Supersymmetric structure of the induced W gravities

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Abstract

We derive the supersymmetric structure present in W-gravities which has been already observed in various contexts as Yang-Mills theory, topological field theories, bosonic string and chiral W_3-gravity. This derivation which is made in the geometrical framework of Zucchini, necessitates the introduction of an appropriate new basis of variables which replace the canonical fields and their derivatives. This construction is used, in the W_2-case, to deduce from the Chern-Simons action the Wess-Zumino-Polyakov action.

1 Introduction

In this article we show that the supersymmetric structure found in Chern-Simons theory quantized in the Landau gauge [1] (so-called since the anti-commuting relations of the generators describe a super algebra of the Wess-Zumino type) is also present in the induced W_n-gravities. These theories [2] are higher spin generalizations of 2-dim gravity whose symmetries are the classical W_n-algebras, where n indicates the highest spin of the currents involved. The geometrical description[3] is based on a straightforward generalization of the notion of projective coordinates. It reproduces the results obtained in the more conventional approaches based on $sl(n, R)$ algebra

[4, 5, 6] . The reference system of complex coordinates $(z, \overline{z})$ corresponds to a complex structure defined on the connected, 2-topological manifold on which we are working. Then the underlying models of the W-gravities are described by a connection $\mathcal{A} = \Omega dz + \Omega^* d\overline{z}$, with zero-curvature condition

$$\mathcal{F} = d\mathcal{A} - \mathcal{A}\mathcal{A} = 0 \tag{1}$$

. The pair of matrices (Ω, Ω^*) contains respectively the currents and the gauge fields of the theory which, in the Virasoro case $n = 2$, correspond respectively to the spin 2 stress-energy tensor ρ_{zz} and the Beltrami coefficient $\mu_{\overline{z}}^{z}$. The fields involved here are smooth functions of the holomorphic coordinates $(z, \overline{z})$. They are built through a basic object, an unimodular matrix W (we denote by $(\partial, \overline{\partial})$ the partial derivatives with respect to $(z, \overline{z})$),

$$\Omega = \partial W W^{-1} \quad ; \quad \Omega^* = \overline{\partial} W W^{-1} \tag{2}$$

The condition (1) becomes[1]

$$\overline{\partial}\Omega - \partial\Omega^* + [\Omega, \Omega^*] = 0 \tag{3}$$

allowing us to determine the elements of Ω^* in terms of the elements of Ω and to give the holomorphy conditions obeyed by the currents which are, in fact, the Ward identities of the theory. The main advantage of [3] is to derive easily the off-shell nilpotent BRST algebra which expresses the invariance of the theory.

$$s\mathcal{A} = -dC + [\mathcal{A}, C] \quad ; \quad sC = CC \tag{4}$$

where the ghost matrix C is traced from Ω^* by substituting the gauge fields by the ghost fields. These laws are formally the BRST transformations of the Yang-Mills (Y-M) connection (d being the usual derivative operator) and of the Faddeev-Popov ghost. The W_n-anomaly has been obtained in this framework[3, 7] . After a general discussion of the supersymmetric structure, we give explicit results in the case of W_2 and W_3-theories. Then a link with the group properties underlying the formalism is made, allowing in principle a general explicitation for W_n. Some general remarks on the global properties of this formulation are given. Finally the formalism is used to deduce from the Chern-Simons action the Wess-Zumino-Polyakov action.

[1] We adopt the convention that $[A, B]$ stands for the anticommutator if both A and B are grassmannian, else it is a commutator.

2 Supersymmetric structure

A new and simple way of solving descent equations has been recently presented in ref.[8] where the case of the Y-M theory has been treated. The method relies on the introduction of an operator Δ which allows one to decompose the exterior derivative d as a BRST commutator

$$d = -[s, \Delta]. \tag{5}$$

The closure of the algebra among $d, s,$and Δ requires, in addition, the introduction of a nilpotent operator G such that the following relations are obeyed

$$[d, \Delta] = 2G \quad ; \quad [d, G] = 0 \quad ; \quad [s, G] = 0 \quad ; \quad [\Delta, G] = 0 \tag{6}$$

This algebraic structure has already been found in topological field theories such as the BF system [9], the cohomological Witten's models[10] or the topological Yang-Mills theory[11] , in bosonic string theory in the Beltrami parametrization [12] and in chiral W_3-gravity [13] . In fact as we shall see now it appears also as a general characteristic of W_n-induced gravities, following at once from the parallelism with Y-M theory.

As in the Y-M case we define the linear operators of even and odd degrees respectively

$$\Delta = \mathcal{A}\frac{\delta}{\delta C} + (2\mathcal{A}\mathcal{A} - d\mathcal{A})\frac{\delta}{\delta(dC)}. \tag{7}$$

$$G = (d\mathcal{A} - \mathcal{A}\mathcal{A})\frac{\delta}{\delta C} + d(\mathcal{A}\mathcal{A})\frac{\delta}{\delta(dC)} \tag{8}$$

and we choose as independent variables the set $(\mathcal{A},\ d\mathcal{A},\ C,\ dC)$. Note that the derivative of the connection $\mathcal{A}$ is taken as a variable. On the local matrix space thus defined, the BRST operator s and the exterior derivative d act as ordinary differential operators. Explicitly they read

$$s = (-dC + [\mathcal{A}, C])\frac{\delta}{\delta\mathcal{A}} + C^2\frac{\delta}{\delta C} - [dC, C]\frac{\delta}{\delta(dC)} - d[C, \mathcal{A}]\frac{\delta}{\delta(d\mathcal{A})}, \tag{9}$$

$$d = dC\frac{\delta}{\delta C} + d\mathcal{A}\frac{\delta}{\delta(\mathcal{A})} \tag{10}$$

where the operator [...]$\delta/\delta\Phi$ replaces the Φ field by the expression [...] by the expression in the right hand side, and obeys the usual rules of derivatives in grassmannian space.

In the case of bilinear conformal theories, the system of descent equations relates the cocycles of the cohomology in the following way

$$sT_2^1 + dT_1^2 = 0 \;\; ; \;\; sT_1^2 + dT_0^3 = 0 \;\; ; \;\; s\,T_0^3 = 0 \tag{11}$$

where the lower index denotes the form degree and the upper index the ghost number.

Now starting from the trace of the C monomial of degree three, $T_0^3 = Tr(CCC)$, the operator Δ generates the tower of descent equations by giving the following solution for the cocycles

$$T_1^2 \;=\; \Delta T_0^3 = 3Tr(CC\mathcal{A}) \tag{12}$$

$$T_2^1 \;=\; \frac{1}{2}\Delta\Delta T_0^3 = 3Tr(C\mathcal{A}\mathcal{A}) \tag{13}$$

To get the last relation we have used the condition (1). T_2^1 appears as a possible candidate for the non-integrated anomaly which satisfies the Wess-Zumino consistency condition. Obviously this is in agreement with the results[7] obtained in the usual way, through the Chern-Weil polynomial.

3 The $n = 2$ example

In the simplest case of the W_2-model the connection $\mathcal{A}$ reads[4, 5]

$$\mathcal{A} = \begin{pmatrix} -\frac{1}{2}\partial\mu_{\bar{z}}^{z} d\bar{z} & \varkappa \\ -\frac{1}{2}\partial^2\mu_{\bar{z}}^{z} d\bar{z} + \rho_{zz} ds & \frac{1}{2}\partial\mu_{\bar{z}}^{z} d\bar{z} \end{pmatrix} . \tag{14}$$

where $\varkappa = dz + \mu_{\bar{z}}^{z} d\bar{z}$. The set of fields appearing in (14) are the Beltrami coefficient $\mu_{\bar{z}}^{z}$ ($|\mu_{\bar{z}}^{z}| \leqslant 1$) and the projective connection ρ_{zz} . The ghost matrix is obtained from the Ω^* matrix by substituting for the Beltrami coefficient $\mu_{\bar{z}}^{z}$ and the form degree the ghost field c^z and the ghost degree respectively

$$C = \begin{pmatrix} -\frac{1}{2}\partial c^z & c^z \\ -\frac{1}{2}\partial^2 c^z + \rho_{zz} c^z & \frac{1}{2}\partial c^z \end{pmatrix} . \tag{15}$$

It is worthwhile to note that the differential form $\varkappa$appears in the well-known local relation involving the holomorphic coordinates $(Z, \overline{Z})$, corresponding to the structure parametrized by $\mu_{\bar{z}}^{z}$

$$dZ = \partial Z(dz + \mu_{\bar{z}}^{z} d\bar{z}) = \varkappa \partial Z$$

Now the fields and their derivatives have to be considered as independent variables. An adequate set of variables $\{a_1, a_2, a_3; c_1, c_2, c_3\}$ is dictated by the expression of the matrices (14,15) and is given by

$$a_1 = \varkappa \ ; \quad a_2 = \partial \mu_{\bar{z}}^{z} d\bar{z} \ ; \quad a_3 = \rho_{zz} \varkappa - \tfrac{1}{2} \partial^2 \mu_{\bar{z}}^{z} d\bar{z} \ ; \tag{16}$$

$$c_1 = c^z \ ; \quad c_2 = \partial c^z \ ; \quad c_3 = \rho_{zz} c^z - \tfrac{1}{2} \partial^2 c^z \ ; \tag{17}$$

On the local space defined by these fields, the BRST operator s and the exterior derivative d act as ordinary differential operators and are given by

$$d = \sum_{i=1}^{3} \left(dc_i \frac{\delta}{\delta c_i} + da_i \frac{\delta}{\delta a_i} \right),$$

$$s = \sum_{i=1}^{3} \left(sc_i \frac{\delta}{\delta c_i} + sa_i \frac{\delta}{\delta a_i} + sdc_i \frac{\delta}{\delta dc_i} + sda_i \frac{\delta}{\delta da_i} \right)$$

.

Note that the dimensional constraints coming from matching conformal indices and the dimension of the matrices fix the number of higher order field derivatives to two. The explicit forms of the BRST transformations of these new fields are easily deduced from the matrix laws (4)

$$sa_1 = -dc_1 + a_1 c_2 + c_1 a_2, sa_2 = -dc_2 + 2a_3 c_1 + 2c_3 a_1, \tag{18}$$

$$sa_3 = -dc_3 + a_2 c_3 + c_2 a_3, \tag{19}$$

$$sc_1 = c_1 c_2, \qquad sc_2 = 2c_3 c_1, \qquad sc_3 = c_2 c_3. \tag{20}$$

whereas the Δ operator (acting on the space $\{c_i, dc_i\}$) is given by

$$\Delta = \sum_{i=1}^{3} \left(a_i \frac{\delta}{\delta c_i} + \Delta dc_i \frac{\delta}{\delta dc_i} \right), \tag{21}$$

with $\quad \Delta dc_1 = 2a_1 a_2 - da_1 \; ; \Delta dc_2 = -4a_1 a_3 - da_2 \; ; \Delta dc_3 = 2a_2 a_3 - da_3.$ These equalities betray the intrinsic relation between the Δ operator and the BRST operator s in this framework, since the relation between these equalities and (18) is obvious.

Finally, the algebra closes on

$$\mathcal{G} = \sum_{i=1}^{3} \left(\mathcal{G} c_i \frac{\delta}{\delta c_i} + \mathcal{G} dc_i \frac{\delta}{\delta dc_i} \right),$$

with $\quad \mathcal{G} c_1 = da_1 - a_1 a_2 \; ; \mathcal{G} c_2 = da_2 + 2a_1 a_3 \; ; \mathcal{G} c_3 = da_3 - a_2 a_3;$
and $\quad \mathcal{G} dc_1 = d(a_1 a_2) \; ; \mathcal{G} dc_2 = -2d(a_1 a_3) \; ; \mathcal{G} dc_3 = d(a_2 a_3).$

Concerning the local cohomology of the BRST operator, we get from $T_0^3 = Tr(CCC)$ and (12,13) the cocycles in the zero and one form sectors with ghost numbers three and two respectively which are already known[12] and the non-integrated anomaly yielding the usual expression of the diffeomorphism anomaly $T_2^1 = -\frac{3}{2}(\partial c^z \partial^2 \mu_{\bar{z}}^z - \partial^2 c^z \partial \mu_{\bar{z}}^z) dz d\bar{z}$, for the bosonic string in the Beltrami parametrization[14] .

Up to now we have ignored the fact that the fields have to obey eq.(3). In fact these constraints are incorporated in the explicit expressions (16,17) of the new fields as functions of the canonical ones. However if we want to derive this cohomology by applying straightforwardly the BRST operator to the a fields we have to use explicitely the zero curvature condition, namely

$$da_1 = a_1 a_2 \; ; \; da_2 = 2a_3 a_1 \; ; \; da_3 = a_2 a_3 \tag{22}$$

Note that the above first two conditions are used to determine the elements of Ω^* in terms of the elements of Ω , whereas the third condition is the Ward identity of the theory.

4 The induced W_3-gravity

Now we present the way to derive the set of independent amplitudes corresponding to the W_3-algebra. We consider, for instance, the ghost matrix

$\{C_{ij}\}$which in this case reads[7]

$$C = \begin{pmatrix} \frac{1}{6}\partial^2 c^{zz} - \frac{2}{3} c^{zz}\rho_{zz} - \partial c^z & c^z - \frac{1}{2}\partial c^{zz} & c^{zz} \\ \partial C_{11} - \frac{1}{2} c^{zz}\partial\rho_{zz} & -\frac{1}{3}\left(\partial^2 c^{zz} - c^{zz}\rho_{zz}\right) & c^z + \frac{1}{2}\partial c^{zz} \\ \begin{array}{c}\partial C_{21} + \partial\left(c^{zz}\rho_{zzz}\right) + c^z\rho_{zzz} \\ +\frac{1}{2} c^z \partial\rho_{zz}\end{array} & \begin{array}{c}\frac{1}{2}\partial C_{22} - \partial^2 c^z + c^z\rho_{zz} \\ +c^{zz}\rho_{zzz}\end{array} & \partial C_{23} + C_{22} \end{pmatrix} \tag{23}$$

From this matrix we have to determine eight independent ghost fields, three of them corresponding to the fields of the W_2-algebra (which start with a linear term of the form $\partial^n c^z$, $n = 0, 1, 2$), while the five remaining fields have the linear terms $\partial^n c^{zz}, n = 0..4$ respectively. Indeed, the formulation of the W_3-gravity has to contain the W_2 results as a by-product. For instance, in the expression of the W_3-anomaly appears the W_2-anomaly. However, as recently shown [15], this extension is partially formal and not well understood; it does not provide a true Beltrami differential since now the modulus of the Beltrami coefficient $\mu_{\bar{z}}^{z}$ appearing in the W_3-formalism, is no more necessarily less than one.

The issues of the band matrix extracted from (23) by considering the main diagonal and the two adjacent ones allow the determination of six fields, whereas the two remaining fields are given by the two remaining issues C_{13} and C_{31} of the matrix.

$$C = \begin{pmatrix} c_2^2 - c_1^1 & c_1^0 - c_2^1 & c_2^0 \\ c_2^3 - c_1^2 & -2c_2^2 & c_1^0 + c_2^1 \\ c_2^4 & -c_2^3 - c_1^2 & c_2^2 + c_1^1 \end{pmatrix} \tag{24}$$

In c_i^j the indices refer to the linear term contained in its expression: j is the derivative power appearing in this term while i means that the ghost on which this derivative acts concerns the W_2-algebra ($i = 1$) or the W_3-algebra ($i = 2$). For instance c_2^1 contains ∂c^{zz}.

The fields corresponding to the connection matrix $\{\mathcal{A}_{ij}\}$ can be obtained in the same way as above. Starting with[7]

$$\mathcal{A} = \left(\begin{array}{ccc} \left(\frac{1}{6}\partial^2\mu_{\overline{z}}^{zz} - \frac{2}{3}\mu_{\overline{z}}^{zz}\rho_{zz} - \partial\mu_{\overline{z}}^{z}\right) d\overline{z} & \varkappa - \frac{1}{2}\partial\mu_{\overline{z}}^{zz} d\overline{z} & \mu_{\overline{z}}^{zz} d\overline{z} \\ \partial\mathcal{A}_{11} - \frac{1}{2}\mu_{\overline{z}}^{zz}\partial\rho_{zz} d\overline{z} & -\frac{1}{3}\left(\partial^2\mu_{\overline{z}}^{zz} - \mu_{\overline{z}}^{zz}\rho_{zz}\right) d\overline{z} & \varkappa + \frac{1}{2}\partial\mu_{\overline{z}}^{zz} d\overline{z} \\ \begin{array}{c} \partial\mathcal{A}_{21} + +\rho_{zzz}\varkappa \\ + \left(\partial\left(\mu_{\overline{z}}^{zz}\rho_{zzz}\right) + \frac{1}{2}\mu_{\overline{z}}^{z}\partial\rho_{zz}\right) d\overline{z} \end{array} & \begin{array}{c} \frac{1}{2}\partial\mathcal{A}_{22} + \rho_{zz}\varkappa \\ -\left(\partial^2\mu_{\overline{z}}^{z} - \mu_{\overline{z}}^{zz}\rho_{zzz}\right) d\overline{z} \end{array} & \partial\mathcal{A}_{23} + \mathcal{A}_{22} \end{array} \right),$$

and defining

$$\mathcal{A} = \left(\begin{array}{ccc} a_2^2 - a_1^1 & a_1^0 - a_2^1 & a_2^0 \\ a_2^3 - a_1^2 & -2a_2^2 & a_1^0 + a_2^1 \\ a_2^4 & -a_2^3 - a_1^2 & a_2^2 + a_1^1 \end{array} \right) \tag{25}$$

it is straightforward to read off the explicit expressions of a_i. The indices have the same meaning as before, the upper indices refering now to the linear terms of the μ's fields[2]. For instance the expression of a_2^1 contains the term $\partial\mu_{\overline{z}}^{zz}$. The upper index corresponds to the power of the derivative whereas the index 2 indicates that $\mu_{\overline{z}}^{zz}$ is a field of the W_3 algebra.

Having identified the basic fields in the connection and ghost field sectors we give now their BRST transformations.

$$sc_1^0 = 3c_2^1c_2^2 + c_2^3c_2^0 + c_1^0c_1^1 \qquad sc_1^1 = c_2^4c_2^0 + c_1^0c_1^2 + c_2^1c_2^3 \qquad sc_1^2 = 3c_2^2c_2^3 + c_2^4c_2^1 + c_1^1c_1^2$$

$$sc_2^0 = 2\left(c_2^0c_1^1 + c_1^0c_2^1\right) \qquad sc_2^1 = 3c_1^0c_2^2 + c_2^0c_1^2 + c_2^1c_1^1 \qquad sc_2^2 = c_1^0c_2^3 + c_2^1c_1^2$$

$$sc_2^3 = 3c_2^2c_1^2 + c_1^1c_2^3 + c_1^0c_2^4 \qquad sc_2^4 = 2\left(c_2^3c_1^2 + c_1^1c_2^4\right)$$

$$sa_1^0 = -dc_1^0 + 3\left[a_2^1, c_2^2\right] + \left[a_1^0, c_1^1\right] + \left[a_2^3, c_2^0\right] \qquad sa_1^1 = -dc_1^1 + \left[a_2^1, c_2^3\right] + \left[a_1^0, c_1^2\right] + \left[a_2^4, c_2^0\right]$$

$$sa_1^2 = -dc_1^2 + 3\left[a_2^2, c_2^3\right] + \left[a_1^1, c_1^2\right] + \left[a_2^4, c_2^1\right]$$

$$sa_2^0 = -dc_2^0 + 2\left[a_2^0, c_1^1\right] + 2\left[a_1^0, c_2^1\right] \qquad sa_2^1 = -dc_2^1 + 3\left[a_1^0, c_2^2\right] + \left[a_2^1, c_1^1\right] + \left[a_2^0, c_1^2\right]$$

$$sa_2^2 = -dc_2^2 + \left[a_1^0, c_2^3\right] + \left[a_2^1, c_1^2\right] \qquad sa_2^3 = -dc_2^3 + 3\left[a_2^2, c_1^2\right] + \left[a_1^1, c_2^3\right] + \left[a_1^0, c_2^4\right]$$

[2] A careful reader would note that the expression of the a_1^2 field is different from the expression of the a_3 field given previously in the W_2-formalism since $a_1^2 = -\partial^2\mu_{\overline{z}}^{z}d\overline{z} + \frac{1}{2}\rho_{zz}\varkappa$. This is due to a trivial redefinition of the fields in the W_3-framework.

$$sa_2^4 = -dc_2^4 + 2\,[a_1^1, c_2^4] + 2\,[a_2^3, c_1^2]\,.$$

The brackets are defined by $\left[a_j^i, c_l^k\right] = a_j^i c_l^k + c_j^i a_l^k$ and obey $\left[a_j^i, c_l^k\right] = \left[c_j^i, a_l^k\right] = -\left[c_l^k, a_j^i\right]$. The Δ operator has the same form as in equation (21) with $\Delta c_i = a_i$. The expression of Δdc_i is simply deduced from the BRST transformations above by replacing the c_i^j's by the a_i^j's.

Let us proceed to give the construction of the anomaly. Starting from the cocycle

$$T_0^3 = tr(C^3) = 6(-3c_1^0c_2^2c_2^3+3c_2^1c_1^2c_2^2+c_1^0c_1^2c_1^1+c_1^1c_2^1c_2^3+c_1^0c_2^1c_2^4-c_2^0c_1^2c_2^3+c_2^0c_1^1c_2^4), \tag{26}$$

the cocycles of the descent equations are obtained by the action of the operator Δ and the following expression of the anomaly is easily deduced

$$\begin{gathered}
T_2^1 = 6(-3a_1^0a_2^2c_2^3 + 3a_2^1a_1^2c_2^2 + a_1^0a_1^2c_1^1 + a_1^0a_2^1c_2^4 - a_2^0a_1^2c_2^3 - 3c_1^0a_2^2a_2^3 \\
+3c_2^1a_1^2a_2^2 + c_1^0a_1^2a_1^1 + c_1^1a_2^1a_2^3 + c_1^0a_2^1a_2^4 - c_2^0a_1^2a_2^3 + c_2^0a_1^1a_2^4 - 3a_1^0c_2^2a_2^3 \\
+3a_2^1c_1^2a_2^2 + a_1^0c_1^2a_1^1 + a_1^1c_2^1a_2^3 + a_1^0c_2^1a_2^4 - a_2^0c_1^2a_2^3 + a_2^0c_1^1a_2^4).
\end{gathered}$$

The final form of this quantity in terms of the basic fields $\mu_{\bar{z}}^{z}$, ρ_{zz}, $\mu_{\bar{z}}^{zz}$, ρ_{zzz} is straightforwardly available from the explicit expressions of the a and c fields and is not given here since it is already known [7, 16] . The only point we want to discuss is that the terms $6(a_1^0a_1^2c_1^1 + a_1^0c_1^2a_1^1)$ and $-6(3a_1^0a_2^2c_2^3 + 3a_1^0c_2^2a_2^3 - a_1^0a_2^1c_2^4 - a_1^0c_2^1a_2^4)$ contains respectively the leading terms $\partial^2\mu_{\bar{z}}^{z}\partial c^z - \partial^2 c^z\partial\mu_{\bar{z}}^{z}$ and $\partial^2\mu_{\bar{z}}^{zz}\partial^3 c^{zz} - \partial^3 c^{zz}\partial^2\mu_{\bar{z}}^{zz}$ (called universal anomalies by Hull [19]).

5 The general formulation

From the examples of the W_2 and W_3 -models we can draw some general lessons. There are n^2-1 fields (and n^2-1 ghosts) necessary to describe the W_n-model. They are decomposed in the following way :

$$n^2 - 1 = \sum_{i=2}^{n}(2i-1) \tag{27}$$

where each term in the sum corresponds to the subset of fields describing the W_i-model. This express the fact that the W_n-algebra contains the nested set of subalgebras W_k , $k = 2, .., n-1$: $W_2 \subset W_3 \subset .. \subset W_{n-1} \subset W_n$, where the inclusion symbol means that the formulation of W_i can be obtained from W_{i+1} by setting to zero the fields occuring at the level $i+1$. Finally the fields of the W_n-model are deduced from the $2(n-1)$ non-principal diagonals following the decomposition $2\sum_{i=1}^{n-1} i$, the main diagonal giving the $n-1$ remaining fields. We note that the constraints imposed by the conformal indices and the size of the matrices, building blocks of the W_n-model, imply that the degree of derivatives appearing in the expressions of the fields must be $2n-2$ at most.

More importantly, the BRST algebra of the ghost fields of the W_n-models expressed in terms of the new fields (see eq.(20) for the example of W_2 and the expressions given in sect.4. for W_3) reflects the group symmetry $sl(n, R)$. To prove this let us remember the link between a Lie algebra and an antiderivation operator. Let $\mathcal{G}$ be a vector space of dimension N with a basis (T_α, $\alpha = 1, N$). The corresponding Lie algebra structure is defined by writing a commutator of two generators of $\mathcal{G}$ as $[T_\alpha, T_\beta] = f^\gamma_{\alpha\beta} T_\gamma$, where $f^\gamma_{\alpha\beta}$ are the antisymmetric structure constants. Moreover the commutators satisfy the Jacobi identity $[[T_\alpha, T_\beta], T_\gamma] + \textit{cyclic permutations} = 0$. The Lie algebra may be also defined in the dual space $\mathcal{G}^*$ of $\mathcal{G}$ (with the wedge product $\wedge$ between its elements). In a dual basis (C^α, $\alpha = 1, ..., N$) the antiderivation s of degree 1 is defined by

$$sC^\alpha = \frac{1}{2} f^\alpha_{\beta\gamma} C^\beta \wedge C^\gamma \tag{28}$$

It is easy to verify that the nilpotency condition of s results from the Jacobi identity and that the algebra (20) corresponds to the Lie brackets satisfied by the generators of the groups $sl(2, R)$ when they are identified with (28). The transformations of the gauge fields are those of a Yang-Mills theory of the $sl(2, R)$ group:

$$sa^\alpha = -dC^\alpha + \frac{1}{2} f^\alpha_{\beta\gamma} a^\beta C^\gamma$$

The generalization to higher W_n-models requires to take into account the nested structure mentioned before. If $\underline{2j+1}_2$ denotes the $(2j+1)-$dimensional irreducible representation of $sl(2,R)$, the $(n^2-1)-$dimensional adjoint representation of $sl(n,R)$, $\underline{ad}_n$ has the branching rule

$$\underline{ad}_n \backsimeq \underline{3}_2 \oplus \underline{5}_2 \oplus \ldots \oplus \underline{2n-1}_2$$

From this follows immediately the decomposition (27). Without going into details[3] we can give the corresponding basis of $sl(3,R)$

$$J^a t_a = \begin{pmatrix} \frac{J^4}{6}+\frac{J^5}{2} & \frac{J^6}{2}+\frac{J^7}{2} & J^8 \\ \frac{J^2}{2}+\frac{J^3}{2} & -\frac{J^4}{3} & -\frac{J^6}{2}+\frac{J^7}{2} \\ J^1 & \frac{J^2}{2}-\frac{J^3}{2} & \frac{J^4}{6}-\frac{J^5}{2} \end{pmatrix}$$

where the $sl(2,R)$ subalgebra is given by J^2, J^5 and J^7. The comparison with (24) and (25) is obvious. Finally in order to illustrate the construction in a less simple case we give the example of $sl(4,R)$:

$$J^a t_a = \begin{pmatrix} \frac{J^7}{2}+J^8+J^9 & J^5+J^6 & J^2+J^3 & J^1 \\ J^{10}+J^{11} & \frac{J^7}{2}-J^8-J^9 & J^4 & J^2-J^3 \\ J^{12}+J^{13} & J^{14} & -\frac{J^7}{2}+J^8-J^9 & J^5-J^6 \\ J^{15} & J^{12}-J^{13} & J^{10}-J^{11} & -\frac{J^7}{2}-J^8+J^9 \end{pmatrix}$$

where now the $sl(2,R)$ subalgebra is given by J^2, J^7 and J^{12} and the $sl(3,R)$ subalgebra is given by J^3, J^6, J^8, J^{10} and J^{13}.

6 Global aspects of the formulation

Clearly the formalism described up to now, is valid only on the plane and on the sphere. When considering a Riemann surface of higher genus a global formulation of the anomaly and of the cocycles linked to it by the cohomology of the BRST operator is possible[18, 7]. However the field used to render the expressions valid on any local coordinate chart is the ρ_{zz} field appearing in (14) and (15). Indeed this field transforms with the Schwarzian derivative under a conformal change of coordinates but is not a true projective connection since it is not locally holomorphic ($\overline{\partial}\rho_{zz} \neq 0$). In fact it obeys to the

[3]The technology of *sl* representations has been recently reviewed in [17]. We borrow the group notations and conventions to these publications

holomorphic condition (for the W_2-model)

$$(\overline{\partial} - \mu_{\overline{z}}^{z}\partial - 2\partial\mu_{\overline{z}}^{z})\rho_{zz} = -\frac{1}{2}\partial^3\mu_{\overline{z}}^{z} \tag{29}$$

which is formally the anomalous Ward identity of the induced 2-dim. conformal gravity. This results in a serious drawback of this formulation since the underlying field theory is ill defined, the holomorphic condition (29) linking in a non-local way the $\mu_{\overline{z}}^{z}$ and the ρ_{zz} fields. The situation is even worse for the W_3-model since then the ρ_{zz} field appears explicitly in the BRST transformations of the c^z and $\mu_{\overline{z}}^{z}$ whereas the known expressions of the non-integrated anomaly [7, 16] display terms containing the ρ_{zz} and ρ_{zzz} fields. In fact this defect is inherent to any calculation starting from the zero curvature condition (3) and being self-contained in the sense that the field used to glue the expressions from different charts is provided by the framework itself.

In [20] a different point of view was adopted. Working with a holomorphic projective connection and thus avoiding to introduce non-locality in the theory, Lazzarini and Stora derived the form of the Virasoro Ward identity (29) on arbitrary Riemann surfaces. Unfortunately their work cannot be carried through for the W_3-model. Indeed replacing the ρ fields appearing in the Ω matrix by holomorphic fields (and thus BRST inerts) obviously breaks the BRST invariance of the integrated (or non-integrated) anomaly as well as the nilpotency of the BRST transformations of c^z and $\mu_{\overline{z}}^{z}$. The systematic way to formulate the theory provided by the geometrical framework of [3] or given by some zero curvature condition [5, 6, 16, 18] is lost.

7 The Wess-Zumino-Polyakov action

In this section we derive the Wess-Zumino-Polyakov action from the Chern-Simons action for the W_2-model. Following the formalism at work in Y-M theory[21] the Chern-Simons action is given by

$$S = \frac{k}{4\pi}\int_Y Tr\left(\widetilde{\mathcal{A}}\widetilde{d}\widetilde{\mathcal{A}} - \frac{2}{3}\widetilde{\mathcal{A}}\widetilde{\mathcal{A}}\widetilde{\mathcal{A}}\right)$$

This action is defined on a 3-dimensional manifold Y whose boundary is the two dimensional space $(z, \overline{z})$. More precisely we assume for Y a "space-time" splitting of the form $\Sigma \times R$ where Σ is a Riemann surface (eventually with boundary). Then the exterior derivative $\widetilde{d} = dt\partial_t + d$ ($\partial_t \equiv \frac{\partial}{\partial t}$) and the matrix form $\widetilde{\mathcal{A}} = \mathcal{A}_t + \mathcal{A}$ are decomposed into a "time" component and the usual space components $(z, \overline{z})$. The action above becomes

$$S = -\frac{k}{4\pi}\int_Y Tr\left(\mathcal{A}\partial_t\mathcal{A}dt\right) + \frac{k}{2\pi}\int_Y Tr\left(\mathcal{A}_t(d\mathcal{A} - \mathcal{A}\mathcal{A})\right) \tag{30}$$

Here $\mathcal{A}_t$ can be viewed as a Lagrange multiplier enforcing the constraints (1) in the space direction. An effective action can then be derived by substituting the expression of $\mathcal{A}$ in terms of the variables $\{a_i\}$into (30).

$$S = \int_Y \left(a_{1\overline{z}}\partial_t a_{3z} - a_{3z}\partial_t a_{1\overline{z}} + a_{3\overline{z}}\partial_t a_{1z} - a_{1z}\partial_t a_{3\overline{z}} + \frac{1}{2}a_{2\overline{z}}\partial_t a_{2z} - \frac{1}{2}a_{2z}\partial_t a_{2\overline{z}}\right) \tag{31}$$

The equations of motion are the zero curvature conditions (22). Now let $\Sigma = D$ be a disk; these constraints are automatically satisfied when the gauge matrix field $\mathcal{A}$ is locally expressed as a pure gauge:

$$\mathcal{A} = dWW^{-1} = \begin{pmatrix} 0 & 1 \\ \partial^2 g\partial h - \partial^2 h\partial g & 0 \end{pmatrix} dz + \begin{pmatrix} \overline{\partial} g\partial h - \partial g\overline{\partial} h & \overline{\partial} hg - h\overline{\partial} g \\ \partial\overline{\partial} g\partial h - \partial\overline{\partial} h\partial g & -\overline{\partial} g\partial h + \partial g\overline{\partial} h \end{pmatrix} d\overline{z}$$

where $W = \begin{pmatrix} g & h \\ \partial g & \partial h \end{pmatrix}$ satisfies to

$$g\partial h - h\partial g = 1 \tag{32}$$

The peculiar structure of the matrix W, a Wronskian structure, determines the form of the matrix Ω as a function of the two independent fields $(\mu_{\overline{z}}^{z}, \rho_{zz})$. The choice of two (-$\frac{1}{2}$) differentials $g = \frac{1}{\sqrt{\partial Z}}$, $h = \frac{Z}{\sqrt{\partial Z}}$ makes the link with the theory of two-dimensional Riemann surfaces[3, 4], since Z is in fact, a generic coordinate of the conformal structure $\mathbf{A}(\mu)$ associated to any Beltrami differential μ . It is a local solution of the Beltrami equation and ρ_{zz} is the Schwarzian derivative of Z.

The appearance of a third coordinate requires some explanation. It is assumed that the matrix field $W(t, z, \overline{z})$ is defined on a three-dimensional hemisphere whose boundary t_B coincides with the two-dimensional plane where the original theory is defined in such a way that $W(t_B, z, \overline{z}) \equiv W(z, \overline{z})$.

Now by following the canonical formalism of Witten it is possible to relate the action (30) to the theory in two dimensions. Integrating by parts we get

$$\begin{array}{c} S = -\frac{k}{4\pi}\int dtd\overline{z}Tr(\overline{\partial}WW^{-1}\partial_t WW^{-1}) - \frac{k}{4\pi}\int dtdzTr(\partial WW^{-1}\partial_t WW^{-1}) \\ +\frac{k}{12\pi}\int_Y Tr(dWW^{-1})^3 \end{array} \tag{33}$$

The change of variables from $\mathcal{A}$ to W involves a unit Jacobian since the measure $d\widetilde{\mathcal{A}}\delta(\mathcal{F}) \equiv dW$. In this way we have a path integral of the action (33) which looks like the chiral version of the Wess-Zumino-Witten path integral.

This action depends only on the boundary values of W on the conformal space $(z, \overline{z})$. The third term on the r.h.s of (33) is the generalization of the usually called Wess-Zumino term [22], and in fact does not depend on the t variable being a total derivative. To prove this latter statement it is worthwhile to restore in its expression the full symmetry with respect to the three variables $z, \overline{z}$ and t . This results in

$$\begin{array}{c} \int_Y Tr(dWW^{-1})^3 = \int_Y [(\partial\overline{\partial}g\partial h - \partial\overline{\partial}h\partial g)(\partial_t g\partial h - \partial g\partial_t h) \\ -(\partial\partial_t g\partial h - \partial\partial_t h\partial g)(\overline{\partial}g\partial h - \partial g\overline{\partial}h) + (\partial^2 g\partial h - \partial^2 h\partial g)(\overline{\partial}g\partial_t h - \partial_t g\overline{\partial}h)] \end{array}$$

With the help of the condition (32) we see that this three dimensional integral depends on boundary values only

$$\begin{array}{c} \int_Y Tr(dWW^{-1})^3 = \int_Y \partial_t[\ln g(\partial\overline{\partial}g\partial h - \overline{\partial}h\partial^2 g)] + \int_Y \partial[\ln g(\partial^2 g\partial_t h - \partial h\partial\partial_t g)] \\ + \int_Y \overline{\partial}[\ln g(\partial\partial_t g\overline{\partial}h - \partial_t h\partial\overline{\partial}g)] \end{array}$$

The formal expression $\ln g$ which appears in the formula above has been obtained by partial integration of $\frac{\partial g}{g}$. Now we assume that g and h satisfy on the boundary of the space (z,$\overline{z}$) the relation $h = \partial g$ (and $g\partial^2 g - (\partial g)^2 = 1$ which preserves (32)). The three dimensional integral is reduced to

$$\int_Y Tr(dWW^{-1})^3 = \int dzd\overline{z}\ln g(\partial\overline{\partial}g\partial h - \overline{\partial}h\partial^2 g)$$

On the Riemann surface the result is (up to some integration by part) the Wess-Zumino-Polyakov action

$$\int_Y Tr(dWW^{-1})^3 = \int dzd\overline{z}\frac{\overline{\partial}Z}{\partial Z}\partial^2 \ln\partial Z$$

which solves the Ward identity. In particular this is a non-local functional of the Beltrami coefficient obeying

$$\frac{\delta}{\delta\mu}\int dz d\overline{z}\frac{\overline{\partial}Z}{\partial Z}\partial^2 \ln \partial Z = 2S(Z,z)$$

where S is the Schwarzian derivative.

Now we explain how the above results are related to other calculations of the Wess-Zumino-Polyakov action. In fact starting from [16, 18]

$$\int dz d\overline{z}\frac{\overline{\partial}Z}{\partial Z}\partial^2 \ln \partial Z = \int dz d\overline{z} Tr(\partial g g^{-1}\overline{\partial} g g^{-1}) - \int dz d\overline{z} Tr(\Lambda g^{-1}\overline{\partial} g) + \int_Y Tr(dWW^{-1})^3 \tag{34}$$

where Λ is the constant matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the expressions in the r.h.s of (34) can be reexpressed as

$$\begin{gathered}\int dz d\overline{z}\tfrac{\overline{\partial}Z}{\partial Z}\partial^2 \ln \partial Z = \tfrac{1}{4}\int dz d\overline{z}(2\mu_{\overline{z}}^{z}\rho_{zz} - \tfrac{1}{2}\partial^2\mu_{\overline{z}}^{z}) - \tfrac{1}{2}\int dz d\overline{z}(\mu_{\overline{z}}^{z}\rho_{zz} - \tfrac{1}{2}\partial^2\mu_{\overline{z}}^{z}) \\ + \int_Y Tr(dWW^{-1})^3\end{gathered}$$

Since the first two terms in the r.h.s reduce to a total derivative, we obtain the desired result.

We have clarified how the Wess-Zumino-Witten action can be obtained in this framework. We hope that this derivation will be generalizable to other bi-dimensional conformal models.

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