Ansatz

In our Ansatz we will deform the projector in 12 to:

$$\Pi_0 \equiv \frac{1}{2} (\mathbf{1} - p_1 \Gamma^{123} + p_2 \Gamma^*), \quad (14)$$

where p_1 and p_2 are functions, and Γ^* is a product of gamma-matrices satisfying $(\Gamma^*)^2 = \mathbf{1}$. For Π_0 to be a projector one must have:

$$p_1^2 + p_2^2 = 1. \quad (15)$$

One can obviously generalize 14 to involve several functions and sums of products of gamma matrices (see, for example [13]). However, 14 will be sufficient for our purposes here. The obvious issue is the gamma-matrices that make up Γ^* . We will return to this after fixing the other projector, but we note that Lorentz invariance on the brane means that Γ^* is made out of the eight gamma-matrices²: $\Gamma^4, \dots, \Gamma^{11}$. Moreover, reality of the Killing spinor and the requirement that $(\Gamma^*)^2 = \mathbf{1}$ means that Γ^* must involve either one, four, five, or seven of the gamma-matrices $\Gamma^4, \dots, \Gamma^{11}$.

Since we wish to focus on flows with eight supersymmetries, we will need a second projection matrix, which we will take to be of the form:

$$\Pi_1 \equiv \frac{1}{2} (\mathbf{1} + \widehat{\Gamma}), \quad (16)$$

where $\widehat{\Gamma}$ is also a product of gamma-matrices with $(\widehat{\Gamma})^2 = \mathbf{1}$. The choice of $\widehat{\Gamma}$ is fixed by the projection condition on the supersymmetries when they are restricted to the moduli space of brane probes. In this instance we will take $\widehat{\Gamma}$ to be the product of all gamma-matrices parallel to the brane-probe moduli space. For flows with eight supersymmetries in M-theory, the moduli space will be a four-dimensional hyper-Kähler manifold and choosing Π_1 in this manner will impose the proper “half-flat” chirality condition on the Killing

²It could involve the product Γ^{123} , but then we would multiply 14 by Γ^{123} to get rid of this product in Γ^* .

spinors. More generally, for four supersymmetries one will get a Kähler moduli space and one will need to impose further projection conditions.

We will denote the coordinates parallel to the moduli-space by x^5, x^9, x^{10}, x^{11} , and hence we will take:

$$\Pi_1 \equiv \frac{1}{2} (\mathbf{1} + \Gamma^{591011}). \quad (17)$$

We have now completely fixed the space of Killing spinors, ϵ^j , $j = 1, \dots, 8$. The standard projector argument using $\hat{\Gamma}$ shows that $K_{(ij)}^\mu$ can *never* be parallel to the moduli space. Now recall that solutions with eight supersymmetries in eleven dimensions have an $SU(2) \times SU(2)$ \mathcal{R} -symmetry. For the flows considered here, one of these $SU(2)$'s acts on the family of complex structures in the hyper-Kähler moduli space, while the second $SU(2)$ acts on the geometry transverse to both the brane and moduli space. Additional isometries of the metric transverse to the brane amount to additional global symmetries of the field theory on the brane. Thus the second $SU(2)$ of the \mathcal{R} symmetry will act in the x^4, x^6, x^7, x^8 directions. Indeed, we will assume that the metric involves the right-invariant 1-forms, σ_j , of $SU(2)$ and that the frames e^6, e^7, e^8 will be proportional to the $\sigma_1, \sigma_2, \sigma_3$ respectively.

To fix the deformation of Π_0 we first require that it be compatible (commute) with Π_1 . This means that it must contain an even number of the gamma-matrices $\Gamma^5, \Gamma^9, \Gamma^{10}, \Gamma^{11}$. More fundamentally, one must decide how the Killing vectors of 13 relate to the \mathcal{R} -symmetries and to any additional global symmetries of the field theory. Here we are going to require 13 generates only one Killing vector *transverse* to the brane. In particular, we will require that 13 generate none of the \mathcal{R} -symmetries, but instead generate a single, $U(1)$, global symmetry of the underlying field theory. This symmetry will be taken to be a translation parallel to the x^8 -coordinate (and it will thus act by rotating σ_1 and σ_2 into one another). Given that anything not forbidden will appear in 13, we must construct the projector Π_0 so that it forbids the proper things via the standard projector argument. Since the M-theory solution will generically depend on all the non-symmetry coordinates $(x^4, x^5, x^9, x^{10}, x^{11})$ these will not generally be Killing directions, and hence must be forbidden from appearing in 13. This means that Γ^* must contain Γ^4 . Similarly, since we required 13 not to yield an \mathcal{R} -symmetry, Γ^* must contain Γ^{67} . The matrix Γ^* must then be completed by any two of $\Gamma^5, \Gamma^9, \Gamma^{10}, \Gamma^{11}$. Since, as yet, there is no special meaning to any of the coordinate labels on the moduli space, we will take:

$$\Gamma^* \equiv \Gamma^{456711} \quad \Rightarrow \quad \Pi_0 \equiv \frac{1}{2} (\mathbf{1} - p_1 \Gamma^{123} + p_2 \Gamma^{456711}). \quad (18)$$

With this choice, 13 will generate Killing vectors parallel to the brane, and a Killing vector, L^μ where $L^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x^8}$. Using this last fact we can fix p_2 in 14 in terms of the Ansatz for the frame e^8_μ , and then the function p_1 is fixed by 15. Finally, the fact that 13 generates Killing vectors parallel to the brane fixes the normalizations of the Killing spinors in terms of the warp factor (*i.e.* the Ansatz for the frames $e^j_\mu, j = 1, 2, 3$) exactly as it does for the harmonic distribution of frames.

We have thus *entirely determined* the Killing spinors in terms of the metric Ansatz, and our task is to simply reconstruct everything else. Indeed, we invert the usual perspective: The Killing spinor equation 1 is no longer a differential equation for ϵ , it is an *algebraic* equation for $F_{\mu\nu\rho\sigma}$.

3 Solving the Ansatz: An example

We are going to generalize the result of the [9] by considering a metric of the same general type, but with metric coefficients that are arbitrary functions of two variables. We thus take:

$$ds_{11}^2 = e^{2A_0} (-dt^2 + dx_1^2 + dx_2^2) + e^{2A_1} du^2 + e^{2A_2} dv^2 + e^{2A_3} (\sigma_1^2 + \sigma_2^2) + e^{2A_4} \sigma_3^2 + e^{2A_5} (\tau_1^2 + \tau_2^2) + e^{2A_6} \tau_3^2. \quad (19)$$

The σ_j and τ_j are two independent sets of $SU(2)$ right-invariant one-forms, which we will parametrize by Euler angles φ_j and ϕ_j respectively. They are normalized so that $d\sigma_1 = \sigma_2 \wedge \sigma_3$ and similarly for τ_j . The functions, $A_a = A_a(u, v), a = 0, \dots, 6$ will be taken to be arbitrary functions of the coordinates (u, v) . We have now made a rather special choice: the x^9, x^{10} directions coincide with another $SU(2)$ symmetry action, and so the combined choices of 19 and 18 are no longer so general. We have, however made these choices so as to parallel, and generalize the results of [9]. We now need to understand how the Killing spinors depend upon the coordinates along which symmetries act.

First, the Poincaré invariance along the brane means that $\partial_\mu \epsilon = 0, \mu = 1, 2, 3$. The metric Ansatz has a manifest $SU(2)_\sigma \times U(1)_\sigma$ and $SU(2)_\tau \times U(1)_\tau$ symmetry, where the subscripts denote the relevant 1-forms in 19. Following [9], the Killing spinors that we seek are going to singlets under $SU(2)_\tau \times U(1)_\sigma$, and they will transform as $\mathbf{2}_{\pm 1}$ under $SU(2)_\sigma \times U(1)_\tau$. The triviality of the $SU(2)_\tau$ action will mean that $\partial_\mu \epsilon = 0, \mu = 9, 10, 11$, while the non-trivial action of the $SU(2)$ \mathcal{R} -symmetry in the x^6, x^7, x^8 directions

enforces the appropriate doublet action of the isometry. Define

$$\mathcal{R}_X(\varphi) = \cos(\tfrac{1}{2}\varphi) \mathbb{1} - \sin(\tfrac{1}{2}\varphi) \Gamma^X, \quad (20)$$

and let

$$g \equiv \mathcal{R}_{67}(\varphi_3) \mathcal{R}_{12378}(\varphi_1) \mathcal{R}_{67}(\varphi_2). \quad (21)$$

By construction one has:

$$dg g^{-1} = -\tfrac{1}{2}(\sigma_1 \Gamma^{12378} - \sigma_2 \Gamma^{12368} + \sigma_3 \Gamma^{67}). \quad (22)$$

The Killing spinors are then of the form:

$$\epsilon = g \hat{\epsilon}_0, \quad \text{where } \partial_\mu \epsilon_0 = 0, \quad \mu = 1, 2, 3, 6, 7, 8, 9, 10, 11. \quad (23)$$

The slightly unusual feature is presence of the Γ^{123} in 21, which means that the Clifford algebra for the $SU(2)_{\mathcal{R}}$ is generated by Γ^{1236} , Γ^{1237} and Γ^8 . However, the presence of the extra Γ^{123} factors is essential for these rotation matrices to commute with the projector, Π_0 , defined by 18. The slightly unusual $SU(2)_{\mathcal{R}}$ Clifford algebra is thus mandated by the deformation of the fundamental projector.

Using the Killing vector conditions on the Killing spinors one obtains:

$$\epsilon = e^{\frac{1}{2}A_0} g \epsilon_0, \quad p_2 = \beta_0 e^{A_4 - A_0}, \quad p_1 = \sqrt{1 - p_2^2}. \quad (24)$$

where β_0 is a constant, and ϵ_0 is a constant spinor. The parameter, β_0 represents the strength of the deformation.

Rather than go through the general procedure outlined in the previous section, we now focus the calculation towards generalizing the results of [9]. To that end, we will require that the only non-zero components of F be:

$$F_{1234}, F_{1235}, F_{45811}, F_{67910}, F_{46711}, F_{56711}, F_{48910}, F_{58910}. \quad (25)$$

This is motivated by the general form of the solution in [9], but here we will allow them to be general functions of u and v .

Using this Ansatz one can easily see that the linear combinations:

$$\Gamma^1 \delta\psi_1 + \Gamma^6 \delta\psi_6 + \Gamma^8 \delta\psi_8 = 0, \quad \Gamma^1 \delta\psi_1 + \Gamma^9 \delta\psi_9 + \Gamma^{11} \delta\psi_{11} = 0, \quad (26)$$

where all indices are frame indices, have the remarkable property that the F -tensor terms cancel out. The vanishing of these combinations of gravitino

variations lead to four conditions on the functions, A_a . First, one finds that $A_0 + A_3 + A_4$ must be purely a function of u and $A_0 + A_5 + A_6$ must be purely a function of v . Since both u and v are each arbitrary up to redefinitions $u \rightarrow f_1(u)$, $v \rightarrow f_2(v)$, for some arbitrary functions, f_j , we now fix this freedom completely by imposing:

$$e^{A_0+A_3+A_4} = \frac{1}{4} u^2, \quad e^{A_0+A_5+A_6} = \frac{1}{4} v^2. \quad (27)$$

Having done this, the other two conditions imposed by 26 mean that the functions A_a can all be expressed in terms of three functions B_0, B_1, B_2 :

$$\begin{aligned} A_0 &= B_0, & A_1 &= B_1 - \frac{1}{2}B_0, & A_2 &= B_2 - \frac{1}{2}B_0, \\ A_3 &= B_1 - \frac{1}{2}B_0 + \log\left(\frac{1}{2}u\right), & A_4 &= -B_1 - \frac{1}{2}B_0 + \log\left(\frac{1}{2}u\right), \\ A_5 &= -B_2 - \frac{1}{2}B_0 + \log\left(\frac{1}{2}v\right), & A_6 &= B_2 - \frac{1}{2}B_0 + \log\left(\frac{1}{2}v\right). \end{aligned} \quad (28)$$

This means that the metric can be recast in the form:

$$\begin{aligned} ds_{11}^2 &= H^{-\frac{2}{3}}(-dt^2 + dx_1^2 + dx_2^2) + H^{\frac{1}{3}}\left[V_1\left(du^2 + \frac{1}{4}u^2(\sigma_1^2 + \sigma_2^2)\right)\right. \\ &\quad \left.+ V_1^{-1}\left(\frac{1}{4}u^2\sigma_3^2\right) + V_2\left(dv^2 + \frac{1}{4}v^2\tau_3^2\right) + V_2^{-1}\left(\frac{1}{4}v^2(\tau_1^2 + \tau_2^2)\right)\right]. \end{aligned} \quad (29)$$

where

$$H \equiv e^{-3B_0}, \quad V_1 \equiv e^{2B_1}, \quad V_2 \equiv e^{2B_2}. \quad (30)$$

Using other parts of the gravitino variation, one finds a simple differential equation that relates B_1 and B_2 . This may be conveniently written as:

$$e^{2(B_1+B_2)} = \frac{1}{2v} \frac{\partial}{\partial v} (v^2 e^{2(B_1-B_2)}). \quad (31)$$

In other words, once one knows $(B_1 - B_2)$, one can use this to find $(B_1 + B_2)$. One can also algebraically determine the components of the Maxwell tensor, 25. The Bianchi identities then give equations of motion for B_0 and $(B_1 - B_2)$. Rather than try to give an exhaustive classification of the solutions of our more limited Ansatz, we will focus on one particular family of solutions.

We take the 3-form potential, $A^{(3)}$ to have the form:

$$A^{(3)} = q_1 dt \wedge dx_1 \wedge dx_2 + q_2 \sigma_1 \wedge \sigma_2 \wedge \tau_3 + q_3 \sigma_3 \wedge J, \quad (32)$$

where the q_j are arbitrary functions of u and v , while the 2-form J is given by:

$$J = \frac{1}{2} e^{2B_2} v dv \wedge \tau_3 + \frac{1}{4} e^{-2B_2} v^2 \tau_1 \wedge \tau_2. \quad (33)$$

This Ansatz for $A^{(3)}$ is, in fact deduced from relations 5. Indeed, for suitably chosen i, j , the 2-form components, $J_{\mu\nu}$, can be extracted as part of $\Omega_{\mu\nu}^{ij}$. One can use 1 with 32 to fix the q_j , or one simply apply 5. This leads immediately to

$$q_1 = \frac{1}{2} e^{3B_0+2B_1}, \quad q_3 = -\frac{1}{2\beta_0} e^{-2B_1}, \quad (34)$$

where β_0 is the constant in 24. The only part of $A^{(3)}$ that is not fixed by 5 is q_2 . This *must* be obtained from 1, and one gets three equations for it. Two of these are equivalent via 31, and one of these two yields:

$$q_2 = -\frac{1}{16\beta_0} u^3 v^2 \frac{\partial}{\partial u} \left(\frac{1}{u^2} e^{2(B_1-B_2)} \right). \quad (35)$$

The third equation for q_2 is:

$$\frac{\partial q_2}{\partial u} = \frac{1}{8\beta_0} u v \frac{\partial}{\partial v} (e^{4B_1}). \quad (36)$$

Write $e^{4B_1} = e^{2(B_1-B_2)} e^{2(B_1+B_2)}$ and use 31 to eliminate $e^{2(B_1+B_2)}$, and one gets:

$$\frac{\partial q_2}{\partial u} = \frac{1}{8\beta_0} u v \frac{\partial}{\partial v} \left(\frac{1}{2v} e^{2(B_1-B_2)} \frac{\partial}{\partial v} (v^2 e^{2(B_1-B_2)}) \right). \quad (37)$$

Comparing this with 35, one sees that all the conditions on q_2 are satisfied if and only if:

$$\frac{1}{u^3} \frac{\partial}{\partial u} \left(u^3 \frac{\partial}{\partial u} \left(\frac{1}{u^2} e^{2(B_1-B_2)} \right) \right) = -\frac{1}{v} \frac{\partial}{\partial v} \left(e^{2(B_1-B_2)} \frac{1}{v} \frac{\partial}{\partial v} \left(\frac{v^2}{u^2} e^{2(B_1-B_2)} \right) \right), \quad (38)$$

which is precisely 6. Finally, the complete solution of 1 determines B_0 algebraically:

$$e^{-3B_0} = \frac{4}{\beta_0 u^2} (e^{2B_1} - e^{-2B_1}). \quad (39)$$

Thus, a complete solution to 1 is obtained by solving 38. From the solution one can obtain B_1 and B_2 independently using 31. Then B_0 is obtained from 39. The functions q_j are then given by 34 and 35. This completely fixes the solution.

One can also check that this solution solves all the equations of motion 9, and not merely a subset of them.

In deriving the solution above we did not try to find the most general solution: We imposed some special Ansätze and discarded some constants of integration. Our purpose was to illustrate the power of the general method. It is interesting to note that every function, save one (q_2) was fixed algebraically in terms of the metric coefficients, and that it was the non-algebraic function that gave rise to the only differential equation that needs to be satisfied.

To conclude this section, we obtain some special solutions to the foregoing procedure. First, note that we can easily recover the result of [9] via the change of coordinates:

$$u = \frac{\rho}{\sqrt{\sinh(2\chi)}} \cos \theta, \quad v = \frac{1}{\sqrt{\sinh(2\chi)}} \sin \theta, \quad (40)$$

where the quantities ρ, χ, θ are defined in [9]. In particular, $e^{2(B_1-B_2)} = \cosh(2\chi)$ is a solution to 38. The form of $A^{(3)}$ presented here is not exactly the same as that of [9], but it is gauge equivalent.

It is also interesting to find the “separable” solutions in which one seeks solutions of 38 with $e^{2(B_1-B_2)} = h_1(u)h_2(v)$ for some functions h_1 and h_2 . Obviously separation of variables does not work in general due to the non-linearity, but we do find a solution to 38 and 31 with:

$$e^{2B_1} = \mu(1 + bu^2), \quad e^{2B_2} = \left(1 \pm \left(\frac{a}{v}\right)^4\right)^{-\frac{1}{2}}, \quad (41)$$

for some constants, a, b and μ . One can then go on to find the expressions for e^{3B_0} and the q_j , and thus obtain a three-parameter family of solutions. The asymptotics at large distances are determined by b and μ : If $b = 0$ then the metric in the u -direction that of \mathbb{R}^4 , but with a stretched Hopf fiber over the S^3 at fixed v . The amount of stretching is given by μ . If $b \neq 0$ then the metric is that of $\mathbb{R}^3 \times S^1$. If one wants the metric to asymptote to that of $AdS_4 \times S^7$ then one must set $\mu = 1$ and $b = 0$. However, this limit must be taken carefully: to get a finite result for e^{3B_0} from 39 one has to set $\mu = 1 + \epsilon^2$ and $\beta_0 = \alpha \epsilon$, for some constant, α , and then take the limit as $\epsilon \rightarrow 0$. Thus the projector, Π_0 , becomes the standard one parallel to the M_2 branes (*i.e.* it has $p_1 = 1$ and $p_2 = 0$). One then finds a solution in which the q_j 's are non-zero, but all the “internal” components are *pure gauge*: That is, the only non-zero component of the field strength is F_{1234} . One also finds that $H = e^{-3B_0} = \frac{4}{\alpha^2 u^2}$. The solution is therefore that of M_2 branes uniformly spread over the \mathbb{R}^4 in the (5, 9, 10, 11) directions. There is, however, a small variation from the usual harmonic-brane story: For $a \neq 0$, the metric in direction of brane spreading is not flat, but is the

Eguchi-Hanson metric. This can be seen more directly by taking the + -sign choice in 41, and changing variables to $w \equiv (v^4 + a^4)^{1/4}$.

While we have constructed a set of rather special solutions, we would like to stress that, as pointed out in the introduction, there is an infinite family of solutions to 38. These solutions can at least be generated by perturbation theory, and presumably correspond to a general, rotationally symmetric, v -dependent distributions of $M2$ -branes in the x^5, x^9, x^{10}, x^{11} directions. It is also important to note that in [9] the “master function” is given by $e^{2(B_1 - B_2)} = \cosh(2\chi)$ and that χ is the gauged supergravity scalar dual to the fermion mass. Thus the entire flow is determined by the flow of this mass term.

4 Some comments on the geometry of the transverse eight-manifold

The metric 29 has quite a number of interesting geometric features that we will expand upon in [14]. Consider the metric, ds_8^2 , in the square brackets:

$$\begin{aligned}
 ds_8^2 = ds_4^2 + d\hat{s}_4^2 \equiv & V_1 \left(du^2 + \frac{1}{4} u^2 (\sigma_1^2 + \sigma_2^2) \right) + V_1^{-1} \left(\frac{1}{4} u^2 \sigma_3^2 \right) \\
 & + V_2 \left(dv^2 + \frac{1}{4} v^2 \tau_3^2 \right) + V_2^{-1} \left(\frac{1}{4} v^2 (\tau_1^2 + \tau_2^2) \right),
 \end{aligned}
 \tag{42}$$

where the split into ds_4^2 and $d\hat{s}_4^2$ corresponds to the split into u, σ_j and v, τ_j respectively.

The form of ds_4^2 is very reminiscent the Gibbons-Hawking ALE metrics [15], and to get it to the same form one simply needs to make a change of variable to remove a conical singularity at $u = 0$. That is, set $u = \sqrt{w}$, and define $\tilde{V}_1 = u^{-2} V_1$, and one then finds:

$$ds_4^2 = \frac{1}{4} \left[\tilde{V}_1 \left(dw^2 + w^2 (\sigma_1^2 + \sigma_2^2) \right) + \tilde{V}_1^{-1} \left(w^2 \sigma_3^2 \right) \right]
 \tag{43}$$

The metric in parentheses is now precisely that of a flat \mathbb{R}^3 , and there is the S^1 fibration over this \mathbb{R}^3 . The only difference with the Gibbons-Hawking form is that the function, \tilde{V}_1 , does not appear to be harmonic on the \mathcal{R}^3 . The function $V_2^{-1} \tilde{V}_1$ satisfies 38, whose left-hand side involves:

$$\frac{1}{w^2} \frac{\partial}{\partial w} \left(w^2 \frac{\partial}{\partial w} \left(V_2^{-1} \tilde{V}_1 \right) \right),$$

which is the Laplacian on \mathbb{R}^3 . As yet, we do not know if 38 can then be translated into some interesting generalization of the harmonic condition on \tilde{V}_1 .

The appearance of this S^1 fibration over a flat \mathbb{R}^3 is a significant new feature of our formulation here. This four-dimensional space is where the \mathcal{R} -symmetry acts, and it lies transverse to both the branes and to their moduli space. We believe that the geometry in this direction will always have the same form, independent of how the branes spread. Indeed, the same geometry appears in the $\mathcal{N} = 2^*$ flows (with eight supersymmetries) in IIB supergravity [10]. The description of ds_4^2 given here is much simpler, and certainly more intuitive than that given in [9]. This is because the coordinates 40 are more naturally adapted to the decomposition of the geometry of ds_8^2 transverse and parallel to the moduli space of the branes. In particular, the selection of the coordinates, (u, v) , is directly linked to the choice of the projector, Π_2 , in 17.

The geometry of ds_4^2 is equally tantalizing: We know that at $u = 0$ it is hyper-Kähler with three harmonic 2-forms. It turns out that these have natural extensions to the whole space. To see this, we first introduce the usual set of Euler angles to define the left-invariant 1-forms, τ_j :

$$\begin{aligned}\tau_1 &\equiv \cos \phi_3 d\phi_1 + \sin \phi_3 \sin \phi_1 d\phi_2, \\ \tau_2 &\equiv \sin \phi_3 d\phi_1 - \cos \phi_3 \sin \phi_1 d\phi_2, \\ \tau_3 &\equiv \cos \phi_1 d\phi_2 + d\phi_3,\end{aligned}\tag{44}$$

The 2-form, J , in 33 is a complex structure for ds_4^2 , but it is not Kähler. The corresponding complex coordinates are simply:

$$\zeta_1 \equiv v \cos(\tfrac{1}{2}\phi_1) e^{\tfrac{i}{2}(\phi_2+\phi_3)}, \quad \zeta_2 \equiv v \sin(\tfrac{1}{2}\phi_1) e^{-\tfrac{i}{2}(\phi_2-\phi_3)}.\tag{45}$$

The other two 2-forms are given by the real and imaginary parts of:

$$\Omega \equiv \tfrac{1}{2}(dv + \tfrac{i}{2}v\tau_3) \wedge v(\tau_1 + i\tau_2) = d\zeta_1 \wedge d\zeta_2.\tag{46}$$

These are global, harmonic 2-forms on the whole of ds_8^2 .

It is relatively straightforward to check that the differential forms, J and Ω play a significant role in the underlying G -structure of our solution.

5 Final comments

There are now many interesting flow solutions that have been constructed using gauged supergravity theories. The advantage of such an approach

has been the simplicity of the equations of motion in the lower dimensional theory. However, if one reviews the solutions in four and five dimensions for which the M -theory or IIB “uplift” is known, then one encounters what appear to be some extremely involved solutions. Even if there are high levels of supersymmetry, these solutions are still very complicated and they have metric and tensor gauge fields that appear to defy simple classification (see, for example, [16][17][18][19]– [20],[9]). One of the important messages of this paper is that the complexity of these previously known solutions lies in the fact that one is focusing on the wrong object: The solutions are very simple if one focuses on the Killing spinors. The contrast between the solution presented here and that described in [9] illustrates this graphically, and the solution presented here is considerably more general.

Our strategy for finding solutions is to make a very general Ansatz for the metric, based upon the Poincaré and \mathcal{R} -symmetries, and upon the presence of a moduli space for the brane. We then make an Ansatz for the Killing spinors by defining projectors *algebraically* in terms of the metric Ansatz. There are several crucial inputs into the projector Ansatz: (i) Deforming the canonical “Dirichlet projector” parallel to the branes, (ii) The \mathcal{R} -symmetries, (iii) Poincaré invariance along the brane, (iv) Spinor bilinears generate Killing vectors, and (v) Projectors on the moduli space of the branes. Once an projector Ansatz is made, any arbitrary functions are fixed (algebraically) in terms of the metric Ansatz because of the Killing vector condition on spinor bilinears. One then uses the supersymmetry variations of the fermions to fix (algebraically) the field strengths of the tensor gauge fields. This system of equations is generically highly over-determined and so it also gives rise to first-order differential equations for some of the metric coefficients. The complete set of equations for the solution are then given by the Bianchi identities of the field strengths. One should then verify that the solution to the supersymmetry variations does indeed satisfy the equations of motion. However, we believe that the equations coming from the supersymmetry variations will generically be sufficient to solve all the equations of motion except in highly specialized, algebraically simple solutions like the pure “harmonic brane” distributions. In these special circumstances, commutators of supersymmetry variations generate combinations of equations of motion in which there are non-trivial cancellations, and the result is only a subset of the constraints on the metric functions. This is the one circumstance where complexity helps: If the solution is sufficiently complex, such cancellations do not happen, and solving the supersymmetry variations will capture everything.

The result presented here fits very nicely with what one expects from the RG flow of the theory on the brane. Specifically, the flow is driven by the

fermion mass parameter because this is the leading term in the field theory Lagrangian, with the bosonic mass term fixed by supersymmetry. The complete holographic flow solution presented here is generated from a single function, $g = e^{2(B_1 - B_2)}$, that is obtained as a solution to the differential equation 6. This leading part of this function is also dual to the fermion mass term, and the entire solution is being determined essentially by the flow of this one term. We also suspect that this observation lies at the root of the success of the “Algebraic Killing Spinor” Ansatz. That is, from the field theory perspective these flows are very simple because they are driven by a single relevant operator in the (supersymmetric) Lagrangian combined with a deformation along the Coulomb branch. In holography, the Coulomb branch is at the root of the “harmonic rule,” while the fermion mass lies at the root of the deformation of the standard Dirichlet projector. All we have really done here is to try to combine these ideas.

We suspect that the ideas presented in this paper will have many applications in generating not just holographic flow solutions, but also in obtaining more general supersymmetric compactifications with non-trivial fluxes. There are some obvious questions about using this to find more general classes of flow solution. There are open issues about classifying the possible supersymmetric deformations of the Dirichlet projectors, and then there are questions about the mathematical structures that underlie the more general classes of solution. What we have done here is to use projectors to define the supersymmetries as a special sub-bundle of the spin bundle. One should be able to find an intrinsic classification of this special sub-bundle, and since the projectors are algebraic in the metric Ansatz one should probably try to classify the total space of this spinor sub-bundle. Then there is an issue of the underlying G -structures and how they fit into the scheme presented here. Finally, there is an interesting question about how boundary conformal field theory relates to the deformation of the Dirichlet projectors.

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