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Supersymmetric $U(N)$ Gauge Model and Partial Breaking of $\mathcal{N}=2$ Supersymmetry

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Guided by the gauging of $U(N)$ isometry associated with the special Kähler geometry, and the discrete R symmetry, we construct the $\mathcal{N}=2$ supersymmetric action of a $U(N)$ invariant nonabelian gauge model in which rigid $\mathcal{N}=2$ supersymmetry is spontaneously broken to $\mathcal{N}=1$. This generalizes the abelian model considered by Antoniadis, Partouche and Taylor. We shed light on complexity of the supercurrents of our model associated with a broken $\mathcal{N}=2$ supermultiplet of currents, and discuss the spontaneously broken supersymmetry as an approximate fermionic shift symmetry.

§1. Introduction

Continuing investigations have been made for more than two decades on supersymmetric field theories,[†)] hoping to obtain realistic description of nature by broken $\mathcal{N}=1$ supersymmetry at an observable energy scale. On the other hand, it is most natural to view that physics beyond this energy scale is controlled by string theory, which, without nontoroidal backgrounds, produces extended supersymmetries in four dimensions. Breaking of extended supersymmetries in this vein provides a bridge between gauge field theory and string theory. String theory does not possess genuine coupling constants: instead, they are the vacuum expectation values of some supersymmetry preserving moduli fields. We are thus led to search for the possibility of spontaneous partial breaking of extended supersymmetries in four dimensions.

In the context of $\mathcal{N}=2$ supergravity,[4)] spontaneous breaking of local $\mathcal{N}=2$ supersymmetry to its $\mathcal{N}=1$ counterpart has been accomplished by the simultaneous realization of the Higgs and the super Higgs mechanisms. Sizable amount of literature has been accumulated till today along this direction.[5)–7)] There have been active researches carried out on nonlinear realization of extended supersymmetries in the partially broken phase.[8)–13)] These are closely related to the effective description of string theory,[14)] brane dynamics[15)–20)] and domain walls.[21)]

After Refs. 8) and 9) and prior to the remainder of the works on nonlinear realization, there was a work within the linear realization done by Antoniadis, Par-

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touche and Taylor[22] who constructed an $\mathcal{N}=2$ supersymmetric, self-interacting $U(1)$ model with one (or several) abelian $\mathcal{N}=2$ vector multiplet(s)[23] which breaks $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$ spontaneously. See also Refs. 24) and 25). The partial breaking of supersymmetry is accomplished by the simultaneous presence of the electric and magnetic Fayet-Iliopoulos terms, which is a generalization of Ref. 26). In the present paper, generalizing the work of Ref. 22), we construct the $\mathcal{N}=2$ supersymmetric action of a $U(N)$ invariant nonabelian gauge model in which rigid $\mathcal{N}=2$ supersymmetry is spontaneously broken to $\mathcal{N}=1$. The gauging of $U(N)$ isometry associated with the special Kähler geometry, and the discrete R symmetry are the primary ingredients of our construction.

Let us recall that partial breaking of extended rigid supersymmetries appears not possible on the basis of the positivity of the supersymmetry charge algebra:

$$\left\{\bar{Q}^i_{\alpha}, Q_{j\dot{\alpha}}\right\} = 2(\mathbf{1})_{\alpha\dot{\alpha}}\delta^i{}_j H. \tag{1·1}$$

In fact, if $Q_1|0\rangle = 0$, one concludes $H|0\rangle = 0$ and $Q_i|0\rangle = 0$ for all i. If $Q_1|0\rangle \neq 0$, then $H|0\rangle = E|0\rangle$ with $E>0$ and $Q_i|0\rangle \neq 0$ for all i. The loophole to this argument is that the use of the local version of the charge algebra is more appropriate in spontaneously broken symmetries and the most general supercurrent algebra is

$$\left\{\bar{Q}^j_{\dot{\alpha}}, \mathcal{S}^m_{\alpha i}(x)\right\} = 2(\sigma^n)_{\alpha\dot{\alpha}}\delta^j{}_i T^m_n(x) + (\sigma^m)_{\alpha\dot{\alpha}} C^j_i, \tag{1·2}$$

where $\mathcal{S}^m_{\alpha i}$ and T^m_n are the supercurrents and the energy momentum tensor respectively. We have denoted by C^j_i a field independent constant matrix permitted by the constraints from the Jacobi identity.[27] This last term does not modify the supersymmetry algebra acting on the fields. The abelian model of Ref. 22) and our nonabelian generalization provide a concrete example of this local algebra within linear realization from the point of view of the action principle.

The Lagrangian of our model has noncanonical kinetic terms coming from the nontrivial Kähler potential and does not fall into the class of renormalizable Lagrangians. As a model with spontaneously broken $\mathcal{N}=2$ supersymmetry, the prepotential $\mathcal{F}$ is present from the beginning of our construction. This is in contrast with breaking $\mathcal{N}=2$ to $\mathcal{N}=1$ by the operator (superpotential) $W(\Phi)$, where $\mathcal{F}$ appears aposteriori according to the recent developments beginning with Dijkgraaf and Vafa.[28] The model has a $U(1)$ sector interacting with an $SU(N)$ sector and the spontaneously broken supersymmetry acts as an approximate fermionic shift symmetry. Piecing through all these properties, we conclude that the action of the model should be regarded as a low energy effective action which applies to various processes and that the dynamical effects including those of (fractional) instantons are to be contained in the prepotential as an input. This input should be supplied by a separate means of calculation. The connection with the exact determination of the prepotential via Refs. 29) and 30) and from integrable systems[31),32] offers a new avenue of thoughts with this regard.

In §2, we provide the construction of the $\mathcal{N}=2$ supersymmetric action of the $U(N)$ invariant nonabelian gauge model which is equipped with the Fayet-Iliopoulos D term and a specific superpotential. Gauging of the noncanonical kinetic terms coming from the Kähler potential is a necessary step to complete the action. In §3, we provide the transformation law of the extended supersymmetries associated with the model. We note that the $SU(2)$ automorphism of $\mathcal{N}=2$ supersymmetry has been fixed in the parameter space. In §4, we fix the form of the prepotential and determine the vacuum with unbroken gauge symmetry. We exhibit partial breaking of $\mathcal{N}=2$ supersymmetry and discuss a mechanism which enables this. In §5, we examine a broken $\mathcal{N}=2$ supermultiplet of currents[33] associated with the model. The $U(1)_R$ current is not conserved except for the case where the prepotential has an R-weight two. Despite this, we show that the broken $\mathcal{N}=2$ supermultiplet of currents provides a useful means to construct the extended supercurrents. We shed light upon their complexity. In §6, we discuss a role played by the spontaneously broken supersymmetry. We see that it acts as a approximate $U(1)$ fermionic shift symmetry in the limit of letting the magnetic Fayet-Iliopoulos term large relative to the electric one. Our discussion in section two and that in section three leading to $\mathcal{N}=2$ supersymmetric Lagrangian exploit an algebraic operation denoted by $\mathfrak{R}$. This operation is defined by including the sign flip of the Fayet-Iliopoulos parameter $\xi \to -\xi$ into the standard discrete canonical transformation R. It is a legitimate algebraic process to use $\mathfrak{R}$ to demonstrate the second supersymmetry and in section three we obtain $\mathcal{N}=2$ supersymmetry transformation by demanding the covariance under $\mathfrak{R}$. In Appendix A, we give a more pedagogical proof of $\mathcal{N}=2$ supersymmetry of our action, using the canonical R. The two approaches are thus shown to be equivalent. In Appendix B, we reexamine the $\mathcal{N}=1$ current supermultiplet[34] in the Wess-Zumino model.

§2. $\mathcal{N}=2$ $U(N)$ gauge model

Let us first state our strategy to obtain the $\mathcal{N}=2$ supersymmetric action with nonabelian $U(N)$ gauge symmetry. We adopt the $\mathcal{N}=1$ superspace formalism to write down a $U(N)$ invariant action consisting of a set of $\mathcal{N}=1$ $U(N)$ chiral superfields and vector superfields in the adjoint representation. The action at this level is equipped with the terms required for the gauging, the Fayet-Iliopoulos D term, and a generic superpotential. Imposing the discrete element of $SU(2)$ automorphism of $\mathcal{N}=2$ supersymmety algebra as symmetry of our action,[2),22)] we obtain the action mentioned in the introduction.

What is meant by this last procedure is, however, a little more subtle than one might first think and we pause to explain this here in more detail. In the presence of the Fayet-Iliopoulos D term with its coefficient ξ, $\mathcal{N}=1$ Lagrangian is in general not invariant under the discrete R symmetry. (See Eq. (2·39).) Best one can do is therefore to consider simultaneously an inversion of the parameter ξ. (See Eq. (2·49).) Under this extended operation denoted by $\mathfrak{R}$, we will find

$$\mathfrak{R}:\mathcal{L}\to\mathcal{L},\quad \mathfrak{R}:\mathcal{L}'\to\mathcal{L}'. \tag{2·1}$$

(See Eqs. (2·26) and (2·33).) Combining this with the algebra

$$\mathfrak{R}\delta_1\mathfrak{R}^{-1}=\delta_2, \tag{2·2}$$

we conclude that our final actions (2·33) and (2·64) with (2·45) and (2·48) are invariant under $\mathcal{N}=2$ supersymmetry. Here we denote by δ_1 and δ_2, the transformation of the first supersymmetry and that of the second supersymmetry respectively. This definition $\mathfrak{R}$ turns out to be consistent with an interpretation that full rigid $SU(2)$ symmetry has been fixed in the parameter space. This is discussed in §3.

2.1. *$U(N)$ gauge model*

Let us introduce a set of $\mathcal{N}=1$ chiral superfields

$$\Phi(x^m,\theta)=\sum_{a=0}^{N^2-1}\Phi^a t_a\ . \tag{2·3}$$

Here, t_a, $a=0,1,\cdots,(N^2-1)$, are $N\times N$ hermitian matrices which generate $u(N)$ algebra, and $t_{\hat{a}}$, $\hat{a}=1,\cdots,(N^2-1)$, generate $su(N)$ algebra

$$[t_{\hat{a}},t_{\hat{b}}]=if_{\hat{a}\hat{b}}^{\hat{c}}t_{\hat{c}}. \tag{2·4}$$

The index 0 refers to the overall $u(1)$ generator. The scalar fields $A=A^a t_a$ in Φ undergo the adjoint action

$$A\to UAU^\dagger, \tag{2·5}$$

under $U(N)$.

The kinetic term for A is generated by

$$\mathcal{L}_K=\int d^2\theta d^2\bar{\theta}\ K(\Phi^a,\Phi^{*a}), \tag{2·6}$$

where $K(A^a,A^{*a})$ is the Kähler potential. The Kähler potential we employ is given by

$$K(A^a,A^{*a})=\frac{i}{2}(A^a\mathcal{F}_a^*-A^{*a}\mathcal{F}_a), \tag{2·7}$$

where $\mathcal{F}_a=\partial_a\mathcal{F}=\frac{d}{dA^a}\mathcal{F}$ and $\mathcal{F}$ is an analytic function of A.†) The Kähler potential can be written using a hermitian metric on the bundle compatible with the symplectic structure as

$$K=-\frac{i}{2}\left\langle\Omega|\bar{\Omega}\right\rangle,\qquad \left\langle\Omega|\bar{\Omega}\right\rangle=-\Omega^T\begin{pmatrix}0&\mathbb{I}\\-\mathbb{I}&0\end{pmatrix}\Omega^*\ . \tag{2·8}$$

The Kähler metric

$$g_{ab^*}=\partial_a\partial_{b^*}K=\mathrm{Im}\,\mathcal{F}_{ab} \tag{2·9}$$

constructed this way always admits a $U(N)$ isometry. The holomorphic Killing vectors $k_a={k_a}^b\partial_b$ are generated by the Killing potential $\mathfrak{D}_a$, to be introduced shortly, as

$${k_a}^b=-ig^{bc^*}\partial_{c^*}\mathfrak{D}_a,\quad k_a^{*b}=ig^{cb^*}\partial_c\mathfrak{D}_a. \tag{2·10}$$

†) The $\Omega=\binom{A^a}{\mathcal{F}_b}$ can be regarded as a section of a holomorphic symplectic bundle on a special Kähler geometry (see Ref. 34) and references therein). We work in special coordinates in this paper.

These form an algebra $[k_a, k_b] = -f_{ab}^c k_c$. The A^a and $\mathcal{F}_a$ transform in the adjoint representation of $U(N)$

$$\delta_b A^a = -f_{bc}^a A^c, \qquad \delta_b \mathcal{F}_a = -f_{ab}^c \mathcal{F}_c \ . \tag{2·11}$$

One finds that the commutator of two of δ_a is given by $[\delta_a, \delta_b] = f_{ab}^c \delta_c$. Comparing this with the commutator of two Killing vectors, we are able to identify δ_a with $-k_a$. Equation (2·11) is rewritten as

$$k_b{}^c \partial_c A^a = f_{bc}^a A^c, \qquad k_b{}^c \partial_c \mathcal{F}_a = -f_{ba}^c \mathcal{F}_c \ . \tag{2·12}$$

The isometry group can be embedded in the symplectic group, and the $\mathfrak{D}_a$ is given by

$$\mathfrak{D}_a = -\frac{1}{2}\left\langle \Omega | T_a \bar{\Omega} \right\rangle = -\frac{1}{2}(\mathcal{F}_b f_{ac}^b A^{*c} + \mathcal{F}_b^* f_{ac}^b A^c), \qquad T_a = \begin{pmatrix} f_{ac}^b & 0 \\ 0 & -f_{ac}^b \end{pmatrix}. \tag{2·13}$$

Note that $\mathfrak{D}_{\hat{a}}$ are completely determined by this formula while $\mathfrak{D}_0$ is determined up to a constant.

In order to gauge the $U(N)$ isometry, we introduce a set of $\mathcal{N} = 1$ vector superfields

$$V(x^m, \theta, \bar{\theta}) = \sum_{a=0}^{N^2-1} V^a t_a. \tag{2·14}$$

The $U(N)$ gauging of $\mathcal{L}_K$ is accomplished[35] by adding

$$\mathcal{L}_\Gamma = \int d^2\theta d^2\bar{\theta}\Gamma, \qquad \Gamma = \left[\int_0^1 d\alpha e^{\frac{i}{2}\alpha v^a(k_a - k_a^*)} v^c \mathfrak{D}_c\right]_{v^a \to V^a}, \tag{2·15}$$

where $[\cdots]_{v^a \to V^a}$ means the replacement of v^a by V^a after evaluating $\cdots$. Combining $\mathcal{L}_K$ with $\mathcal{L}_\Gamma$, we obtain

$$\begin{aligned} \mathcal{L}_K + \mathcal{L}_\Gamma = &-g_{ab^*}\mathcal{D}_m A^a \mathcal{D}^m A^{*b} - \frac{i}{2} g_{ab^*}\psi^a \sigma^m \mathcal{D}'_m \bar{\psi}^b + \frac{i}{2} g_{ab^*}\mathcal{D}'_m \psi^a \sigma^m \bar{\psi}^b \\ &+ g_{ab^*} F^a F^{*b} - \frac{1}{2} g_{ab^*,c^*} F^a \bar{\psi}^b \bar{\psi}^c - \frac{1}{2} g_{bc^*,a} F^{*c} \psi^a \psi^b \\ &+ \frac{1}{\sqrt{2}} g_{ab^*}(\lambda^c \psi^a k_c^{*b} + \bar{\lambda}^c \bar{\psi}^b k_c{}^a) + \frac{1}{2} D^a \mathfrak{D}_a \ , \end{aligned} \tag{2·16}$$

where we have exploited $\frac{1}{4} g_{ac^*,bd^*}\psi^a\psi^b\bar{\psi}^c\bar{\psi}^d = 0$ as $g_{ac^*,bd^*} = 0$ for the choice of K in (2·7). The covariant derivatives are defined as

$$\mathcal{D}_m A^a = \partial_m A^a - \frac{1}{2} v_m^b k_b{}^a, \tag{2·17}$$

$$\mathcal{D}'_m \psi^a = \mathcal{D}_m \psi^a + \Gamma_{bc}^a \mathcal{D}_m A^b \psi^c, \tag{2·18}$$

$$\mathcal{D}_m \psi^a = \partial_m \psi^a - \frac{1}{2} v_m^b \partial_c k_b{}^a \psi^c \ , \tag{2·19}$$

where $\Gamma^a_{bc} = g^{ad^*} g_{bd^*,c}$.

The gauged kinetic action for the vector superfield V is given by

$$\mathcal{L}_{\mathcal{W}^2} = -\frac{i}{4}\int d^2\theta \tau_{ab}\mathcal{W}^a\mathcal{W}^b + \text{c.c.}\ , \qquad \mathcal{W}_\alpha = -\frac{1}{4}\bar{D}\bar{D}e^{-V}D_\alpha e^V = \mathcal{W}^a_\alpha t_a\ , \quad (2\cdot20)$$

where $\tau_{ab} = (\tau_1)_{ab} + i(\tau_2)_{ab}$ is an analytic function of Φ, and will be determined by requiring $\mathcal{N} = 2$ supersymmetry. The $\mathcal{L}_{\mathcal{W}^2}$ is evaluated as

$$\begin{aligned}\mathcal{L}_{\mathcal{W}^2} = &-\frac{1}{2}\tau_{ab}\lambda^a\sigma^m\mathcal{D}_m\bar{\lambda}^b - \frac{1}{2}\bar{\tau}_{ab}\mathcal{D}_m\lambda^a\sigma^m\bar{\lambda}^b - \frac{1}{4}(\tau_2)_{ab}v^a_{mn}v^{bmn} - \frac{1}{8}(\tau_1)_{ab}\epsilon^{mnpq}v^a_{mn}v^b_{pq} \\ &-i\frac{\sqrt{2}}{8}(\partial_c\tau_{ab}\psi^c\sigma^n\bar{\sigma}^m\lambda^a - \partial_{c^*}\tau^*_{ab}\bar{\lambda}^a\bar{\sigma}^m\sigma^n\bar{\psi}^c)v^b_{mn} \\ &+\frac{1}{2}(\tau_2)_{ab}D^aD^b + \frac{\sqrt{2}}{4}(\partial_c\tau_{ab}\psi^c\lambda^a + \partial_{c^*}\tau^*_{ab}\bar{\psi}^c\bar{\lambda}^a)D^b + \frac{i}{4}\partial_c\tau_{ab}F^c\lambda^a\lambda^b \\ &-\frac{i}{4}\partial_{c^*}\tau^*_{ab}F^{*c}\bar{\lambda}^a\bar{\lambda}^b - \frac{i}{8}\partial_c\partial_d\tau_{ab}\psi^c\psi^d\lambda^a\lambda^b + \frac{i}{8}\partial_{c^*}\partial_{d^*}\tau^*_{ab}\bar{\psi}^c\bar{\psi}^d\bar{\lambda}^a\bar{\lambda}^b, \end{aligned} \quad (2\cdot21)$$

where we have defined

$$v^a_{mn} = \partial_m v^a_n - \partial_n v^a_m - \frac{1}{2}f^a_{bc}v^b_m v^c_n, \quad (2\cdot22)$$

$$\mathcal{D}_m\lambda^a = \partial_m\lambda^a - \frac{1}{2}f^a_{bc}v^b_m\lambda^c. \quad (2\cdot23)$$

In addition, we include the superpotential term

$$\begin{aligned}\mathcal{L}_W &= \int d^2\theta W(\Phi) + \text{c.c.} \\ &= F^a\partial_a W - \frac{1}{2}\partial_a\partial_b W\psi^a\psi^b + F^{*a}\partial_{a^*}W^* - \frac{1}{2}\partial_{a^*}\partial_{b^*}W^*\bar{\psi}^a\bar{\psi}^b\ , \end{aligned} \quad (2\cdot24)$$

and the Fayet-Iliopoulos D-term[26)]

$$\mathcal{L}_D = \xi\int d^2\theta d^2\bar{\theta}V^0 = \sqrt{2}\xi D^0. \quad (2\cdot25)$$

The superpotential W will be determined by requiring $\mathcal{N} = 2$ supersymmetry. Finally, putting all these together, the total action is given as

$$\mathcal{L} = \mathcal{L}_K + \mathcal{L}_\Gamma + \mathcal{L}_{\mathcal{W}^2} + \mathcal{L}_W + \mathcal{L}_D. \quad (2\cdot26)$$

For the sake of our discussion in the next subsection, we present the on-shell action, eliminating the auxiliary fields by using the equations of motion

$$D^a = \hat{D}^a - (\tau_2^{-1})^{ab}\left(\frac{1}{2}\mathfrak{D}_b + \sqrt{2}\xi\delta^0_b\right), \quad (2\cdot27)$$

$$F^a = \hat{F}^a - g^{ab^*}\partial_{b^*}W^*, \quad (2\cdot28)$$

$$F^{*a} = \hat{F}^{*a} - g^{ba^*}\partial_b W\ , \quad (2\cdot29)$$

where

$$\hat{D}^a = -\frac{\sqrt{2}}{4}(\tau_2^{-1})^{ab}\left(\partial_d\tau_{bc}\psi^d\lambda^c + \partial_{d^*}\tau^*_{bc}\bar{\psi}^d\bar{\lambda}^c\right), \tag{2·30}$$

$$\hat{F}^a = -g^{ab^*}\left(-\frac{i}{4}\partial_{b^*}\tau^*_{cd}\bar{\lambda}^c\bar{\lambda}^d - \frac{1}{2}g_{cb^*,d}\psi^c\psi^d\right), \tag{2·31}$$

$$\hat{F}^{*a} = -g^{ba^*}\left(\frac{i}{4}\partial_b\tau_{cd}\lambda^c\lambda^d - \frac{1}{2}g_{bc^*,d^*}\bar{\psi}^c\bar{\psi}^d\right). \tag{2·32}$$

The action $\mathcal{L}$ takes the following form:

$$\mathcal{L}' = \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm pot} + \mathcal{L}_{\rm Pauli} + \mathcal{L}_{\rm mass} + \mathcal{L}_{{\rm fermi}^4}, \tag{2·33}$$

where

$$\begin{aligned}
\mathcal{L}_{\rm kin} =& -g_{ab^*}\mathcal{D}_m A^a \mathcal{D}^m A^{*b} - \frac{1}{4}(\tau_2)_{ab}v^a_{mn}v^{bmn} - \frac{1}{8}(\tau_1)_{ab}\epsilon^{mnpq}v^a_{mn}v^b_{pq} \qquad (2·34)\\
&-\frac{1}{2}\tau_{ab}\lambda^a\sigma^m\mathcal{D}_m\bar{\lambda}^b - \frac{1}{2}\tau^*_{ab}\mathcal{D}_m\lambda^a\sigma^m\bar{\lambda}^b\\
&-\frac{i}{2}g_{ab^*}\psi^a\sigma^m\mathcal{D}_m\bar{\psi}^b + \frac{i}{2}g_{ab^*}\mathcal{D}_m\psi^a\sigma^m\bar{\psi}^b,\\
\mathcal{L}_{\rm pot} =& -\frac{1}{2}\left(\tau_2^{-1}\right)^{ab}\left(\frac{1}{2}\mathfrak{D}_a + \sqrt{2}\xi\delta^0_a\right)\left(\frac{1}{2}\mathfrak{D}_b + \sqrt{2}\xi\delta^0_b\right) - g^{ab^*}\partial_a W\partial_{b^*}W^*, \quad (2·35)\\
\mathcal{L}_{\rm Pauli} =& -i\frac{\sqrt{2}}{8}\partial_c\tau_{ab}\psi^c\sigma^n\bar{\sigma}^m\lambda^a v^b_{mn} + i\frac{\sqrt{2}}{8}\partial_{c^*}\tau^*_{ab}\bar{\lambda}^a\bar{\sigma}^m\sigma^n\bar{\psi}^c v^b_{mn}, \qquad (2·36)\\
\mathcal{L}_{\rm mass} =& -\frac{1}{2}\partial_a\partial_b W\psi^a\psi^b - g^{ab^*}\partial_a W\left(-\frac{i}{4}\partial_{b^*}\tau^*_{cd}\bar{\lambda}^c\bar{\lambda}^d - \frac{1}{2}g_{cb^*,d}\psi^c\psi^d\right)\\
&-\frac{1}{2}\partial_{a^*}\partial_{b^*}W^*\bar{\psi}^a\bar{\psi}^b - g^{ab^*}\left(\frac{i}{4}\partial_a\tau_{cd}\lambda^c\lambda^d - \frac{1}{2}g_{ac^*,d^*}\bar{\psi}^c\bar{\psi}^d\right)\partial_{b^*}W^*\\
&+\frac{1}{\sqrt{2}}g_{ab^*}\left(\bar{\lambda}^c\bar{\psi}^b k_c{}^a + \lambda^c\psi^a k^{*b}_c\right)\\
&-\frac{\sqrt{2}}{4}\left(\tau_2^{-1}\right)^{ab}\left(\frac{1}{2}\mathfrak{D}_a + \sqrt{2}\xi\delta^0_a\right)\left(\partial_d\tau_{bc}\psi^d\lambda^c + \partial_{d^*}\tau^*_{bc}\bar{\psi}^d\bar{\lambda}^c\right), \qquad (2·37)\\
\mathcal{L}_{{\rm fermi}^4} =& -\frac{i}{8}\partial_c\partial_d\tau_{ab}\psi^c\psi^d\lambda^a\lambda^b + \frac{i}{8}\partial_{c^*}\partial_{d^*}\tau^*_{ab}\bar{\psi}^c\bar{\psi}^d\bar{\lambda}^a\bar{\lambda}^b\\
&-\frac{1}{16}\left(\tau_2^{-1}\right)^{ab}\left(\partial_d\tau_{ac}\psi^d\lambda^c + \partial_{d^*}\tau^*_{ac}\bar{\psi}^d\bar{\lambda}^c\right)\left(\partial_f\tau_{be}\psi^f\lambda^e + \partial_{f^*}\tau^*_{be}\bar{\psi}^f\bar{\lambda}^e\right)\\
&-g^{ab^*}\left(\frac{i}{4}\partial_a\tau_{cd}\lambda^c\lambda^d - \frac{1}{2}g_{ac^*,d^*}\bar{\psi}^c\bar{\psi}^d\right)\left(-\frac{i}{4}\partial_{b^*}\tau^*_{ef}\bar{\lambda}^e\bar{\lambda}^f - \frac{1}{2}g_{eb^*,f}\psi^e\psi^f\right).
\end{aligned}$$

(2·38)

2.2. *Discrete R-symmetry*

We shall show that our Lagrangian (2·33) $\mathcal{L}'$ can be made invariant under the action

$$\mathfrak{R}: \begin{pmatrix} \lambda^a \\ \psi^a \end{pmatrix} \to \begin{pmatrix} \psi^a \\ -\lambda^a \end{pmatrix}, \tag{2·39}$$

which is a discrete element of the $SU(2)$ R-symmetry that acts as an automorphism of $\mathcal{N}=2$ supersymmetry.

First, we examine the invariance of $\mathcal{L}_{\rm Pauli}$, $\mathcal{L}_{{\rm fermi}^4}$ and $\mathcal{L}_{\rm kin}$ under the action (2·39). The invariance of $\mathcal{L}_{\rm Pauli}$ and that of $\mathcal{L}_{{\rm fermi}^4}$ under (2·39) require

$$\partial_c \tau_{ab} = \partial_a \tau_{cb}, \tag{2·40}$$

and

$$\partial_c \partial_d \tau_{ab} = \partial_a \partial_b \tau_{cd}\ , \quad \partial_c \tau_{ab} = \mathcal{F}_{abc}, \tag{2·41}$$

respectively. In addition, the invariance of the fermion kinetic terms in $\mathcal{L}_{\rm kin}$ implies that

$$\mathrm{Im}(\tau_{ab}) = \mathrm{Im}(\mathcal{F}_{ab}) \tag{2·42}$$

and

$$-2\partial_a \partial_{b^*} \mathfrak{D}_c = \tau_{ad} f^d_{cb} + \tau^*_{bd} f^d_{ca}, \tag{2·43}$$

as well as the last condition in (2·41) which comes from that the terms with a derivative of A^* vanish. The first condition (2·42) comes from the terms with a derivative of λ or ψ while the second one (2·43) from those including v^a_m. For the boson kinetic terms in $\mathcal{L}_{\rm kin}$, the invariance is obvious because they do not contain fermionic fields. From the conditions (2·41) and (2·42), we conclude that

$$\tau_{ab} = \mathcal{F}_{ab}, \tag{2·44}$$

so that $g_{ab^*} = (\tau_2)_{ab}$. It is easy to show that the Killing potential $\mathfrak{D}_a$ defined in (2·13) solves the condition (2·43).

Secondly, we examine the invariance of the $\lambda\lambda$ and $\psi\psi$ mass terms in $\mathcal{L}_{\rm mass}$ under (2·39). The key relation required for this invariance is

$$-\frac{i}{4} g^{cd^*} \partial_c \tau_{ab} \partial_{d^*} W^* = \frac{1}{2} g^{cd^*} \partial_c W g_{ad^*,b} - \frac{1}{2} \partial_a \partial_b W. \tag{2·45}$$

Writing the $U(N)$ invariant function W as $W = eA^0 + m\phi(A)$, where the e and m are real constants, it reduces to

$$\mathcal{F}_{abc} \left(\frac{1}{\mathcal{F} - \mathcal{F}^*} \right)^{cd} (\partial_d \phi - \partial_{d^*} \phi^*) = \partial_a \partial_b \phi, \tag{2·46}$$

which can be solved by $\phi = \mathcal{F}_0 + \text{const}$. Thus we can choose

$$W = eA^0 + m\mathcal{F}_0, \tag{2·47}$$

up to an irrelevant constant.

Thirdly, we examine the $\psi\lambda$ terms in $\mathcal{L}_{\rm mass}$. Because $\psi^a \lambda^b$ is odd under the action (2·39), the coefficient, $\frac{1}{\sqrt{2}} g_{ac^*} k^{*c}_b - \frac{\sqrt{2}}{8} (\tau_2^{-1})^{cd} \partial_a \tau_{cb} (\mathfrak{D}_d + 2\sqrt{2}\xi \delta^0_d)$, must be odd. This implies the key relation for the invariance

$$i\partial_a \mathfrak{D}_b + i\partial_b \mathfrak{D}_a - \frac{1}{2} (\tau_2^{-1})^{cd} \partial_a \tau_{cb} \mathfrak{D}_d = 0, \tag{2·48}$$

as well as

$$\mathfrak{R};\ \xi \to -\xi. \tag{2·49}$$

Equation (2·48) can be proven as follows. First, we note that

$$\mathcal{F}_{ac} f^{c}_{db} + \mathcal{F}_{bc} f^{c}_{da} = -\mathcal{F}_{abc} f^{c}_{de} A^{e}, \tag{2·50}$$

which is derived as a derivative of the second relation in (2·12). Using this relation and the definition (2·13), one finds that

$$i\partial_a \mathfrak{D}_b + i\partial_b \mathfrak{D}_a = -\frac{i}{2}\mathcal{F}_{abc} f^{c}_{de} A^{*d} A^{e}. \tag{2·51}$$

On the other hand, the Killing potential is shown to be rewritten as

$$\mathfrak{D}_a = \frac{1}{2} f^{b}_{cd} A^{*c} A^{d} (\mathcal{F}^{*}_{ab} - \mathcal{F}_{ab}) = -i g_{ab} f^{b}_{cd} A^{*c} A^{d} \tag{2·52}$$

by using the second relation in (2·12). Equations (2·51) and (2·52) are enough to see that Eq. (2·48) is true.

Lastly, we examine $\mathcal{L}_{\rm pot}$. The invariance of $\mathcal{L}_{\rm pot}$ under (2·49) follows from the fact that the term linear in ξ in $\mathcal{L}_{\rm pot}$ vanishes:

$$-\frac{1}{2}(\tau_2^{-1})^{ab}\mathfrak{D}_a \sqrt{2}\xi\delta^0_b = -\frac{\sqrt{2}}{2}\xi g^{a0}(-i g_{ab} f^{b}_{cd} A^{*c} A^{d}) = i\frac{\sqrt{2}}{2}\xi f^{0}_{cd} A^{*c} A^{d} = 0, \tag{2·53}$$

where we have used (2·44) and (2·52).

In summary, we have shown that our on-shell action (2·33) admits the discrete $\mathfrak{R}$-symmetry (2·39) and (2·49) if we choose τ_{ab} as (2·44) and W as (2·47).

We will show that the discrete $\mathfrak{R}$-symmetry can be realized in the off-shell action (2·26) with (2·44) and (2·47). In an ungauged theory without a superpotential, the discrete action on the auxiliary fields is $D^a \to -D^a$ and $F^a \to F^{*a}$. In our model, this is modified as is seen below. The terms which need to be checked are those including auxiliary fields. First, we examine bosonic terms including F^a and F^{*a},

$$g_{ab^*} F^a F^{*b} + F^a \partial_a W + F^{*a} \partial_{a^*} W^*. \tag{2·54}$$

Apparently, this is not invariant under $F \to F^*$. Rewriting it as

$$g_{ab^*}(F^a + g^{ac^*}\partial_{c^*} W^*)(F^{*b} + g^{db^*}\partial_d W) - g^{ab^*}\partial_a W \partial_{b^*} W^*, \tag{2·55}$$

one finds that the action

$$\mathfrak{R} : F^a + g^{ac^*}\partial_{c^*} W^* \to F^{*b} + g^{db^*}\partial_d W \tag{2·56}$$

is a symmetry. Secondly, we consider the $\psi\psi$ and $\lambda\lambda$ mass terms in (2·16), (2·24) and (2·21). Under the action (2·39) and (2·56) the $\psi\psi$ mass terms become

$$\left(\frac{i}{4}\mathcal{F}_{abc}(F^c + g^{cd^*}\partial_{d^*} W^*) - \frac{i}{4}\mathcal{F}_{abc} g^{dc^*}\partial_d W - \frac{1}{2}\partial_a\partial_b W\right)\lambda^a\lambda^b. \tag{2·57}$$

Equating it with the original $\lambda\lambda$ mass term, $\frac{i}{4}\partial_c\tau_{ab}F^c\lambda^a\lambda^b$, we find that the invariance implies

$$\frac{i}{4}\mathcal{F}_{abc}(g^{cd^*}\partial_{d*}W^* - g^{dc^*}\partial_d W) - \frac{1}{2}\partial_a\partial_b W = 0. \tag{2·58}$$

It is easy to see that the superpotential (2·47) solves this equation. Thirdly, we examine the $\psi\lambda$ mass term in (2·16) and (2·21)

$$\frac{1}{\sqrt{2}}\left(g_{ac^*}k_b^{*c} + \frac{1}{2}\partial_a\tau_{bc}D^c\right)\psi^a\lambda^b. \tag{2·59}$$

We rewrite it as

$$\frac{1}{\sqrt{2}}\left(g_{ac^*}k_b^{*c} - \frac{1}{4}\partial_a\tau_{bc}g^{cd}\mathfrak{D}_d\right)\psi^a\lambda^b + \frac{1}{2\sqrt{2}}\partial_a\tau_{bc}\left(D^c + \frac{1}{2}g^{cd}\mathfrak{D}_d\right)\psi^a\lambda^b. \tag{2·60}$$

The invariance of the first term is guaranteed by (2·48), and thus we find

$$\mathfrak{R} : D^c + \frac{1}{2}g^{cd}\mathfrak{D}_d \to -\left(D^c + \frac{1}{2}g^{cd}\mathfrak{D}_d\right) \tag{2·61}$$

for the invariance. Lastly, let us turn to the bosonic terms including D^a

$$\frac{1}{2}(\tau_2)_{ab}D^aD^b + \frac{1}{2}D^a\left(\mathfrak{D}_a \ + 2\sqrt{2}\xi\delta_a^0\right). \tag{2·62}$$

We rewrite it as

$$\begin{aligned}&\frac{1}{2}g_{ab}\left(D^a + \frac{1}{2}g^{ac}(\mathfrak{D}_c + 2\sqrt{2}\xi\delta_c^0)\right)\left(D^b + \frac{1}{2}g^{bd}(\mathfrak{D}_d + 2\sqrt{2}\xi\delta_d^0)\right)\\ &\quad - \frac{1}{8}g^{ab}\left(\mathfrak{D}_a + 2\sqrt{2}\xi\delta_a^0\right)\left(\mathfrak{D}_b + 2\sqrt{2}\xi\delta_b^0\right).\end{aligned} \tag{2·63}$$

The first term in (2·63) is obviously invariant under the action (2·49) and (2·61). The last term is also invariant under the action (2·49) because the term linear in ξ vanishes as is shown in (2·53).

As a result, we have found that the off-shell action $\mathcal{L}$ (2·26) is invariant under the discrete $\mathfrak{R}$-symmetry (2·39), (2·49), (2·56) and (2·61) if we choose τ_{ab} as (2·44) and W as (2·47). For completeness, we present the off-shell action of our $U(N)$ gauge model which is invariant under the discrete $\mathfrak{R}$-symmetry:

$$\begin{aligned}\mathcal{L} = &-g_{ab^*}\mathcal{D}_mA^a\mathcal{D}^mA^{*b} - \frac{1}{4}g_{ab}v_{mn}^a v^{bmn} - \frac{1}{8}\mathrm{Re}(\mathcal{F}_{ab})\epsilon^{mnpq}v_{mn}^a v_{pq}^b\\ &-\frac{1}{2}\mathcal{F}_{ab}\lambda^a\sigma^m\mathcal{D}_m\bar{\lambda}^b - \frac{1}{2}\mathcal{F}_{ab}^*\mathcal{D}_m\lambda^a\sigma^m\bar{\lambda}^b - \frac{1}{2}\mathcal{F}_{ab}\psi^a\sigma^m\mathcal{D}_m\bar{\psi}^b - \frac{1}{2}\mathcal{F}_{ab}^*\mathcal{D}_m\psi^a\sigma^m\bar{\psi}^b\\ &+g_{ab^*}F^aF^{*b} + F^a\partial_aW + F^{*a}\partial_{a^*}W^* + \frac{1}{2}g_{ab}D^aD^b + \frac{1}{2}D^a\left(\mathfrak{D}_a + 2\sqrt{2}\xi\delta_a^0\right)\\ &+\left(\frac{i}{4}\mathcal{F}_{abc}F^{*c} - \frac{1}{2}\partial_a\partial_bW\right)\psi^a\psi^b + \frac{i}{4}\mathcal{F}_{abc}F^c\lambda^a\lambda^b + \frac{1}{\sqrt{2}}\left(g_{ac^*}k_b^{*c} + \frac{1}{2}\mathcal{F}_{abc}D^c\right)\psi^a\lambda^b\end{aligned}$$

$$
\begin{aligned}
&+\left(-\frac{i}{4}\mathcal{F}^*_{abc}F^c-\frac{1}{2}\partial_{a^*}\partial_{b^*}W^*\right)\bar\psi^a\bar\psi^b-\frac{i}{4}\mathcal{F}^*_{abc}F^{*c}\bar\lambda^a\bar\lambda^b\\
&+\frac{1}{\sqrt2}(g_{ca^*}k_b{}^c+\frac{1}{2}\mathcal{F}^*_{abc}D^c)\bar\psi^a\bar\lambda^b\\
&-i\frac{\sqrt2}{8}(\mathcal{F}_{abc}\psi^c\sigma^n\bar\sigma^m\lambda^a-\mathcal{F}^*_{abc}\bar\lambda^a\bar\sigma^m\sigma^n\bar\psi^c)v^b_{mn}\\
&-\frac{i}{8}\mathcal{F}_{abcd}\psi^c\psi^d\lambda^a\lambda^b+\frac{i}{8}\mathcal{F}^*_{abcd}\bar\psi^c\bar\psi^d\bar\lambda^a\bar\lambda^b, \qquad (2\cdot64)
\end{aligned}
$$

where $g_{ab^*}=\mathrm{Im}(\mathcal{F}_{ab})$ and $W=eA^0+m\mathcal{F}_0$. In the above expression, the covariant derivatives are defined as

$$\mathcal{D}_m\Psi^a=\partial_m\Psi^a-\frac{1}{2}f^a_{bc}v^b_m\Psi^c, \qquad \Psi^a=\{A^a,\psi^a,\lambda^a\}, \qquad (2\cdot65)$$

$$v^a_{mn}=\partial_m v^a_n-\partial_n v^a_m-\frac{1}{2}f^a_{bc}v^b_m v^c_n. \qquad (2\cdot66)$$

By the reasoning we explained at the beginning of this section, our action (2·33) and (2·64) are invariant under $\mathcal{N}=2$ supersymmetry.

§3. Extended supersymmetry transformation

Our action is manifestly invariant under the $\mathcal{N}=1$ supersymmetry transformation. We have made our action invariant under the discrete transformation $\mathfrak{R}$, and the algebra of extended supersymmetry permits us to argue for the invariance of our action under the extended $\mathcal{N}=2$ supersymmetry transformation. In this section, we will first lift the $\mathcal{N}=1$ supersymmetry transformation

$$
\begin{aligned}
\delta_{\eta_1}A^a&=\sqrt2\eta_1\psi^a, &(3\cdot1)\\
\delta_{\eta_1}\psi^a&=i\sqrt2\sigma^m\bar\eta_1\mathcal{D}_mA^a+\sqrt2\eta_1F^a, &(3\cdot2)\\
\delta_{\eta_1}v^a_m&=i\eta_1\sigma_m\bar\lambda^a-i\lambda^a\sigma_m\bar\eta_1, &(3\cdot3)\\
\delta_{\eta_1}\lambda^a&=\sigma^{mn}\eta_1v^a_{mn}+i\eta_1D^a, &(3\cdot4)
\end{aligned}
$$

to its $\mathcal{N}=2$ counterpart by exploiting the discrete symmetry $\mathfrak{R}$. We will subsequently examine $SU(2)$ covariance of the $\mathcal{N}=2$ supersymmetry transformation obtained.

Let us first form a following doublet of fermions:

$$\boldsymbol{\lambda}_i{}^a\equiv\begin{pmatrix}\lambda^a\\ \psi^a\end{pmatrix}, \qquad \boldsymbol{\lambda}^{ia}\equiv\epsilon^{ij}\boldsymbol{\lambda}_j{}^a=\begin{pmatrix}+\psi^a\\ -\lambda^a\end{pmatrix}, \qquad (3\cdot5)$$

$$\bar{\boldsymbol{\lambda}}^{ia}\equiv\overline{\boldsymbol{\lambda}_i{}^a}=\begin{pmatrix}\bar\lambda^a\\ \bar\psi^a\end{pmatrix}, \qquad \bar{\boldsymbol{\lambda}}_i{}^a\equiv\epsilon_{ik}\bar{\boldsymbol{\lambda}}^{ka}=\begin{pmatrix}-\bar\psi^a\\ +\bar\lambda^a\end{pmatrix}=-\overline{\boldsymbol{\lambda}^{ia}}. \qquad (3\cdot6)$$

We carry out the raising and the lowering of i,j indices by ϵ^{ij} ; $\epsilon^{12}=\epsilon_{21}=1$, $\epsilon^{21}=\epsilon_{12}=-1$. Recall the action of $\mathfrak{R}$:

$$\mathfrak{R}:\boldsymbol{\lambda}_i{}^a=\begin{pmatrix}+\lambda^a\\ +\psi^a\end{pmatrix}\longrightarrow\boldsymbol{\lambda}^{ia}=\begin{pmatrix}+\psi^a\\ -\lambda^a\end{pmatrix},$$

$$\bar{\boldsymbol{\lambda}}_i{}^a = \begin{pmatrix} -\bar{\psi}^a \\ +\bar{\lambda}^a \end{pmatrix} \longrightarrow \bar{\boldsymbol{\lambda}}^{ia} = \begin{pmatrix} +\bar{\lambda}^a \\ +\bar{\psi}^a \end{pmatrix}, \tag{3·7}$$

and therefore the terms $\hat{F}^a$ in (2·31) and $\hat{D}^a$ in (2·30) which are bilinear in fermions undergo the action:

$$\mathfrak{R}: \begin{array}{l} \hat{F}^a \longrightarrow \hat{F}^{*a} \\ \hat{D}^a \longrightarrow -\hat{D}^a \end{array} . \tag{3·8}$$

Note that this is nothing but (2·61) and (2·56). The bosonic fields A^a, v_m^a are invariant under $\mathfrak{R}$. So from (3·1) and (3·3), we see that the grassman parameter η_2 for the second supersymmetry forms a doublet with η_1 such that

$$\mathfrak{R}: \boldsymbol{\eta}_i \equiv \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \longrightarrow \begin{pmatrix} +\eta_2 \\ -\eta_1 \end{pmatrix} \equiv \boldsymbol{\eta}^{\boldsymbol{i}} \equiv \epsilon^{ij}\boldsymbol{\eta}_j . \tag{3·9}$$

Demanding the covariance under $\mathfrak{R}$, we obtain the extended supersymmetry transformation:

$$\boldsymbol{\delta} A^a = \sqrt{2}\boldsymbol{\eta}_j \boldsymbol{\lambda}^{ja}, \tag{3·10}$$

$$\begin{aligned} \boldsymbol{\delta}\boldsymbol{\lambda}_j{}^a &= \sigma^{mn}\boldsymbol{\eta}_j v_{mn}^a + \sqrt{2}i\left(\sigma^m \bar{\boldsymbol{\eta}}_j\right)\mathcal{D}_m A^a \\ &\quad + i\begin{pmatrix} \hat{D}^a & +i\sqrt{2}\hat{F}^{*a} \\ -i\sqrt{2}\hat{F}^a & -\hat{D}^a \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} - \frac{i}{2}\boldsymbol{\eta}_j g^{ab}\mathfrak{D}_b \\ &\quad -\sqrt{2}ig^{ab^*}\frac{\partial}{\partial A^{*b}}\begin{pmatrix} \xi A^{*0} & +i(eA^{*0} + m\mathcal{F}_0^*) \\ -i(eA^{*0} + m\mathcal{F}_0^*) & -\xi A^{*0} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \end{aligned} \tag{3·11}$$

$$\boldsymbol{\delta} v_m^a = i\boldsymbol{\eta}_j \sigma_m \bar{\boldsymbol{\lambda}}^{ja} - i\boldsymbol{\lambda}_j{}^a \sigma_m \bar{\boldsymbol{\eta}}^j . \tag{3·12}$$

Here

$$\bar{\boldsymbol{\eta}}^j \equiv \begin{pmatrix} \bar{\eta}_1 \\ \bar{\eta}_2 \end{pmatrix} \quad \text{and} \quad \bar{\boldsymbol{\eta}}_j \equiv \epsilon_{ji}\bar{\boldsymbol{\eta}}^i = \begin{pmatrix} -\bar{\eta}_2 \\ +\bar{\eta}_1 \end{pmatrix}. \tag{3·13}$$

The transformation (3·11) is further recast into the following form:

$$\boldsymbol{\delta}\boldsymbol{\lambda}_j{}^a = (\sigma^{mn}\boldsymbol{\eta}_j)v_{mn}^a + \sqrt{2}i(\sigma^m\bar{\boldsymbol{\eta}}_j)\mathcal{D}_m A^a + i(\boldsymbol{\tau}\cdot\boldsymbol{D}^a)_j{}^k\boldsymbol{\eta}_k - \frac{1}{2}\boldsymbol{\eta}_j f^a_{\ bc}A^{*b}A^c, \tag{3·14}$$

$$\boldsymbol{\delta}\bar{\boldsymbol{\lambda}}^{ja} = -(\bar{\boldsymbol{\eta}}^j\bar{\sigma}^{mn})v_{mn}^a - \sqrt{2}i(-\boldsymbol{\eta}^j\sigma^m)\mathcal{D}_m A^{*a} - i\bar{\boldsymbol{\eta}}^k(\boldsymbol{\tau}\cdot\boldsymbol{D}^{*a})_k^{\ j} - \frac{1}{2}\bar{\boldsymbol{\eta}}^j f^a_{\ bc}A^b A^{*c}, \tag{3·15}$$

$$\boldsymbol{D}^a = \hat{\boldsymbol{D}}^a - \sqrt{2}g^{ab^*}\frac{\partial}{\partial A^{*b}}\left(\boldsymbol{\mathcal{E}}A^{*0} + \boldsymbol{\mathcal{M}}\mathcal{F}_0^*\right). \tag{3·16}$$

Here

$$\hat{\boldsymbol{D}}^a = (\hat{D}_1^a,\ \hat{D}_2^a,\ \hat{D}_3^a), \qquad \left\{ \begin{array}{rcl} \hat{D}_1^a + i\hat{D}_2^a &=& -i\sqrt{2}\hat{F}^a, \\ \hat{D}_1^a - i\hat{D}_2^a &=& +i\sqrt{2}\hat{F}^{*a}, \\ \hat{D}_3^a &=& \hat{D}^a, \end{array} \right. \tag{3·17}$$

$$\boldsymbol{\mathcal{E}} = (0,\ -e,\ \xi), \tag{3·18}$$

$$\boldsymbol{\mathcal{M}} = (0,\ -m,\ 0), \tag{3·19}$$

and $\boldsymbol{\tau}$ are the Pauli matrices. We have used (2·52) in the last term of (3·14) and that of (3·15). Finally, we can easily check that (2·2) in fact holds in these transformations.

Let us now examine the $SU(2)$ covariance of the extended susy transformation given by (3·10), (3·11) and (3·12). All except the last term in (3·11) are manifestly covariant under the rigid $SU(2)$ transformations. In particular, $\hat{\boldsymbol{D}}^a$, given by (2·30)–(2·32) which are bilinear in fermions, transforms as a real triplet under $SU(2)$,

$$
\begin{aligned}
i\boldsymbol{\tau}\cdot\hat{\boldsymbol{D}}^a &= i\sqrt{2}\begin{pmatrix} \hat{D}^a & i\sqrt{2}\hat{F}^{*a} \\ -i\sqrt{2}\hat{F}^a & -\hat{D}^a \end{pmatrix} \\
&= g^{ab^*}g_{cb^*,d}\boldsymbol{\lambda}_j^{\{c}\boldsymbol{\lambda}^{d\}k} + g^{ab^*}g_{cb^*,d^*}\bar{\boldsymbol{\lambda}}_j^{\{c}\bar{\boldsymbol{\lambda}}^{d\}k}.
\end{aligned}
\tag{3·20}
$$

The last term in (3·11) is $SU(2)$ covariant provided the two three-dimensional real vectors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{M}}$ transform as triplets. Their actual form (3·18) and (3·19) tell us that the rigid $SU(2)$ has been gauge fixed in this six-dimensional parameter space of $(\boldsymbol{\mathcal{E}},\, \boldsymbol{\mathcal{M}})$, by making these two vectors point to a specific direction. The manifest $SU(2)$ covariance is lost at this point. The transformation law we have exhibited generalizes the one seen in the literature[3)] by the inclusion of the ξ term and the superpotential.

A very important property of the triplet of the auxiliary fields $\boldsymbol{D}^a$ is that it is complex as opposed to be real. Indeed, it has a constant imaginary part:

$$
\mathrm{Im}\ \boldsymbol{D}^a = \delta^a_{\ 0}(-\sqrt{2}m)\,(0,1,0)\,. \tag{3·21}
$$

This supplies an essential ingredient for partial breaking of $\mathcal{N}=2$ supersymmetry in the next section.

The supersymmetry transformation law for the auxiliary fields is determined by requiring the closure of the η_1- and η_2-supersymmetries:

$$
\begin{aligned}
\delta F^a &= -i\sqrt{2}\mathcal{D}_m\psi^a\sigma^m\bar{\eta}_1 - \bar{\eta}_1\bar{\lambda}^b k_b{}^a \\
&\quad + \delta_{\eta_2}(g^{ab}\partial_b W - g^{ab}\partial_{b^*}W^*) + i\sqrt{2}\eta_2\sigma^m\mathcal{D}_m\bar{\lambda}^a + \eta_2\psi^b k^{*a}_b,
\end{aligned}
\tag{3·22}
$$

$$
\begin{aligned}
\delta D^a &= -\eta_1\sigma^m\mathcal{D}_m\bar{\lambda}^a - \mathcal{D}_m\lambda^a\sigma^m\bar{\eta}_1 \\
&\quad - \delta_{\eta_2}(g^{ab}\mathfrak{D}_b) - \eta_2\sigma^m\mathcal{D}_m\bar{\psi}^a - \mathcal{D}_m\psi^a\sigma^m\bar{\eta}_2,
\end{aligned}
\tag{3·23}
$$

where the $\mathcal{D}_m$ represents the gauge covariant derivative (2·65). The supersymmetry transformation forms the algebra

$$
[\delta_\eta, \delta_{\eta'}]\varPsi^a = -2i(\eta\sigma^m\bar{\eta}' - \eta'\sigma^m\bar{\eta})\mathcal{D}_m\varPsi^a, \quad \varPsi^a = \{A^a, \psi^a, F^a, v^a_{mn}, \lambda^a, D^a\}, \tag{3·24}
$$

where $(\eta, \eta') = (\eta_1, \eta_1')$ or (η_2, η_2').

§4. Some properties of the vacuum

In order to discuss properties of our model, let us fix the form of $\mathcal{F}$. The first equation in (2·12) implies that $k_a{}^b = f^b_{ac}A^c$ and thus $k_0{}^a = k_a{}^0 = 0$, while the second equation in (2·12) implies that

$$
k_{\hat{a}}{}^{\hat{b}}\partial_{\hat{b}}\mathcal{F}_0 = 0, \tag{4·1}
$$

as well as $k_{\hat{a}}{}^{\hat{b}}\partial_{\hat{b}}\mathcal{F}_{\hat{c}} = -f^{\hat{b}}_{\hat{a}\hat{c}}\mathcal{F}_{\hat{b}}$. An obvious solution to (4·1) is

$$\mathcal{F} = f(A^0) + cA^0\ \mathcal{G}(\hat{B}) + \hat{\mathcal{F}}(\hat{A}), \tag{4·2}$$

where $f(A^0)$, $\mathcal{G}(\hat{B})$ and $\hat{\mathcal{F}}(\hat{A})$ are analytic functions of A^0, $\hat{B} = \mathrm{Tr}(\hat{A}^2)/2c_2$ and a trace function of $\hat{A} = A^{\hat{a}}t_{\hat{a}}$, respectively. We can choose $\mathcal{G}(0) = 0$ without loss of generality. The constant c_2 is the quadratic Casimir defined by $\mathrm{Tr}(t_{\hat{a}}t_{\hat{b}}) = c_2\delta_{\hat{a}\hat{b}}$. One finds that for this prepotential the Kähler metric becomes

$$g_{00^*} = \mathrm{Im}(f_{00}),\ g_{\hat{a}0^*} = \delta_{\hat{a}\hat{b}}\mathrm{Im}(\mathcal{G}'cA^{\hat{b}}),\ g_{\hat{a}\hat{b}^*} = \mathrm{Im}\left(cA^0(\delta_{\hat{a}\hat{b}}\mathcal{G}' + \mathcal{G}''\delta_{\hat{a}\hat{c}}\delta_{\hat{b}\hat{d}}A^{\hat{c}}A^{\hat{d}}) + \hat{\mathcal{F}}_{\hat{a}\hat{b}}\right). \tag{4·3}$$

Note that the $U(1)$ part and the $SU(N)$ part have non-trivial mixings as long as $c \neq 0$. In the following we examine the model specified by (4·2).

Let us first examine the local minimum of the scalar potential $\mathcal{V} \equiv -\mathcal{L}_{\rm pot}$

$$\begin{aligned}\mathcal{V} &= g^{ab}\left(\frac{1}{8}\mathfrak{D}_a\mathfrak{D}_b + \xi^2\delta^0_a\delta^0_b + \partial_a W\partial_{b^*}W^*\right)\\ &= g^{ab}\left(\frac{1}{8}\mathfrak{D}_a\mathfrak{D}_b + \partial_a\left(\boldsymbol{\mathcal{E}}A^0 + \boldsymbol{\mathcal{M}}\mathcal{F}_0\right)\cdot\partial_{b^*}\left(\boldsymbol{\mathcal{E}}A^0 + \boldsymbol{\mathcal{M}}\mathcal{F}_0\right)^*\right),\end{aligned} \tag{4·4}$$

where we have used (2·53). Here, we consider the unbroken $SU(N)$ phase at which the $A^{\hat{a}}$ do not acquire vacuum expectation values. Substituting $A^{\hat{a}} = 0$ into the equation

$$\begin{aligned}0 = \delta\mathcal{V}/\delta A^a = &-g^{bd}\partial_a g_{de}g^{ec}\left(\frac{1}{8}\mathfrak{D}_b\mathfrak{D}_c + \xi^2\delta^0_b\delta^0_c + \partial_b W\partial_{c^*}W^*\right)\\ &+g^{bc}\left(\frac{1}{4}\mathfrak{D}_b\partial_a\mathfrak{D}_c + \partial_a\partial_b W\partial_{c^*}W^*\right),\end{aligned} \tag{4·5}$$

we obtain

$$\frac{i}{2}f_{000}g_{00}^{-2}\delta^0_a\left(\xi^2 + (e+mf_{00})(e+mf^*_{00})\right) + g_{00}^{-1}mf_{000}\delta^0_a(e+mf^*_{00}) = 0. \tag{4·6}$$

Here we have derived

$$\langle\mathfrak{D}_a\rangle = 0,\quad \langle\partial_a W\rangle = \delta^0_a(e+mf_{00}),\quad \partial_0\partial_a W = \delta^0_a mf_{000},\quad \partial_a g_{00} = -\frac{i}{2}f_{000}\delta^0_a, \tag{4·7}$$

as well as

$$\left\langle g^{00}\right\rangle = g_{00}^{-1},\qquad \left\langle g^{0\hat{a}}\right\rangle = 0. \tag{4·8}$$

The expressions with bracket $\langle\cdots\rangle$ imply $\cdots$ evaluated at $A^{\hat{a}} = 0$. It is obvious that (4·6) is satisfied when $f_{000} = 0$, but it is a saddle point because $\langle\partial_0\partial_{0^*}\mathcal{V}\rangle = 0$, and thus does not represent a stable vacuum. The stable minimum is at

$$f_{00} = -\frac{e}{m} \pm i\frac{\xi}{m}. \tag{4·9}$$

We shall show that at the stable minimum (4·9) massless fermions emerge. For this purpose, we examine the fermion mass term

$$\mathcal{L}_{\rm mass} = -\frac{i}{4} g^{cd^*} \partial_c \tau_{ab} \partial_{d^*} W^* (\psi^a \psi^b + \lambda^a \lambda^b) + \frac{1}{2\sqrt{2}} \left(g_{ac^*} k_b^{*c} - g_{bc^*} k_a^{*c} - \sqrt{2}\xi \delta_c^0 (\tau_2^{-1})^{cd} \partial_a \tau_{bd} \right) \psi^a \lambda^b \quad + \text{c.c.} \quad . \tag{4·10}$$

Substituting $A^{\hat{a}} = 0$ into this mass term $\mathcal{L}_{\rm mass}$, we find that the $U(1)$ fermions and the $SU(N)$ fermions decouple because $\langle \mathcal{F}_{00\hat{a}} \rangle = 0$,

$$\begin{aligned} \mathcal{L}_{\rm mass} &= \frac{1}{2} \boldsymbol{\lambda}_i{}^0 M^{ij}_{U(1)} \boldsymbol{\lambda}_j{}^0 + \frac{1}{2} \delta_{\hat{a}\hat{b}} \boldsymbol{\lambda}_i{}^{\hat{a}} M^{ij}_{SU(N)} \boldsymbol{\lambda}_j{}^{\hat{b}} \quad + \text{c.c.}, \\ M^{ij}_{U(1)} &= -\frac{i}{2} g_{00}^{-1} f_{000} \begin{pmatrix} e + m f_{00}^* & -i\xi \\ -i\xi & e + m f_{00}^* \end{pmatrix}, \\ M^{ij}_{SU(N)} &= -\frac{i}{2} g_{00}^{-1} c \langle \mathcal{G}' \rangle \begin{pmatrix} e + m f_{00}^* & -i\xi \\ -i\xi & e + m f_{00}^* \end{pmatrix}. \end{aligned} \tag{4·11}$$

It is easy to diagonalize these mass matrices and one finds that the $U(1)$ fermions $\frac{1}{\sqrt{2}}(\lambda^0 \pm \psi^0)$ acquire masses $\left| -\frac{i}{2} g_{00}^{-1} f_{000} (e + m f_{00}^* \mp i\xi) \right|$, while the $SU(N)$ fermions $\frac{1}{\sqrt{2}}(\lambda^{\hat{a}} \pm \psi^{\hat{a}})$ acquire masses $\left| -\frac{i}{2} g_{00}^{-1} c \langle \mathcal{G}' \rangle (e + m f_{00}^* \mp i\xi) \right|$.

At the stable minimum $f_{00} = -\frac{e}{m} \pm i\frac{\xi}{m}$, the $U(1)$ fermion $\frac{1}{\sqrt{2}}(\lambda^0 \mp \psi^0)$ and the $SU(N)$ fermions $\frac{1}{\sqrt{2}}(\lambda^{\hat{a}} \mp \psi^{\hat{a}})$ remain massless, while the $U(1)$ fermion $\frac{1}{\sqrt{2}}(\lambda^0 \pm \psi^0)$ and the $SU(N)$ fermions $\frac{1}{\sqrt{2}}(\lambda^{\hat{a}} \pm \psi^{\hat{a}})$ become massive with masses, $|-m \langle\!\langle f_{000} \rangle\!\rangle|$ and $|-mc \langle\!\langle \mathcal{G}' \rangle\!\rangle|$, respectively. Here, $\langle\!\langle \cdots \rangle\!\rangle$ is the expectation value of $\cdots$ at the vacuum. The $U(1)$ massless fermion is regarded as the Nambu-Goldstone fermion.

Let us demonstrate this last statement from the transformation law (3·11). Taking the expectation value, we see

$$\begin{aligned} \langle\!\langle \boldsymbol{\delta\lambda}^0 \rangle\!\rangle &= -\sqrt{2} i \langle\!\langle g^{00} \rangle\!\rangle \begin{pmatrix} \xi & i \langle\!\langle e + m f_{00}^* \rangle\!\rangle \\ -i \langle\!\langle e + m f_{00}^* \rangle\!\rangle & -\xi \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \\ &= \mp \sqrt{2} i m \begin{pmatrix} 1 & \pm 1 \\ \mp 1 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \end{aligned} \tag{4·12}$$

$$\left\langle\!\!\left\langle \boldsymbol{\delta\lambda}^{\hat{a}} \right\rangle\!\!\right\rangle = 0. \tag{4·13}$$

We have used (4·8) and (4·9). Therefore,

$$\begin{aligned} \left\langle\!\!\left\langle \frac{\delta(\lambda^0 \mp \psi^0)}{\sqrt{2}} \right\rangle\!\!\right\rangle &= \mp 2mi(\eta_1 \pm \eta_2), \\ \left\langle\!\!\left\langle \frac{\delta(\lambda^0 \pm \psi^0)}{\sqrt{2}} \right\rangle\!\!\right\rangle &= 0. \end{aligned} \tag{4·14}$$

One linear combination of the $U(1)$ fermion, $\frac{1}{\sqrt{2}}(\lambda^0 \mp \psi^0)$, is in fact the Nambu-Goldstone fermion.

Finally, let us discuss a mechanism which is responsible for partial breaking of $\mathcal{N}=2$ supersymmetry to be realized. We see that partial breaking requires that the 2×2 matrix $\langle\!\langle \boldsymbol{\tau}\cdot\boldsymbol{D}^a\rangle\!\rangle$ in (3·14) has one nonvanishing eigenvalue for some a. We obtain

$$\begin{aligned}
-\langle\!\langle \det\boldsymbol{\tau}\cdot\boldsymbol{D}^a\rangle\!\rangle &= \langle\!\langle \boldsymbol{D}^a\cdot\boldsymbol{D}^a\rangle\!\rangle \\
&= \langle\!\langle \mathrm{Re}\,\boldsymbol{D}^a\cdot\mathrm{Re}\,\boldsymbol{D}^a\rangle\!\rangle - \langle\!\langle \mathrm{Im}\,\boldsymbol{D}^a\cdot\mathrm{Im}\,\boldsymbol{D}^a\rangle\!\rangle + 2i\,\langle\!\langle \mathrm{Re}\,\boldsymbol{D}^a\cdot\mathrm{Im}\,\boldsymbol{D}^a\rangle\!\rangle \\
&= 0, \qquad (4\cdot 15)
\end{aligned}$$

which implies that partial breaking is certainly not possible without nonvanishing imaginary part of $\boldsymbol{D}^a$. Using (3·21), we convert this condition into

$$\begin{aligned}
\left\langle\!\!\left\langle \mathrm{Re}\,\boldsymbol{D}^{\hat{a}}\right\rangle\!\!\right\rangle &= 0, \\
\|\,\langle\!\langle \mathrm{Re}\,\boldsymbol{D}^0\rangle\!\rangle\,\| = \|\,\mathrm{Im}\,\boldsymbol{D}^0\| &= \sqrt{2}m, \\
\langle\!\langle \mathrm{Re}\,\boldsymbol{D}^0\rangle\!\rangle\cdot\mathrm{Im}\,\boldsymbol{D}^0 &= 0. \qquad (4\cdot 16)
\end{aligned}$$

Coming back to the extremum condition (4·5) of the scalar potential at the unbroken $SU(N)$ phase, we see that it can also be converted as

$$0=\frac{\delta\mathcal{V}}{\delta A^a}=\frac{i}{4}\,\langle\!\langle f_{000}\delta_a^0\rangle\!\rangle\,\langle\!\langle \boldsymbol{D}^0\rangle\!\rangle\cdot\langle\!\langle \boldsymbol{D}^0\rangle\!\rangle\,. \qquad (4\cdot 17)$$

The condition for a stable vacuum is obviously equivalent to that of partial supersymmetry breaking (4·16). Note that at the vacuum

$$\langle\!\langle \mathcal{V}\rangle\!\rangle=\langle\!\langle g_{00}^{-1}\rangle\!\rangle\left(\xi^2+\left\langle\!\!\left\langle |e+mf_{00}|^2\right\rangle\!\!\right\rangle\right)=\pm 2m\xi\neq 0\ . \qquad (4\cdot 18)$$

§5. $\mathcal{N}=2$ supercurrents

In the previous section, the rigid $SU(2)$ symmetry, in particular, its discrete element $\mathfrak{R}$ has been exploited to provide $N=2$ supersymmetry of our model. In this section, we discuss another rigid transformation, namely, the one associated with the $U(1)_R$ transformation and the attendant supermultiplet of currents.

It is well known that the Wess-Zumino model consisting of the scalar superfield with a superpotential permits the $U(1)_R$ current, the supercurrent and the energy momentum tensor as a supermultiplet of currents when the superpotential is a monomial in scalar superfield.[36)] It is then possible to assign R weight one to the superpotential. (Extended) supermultiplet of currents exists for ($\mathcal{N}=2$) super Yang-Mills as well [36),33)] Starting from the $U(1)_R$ current, we can use this multiplet structure to derive the form of the supercurrent and the energy momentum tensor and to check the consistency of supersymmetry algebra. We illustrate this in the Wess-Zumino model in the Appendix A. Our model has $N=2$ supermultiplet of Noether currents when it is possible to assign R weight two to the prepotential $\mathcal{F}$. We show how this is used to derive the $\mathcal{N}=2$ supercurrents for generic $\mathcal{F}$.

The R transformation is given by

$$R:\quad \Phi(x,\theta,\bar{\theta})\rightarrow e^{i\alpha}\Phi(x,e^{-\frac{i\alpha}{2}}\theta,\bar{\theta}),$$

$$\mathcal{W}_\alpha(x,\theta,\bar{\theta}) \to \mathcal{W}_\alpha(x,e^{-\frac{i\alpha}{2}}\theta,\bar{\theta}), \tag{5·1}$$

$$\begin{aligned} R: A &\to e^{i\alpha}A, \quad v_m \to v_m, \\ \psi &\to e^{\frac{i\alpha}{2}}\psi, \quad \lambda \to e^{\frac{i\alpha}{2}}\lambda, \\ F &\to F, \qquad D \to D. \end{aligned} \tag{5·2}$$

We assume that the prepotential $\mathcal{F}$ is transformed as weight two under R

$$\mathcal{F} \to e^{2i\alpha}\mathcal{F}. \tag{5·3}$$

The $U(1)_R$ current associated is

$$\theta J\bar{\theta} \equiv (\tau_2)_{ab}\left(\bar{\theta}\bar{\boldsymbol{\lambda}}^{ia}\boldsymbol{\lambda}_i{}^b\theta + iA^{*^a}\theta\overleftrightarrow{\mathcal{D}}\cdot\sigma\bar{\theta}A^b\right) \tag{5·4}$$

$$\equiv (\tau_2)_{ab}\left(\theta j^{ab}\bar{\theta} + 2\theta\Delta j^{ab}\bar{\theta}\right) \ . \tag{5·5}$$

The second term is known as the improvement term. Using the transformation law of rigid $\mathcal{N}=2$ supersymmetry in §3, we obtain

$$\theta\boldsymbol{\delta}J\bar{\theta} = (\tau_2)_{ab}\left(\theta\boldsymbol{\delta}j^{ab}\bar{\theta} + 2\theta\boldsymbol{\delta}(\Delta j)^{ab}\bar{\theta}\right) + \boldsymbol{\delta}(\tau_2)_{ab}\left(\theta j^{ab}\bar{\theta} + 2\theta(\Delta j)^{ab}\bar{\theta}\right), \tag{5·6}$$

where

$$\begin{aligned} \theta\boldsymbol{\delta}j^{ab}\bar{\theta} = \bar{\theta}\bar{\boldsymbol{\lambda}}^{jb}\Big(&(\theta\sigma^{mn}\boldsymbol{\eta}_j)v_{mn}^{\ a} + \sqrt{2}i(\theta\sigma^m\bar{\boldsymbol{\eta}}_j)\mathcal{D}_m A^a \\ &+ i(\boldsymbol{\tau}\cdot\boldsymbol{D}^a)_j{}^k(\theta\boldsymbol{\eta}_k) - \frac{1}{2}(\theta\boldsymbol{\eta}_j)f^a_{\ cd}A^{*c}A^d\Big) \\ &- \Big((\bar{\boldsymbol{\eta}}^j\bar{\sigma}^{mn}\bar{\theta})v_{mn}^{\ b} + \sqrt{2}i(-\boldsymbol{\eta}^j\sigma^m\bar{\theta})\mathcal{D}_m A^{*b} \\ &+ i(\bar{\boldsymbol{\eta}}^k\bar{\theta})(\boldsymbol{\tau}\cdot\boldsymbol{D}^{*b})_k{}^j + \frac{1}{2}(\bar{\boldsymbol{\eta}}^j\bar{\theta})f^b_{\ ef}A^e A^{*f}\Big)\,\theta\lambda_j{}^a, \end{aligned} \tag{5·7}$$

$$\begin{aligned} \theta\boldsymbol{\delta}(\Delta j^{ab})\bar{\theta} = &\frac{\sqrt{2}}{2}iA^{*a}\theta\overleftrightarrow{\mathcal{D}}_m\sigma^m\bar{\theta}\boldsymbol{\eta}_j\boldsymbol{\lambda}^{jb} + \frac{\sqrt{2}}{2}i\bar{\boldsymbol{\eta}}^j\bar{\boldsymbol{\lambda}}_j{}^a\theta\overleftrightarrow{\mathcal{D}}_m\sigma^m\bar{\theta}A^b \\ &+ \frac{i}{2}A^{*a}\theta\boldsymbol{\delta}\overleftrightarrow{\mathcal{D}}_m\sigma^m\theta A^b, \end{aligned} \tag{5·8}$$

$$2i\boldsymbol{\delta}(\tau_2)_{ab} = \sqrt{2}\left(\tau_{abc}(A^d)\boldsymbol{\eta}_i\boldsymbol{\lambda}^{ic} - \tau^*_{abc}(A^{*d})\bar{\boldsymbol{\eta}}^i\bar{\boldsymbol{\lambda}}_i{}^c\right). \tag{5·9}$$

In the case where the prepotential is a degree two polynomial in A^a, $\boldsymbol{\delta}(\tau_2)_{ab}=0$ and Eq. (5·6) provides construction of $\mathcal{N}=2$ improved supercurrents which are conserved:

$$\boldsymbol{\eta}_j\boldsymbol{\mathcal{S}}^{(j)m} + \bar{\boldsymbol{\eta}}^j\bar{\boldsymbol{\mathcal{S}}}_{(j)}^{\ m} \equiv -\frac{1}{2}(\tau_2)_{ab}\,\mathrm{tr}\,\bar{\sigma}^m\left(\boldsymbol{\delta}(j^{ab}) + 2\boldsymbol{\delta}(\Delta j^{ab})\right). \tag{5·10}$$

Here “tr” implies a trace in the spinor space.

The R current is not conserved when $\mathcal{F}$ is not a degree two polynomial in A and the above construction would appear not useful for the general construction of the conserved supercurrents. We will show below that this is not the case. Let us write the prepotential $\mathcal{F}$ generically as

$$\mathcal{F} = \sum_{n,j} h_j^{(n)} C_j^{(n)}(A^a). \tag{5·11}$$

Here $C_j^{(n)}(A^a)$ are n-th order $U(N)$ invariant polynomials in A^a properly normalized and labelled by the index j, and $h_j^{(n)}$ are their coefficients. We first observe that we can assign weight two to $\mathcal{F}$ in (5·11) provided $h_j^{(n)}$ transform as weight $-(n-2)$. Let us consider the local version of the $U(1)_R$ transformation (5·2), replacing α by $\alpha(x)$. We obtain

$$\begin{aligned}
&S[Ae^{i\alpha(x)},\ \boldsymbol{\lambda}_j e^{\frac{i\alpha(x)}{2}},\ \cdots] - S[A,\ \boldsymbol{\lambda}_j,\ \cdots] \\
&= \int d^4x\ \partial_m \left(\alpha(x) \left(-\frac{1}{2} \right) \mathrm{tr}\, \bar{\sigma}^m J \right) + \int d^4x\ \alpha(x) \partial_m \left(\frac{1}{2}\, \mathrm{tr}\, \bar{\sigma}^m J \right) \\
&\quad + \int d^4x\ i\alpha(x) \sum_{n,j} (n-2) \frac{\partial}{\partial h_j^{(n)}} \mathcal{L}.
\end{aligned} \tag{5·12}$$

Here $\mathcal{L}$ and S are the Lagrangian and the action of our model respectively. The left-hand side vanishes by the equation of motion, and we obtain

$$\partial_m \left(-\frac{1}{2}\, \mathrm{tr}\, \bar{\sigma}^m J \right) = i \left(\sum_{n,j} (n-2) \frac{\partial}{\partial h_j^{(n)}} \right) \mathcal{L} \equiv \varDelta_h \mathcal{L}. \tag{5·13}$$

Taking the supersymmetry variation of this equation, we obtain

$$\partial_m \left(-\frac{1}{2}\, \mathrm{tr}\, \bar{\sigma}^m \boldsymbol{\delta} J \right) = \varDelta_h \boldsymbol{\delta} \mathcal{L}. \tag{5·14}$$

As our action is $\mathcal{N}=2$ supersymmetric, the right hand side is written as

$$\varDelta_h \partial_m X^m = \partial_m \varDelta_h X^m, \tag{5·15}$$
$$X^m = \boldsymbol{\eta}_j \boldsymbol{y}^j + \bar{\boldsymbol{\eta}}^j \bar{\boldsymbol{y}}_j, \tag{5·16}$$

for some operator X^m linear in $\boldsymbol{\eta}_i$ and $\bar{\boldsymbol{\eta}}^i$. Hence

$$\partial_m \left(-\frac{1}{2}\, \mathrm{tr}\, \bar{\sigma}^m \boldsymbol{\delta} J - \varDelta_h X^m \right) = 0. \tag{5·17}$$

This provides a general construction of the conserved $\mathcal{N}=2$ supercurrents of our model:

$$\boldsymbol{\eta}_j \boldsymbol{\mathcal{S}}^{(j)m} + \bar{\boldsymbol{\eta}}^j \bar{\boldsymbol{\mathcal{S}}}_{(j)}^{\ m} \equiv -\frac{1}{2}\, \mathrm{tr}(\bar{\sigma}^m \boldsymbol{\delta} J) - \varDelta_h X^m. \tag{5·18}$$

The form of the supercurrents given in Eq. (5·18) tells us that our model does not permit a universal coupling to $\mathcal{N}=2$ supergravity. The piece $-\Delta_h X^m$ is not generic and depends on the functional form of the prepotential $\mathcal{F}(A)$ in A. This and the previous analysis in Refs. 22) and 24) support the point of view that $\mathcal{N}=2$ supersymmetric gauge models with nontrivial Kähler potential should be viewed as a low energy effective action.

Let us now further transform (5·18)

$$\boldsymbol{\delta}\left(\boldsymbol{\eta}_j \boldsymbol{\mathcal{S}}^{(j)m} + \bar{\boldsymbol{\eta}}^j \bar{\boldsymbol{\mathcal{S}}}_{(j)}^{\ \ m}\right) = -\frac{1}{2}\,\mathrm{tr}\,\bar{\sigma}^m \boldsymbol{\delta\delta} J - \Delta_h \boldsymbol{\delta} X^m. \tag{5·19}$$

This generates the $\mathcal{N}=2$ supersymmetry algebra (1·2) quoted in the introduction and at the same time provides its consistency conditions. Let us note that

$$\begin{aligned}\theta\boldsymbol{\delta\delta} J\bar{\theta} = (\tau_2)_{ab}\left(\theta\boldsymbol{\delta\delta} j^{ab}\bar{\theta} + 2\theta\boldsymbol{\delta\delta}(\Delta j)^{ab}\bar{\theta}\right) + 2\boldsymbol{\delta}(\tau_2)_{ab}\left(\theta\boldsymbol{\delta} j^{ab}\bar{\theta} + 2\theta\boldsymbol{\delta}(\Delta j)^{ab}\bar{\theta}\right)\\ +\boldsymbol{\delta\delta}(\tau_2)_{ab}\left(\theta j^{ab}\bar{\theta} + 2\theta(\Delta j)^{ab}\bar{\theta}\right).\end{aligned} \tag{5·20}$$

Denote by δ_{η_j} $(\delta_{\bar{\eta}^j})$ the transformation in which only $\eta_j(\bar{\eta}^j)$ is kept in $\boldsymbol{\delta}$. The conditions

$$\begin{aligned}\delta_{\eta_j}\mathcal{S}^{(j)m} = 0\\ \delta_{\bar{\eta}^j}\bar{\mathcal{S}}^{(j)m} = 0\end{aligned} \qquad \text{with } j \text{ not summed} \tag{5·21}$$

provide

$$-\frac{1}{2}\,\mathrm{tr}\,\bar{\sigma}^m \delta_{\eta_j}\delta_{\eta_j} J - \Delta_h \eta_j \delta_{\eta_j} y^j = 0 \qquad \text{with } j \text{ not summed} \tag{5·22}$$

and its complex conjugate. Their actual expressions are quite involved as one sees from (5·20) and the transformation laws (3·10)–(3·16). We will not discuss Eq. (5·22) further in this paper. In the case where $\mathcal{F}$ is degree two in A, $y^j=0$, and $\boldsymbol{\delta}(\tau_2)_{ab}=0$, Eq. (5·22) can be checked easily as in Ref. 33) and in Appendix (B·18) with the aid of the equations of motion.

Let us finally read off the constant matrix $C_i^{\ j}$ in (1·2) from our algebra (5·19). The only piece in (5·20) which can contribute to $C_i^{\ j}$ is the part in $(\tau_2)_{ab}\boldsymbol{\delta\delta} j^{ab}$ which is linear both in $\boldsymbol{D}^a$ and in $\boldsymbol{D}^{*a}$. This part is computed as

$$2\left(\tau_2\right)_{ab}\boldsymbol{D}^{*b}\cdot\boldsymbol{D}^a\bar{\theta}\left(\bar{\boldsymbol{\eta}}\boldsymbol{\eta}\right)\theta + 2i\left(\tau_2\right)_{ab}\left(\boldsymbol{D}^{*b}\times\boldsymbol{D}^a\right)\cdot\bar{\theta}\bar{\boldsymbol{\eta}}\boldsymbol{\tau}\boldsymbol{\eta}\theta. \tag{5·23}$$

Substituting the expressions (3·16)–(3·19) into this equation, we find that the second term contains $8m\xi\bar{\theta}\bar{\boldsymbol{\eta}}\tau_1\boldsymbol{\eta}\theta$. Translated into (1·2), this implies

$$C_i^{\ j} = +2m\xi(\boldsymbol{\tau}_1)_i^{\ j}. \tag{5·24}$$

This is consistent with (4·18).

§6. Fermionic shift symmetry

Equations (4·12) and (4·13) express the extended supersymmetry transformation of the $SU(2)$ doublet of $U(N)$ fermions on the vacuum as $U(1)$ fermionic shift generated by

$$\boldsymbol{\chi}_i \equiv \sqrt{2}m \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \tag{6·1}$$

Note that the coupling constants e, m, ξ of our model carry dimension two and that $\boldsymbol{\chi}_i$ carry dimension 3/2. The Nambu-Goldstone fermion is the maximal mixing of the $U(1)$ gauge fermion and the $U(1)$ matter fermion.

Restricting our attention to the $U(N)$ field strength gauge superfield $\mathcal{W}_\alpha$, let us recast (4·12) into

$$\langle\!\langle \boldsymbol{\delta}\mathcal{W}_\alpha \rangle\!\rangle = (\mp\chi_1 - \chi_2)\,\mathbf{1}_{N\times N} \equiv 4\pi\chi_\alpha \mathbf{1}_{N\times N}. \tag{6·2}$$

We obtain

$$\langle\!\langle \boldsymbol{\delta} S \rangle\!\rangle = \chi^\alpha \langle\!\langle w_\alpha \rangle\!\rangle\,, \tag{6·3}$$

$$\langle\!\langle \boldsymbol{\delta} w_\alpha \rangle\!\rangle = N\chi_\alpha, \tag{6·4}$$

where

$$S = \frac{1}{32\pi^2}\,\mathrm{tr}\,\mathcal{W}^\alpha\mathcal{W}_\alpha, \qquad w_\alpha = \frac{1}{4\pi}\,\mathrm{tr}\,\mathcal{W}_\alpha. \tag{6·5}$$

In this sense, our spontaneously broken supersymmetry is realized on the vacuum as the $U(1)$ fermionic shift noted by Ref. 37) in the $\mathcal{N}=2$ $U(N)$ super Yang-Mills deformed by the superpotential $W(\Phi)$. See also Ref. 38). As for its transformation acting on the fields or equivalently on a generic state, let us note that

$$\boldsymbol{\delta}\boldsymbol{\lambda}_j{}^a = \left\langle\!\!\left\langle \boldsymbol{\delta}\boldsymbol{\lambda}_j{}^a \right\rangle\!\!\right\rangle + \cdots\,. \tag{6·6}$$

Here $\left\langle\!\left\langle \boldsymbol{\delta}\boldsymbol{\lambda}_j{}^a \right\rangle\!\right\rangle$ is given in (4·12), and $\cdots$ denotes the parts which do not receive the vacuum expectation values. This latter part is to be suppressed by $\frac{1}{m}$ with the replacement $\boldsymbol{\eta}_j \to \frac{\boldsymbol{\chi}_j}{\sqrt{2}m}$ when

$$\frac{e}{m} \ll 1, \quad \frac{\xi}{m} \ll 1, \quad \xi \neq 0 \tag{6·7}$$

for appropriate low energy processes. The spontaneously broken supersymmetry operates as an approximate fermionic $U(1)$ shift symmetry in this regime.

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Appendix A

In the text, we exploited the extended operation $\mathfrak{R}$ which involves the sign change of the parameter ξ as well as the transformation of the two component spinor parameter $\boldsymbol{\eta}_j$ to demonstrate that our action $\mathcal{L}$ or $\mathcal{L}'$ is invariant under $\mathcal{N}=2$ supersymmetry. Though the use of $\mathfrak{R}$ is logical from an algebraic point of view, clearly it is not a symmetry in the sense of Noether. In this appendix, we provide another proof, using the more conventional operation which involves the transformation of the fields alone. To be more specific, let R be a generator such that

$$R\lambda^a R^{-1}=\psi^a,\ R\psi^a R^{-1}=-\lambda^a,\ RA^aR^{-1}=A^a \text{ and } Rv_m^{\ a}R^{-1}=v_m^{\ a}. \qquad (\text{A}\cdot 1)$$

Let us start from the equations of the $\mathcal{N}=1$ transformation laws (3·1)–(3·4), replacing η, by θ and writing F^a and D^a explicitly by (3·16) and (3·17).

$$\delta_\theta^{(1,\xi)}A^a=\sqrt{2}\theta\psi^a, \qquad (\text{A}\cdot 2)$$

$$\delta_\theta^{(1,\xi)}\psi^a=i\sqrt{2}\sigma^m\bar\theta\mathcal{D}_mA^a+\sqrt{2}\theta\left(\hat F^a-\sqrt{2}g^{ab^*}\frac{\partial}{\partial A^{*b}}\left(eA^{*0}+m\mathcal{F}_0^*\right)\right), \qquad (\text{A}\cdot 3)$$

$$\delta_\theta^{(1,\xi)}v_m^{\ a}=i\theta\sigma_m\bar\lambda^a-i\lambda^a\sigma^m\bar\theta, \qquad (\text{A}\cdot 4)$$

$$\delta_\theta^{(1,\xi)}\lambda^a=\sigma^{mn}\theta v_{mn}^a+i\theta\left(\hat D^a-\sqrt{2}g^{ab^*}\frac{\partial}{\partial A^{*b}}\left(\xi A^{*0}\right)\right), \qquad (\text{A}\cdot 5)$$

where $\hat D^a$ and $\hat F^a$ are given in terms of fermion bilinears by (2·30) and (2·31). We have introduced the superscript $(1,\xi)$ to label the transformation fully. Operating R from the left and R^{-1} from the right on (A·5), we obtain

$$\begin{aligned}R\delta_\theta^{(1,\xi)}\lambda^aR^{-1}&=\left(R\delta_\theta^{(1,\xi)}R^{-1}\right)\psi^a\\&=\sigma^{mn}\theta v_{mn}^{\ a}+i\theta\left(-\hat D^a-\sqrt{2}g^{ab^*}\frac{\partial}{\partial A^{*b}}\left(\xi A^{*0}\right)\right),\end{aligned} \qquad (\text{A}\cdot 6)$$

where we have used $R\hat D^aR^{-1}=-\hat D^a$. Eqation (A·6) is compared with $\boldsymbol{\delta}\psi^a$ at $\eta_1=0$ in (3·11) of the text, and we find

$$R\delta_{\eta_1=\theta}^{(1,\xi)}R^{-1}=\boldsymbol{\delta}_{\eta_1=0,\eta_2=\theta}^{(-\xi)}\equiv\delta_{\eta_2=\theta}^{(2,-\xi)} \qquad (\text{A}\cdot 7)$$

on ψ^a. We have introduced the subscript and the superscript to $\boldsymbol{\delta}$ to specify the transformation completely. Proceeding in a similar way on (A·1), we obtain

$$\begin{aligned} R\delta^{(1,\xi)}\psi^a R^{-1} &= R\delta^{(1,\xi)}R^{-1}(-\lambda^a) \\ &= i\sqrt{2}\sigma^m\bar{\theta}\mathcal{D}_m A^a + \sqrt{2}\theta\left(\hat{F}^{*a} - \sqrt{2}g^{ab^*}\frac{\partial}{\partial A^{*b}}\left(eA^{*0} + m\mathcal{F}_0^*\right)\right). \end{aligned} \quad (\mathrm{A{\cdot}8})$$

We see that (A·7) is true on λ^a as well. It is easy to check from (A·2) and (A·4) that (A·7) holds on A^a and on v_m^a. We conclude that (A·7) is valid on all fields.

Once this is established, it is immediate to provide a proof that our action is invariant under $\mathcal{N}=2$ supersymmetry. Let

$$S(\xi) = \int d^4x\mathcal{L}(x) \ \text{ or } \int d^4x\mathcal{L}'(x), \quad (\mathrm{A{\cdot}9})$$

where $\mathcal{L}(x)$ and $\mathcal{L}'(x)$ are given by (2·26) and by (2·33) respectively. $\mathcal{N}=1$ supersymmetry implies

$$\delta^{(1,\xi)}_{\eta_1=\theta}S(\xi) = 0\ . \quad (\mathrm{A{\cdot}10})$$

Multiplying R from left and R^{-1} from right, we obtain

$$\left(R\delta^{(1,\xi)}_{\eta_1=\theta}R^{-1}\right)RS(\xi)R^{-1} = \delta^{(2,-\xi)}_{\eta_2=\theta}S(-\xi) = 0, \ \text{ and thus } \ \delta^{(2,\xi)}_{\eta_2=\theta}S(\xi) = 0, \quad (\mathrm{A{\cdot}11})$$

which is a statement that our action is $\mathcal{N}=2$ supersymmetric.

Appendix B

In this appendix, we reexamine the current supermultiplet in the Wess-Zumino model. While its superfield expression is well-known, we will present this supermultiplet in the component formalism, so that the reasoning here is applicable to the discussion in the text. The action is

$$S = \int d^4x\mathcal{L}\ ,\ \mathcal{L} = \int d^2\theta d^2\bar{\theta}\Phi^*\Phi + \int d^2\theta W(\Phi) + \int d^2\bar{\theta}W^*(\Phi^*) \quad (\mathrm{B{\cdot}1})$$

and the superpotential $W(\Phi)$ (or $W^*(\Phi^*)$) is a monomial of degree k in Φ (or Φ^*). The model possesses $U(1)_R$ symmetry associated with

$$R:\ \Phi(x,\theta,\bar{\theta}) \to e^{i\alpha/k}\Phi(x, e^{-i\alpha/2}\theta, \bar{\theta}), \quad (\mathrm{B{\cdot}2})$$

so that

$$\begin{aligned} R:\ A &\to e^{i\alpha/k}A, \\ \psi &\to e^{i\alpha(\frac{1}{k}-\frac{1}{2})}\psi, \\ F &\to e^{i\alpha(\frac{1}{k}-1)}F. \end{aligned} \quad (\mathrm{B{\cdot}3})$$

The proper Noether current $J_{\alpha\dot{\alpha}}$ is given by

$$\theta J\bar{\theta} = \bar{\psi}\bar{\theta}\theta\psi + c\frac{i}{2}A^*\theta\overleftrightarrow{\partial}\cdot\sigma\bar{\theta}A \quad (\mathrm{B{\cdot}4})$$

$$\equiv \theta j\bar{\theta} + c\theta\Delta j\bar{\theta}, \quad (\mathrm{B{\cdot}5})$$

where

$$c = \frac{1}{1-k/2} \tag{B·6}$$

in accordance with the R weights of the fields which are read off from (B·3). We have introduced grassman coordinates $\theta^\alpha, \bar{\theta}_{\dot\alpha}$ to contract and suppress spinorial indices. The dot implies a contraction of Minkowski indices. The second term $(\Delta j)_{\alpha\dot\alpha}$ is known as the improvement term.

Let us check the supersymmetry transformation of (B·4), which acts as the lowest component of the supermultiplet:

$$\begin{aligned}\theta\delta j\bar{\theta} &\equiv \theta\eta^\alpha s_\alpha\bar{\theta} + \theta\bar{\eta}_{\dot\alpha}\bar{s}^{\dot\alpha}\bar{\theta} \\ &= \left(-i\sqrt{2}\eta\sigma\bar{\theta}\cdot\partial A^* + \sqrt{2}\bar{\theta}\bar{\eta}F^*\right)\theta\psi + \bar{\psi}\bar{\theta}\left(i\sqrt{2}\theta\sigma\bar{\eta}\cdot\partial A + \sqrt{2}\theta\eta F\right), \end{aligned} \tag{B·7}$$

$$\begin{aligned}\theta\delta(\Delta j)\bar{\theta} &\equiv \theta\eta^\alpha(\Delta s)_\alpha\bar{\theta} + \theta\bar{\eta}_{\dot\alpha}(\Delta\bar{s})^{\dot\alpha}\bar{\theta} \\ &= \frac{i}{2}A^*\theta\sigma\cdot\overleftrightarrow{\partial}\bar{\theta}(\sqrt{2}\eta\psi) + \frac{i}{2}(\sqrt{2}\bar{\eta}\bar{\psi})\theta\sigma\cdot\overleftrightarrow{\partial}\bar{\theta}A. \end{aligned} \tag{B·8}$$

The improved supercurrents are

$$-\frac{1}{2}\,\mathrm{tr}\,\bar{\sigma}^m\left(s_\alpha + c(\Delta s)_\alpha\right) \quad \text{and} \quad -\frac{1}{2}\,\mathrm{tr}\,\bar{\sigma}^m\left(\bar{s}^{\dot\alpha} + c(\Delta\bar{s})^{\dot\alpha}\right). \tag{B·9}$$

It is easy to check

$$\theta\left(\delta j + c\delta(\Delta j)\right)\bar{\theta}\,\Big|_{\eta=\theta,\ \bar{\eta}=\bar{\theta}} = 0, \tag{B·10}$$

if and only if $k=3$ and therefore $c=-2$ from (B·6). This is nothing but the condition that the supercurrents (B·9) implement the superconformal constraints, that is, the irreducibility of their spin when the coupling constant in the superpotential is dimensionless.

Let us further transform (B·7) and (B·8) to generate the stress-energy tensor and we check the consistency with the supersymmetry algebra as well:

$$\theta\delta\delta j\bar{\theta} = (\bar{\psi}\bar{\theta})(\theta\delta\delta\psi) + 2(\delta\bar{\psi}\bar{\theta})(\theta\delta\psi) + (\delta\delta\bar{\psi}\bar{\theta})(\theta\psi), \tag{B·11}$$

$$\theta\delta\delta(\Delta j)\bar{\theta} = \frac{i}{2}A^*\theta\overleftrightarrow{\partial}\cdot\sigma\bar{\theta}(\delta\delta A) + i\delta A^*\theta\overleftrightarrow{\partial}\cdot\bar{\theta}\delta A + \frac{i}{2}(\delta\delta A^*)\theta\overleftrightarrow{\partial}\cdot\sigma\bar{\theta}A. \tag{B·12}$$

The fermionic part of (B·11) is

$$\begin{aligned}(\bar{\psi}\bar{\theta})(\theta\delta\delta\psi) + (\delta\delta\bar{\psi}\bar{\theta})(\theta\psi) &= -2i(\theta\eta)(\bar{\theta}\bar{\eta})\bar{\psi}\bar{\sigma}\cdot\overleftrightarrow{\partial}\psi + 2i(\bar{\theta}\bar{\sigma}\theta)\cdot\left((\bar{\psi}\bar{\eta})\overleftrightarrow{\partial}(\psi\eta)\right) \\ &\quad -2i(\bar{\eta}\bar{\sigma}\theta)\cdot\partial\left((\bar{\psi}\bar{\theta})(\eta\psi)\right) + 2i(\bar{\theta}\bar{\sigma}\eta)\cdot\partial\left((\bar{\psi}\bar{\eta})(\theta\psi)\right). \end{aligned} \tag{B·13}$$

The bosonic part of (B·11) is

$$\begin{aligned}2(\delta\bar{\psi}\bar{\theta})(\theta\delta\psi) &= 4(\eta\theta)(\bar{\eta}\bar{\theta})(F^*F - \partial A^*\cdot\partial A) - 4(\theta\sigma\bar{\theta})\cdot\partial A^*(\eta\sigma\bar{\eta})\cdot\partial A \\ &+ 8(\eta\theta)(\bar{\theta}\bar{\sigma}^{mn}\bar{\eta})\partial_m A^*\partial_n A - 2i(\bar{\eta}\bar{\eta})F^*(\theta\sigma\bar{\theta})\cdot\partial A + 2i(\eta\eta)(\theta\sigma\bar{\theta})\cdot\partial A^* F. \end{aligned} \tag{B·14}$$

The fermionic part of (B·12) is

$$i\delta A^*\theta\overleftrightarrow{\partial}\cdot\bar{\theta}\delta A = 2i(\theta\sigma\bar{\theta})\cdot(\bar{\eta}\bar{\psi}\overleftrightarrow{\partial}\eta\psi). \tag{B·15}$$

The bosonic part of (B·12) is

$$\begin{aligned}
&\frac{i}{2}A^*\theta\overleftrightarrow{\partial}\cdot\sigma\bar{\theta}(\delta\delta A)+\frac{i}{2}(\delta\delta A^*)\theta\overleftrightarrow{\partial}\cdot\sigma\bar{\theta}A \\
&\quad=-(\theta\sigma\bar{\theta}\cdot A^*\overleftrightarrow{\partial})(\partial A\cdot\eta\sigma\bar{\eta})+(\eta\sigma\bar{\eta}\cdot\partial A^*)(\overleftrightarrow{\partial}A\cdot\theta\sigma\bar{\theta}) \\
&\qquad+i(\eta\eta)\theta\sigma\bar{\theta}\cdot A^*\overleftrightarrow{\partial}F+i(\bar{\eta}\bar{\eta})(\theta\sigma\bar{\theta})\cdot F^*\overleftrightarrow{\partial}A\,. \qquad \text{(B·16)}
\end{aligned}$$

The consistency of the supersymmetry algebra demands that the $\eta\eta$ term and the $\bar{\eta}\bar{\eta}$ term be absent in $\theta\delta\delta J\bar{\theta}$. Let us check that this is in fact the case. From (B·14) and (B·16), we see that the $\bar{\eta}\bar{\eta}$ term is

$$-2i(\bar{\eta}\bar{\eta})F^*(\theta\sigma\bar{\theta})\cdot\partial A+ci(\bar{\eta}\bar{\eta})(\theta\sigma\bar{\theta})\cdot F^*\overleftrightarrow{\partial}A. \qquad \text{(B·17)}$$

Using equation of motion for auxiliary fields F, F^* and that $W(A)$ is a degree k monomial in A, this is equal to

$$2i(\bar{\eta}\bar{\eta})\left(1-\frac{c}{2}+\frac{c}{2}(k-1)\right)\theta\sigma\bar{\theta}\cdot\partial A, \qquad \text{(B·18)}$$

which vanishes when c is chosen as (B·6).

The remainder of $\theta\delta\delta J\bar{\theta}$ closes into the stress-energy tensor. Using equations of motion, we have checked

$$\theta\delta\delta J\bar{\theta}=-2(c-1)T. \qquad \text{(B·19)}$$

Here

$$\begin{aligned}
T&\equiv\eta\sigma^m\bar{\eta}\theta\sigma^n\bar{\theta}T_{mn} \\
&=-(\eta\sigma\bar{\eta})\cdot\partial A^*(\theta\sigma\bar{\theta})\cdot\partial A-(\eta\sigma\bar{\eta})\cdot\partial A(\theta\sigma\bar{\theta})\cdot\partial A^* \\
&\quad-\frac{i}{2}\left(\theta\sigma\bar{\theta}\cdot\bar{\eta}\bar{\psi}\overleftrightarrow{\partial}\eta\psi+\eta\sigma\bar{\eta}\cdot\bar{\theta}\bar{\psi}\overleftrightarrow{\partial}\theta\psi\right)+2\eta\theta\bar{\eta}\bar{\theta}\mathcal{L}. \qquad \text{(B·20)}
\end{aligned}$$

References

1) J. Wess and J. Bagger, *Supersymmetry and supergravity*, second edition (Princeton University Press, 1992).
2) S. Weinberg, *The quantum theory of fields. Vol. 3: Supersymmetry* (Cambridge University Press, 2000).
3) M. F. Sohnius, Phys. Rep. **128** (1985), 39.
4) B. de Wit, J. W. van Holten and A. Van Proeyen, Nucl. Phys. B **167** (1980), 186.
J. Bagger and E. Witten, Nucl. Phys. B **222** (1983), 1.
B. de Wit, P. G. Lauwers, R. Philippe, S. Q. Su and A. Van Proeyen, Phys. Lett. B **134** (1984), 37.
B. de Wit and A. Van Proeyen, Nucl. Phys. B **245** (1984), 89.
B. de Wit, P. G. Lauwers and A. Van Proeyen, Nucl. Phys. B **255** (1985), 569.
E. Cremmer, C. Kounnas, A. Van Proeyen, J. P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, Nucl. Phys. B **250** (1985), 385.
H. Itoyama, L. D. McLerran, T. R. Taylor and J. J. van der Bij, Nucl. Phys. B **279** (1987), 380.
R. D'Auria, S. Ferrara and P. Fre, Nucl. Phys. B **359** (1991), 705.
L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara and P. Fre', Nucl. Phys. B **476** (1996), 397; hep-th/9603004.
L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fre and T. Magri, J. Geom. Phys. **23** (1997), 111; hep-th/9605032.

5) S. Cecotti, L. Girardello and M. Porrati, Phys. Lett. B **168** (1986), 83.
S. Ferrara, L. Girardello and M. Porrati, Phys. Lett. B **366** (1996), 155; hep-th/9510074.
P. Fre, L. Girardello, I. Pesando and M. Trigiante, Nucl. Phys. B **493** (1997), 231; hep-th/9607032.
M. Porrati, Nucl. Phys. B (Proc. Suppl.) **55** (1997), 240; hep-th/9609073.
L. Andrianopoli, R. D'Auria and S. Ferrara, J. High Energy Phys. **03** (2002), 025; hep-th/0110277.
L. Andrianopoli, R. D'Auria, S. Ferrara and M. A. Lledo, Nucl. Phys. B **640** (2002), 46; hep-th/0202116.
L. Andrianopoli, R. D'Auria and S. Ferrara, Nucl. Phys. B **628** (2002), 387; hep-th/0112192.
J. Louis, hep-th/0203138.
L. Andrianopoli, R. D'Auria, S. Ferrara and M. A. Lledo, J. High Energy Phys. **07** (2002), 010; hep-th/0203206.
L. Andrianopoli, R. D'Auria, S. Ferrara and M. A. Lledo, Nucl. Phys. B **640** (2002), 63; hep-th/0204145.
L. Andrianopoli, R. D'Auria, S. Ferrara and M. A. Lledo, J. High Energy Phys. **01** (2003), 045; hep-th/0212236.
6) R. Altendorfer and J. Bagger, Phys. Lett. B **460** (1999), 127; hep-th/9904213.
7) J. R. David, E. Gava and K. S. Narain, J. High Energy Phys. **06** (2004), 041; hep-th/0311086.
8) J. Hughes and J. Polchinski, Nucl. Phys. B **278** (1986), 147.
J. Hughes, J. Liu and J. Polchinski, Phys. Lett. B **180** (1986), 370.
9) A. Achucarro, J. P. Gauntlett, K. Itoh and P. K. Townsend, Nucl. Phys. B **314** (1989), 129.
10) J. Bagger and A. Galperin, Phys. Rev. D **55** (1997), 1091; hep-th/9608177.
J. A. Bagger, Nucl. Phys. B (Proc. Suppl.) **52** (1997), 362; hep-th/9610022.
J. Bagger and A. Galperin, Phys. Lett. B **412** (1997), 296; hep-th/9707061.
J. Bagger and A. Galperin, Phys. Lett. B **336** (1994), 25; hep-th/9406217.
11) M. Rocek and A. A. Tseytlin, Phys. Rev. D **59** (1999), 106001; hep-th/9811232.
12) Y. Gotoh and T. Uematsu, Phys. Lett. B **420** (1998), 69; hep-th/9707056.
13) F. Gonzalez-Rey, I. Y. Park and M. Rocek, Nucl. Phys. B **544** (1999), 243; hep-th/9811130.
14) E. Kiritsis and C. Kounnas, Nucl. Phys. B **503** (1997), 117; hep-th/9703059.
T. R. Taylor and C. Vafa, Phys. Lett. B **474** (2000), 130; hep-th/9912152.
P. Mayr, Nucl. Phys. B **593** (2001), 99; hep-th/0003198.
G. Curio, A. Klemm, D. Lust and S. Theisen, Nucl. Phys. B **609** (2001), 3; hep-th/0012213.
15) S. Bellucci, E. Ivanov and S. Krivonos, Phys. Lett. B **460** (1999), 348; hep-th/9811244.
E. Ivanov and S. Krivonos, Phys. Lett. B **453** (1999), 237; hep-th/9901003.
S. Bellucci, E. Ivanov and S. Krivonos, Phys. Lett. B **482** (2000), 233; hep-th/0003273.
S. Bellucci, E. Ivanov and S. Krivonos, Phys. Lett. B **502** (2001), 279; hep-th/0012236.
S. Bellucci, E. Ivanov and S. Krivonos, Phys. Rev. D **64** (2001), 025014; hep-th/0101195.
16) P. C. West, J. High Energy Phys. **02** (2000), 024; hep-th/0001216.
17) S. V. Ketov, Mod. Phys. Lett. A **14** (1999), 501; hep-th/9809121.
S. V. Ketov, Class. Quantum. Grav. **17** (2000), L91; hep-th/0005126.
18) S. M. Kuzenko and S. Theisen, J. High Energy Phys. **03** (2000), 034; hep-th/0001068.
S. M. Kuzenko and S. Theisen, Fortsch. Phys. **49** (2001), 273; hep-th/0007231.
19) T. E. Clark, M. Nitta and T. ter Veldhuis, Phys. Rev. D **67** (2003), 085026; hep-th/0208184.
T. E. Clark, M. Nitta and T. ter Veldhuis, Phys. Rev. D **69** (2004), 047701; hep-th/0209142.
20) A. De Castro, L. Quevedo and A. Restuccia, J. High Energy Phys. **06** (2004), 055; hep-th/0405062.
21) For example, see below and references thereof:
M. Cvetic, F. Quevedo and S. J. Rey, Phys. Rev. Lett. **67** (1991), 1836.
E. R. C. Abraham and P. K. Townsend, Nucl. Phys. B **351** (1991), 313.
M. Cvetic, S. Griffies and S. J. Rey, Nucl. Phys. B **381** (1992), 301; hep-th/9201007.
M. Cvetic, S. Griffies and H. H. Soleng, Phys. Rev. D **48** (1993), 2613; gr-qc/9306005.
G. R. Dvali and M. A. Shifman, Nucl. Phys. B **504** (1997), 127; hep-th/9611213.

G. R. Dvali and M. A. Shifman, Phys. Lett. B **396** (1997), 64 [Errata; **407** (1997), 452]; hep-th/9612128.
M. Naganuma and M. Nitta, Prog. Theor. Phys. **105** (2001), 501; hep-th/0007184.
M. Naganuma, M. Nitta and N. Sakai, Phys. Rev. D **65** (2002), 045016; hep-th/0108179.
22) I. Antoniadis, H. Partouche and T. R. Taylor, Phys. Lett. B **372** (1996), 83; hep-th/9512006.
I. Antoniadis and T. R. Taylor, Fortsch. Phys. **44** (1996), 487; hep-th/9604062.
23) R. Grimm, M. Sohnius and J. Wess, Nucl. Phys. B **133** (1978), 275.
M. F. Sohnius, Nucl. Phys. B **136** (1978), 461.
M. de Roo, J. W. van Holten, B. de Wit and A. Van Proeyen, Nucl. Phys. B **173** (1980), 175.
24) S. Ferrara, L. Girardello and M. Porrati, Phys. Lett. B **376** (1996), 275; hep-th/9512180.
H. Partouche and B. Pioline, Nucl. Phys. B (Proc. Suppl.) **56** (1997), 322; hep-th/9702115.
25) E. A. Ivanov and B. M. Zupnik, Phys. Atom. Nucl. **62** (1999), 1043 [Yadern. Fiz. **62** (1999), 1110]; hep-th/9710236.
26) P. Fayet and J. Iliopoulos, Phys. Lett. B **51** (1974), 461.
P. Fayet, Nucl. Phys. B **113** (1976), 135.
27) J. T. Lopuszanski, Rep. Math. Phys. **13** (1978), 37.
28) R. Dijkgraaf and C. Vafa, Nucl. Phys. B **644** (2002), 3; hep-th/0206255.
R. Dijkgraaf and C. Vafa, Nucl. Phys. B **644** (2002), 21; hep-th/0207106.
R. Dijkgraaf and C. Vafa, hep-th/0208048.
29) N. Seiberg and E. Witten, Nucl. Phys. B **426** (1994), 19 [Errata; **430** (1994), 485]; hep-th/9407087.
30) A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, Phys. Lett. B **344** (1995), 169; hep-th/9411048.
P. C. Argyres and A. E. Faraggi, Phys. Rev. Lett. **74** (1995), 3931; hep-th/9411057.
A. Hanany and Y. Oz, Nucl. Phys. B **452** (1995), 283; hep-th/9505075.
M. Matone, Phys. Lett. B **357** (1995), 342; hep-th/9506102.
G. Bonelli, M. Matone and M. Tonin, Phys. Rev. D **55** (1997), 6466; hep-th/9610026.
31) A. Gorsky, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, Phys. Lett. B **355** (1995), 466; hep-th/9505035.
E. J. Martinec and N. P. Warner, Nucl. Phys. B **459** (1996), 97; hep-th/9509161.
T. Nakatsu and K. Takasaki, Mod. Phys. Lett. A **11** (1996), 157; hep-th/9509162.
R. Donagi and E. Witten, Nucl. Phys. B **460** (1996), 299; hep-th/9510101.
T. Eguchi and S. K. Yang, Mod. Phys. Lett. A **11** (1996), 131; hep-th/9510183.
H. Itoyama and A. Morozov, Nucl. Phys. B **477** (1996), 855; hep-th/9511126.
H. Itoyama and A. Morozov, hep-th/9601168.
H. Itoyama and A. Morozov, Nucl. Phys. B **491** (1997), 529; hep-th/9512161.
32) L. Chekhov and A. Mironov, Phys. Lett. B **552** (2003), 293; hep-th/0209085.
H. Itoyama and A. Morozov, Nucl. Phys. B **657** (2003), 53; hep-th/0211245.
H. Itoyama and A. Morozov, Phys. Lett. B **555** (2003), 287; hep-th/0211259.
H. Itoyama and A. Morozov, Prog. Theor. Phys. **109** (2003), 433; hep-th/0212032.
M. Matone, Nucl. Phys. B **656** (2003), 78; hep-th/0212253.
L. Chekhov, A. Marshakov, A. Mironov and D. Vasiliev, Phys. Lett. B **562** (2003), 323; hep-th/0301071.
A. Dymarsky and V. Pestun, Phys. Rev. D **67** (2003), 125001; hep-th/0301135.
H. Itoyama and A. Morozov, Int. J. Mod. Phys. A **18** (2003), 5889; hep-th/0301136.
S. Aoyama and T. Masuda, J. High Energy Phys. **03** (2004), 072; hep-th/0309232.
H. Itoyama and H. Kanno, Nucl. Phys. B **686** (2004), 155; hep-th/0312306.
33) H. Itoyama, M. Koike and H. Takashino, Mod. Phys. Lett. A **13** (1998), 1063; hep-th/9610228.
34) P. Fre, Nucl. Phys. B (Proc. Suppl.) **45** (1996), 59; hep-th/9512043.
A. Van Proeyen, hep-th/9512139.
B. Craps, F. Roose, W. Troost and A. Van Proeyen, Nucl. Phys. B **503** (1997), 565; hep-th/9703082.
D. S. Freed, Commun. Math. Phys. **203** (1999), 31; hep-th/9712042.
35) J. Bagger and E. Witten, Phys. Lett. B **118** (1982), 103.
C. M. Hull, A. Karlhede, U. Lindstrom and M. Rocek, Nucl. Phys. B **266** (1986), 1.

36) S. Ferrara and B. Zumino, Nucl. Phys. B **87** (1975), 207.
37) F. Cachazo, M. R. Douglas, N. Seiberg and E. Witten, J. High Energy Phys. **12** (2002), 071; hep-th/0211170.
38) H. Itoyama and H. Kanno, Phys. Lett. B **573** (2003), 227; hep-th/0304184.