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Supersymmetric Wilson loops in $\mathcal{N} = 4$ super Chern-Simons-matter theory

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Abstract: We investigate the supersymmetric Wilson loops in $d = 3$ $\mathcal{N} = 4$ super Chern-Simons-matter theory obtained from non-chiral orbifold of ABJM theory. We work in both Minkowski spacetime and Euclidean space, and we construct 1/4 and 1/2 BPS Wilson loops. We also provide a complete proof that the difference between 1/4 and 1/2 Wilson loops is Q-exact with Q being some supercharge that is preserved by both the 1/4 and 1/2 Wilson loops. This plays an important role in applying the localization techniques to compute the vacuum expectation values of Wilson loops. We also study the M-theory dual of the 1/2 BPS circular Wilson loop.

Keywords: Wilson, 't Hooft and Polyakov loops, AdS-CFT Correspondence, Chern-Simons Theories

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Contents

1 Introduction

After the discovery of $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence [1–3], people have also been interested in the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence. A superconformal field theory (SCFT) that is dual to M-theory on $\mathrm{AdS}_4 \times S^7$ spacetime is needed, or equivalently an SCFT that describes coinciding M2-branes is needed. The theory was finally constructed in [4] and is known as Aharony-Bergman-Jafferis-Maldacena (ABJM) theory. The ABJM theory is an $\mathcal{N} = 6$ super Chern-Simons-matter (SCSM) theory with gauge group $\mathrm{U}(N) \times \mathrm{U}(N)$ and levels $(k, -k)$, and it is dual to M-theory in $\mathrm{AdS}_4 \times S^7/Z_k$ spacetime, or type IIA superstring theory in $\mathrm{AdS}_4 \times CP^3$ spacetime. When $k = 1, 2$ the supersymmetries are enhanced non-perturbatively to $\mathcal{N} = 8$.

Like the study of Bogomol'nyi-Prasad-Sommerfield (BPS) Wilson loops in $d = 4$ $\mathcal{N} = 4$ super Yang-Mills theory in $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence [5–9], there are also many studies of Wilson loops in ABJM theory in the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence. The simplest BPS Wilson loop in ABJM theory is the 1/6 BPS one that was constructed in [10–12], but the simplest fundamental string solution dual to a Wilson loop in type IIA superstring theory in $\mathrm{AdS}_4 \times CP^3$ spacetime is 1/2 BPS. With more efforts the 1/2 BPS Wilson loop was constructed in [13].[1] When M theory or IIA superstring theory is weakly coupled, ABJM theory is strongly coupled. Then in order to compare with available results in gravity side, one needs to compute the vacuum expectation values (VEVs) of Wilson loops at strong coupling which is usually a hard task. However, using localization techniques [19][2] one can compute these VEVs in ABJM theory at both weak and strong couplings [19, 22, 23]. In applying the localization techniques to 1/2 BPS Wilson loops one needs to check that 1/2 and 1/6 BPS Wilson loops difference by a Q-exact term with Q being some supercharge that is preserved by both the 1/2 and 1/6 BPS Wilson loops and being used in localization [13]. The result of localization techniques for the 1/2 BPS loop has also been checked up to two loops in [24–26].

The construction of the 1/6 BPS Wilson loops in ABJM theory in [10–12] is basically the same as the 1/2 BPS (1/3 BPS) Wilson loop in $d = 3$ $\mathcal{N} = 2$ ($\mathcal{N} = 3$) Chern-Simons-matter theory in [27]. We will call these Wilson loops Gaiotto-Yin (GY) type. When $2 \leq \mathcal{N} \leq 6$, the GY type Wilson loop usually preserves four real supercharges, including the Poncaré supercharges and superconformal ones, if the loop is along a straight line or a circle.[3] The construction of 1/2 BPS Wilson loops in [13] includes essentially the fermionic matter fields and the supergroup structure appears in the ABJM theory. This type of Wilson loops will be called Drukker-Trancanelli (DT) type. It is an interesting question whether DT type BPS Wilson loops exist in theories with fewer supersymmetries. In $\mathcal{N} = 5$ theories [29, 30], such Wilson loops do exist and they are 2/5 BPS [31]. The situation in $\mathcal{N} = 3$ theories is interesting but not completely clear. Based on studies of the dual M2-brane solutions, strong evidence was given in [32] to support the conjecture that

[1] There are also similar constructions of Wilson loops but with fewer supersymmetries in [14–18].

[2] This has been generalized to $\mathcal{N} = 2$ SCSM theories in [20, 21].

[3] However supersymmetry enhancement for some special GY type Wilson loops in $d = 3, \mathcal{N} = 4$ theories has been recently found in [28].

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there are no Wilson loops in $\mathcal{N} = 3$ theories that preserve more that four supercharges. If this conjecture is right, there are two possibilities about the fate of the DT type BPS Wilson loops: one is that there are no DT type BPS Wilson loops in such theories, and the other is that DT type BPS Wilson loops exist in some $\mathcal{N} = 3$ theories and they are at most 1/3 BPS. This direction is interesting to be further studied.

Putting the situations in $\mathcal{N} = 3$ theories aside, we would like now to study the construction of DT type BPS Wilson loops in $\mathcal{N} = 4$ Chern-Simons-matter theories. Among these theories [33, 34], we will focus on the theories that are obtained from the non-chiral orbifold of ABJM theory. From ABJM theory with gauge group $\mathrm{U}(nN) \times \mathrm{U}(nN)$, one can perform a Z_n orbifolding and get an $\mathcal{N} = 4$ SCSM theory [35], and it is dual to M-theory in $\mathrm{AdS}_4 \times S^7/(Z_n \times Z_{nk})$ background [35–37]. The Wilson loops in this $\mathcal{N} = 4$ theory in fundamental representation are dual to M2-branes in $\mathrm{AdS}_4 \times S^7/(Z_n \times Z_{nk})$ spacetime. We consider the simplest embedding of such a membrane. The topology of the membrane worldvolume is $\mathrm{AdS}_2 \times S^1$, where $\mathrm{AdS}_2 \subset \mathrm{AdS}_4$ and S^1 is along the M-theory circle direction in $S^7/(Z_n \times Z_{nk})$. We find that there exists such an M2-brane that preserves half of the supersymmetries of the M-theory in $\mathrm{AdS}_4 \times S^7/(Z_n \times Z_{nk})$ spacetime. This indicates that there should be half-BPS Wilson loops in such $\mathcal{N} = 4$ SCSM theory. Based on experience in ABJM case, this Wilson loop may not be GY type, and may be DT type. One of the main results in this paper is the construction of such DT type half-BPS Wilson loops. We get the Poncaré and conformal supersymmetry (SUSY) transformation of the $\mathcal{N} = 4$ SCSM theory from that of ABJM theory. As a warm-up, we firstly construct the 1/4 BPS GY type Wilson loops. Then we give the details of the construction of the 1/2 BPS DT type Wilson loops. In Minkowski spacetime, we have the 1/4 and 1/2 BPS Wilson loops along a timelike infinite straight line. In Euclidean space, we have the 1/4 and 1/2 BPS Wilson loops along a infinite straight line, as well as 1/4 and 1/2 BPS Wilson loops along a circle. We also provide a complete proof that 1/2 and 1/4 BPS Wilson loops difference by a Q-exact term, with Q being some supercharge that is preserved by both the 1/4 and 1/2 BPS Wilson loops.

The rest of the paper is arranged as follows. In section 2, we study the simplest M2-brane solution dual to a circular Wilson loop in orbifold ABJM theory. We compute its on-shell action with boundary terms included and we also find that this M2-brane can be half BPS. In section 3 we review the basics of this $\mathcal{N} = 4$ SCSM theory and derive its SUSY transformation. In section 4 we consider the 1/4 and 1/2 BPS Wilson loops along a timelike infinite straight line in Minkowski spacetime. In section 5 we consider the 1/4 and 1/2 BPS Wilson loops along an infinite straight line in Euclidean space. In section 6 we construct the circular 1/4 and 1/2 BPS Wilson loops. We conclude with conclusion and discussion in section 7. We review the 1/6 and 1/2 BPS Wilson loops in ABJM theory in appendix A. In appendix B we provide a simple proof of gauge covariance of Wilson loops, from which we also prove a useful statement that has appeared in [13]. In appendix C we explore alternative definitions of Wilson loops for a super connection, but we find no nontrivial ones. In appendix D there are the calculation details of subsection 6.3, and we give a complete proof that the difference between 1/2 and 1/4 BPS Wilson loops is Q-exact.

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Note added. After the paper appears in ArXiv, there appears another paper [28] that has some overlaps with ours. There are more general 1/2 BPS Wilson loops in the orbifold ABJM theory, as well as in other $\mathcal{N} = 4$ SCSM theories. According to terminology of [28], the 1/2 BPS Wilson loops in this paper are ψ_1-loops. For a timelike straight line $x^\mu = \tau\delta_0^\mu$ in Minkowski space if we change the first two equations of the ansatz (4.16) to

$$\bar{\eta}_i^{(2\ell)} = \bar{\eta}^{(2\ell)}\delta_i^2, \quad \eta_{(2\ell)}^i = \eta_{(2\ell)}\delta_2^i, \tag{1.1}$$

we would get the ψ_2-loops. In this case we would have $\hat{m} = \hat{n} = -1$, as well as

$$\gamma_0\eta_{(2\ell)} = -i\eta_{(2\ell)}, \quad \bar{\eta}^{(2\ell)}\gamma_0 = -i\bar{\eta}^{(2\ell)}, \quad \eta_{(2\ell)}\bar{\eta}^{(2\ell)} = -i + \gamma_0. \tag{1.2}$$

The ψ_1- and ψ_2-loops have the same conserved supersymmetries (4.19) and (4.20). There are similar stories for 1/2 BPS Wilson loops along straight lines and circles in Euclidean space. There are a large number of 1/2 BPS Wilson loops, but there are not so many dual 1/2 BPS objects in M-theory. And so it is expected in [28] that these Wilson loops are 1/2 BPS classically, and only some special linear combination of them is 1/2 BPS quantum mechanically.

2 M2-branes in $\mathrm{AdS}_4 \times S^7/(Z_n \times Z_{nk})$ spacetime

The $\mathcal{N} = 4$ theory obtained from orbifolding ABJM theory is dual to M-theory in $\mathrm{AdS}_4 \times S^7/(Z_n \times Z_{nk})$ spacetime. We will denote $Z_n \times Z_{nk}$ as $\Gamma_{n,k}$ below. If we embed a unit S^7 inside $C^4 \cong R^8$ as

$$\sum_{i=1}^4 |z_i|^2 = 1, \quad z_i \in C, \tag{2.1}$$

the action of $\Gamma_{n,k}$ on S^7 is generated by [36]

$$(z_1, z_2, z_3, z_4) \to (\omega_n z_1, \omega_n z_2, z_3, z_4), \tag{2.2}$$

and

$$z_i \to \omega_{nk} z_i, \tag{2.3}$$

where $\omega_m \equiv \exp\left(\frac{2\pi i}{m}\right)$.

We parameterize z_i as

$$\begin{aligned}
z_1 &= \cos\frac{\alpha}{2}\cos\frac{\theta_1}{2}\exp\left[\frac{i}{4}(2\varphi_1 + \chi + \zeta)\right], \\
z_2 &= \cos\frac{\alpha}{2}\sin\frac{\theta_1}{2}\exp\left[\frac{i}{4}(-2\varphi_1 + \chi + \zeta)\right], \\
z_3 &= \sin\frac{\alpha}{2}\cos\frac{\theta_2}{2}\exp\left[\frac{i}{4}(2\varphi_2 - \chi + \zeta)\right], \\
z_4 &= \cos\frac{\alpha}{2}\sin\frac{\theta_2}{2}\exp\left[\frac{i}{4}(-2\varphi_2 - \chi + \zeta)\right],
\end{aligned} \tag{2.4}$$

where $\alpha \in [0,\pi]$, $\zeta \in [0,8\pi]$, $\chi \in [0,4\pi]$, $\theta_{1,2} \in [0,\pi]$, $\varphi_{1,2} \in [0,2\pi]$. Then the action of $\Gamma_{n,k}$ is generated by

$$\chi \to \chi + \frac{4\pi}{n}, \quad \zeta \to \zeta + \frac{4\pi}{n}, \tag{2.5}$$

and

$$\zeta \to \zeta + \frac{8\pi}{nk}. \tag{2.6}$$

The IIA limit of M-theory is obtained by taking $k \to \infty$ while keeping n fixed. In this limit the circle along ζ direction will shrink. So this circle is the M-theory circle, and its circumference is $\frac{8\pi}{nk}$. The metric of unit S^7 is

$$\begin{aligned} ds^2_{S^7} = \frac{1}{4}\Big[& d\alpha^2 + \cos^2\frac{\alpha}{2}\left(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2\right) + \sin^2\frac{\alpha}{2}\left(d\theta_2^2 + \sin^2\theta_2 d\varphi_2^2\right) \\ & + \sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}(d\chi + \cos\theta_1 d\varphi_1 - \cos\theta_2 d\varphi_2)^2 \\ & + \left(\frac{1}{2}d\zeta + \cos^2\frac{\alpha}{2}\cos\theta_1 d\varphi_1 + \sin^2\frac{a}{2}\cos\theta_2 d\varphi_2 + \frac{1}{2}\cos\alpha d\chi\right)^2\Big]. \end{aligned} \tag{2.7}$$

The metric of $\mathrm{AdS}_4 \times S^7/\Gamma_{n,k}$ is

$$ds^2_{11} = R^2\left(\frac{1}{4}ds^2_{\mathrm{AdS}_4} + ds^2_{S^7/\Gamma_{n,k}}\right). \tag{2.8}$$

For Lorentzian signature we choose the following global coordinate on AdS_4,

$$ds^2_{\mathrm{AdS}_4} = \cosh^2 u\left(-\cosh^2\rho dt^2 + d\rho^2\right) + du^2 + \sinh^2 u d\phi^2. \tag{2.9}$$

The four-form field strength on this background is

$$H_4 = \frac{3R^3}{8}\cosh^2 u \sinh u \cosh\rho dt \wedge d\rho \wedge du \wedge d\phi. \tag{2.10}$$

Flux quantization gives

$$R = 2\pi\ell_p\left[\frac{N}{6\mathrm{Vol}(S^7/\Gamma_{n,k})}\right]^{1/6} = \ell_p(32\pi^2 n^2 Nk)^{1/6}, \tag{2.11}$$

where ℓ_p is the eleven-dimensional Planck length, and we have used

$$\mathrm{Vol}(S^7/\Gamma_{n,k}) = \frac{\mathrm{Vol}(S^7)}{n^2 k} = \frac{\pi^4}{3n^2 k}. \tag{2.12}$$

The radius of the ζ circle in Planck unit is of order $R/(nk\ell_p) \propto (n^2Nk)^{1/6}/(nk)$, and so the M-theory description is a good one when $N \gg n^4k^5$.

We consider the probe M2-brane solution in this background. In Lorentzian signature the bosonic part of the M2-brane action is

$$S_{M2} = S^{DBI}_{M2} + S^{WZ}_{M2} = -T_{M2}\left(\int d^3\sigma\sqrt{-\mathrm{det}g_{mn}} + \int P[C_3]\right). \tag{2.13}$$

Here g_{mn} is the induced metric of the membrane worldvolume, T_{M2} is the tension of the M2-brane

$$T_{M2} = \frac{1}{(2\pi)^2 \ell_p^3}, \tag{2.14}$$

and $P[C_3]$ is the pullback of the bulk 3-form gauge potential to the worldvolume of the membrane. The gauge choice for the background 3-form gauge potential C_3 is

$$C_3 = \frac{R^3}{8}(\cosh^3 u - 1)\cosh\rho dt \wedge d\rho \wedge d\phi. \tag{2.15}$$

From the action, one can obtain the membrane equation of motion as[4]

$$\frac{1}{\sqrt{-g}}\partial_m\left(\sqrt{-g}g^{mn}\partial_n X^{\underline{N}}\right)G_{\underline{MN}} + g^{mn}\partial_m X^{\underline{N}}\partial_n X^{\underline{P}}\Gamma^{\underline{Q}}_{\underline{NP}}G_{\underline{QM}} = \frac{1}{3!\sqrt{-g}}\epsilon^{mnp}(P[dC_3])_{\underline{M}mnp}. \tag{2.16}$$

Note that ϵ^{mnp} is a tensor density on the world-volume of the membrane.

Since we want to find the simplest membrane embedding corresponding to a Wilson loop in the dual field theory, we take the topology of the membrane worldvolume to be $\mathrm{AdS}_2 \times S^1$. The AdS_2 is embedded in AdS_4, while S^1 is along the M-theory circle. So we consider the ansatz

$$t = \sigma^0, \quad \rho = \sigma^1, \quad \zeta = \sigma^2, \tag{2.17}$$

where $\sigma^\mu, \mu = 0, 1, 2$ is the coordinates on the worldvolume of M2-brane. One can find that the equations of motion only lead to the constraint that $u = 0$. Then the induced metric of M2-brane is

$$ds^2_{M2} = R^2\left(-\frac{1}{4}\cosh^2\rho dt^2 + \frac{1}{4}d\rho^2 + \frac{1}{16}d\zeta^2\right). \tag{2.18}$$

To compute the on-shell action of the M2-brane whose boundary at infinity is S^1, we work in Euclidean signature and choose the AdS_4 coordinates

$$ds^2_{\mathrm{AdS}_4} = \cosh^2 u\left(\sinh^2\rho d\psi^2 + d\rho^2\right) + du^2 + \sinh^2 u d\phi^2, \tag{2.19}$$

where $\psi \in [0, 2\pi]$. In Euclidean signature the M2-brane action becomes[5]

$$S_{M2} = S^{DBI}_{M2} = T_2 \int d^3\sigma \sqrt{\det g_{mn}}. \tag{2.20}$$

For the M2-brane that is put at

$$\psi = \sigma_1, \quad \rho = \sigma_2, \quad \zeta = \sigma_3, \quad u = 0, \tag{2.21}$$

the on-shell action is

$$S_{M2} = \frac{T_{M2}R^3}{16}\int d\zeta d\rho d\psi \sinh\rho. \tag{2.22}$$

[4] We always use the indices from the beginning (middle) of the alphabet to refer to the frame (coordinate) coordinates, and the underlined indices to refer to the target space ones.

[5] It is easy to see that $S^{WZ}_{M2} = 0$ for the M2-brane solution considered here.

After adding boundary terms to regulate the action as in [8], we get

$$S_{M2}^{\text{total}} = -\frac{\pi T_{M2} R^3}{8} \int d\zeta. \tag{2.23}$$

Using the fact that $\zeta \in [0, \frac{8\pi}{nk}]$, $T_{M2} = 1/(4\pi^2 \ell_p^3)$ and (2.11), we can get

$$S_{M2}^{\text{total}} = -\pi \sqrt{\frac{2N}{k}}. \tag{2.24}$$

Then the holographic prediction for the leading exponential behavior of the VEV of the 1/2 BPS Wilson loop in the large N limit with finite k and n is

$$\langle W \rangle \sim \exp(-S_{M2}^{\text{total}}) = \exp(\pi \sqrt{2N/k}). \tag{2.25}$$

Note that the result is not dependent on n.

In Lorentzian signature the Killing spinor of M-theory in $\mathrm{AdS}_4 \times S^7$ spacetime with the AdS_4 coordinates (2.9) is [10]

$$\epsilon = e^{\frac{\alpha}{4}(\hat{\gamma}\gamma_4 - \gamma_{7\sharp})} e^{\frac{\theta_1}{4}(\hat{\gamma}\gamma_5 - \gamma_{8\sharp})} e^{\frac{\theta_2}{2}(\gamma_{79}+\gamma_{46})} e^{-\frac{\xi_1}{2}\hat{\gamma}\gamma_\sharp} e^{-\frac{\xi_2}{2}\gamma_{58}} e^{-\frac{\xi_3}{2}\gamma_{47}} e^{-\frac{\xi_4}{2}\gamma_{69}} e^{\frac{u}{2}\hat{\gamma}\gamma_2} e^{\frac{\phi}{2}\gamma_{23}} e^{\frac{\rho}{2}\hat{\gamma}\gamma_1} e^{\frac{t}{2}\hat{\gamma}\gamma_0} \epsilon_0, \tag{2.26}$$

where ϵ_0 is a constant eleven-dimensional Majorana spinor which has 32 real degrees of freedom. The definitions of ξ_i with $i = 1, 2, 3, 4$ are

$$\begin{aligned} \xi_1 &= \frac{1}{4}(2\phi_1 + \chi + \zeta), \quad \xi_2 = \frac{1}{4}(-2\phi_1 + \chi + \zeta), \\ \xi_3 &= \frac{1}{4}(2\phi_2 - \chi + \zeta), \quad \xi_4 = \frac{1}{4}(-2\phi_2 - \chi + \zeta). \end{aligned} \tag{2.27}$$

Also γ_0, γ_1, $\cdots$, γ_9, $\gamma_\sharp$ are eleven-dimensional gamma matrices, and $\hat{\gamma} \equiv \gamma^{0123}$. Note that the eleven-dimensional gamma matrices are chosen such that

$$\gamma_{0123456789\sharp} = 1. \tag{2.28}$$

To obtained the Killing spinor of M-theory on $\mathrm{AdS}_4 \times S^7/\Gamma_{n,k}$, we need to impose the conditions

$$\mathcal{L}_{K_1}\epsilon = \mathcal{L}_{K_2}\epsilon = 0, \tag{2.29}$$

where $K_{1,2}$ are the following two Killing vectors

$$K_1 = \partial_\chi + \partial_\zeta, \quad K_2 = \partial_\zeta, \tag{2.30}$$

and they are related to the generators of $\Gamma_{n,k}$. The definition of $\mathcal{L}_K\epsilon$ is

$$\mathcal{L}_K\epsilon \equiv K^{\underline{M}}\nabla_{\underline{M}}\epsilon + \frac{1}{4}(\nabla_{\underline{M}}K_{\underline{N}})\gamma^{\underline{MN}}\epsilon. \tag{2.31}$$

After some computations, we find that the two conditions (2.29) are equivalent to

$$\gamma_{4679}\epsilon_0 = -\epsilon_0. \tag{2.32}$$

So the background is half BPS compared to the maximal possibility, i.e. there are 16 real supercharges. This is consistent with the fact that the dual three-dimensional SCFT is an $\mathcal{N} = 4$ theory.

The supercharges preserved by the probe membrane are determined by the following equation

$$\Gamma_{M2}\epsilon = \epsilon, \tag{2.33}$$

with

$$\Gamma_{M2} = \frac{1}{\sqrt{-g}}\partial_{\sigma_0}X^{\underline{M}}\partial_{\sigma_1}X^{\underline{N}}\partial_{\sigma_2}X^{\underline{P}}e^{A}_{\underline{M}}e^{B}_{\underline{N}}e^{C}_{\underline{P}}\gamma_{\underline{ABC}}. \tag{2.34}$$

For the membrane we just found, we have

$$\Gamma_{M2} = \gamma_{01\sharp}. \tag{2.35}$$

So the supercharges preserved by this probe membrane correspond to the solution of

$$\gamma_{01\sharp}\epsilon = \epsilon. \tag{2.36}$$

At the positions with $\alpha = \theta_1 = 0$, this is equivalent to [10]

$$\gamma_{01\sharp}\epsilon_0 = \epsilon_0. \tag{2.37}$$

Since it is compatible with the projection condition in (2.32), we arrive at the conclusion that the probe M2-brane put at $\alpha = \theta_1 = 0$ is half BPS compared to the supersymmetries of M-theory in $\mathrm{AdS}_4 \times S^7/\Gamma_{n,k}$ spacetime.

3 $\mathcal{N} = 4$ SCSM theory

Orbifolding the ABJM theory with gauge group $\mathrm{U}(nN)\times\mathrm{U}(nN)$ and levels $(k, -k)$ by Z_n, one can get the $\mathcal{N} = 4$ SCSM theory with gauge group $\mathrm{U}(N)^{2n}$ and Chern-Simons levels $(k, -k, \cdots, k, -k)$ [35]. We can get the SUSY transformation of this $\mathcal{N} = 4$ theory from that of ABJM theory (A.1) by the orbifolding. The result is

$$\begin{aligned}
\delta\phi_i^{(2\ell+1)} &= 2i\bar{\chi}_{i\hat{\imath}}\psi^{\hat{\imath}}_{(2\ell+1)}, \quad \delta\phi_{\hat{\imath}}^{(2\ell)} = -2i\bar{\chi}_{i\hat{\imath}}\psi^{i}_{(2\ell)}, \\
\delta\bar{\phi}^i_{(2\ell+1)} &= 2i\bar{\psi}_{\hat{\imath}}^{(2\ell+1)}\chi^{i\hat{\imath}}, \quad \delta\bar{\phi}^{\hat{\imath}}_{(2\ell)} = -2i\bar{\psi}_{i}^{(2\ell)}\chi^{i\hat{\imath}}, \\
\delta A_\mu^{(2\ell+1)} &= \frac{4\pi}{k}\left[\left(\phi_i^{(2\ell+1)}\bar{\psi}_{\hat{\imath}}^{(2\ell+1)} - \phi_{\hat{\imath}}^{(2\ell)}\bar{\psi}_i^{(2\ell)}\right)\gamma_\mu\chi^{i\hat{\imath}} + \bar{\chi}_{i\hat{\imath}}\gamma_\mu\left(\psi^{\hat{\imath}}_{(2\ell+1)}\bar{\phi}^i_{(2\ell+1)} - \psi^i_{(2\ell)}\bar{\phi}^{\hat{\imath}}_{(2\ell)}\right)\right], \\
\delta\hat{A}_\mu^{(2\ell)} &= \frac{4\pi}{k}\left[\left(\bar{\psi}_{\hat{\imath}}^{(2\ell-1)}\phi_i^{(2\ell-1)} - \bar{\psi}_i^{(2\ell)}\phi_{\hat{\imath}}^{(2\ell)}\right)\gamma_\mu\chi^{i\hat{\imath}} + \bar{\chi}_{i\hat{\imath}}\gamma_\mu\left(\bar{\phi}^i_{(2\ell-1)}\psi^{\hat{\imath}}_{(2\ell-1)} - \bar{\phi}^{\hat{\imath}}_{(2\ell)}\psi^i_{(2\ell)}\right)\right], \\
\delta\psi^i_{(2\ell)} &= 2\gamma^\mu\chi^{i\hat{\imath}}D_\mu\phi_{\hat{\imath}}^{(2\ell)} + 2\vartheta^{i\hat{\imath}}\phi_{\hat{\imath}}^{(2\ell)} - \frac{4\pi}{k}\chi^{i\hat{\imath}}\left(\phi_{\hat{\imath}}^{(2\ell)}\bar{\phi}^j_{(2\ell-1)}\phi_j^{(2\ell-1)}\right. \\
&\quad \left.+\phi_{\hat{\imath}}^{(2\ell)}\bar{\phi}^{\hat{\jmath}}_{(2\ell)}\phi_{\hat{\jmath}}^{(2\ell)} - \phi_j^{(2\ell+1)}\bar{\phi}^j_{(2\ell+1)}\phi_{\hat{\imath}}^{(2\ell)} - \phi_{\hat{\jmath}}^{(2\ell)}\bar{\phi}^{\hat{\jmath}}_{(2\ell)}\phi_{\hat{\imath}}^{(2\ell)}\right) \\
&\quad - \frac{8\pi}{k}\chi^{j\hat{\jmath}}\left(\phi_j^{(2\ell+1)}\bar{\phi}^i_{(2\ell+1)}\phi_{\hat{\jmath}}^{(2\ell)} - \phi_{\hat{\jmath}}^{(2\ell)}\bar{\phi}^i_{(2\ell-1)}\phi_j^{(2\ell-1)}\right), \\
\delta\psi^{\hat{\imath}}_{(2\ell+1)} &= -2\gamma^\mu\chi^{i\hat{\imath}}D_\mu\phi_i^{(2\ell+1)} - 2\vartheta^{i\hat{\imath}}\phi_i^{(2\ell+1)} + \frac{4\pi}{k}\chi^{i\hat{\imath}}\left(\phi_i^{(2\ell+1)}\bar{\phi}^j_{(2\ell+1)}\phi_j^{(2\ell+1)}\right. \\
&\quad \left.+\phi_i^{(2\ell+1)}\bar{\phi}^{\hat{\jmath}}_{(2\ell+2)}\phi_{\hat{\jmath}}^{(2\ell+2)} - \phi_j^{(2\ell+1)}\bar{\phi}^j_{(2\ell+1)}\phi_i^{(2\ell+1)} - \phi_{\hat{\jmath}}^{(2\ell)}\bar{\phi}^{\hat{\jmath}}_{(2\ell)}\phi_i^{(2\ell+1)}\right) \\
&\quad - \frac{8\pi}{k}\chi^{j\hat{\jmath}}\left(\phi_j^{(2\ell+1)}\bar{\phi}^{\hat{\imath}}_{(2\ell+2)}\phi_{\hat{\jmath}}^{(2\ell+2)} - \phi_{\hat{\jmath}}^{(2\ell)}\bar{\phi}^{\hat{\imath}}_{(2\ell)}\phi_j^{(2\ell+1)}\right),
\end{aligned} \tag{3.1}$$

$$
\begin{aligned}
\delta\bar\psi_i^{(2\ell)} &= -2\bar\chi_{i\hat\imath}\gamma^\mu D_\mu\bar\phi^{\hat\imath}_{(2\ell)} + 2\bar\vartheta_{i\hat\imath}\bar\phi^{\hat\imath}_{(2\ell)} + \frac{4\pi}{k}\bar\chi_{i\hat\imath}\left(\bar\phi^{\hat\imath}_{(2\ell)}\phi_j^{(2\ell+1)}\bar\phi^j_{(2\ell+1)}\right.\\
&\quad\left.+\bar\phi^{\hat\imath}_{(2\ell)}\phi_{\hat\jmath}^{(2\ell)}\bar\phi^{\hat\jmath}_{(2\ell)} - \bar\phi^j_{(2\ell-1)}\phi_j^{(2\ell-1)}\bar\phi^{\hat\imath}_{(2\ell)} - \bar\phi^{\hat\jmath}_{(2\ell)}\phi_{\hat\jmath}^{(2\ell)}\bar\phi^{\hat\imath}_{(2\ell)}\right)\\
&\quad+\frac{8\pi}{k}\bar\chi_{j\hat\jmath}\left(\bar\phi^j_{(2\ell-1)}\phi_i^{(2\ell-1)}\bar\phi^{\hat\jmath}_{(2\ell)} - \bar\phi^{\hat\jmath}_{(2\ell)}\phi_i^{(2\ell+1)}\bar\phi^j_{(2\ell+1)}\right),\\
\delta\bar\psi_{\hat\imath}^{(2\ell+1)} &= 2\bar\chi_{i\hat\imath}\gamma^\mu D_\mu\bar\phi^i_{(2\ell+1)} - 2\bar\vartheta_{i\hat\imath}\bar\phi^i_{(2\ell+1)} - \frac{4\pi}{k}\bar\chi_{i\hat\imath}\left(\bar\phi^i_{(2\ell+1)}\phi_j^{(2\ell+1)}\bar\phi^j_{(2\ell+1)}\right.\\
&\quad\left.+\bar\phi^i_{(2\ell+1)}\phi_{\hat\jmath}^{(2\ell)}\bar\phi^{\hat\jmath}_{(2\ell)} - \bar\phi^j_{(2\ell+1)}\phi_j^{(2\ell+1)}\bar\phi^i_{(2\ell+1)} - \bar\phi^{\hat\jmath}_{(2\ell+2)}\phi_{\hat\jmath}^{(2\ell+2)}\bar\phi^i_{(2\ell+1)}\right)\\
&\quad+\frac{8\pi}{k}\bar\chi_{j\hat\jmath}\left(\bar\phi^j_{(2\ell+1)}\phi_{\hat\imath}^{(2\ell)}\bar\phi^{\hat\jmath}_{(2\ell)} - \bar\phi^{\hat\jmath}_{(2\ell+2)}\phi_{\hat\imath}^{(2\ell+2)}\bar\phi^j_{(2\ell+1)}\right).
\end{aligned}
$$

Here $\ell = 0, 1, \cdots, n-1$. There are no summations of ℓ here, and would not be summations of ℓ later unless it is given out explicitly. Indices $i, j, \cdots = 1, 2$ and $\hat\imath, \hat\jmath, \cdots = \hat 1, \hat 2$ are those of the SU(2) $\times$ SU(2) R-symmetry. There are definitions of covariant derivatives

$$
\begin{aligned}
D_\mu\phi_{\hat\imath}^{(2\ell)} &= \partial_\mu\phi_{\hat\imath}^{(2\ell)} + iA_\mu^{(2\ell+1)}\phi_{\hat\imath}^{(2\ell)} - i\phi_{\hat\imath}^{(2\ell)}\hat A_\mu^{(2\ell)},\\
D_\mu\phi_i^{(2\ell+1)} &= \partial_\mu\phi_i^{(2\ell+1)} + iA_\mu^{(2\ell+1)}\phi_i^{(2\ell+1)} - i\phi_i^{(2\ell+1)}\hat A_\mu^{(2\ell+2)},\\
D_\mu\bar\phi^{\hat\imath}_{(2\ell)} &= \partial_\mu\bar\phi^{\hat\imath}_{(2\ell)} + i\hat A_\mu^{(2\ell)}\bar\phi^{\hat\imath}_{(2\ell)} - i\bar\phi^{\hat\imath}_{(2\ell)}A_\mu^{(2\ell+1)},\\
D_\mu\bar\phi^i_{(2\ell+1)} &= \partial_\mu\bar\phi^i_{(2\ell+1)} + i\hat A_\mu^{(2\ell+2)}\bar\phi^i_{(2\ell+1)} - i\bar\phi^i_{(2\ell+1)}A_\mu^{(2\ell+1)}.
\end{aligned}
\tag{3.2}
$$

Also $\chi^{i\hat\imath} = \theta^{i\hat\imath} + x^\mu\gamma_\mu\vartheta^{i\hat\imath}$ and $\bar\chi_{i\hat\imath} = \bar\theta_{i\hat\imath} - \bar\vartheta_{i\hat\imath}x^\mu\gamma_\mu$, and $\theta^{i\hat\imath}$, $\bar\theta_{i\hat\imath}$, $\vartheta^{i\hat\imath}$, $\bar\vartheta_{i\hat\imath}$ are Dirac spinors with constraints

$$
\begin{aligned}
(\theta^{i\hat\imath})^* &= \bar\theta_{i\hat\imath}, \quad \bar\theta_{i\hat\imath} = \epsilon_{ij}\epsilon_{\hat\imath\hat\jmath}\theta^{j\hat\jmath},\\
(\vartheta^{i\hat\imath})^* &= \bar\vartheta_{i\hat\imath}, \quad \bar\vartheta_{i\hat\imath} = \epsilon_{ij}\epsilon_{\hat\imath\hat\jmath}\vartheta^{j\hat\jmath}.
\end{aligned}
\tag{3.3}
$$

Symbols ϵ_{ij} and $\epsilon_{\hat\imath\hat\jmath}$ are antisymmetric with $\epsilon_{12} = \epsilon_{\hat 1\hat 2} = 1$. Note that for δ in (3.1) we have

$$
\delta = 2i\left(\bar\theta_{i\hat\imath}P^{i\hat\imath} + \bar\vartheta_{i\hat\imath}S^{i\hat\imath}\right) = 2i\left(\bar P_{i\hat\imath}\theta^{i\hat\imath} + \bar S_{i\hat\imath}\vartheta^{i\hat\imath}\right), \tag{3.4}
$$

with $P^{i\hat\imath}$, $\bar P_{i\hat\imath}$ and $S^{i\hat\imath}$, $S_{i\hat\imath}$ being Poncaré and conformal supercharges that satisfy

$$
\begin{aligned}
(P^{i\hat\imath})^* &= \bar P_{i\hat\imath}, \quad \bar P_{i\hat\imath} = \epsilon_{ij}\epsilon_{\hat\imath\hat\jmath}P^{j\hat\jmath},\\
(S^{i\hat\imath})^* &= \bar S_{i\hat\imath}, \quad \bar S_{i\hat\imath} = \epsilon_{ij}\epsilon_{\hat\imath\hat\jmath}S^{j\hat\jmath}.
\end{aligned}
\tag{3.5}
$$

In Euclidean space, the SUSY transformation is formally identical to (3.1), with $\chi^{i\hat\imath} = \theta^{i\hat\imath} + x^\mu\gamma_\mu\vartheta^{i\hat\imath}$ and $\bar\chi_{i\hat\imath} = \bar\theta_{i\hat\imath} - \bar\vartheta_{i\hat\imath}x^\mu\gamma_\mu$. But now equations (3.3) become

$$
\bar\theta_{i\hat\imath} = \epsilon_{ij}\epsilon_{\hat\imath\hat\jmath}\theta^{j\hat\jmath}, \qquad \bar\vartheta_{i\hat\imath} = \epsilon_{ij}\epsilon_{\hat\imath\hat\jmath}\vartheta^{j\hat\jmath}.
$$

Note the eight spinors $\theta^{i\hat\imath}$, $\vartheta^{i\hat\imath}$ with $i = 1, 2$, $\hat\imath = \hat 1, \hat 2$ are independent Dirac spinors. Now equations (3.4) are invariant but (3.5) become

$$
\bar P_{i\hat\imath} = \epsilon_{ij}\epsilon_{\hat\imath\hat\jmath}P^{j\hat\jmath}, \quad \bar S_{i\hat\imath} = \epsilon_{ij}\epsilon_{\hat\imath\hat\jmath}S^{j\hat\jmath}. \tag{3.6}
$$

4 Straight line in Minkowski spacetime

In Minkowski spacetime there are BPS Wilson loops along null and timelike infinite straight lines [38]. It is easy to construct a null Wilson loop, and it is 1/2 BPS. We only consider the timelike BPS Wilson loops here.

4.1 1/4 BPS Wilson loop

We consider the Wilson loop along a timelike straight line $x^\mu = \tau\delta_0^\mu$ as

$$
\begin{aligned}
W_{1/4}^{(2\ell+1)} &= \mathcal{P}\exp\left(-i\int d\tau \mathcal{A}^{(2\ell+1)}(\tau)\right), \\
\mathcal{A}^{(2\ell+1)} &= A_\mu^{(2\ell+1)}\dot{x}^\mu + \frac{2\pi}{k}\left(M^i{}_j\phi_i^{(2\ell+1)}\bar{\phi}^j_{(2\ell+1)} + M^{\hat{\imath}}{}_{\hat{\jmath}}\phi_{\hat{\imath}}^{(2\ell)}\bar{\phi}^{\hat{\jmath}}_{(2\ell)}\right)|\dot{x}|.
\end{aligned}
\tag{4.1}
$$

For Poncaré SUSY transformation we can get

$$
\begin{aligned}
\delta\mathcal{A}^{(2\ell+1)} = \frac{4\pi}{k}\Big[&\phi_i^{(2\ell+1)}\bar{\psi}_{\hat{\imath}}^{(2\ell+1)}\left(\gamma_0\theta^{i\hat{\imath}} + iM^i{}_j\theta^{j\hat{\imath}}\right) - \phi_{\hat{\imath}}^{(2\ell)}\bar{\psi}_i^{(2\ell)}\left(\gamma_0\theta^{i\hat{\imath}} + iM^{\hat{\imath}}{}_{\hat{\jmath}}\theta^{i\hat{\jmath}}\right) \\
&+\left(\bar{\theta}_{i\hat{\imath}}\gamma_0 + iM^j{}_i\bar{\theta}_{j\hat{\imath}}\right)\psi^{\hat{\imath}}_{(2\ell+1)}\bar{\phi}^i_{(2\ell+1)} - \left(\bar{\theta}_{i\hat{\imath}}\gamma_0 + iM^{\hat{\jmath}}{}_{\hat{\imath}}\bar{\theta}_{i\hat{\jmath}}\right)\psi^i_{(2\ell)}\bar{\phi}^{\hat{\imath}}_{(2\ell)}\Big].
\end{aligned}
\tag{4.2}
$$

We work in the basis of diagonal $M^i{}_j = m_i\delta^i_j$ and $M^{\hat{\imath}}{}_{\hat{\jmath}} = m_{\hat{\imath}}\delta^{\hat{\imath}}_{\hat{\jmath}}$, and then we get

$$
\begin{aligned}
\gamma_0\theta^{i\hat{\imath}} &= -im_i\theta^{i\hat{\imath}} = -im_{\hat{\imath}}\theta^{i\hat{\imath}}, \\
\gamma_0\bar{\theta}_{i\hat{\imath}} &= im_i\bar{\theta}_{i\hat{\imath}} = im_{\hat{\imath}}\bar{\theta}_{i\hat{\imath}}.
\end{aligned}
\tag{4.3}
$$

Supposing $\theta^{1\hat{1}} \neq 0$, we choose without loss of generality

$$
\gamma_0\theta^{1\hat{1}} = i\theta^{1\hat{1}}. \tag{4.4}
$$

Using (3.3) we know $(\theta^{1\hat{1}})^* = \theta^{2\hat{2}}$, and then we get

$$
\gamma_0\theta^{2\hat{2}} = -i\theta^{2\hat{2}}. \tag{4.5}
$$

This means that $m_1 = m_{\hat{1}} = -1$, $m_2 = m_{\hat{2}} = 1$. Then we have

$$
\theta^{1\hat{2}} = \theta^{2\hat{1}} = 0. \tag{4.6}
$$

We can check that the equations (4.3) are consistent. It is similar for conformal SUSY transformation. Thus we get a 1/4 BPS Wilson loop.

Similarly we can construct the 1/4 BPS Wilson loop along $x^\mu = \tau\delta_0^\mu$ that preserves the same supersymmetries

$$
\begin{aligned}
\hat{W}_{1/4}^{(2\ell)} &= \mathcal{P}\exp\left(-i\int d\tau \hat{\mathcal{A}}^{(2\ell)}(\tau)\right), \\
\hat{\mathcal{A}}^{(2\ell)} &= \hat{A}_\mu^{(2\ell)}\dot{x}^\mu + \frac{2\pi}{k}\left(N_i{}^j\bar{\phi}^i_{(2\ell-1)}\phi_j^{(2\ell-1)} + N_{\hat{\imath}}{}^{\hat{\jmath}}\bar{\phi}^{\hat{\imath}}_{(2\ell)}\phi_{\hat{\jmath}}^{(2\ell)}\right)|\dot{x}|, \\
N_i{}^j &= N_{\hat{\imath}}{}^{\hat{\jmath}} = \mathrm{diag}(-1, 1).
\end{aligned}
\tag{4.7}
$$

Also we can combine (4.1) and (4.7) and get the 1/4 BPS Wilson loop

$$\begin{aligned}
W_{1/4} &= \mathcal{P}\exp\left(-i\int d\tau L_{1/4}(\tau)\right), \\
L_{1/4} &= \begin{pmatrix} \mathcal{A} & \\ & \hat{\mathcal{A}} \end{pmatrix}, \\
\mathcal{A} &= \mathrm{diag}(\mathcal{A}^{(1)}, \mathcal{A}^{(3)}, \cdots, \mathcal{A}^{(2n-1)}), \\
\hat{\mathcal{A}} &= \mathrm{diag}(\hat{\mathcal{A}}^{(0)}, \hat{\mathcal{A}}^{(2)}, \cdots, \hat{\mathcal{A}}^{(2n-2)}).
\end{aligned} \tag{4.8}$$

Note that we can also construct the 1/4 BPS Wilson loop in this subsection using the consideration in [27] for general $\mathcal{N}=2$ theories. So this Wilson loop is GY type.

4.2 1/2 BPS Wilson loop

We consider the timelike Wilson loop along $x^\mu = \tau\delta_0^\mu$

$$W_{1/2} = \mathcal{P}\exp\left(-i\int d\tau L_{1/2}(\tau)\right), \tag{4.9}$$

where $L_{1/2}$ is a supermatrix

$$L_{1/2} = \begin{pmatrix} \mathcal{A} & \bar{F}_1 \\ F_2 & \hat{\mathcal{A}} \end{pmatrix}. \tag{4.10}$$

Here we have definitions

$$\begin{aligned}
\mathcal{A} &= \mathrm{diag}(\mathcal{A}^{(1)}, \mathcal{A}^{(3)}, \cdots, \mathcal{A}^{(2n-1)}), \\
\mathcal{A}^{(2\ell+1)} &= A_\mu^{(2\ell+1)}\dot{x}^\mu + \frac{2\pi}{k}\left(M^i{}_j\phi_i^{(2\ell+1)}\bar{\phi}^j_{(2\ell+1)} + M^{\hat{\imath}}{}_{\hat{\jmath}}\phi_{\hat{\imath}}^{(2\ell)}\bar{\phi}^{\hat{\jmath}}_{(2\ell)}\right)|\dot{x}|, \\
\hat{\mathcal{A}} &= \mathrm{diag}(\hat{\mathcal{A}}^{(0)}, \hat{\mathcal{A}}^{(2)}, \cdots, \hat{\mathcal{A}}^{(2n-2)}), \\
\hat{\mathcal{A}}^{(2\ell)} &= \hat{A}_\mu^{(2\ell)}\dot{x}^\mu + \frac{2\pi}{k}\left(N_i{}^j\bar{\phi}^i_{(2\ell-1)}\phi_j^{(2\ell-1)} + N_{\hat{\imath}}{}^{\hat{\jmath}}\bar{\phi}^{\hat{\imath}}_{(2\ell)}\phi_{\hat{\jmath}}^{(2\ell)}\right)|\dot{x}|,
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
\bar{F}_1 &= \begin{pmatrix}
\bar{f}_1^{(0)} & \bar{f}_1^{(1)} & & & \\
& \bar{f}_1^{(2)} & \bar{f}_1^{(3)} & & \\
& & \ddots & \ddots & \\
& & & \bar{f}_1^{(2n-4)} & \bar{f}_1^{(2n-3)} \\
\bar{f}_1^{(2n-1)} & & & & \bar{f}_1^{(2n-2)}
\end{pmatrix}|\dot{x}|, \\
\bar{f}_1^{(2\ell+1)} &= \sqrt{\frac{2\pi}{k}}\bar{\eta}_{\hat{\imath}}^{(2\ell+1)}\psi^{\hat{\imath}}_{(2\ell+1)}, \quad \bar{f}_1^{(2\ell)} = \sqrt{\frac{2\pi}{k}}\bar{\eta}_i^{(2\ell)}\psi^i_{(2\ell)}, \\
F_2 &= \begin{pmatrix}
f_2^{(0)} & & & & f_2^{(2n-1)} \\
f_2^{(1)} & f_2^{(2)} & & & \\
& f_2^{(3)} & \ddots & & \\
& & \ddots & f_2^{(2n-4)} & \\
& & & f_2^{(2n-3)} & f_2^{(2n-2)}
\end{pmatrix}|\dot{x}|, \\
f_2^{(2\ell+1)} &= \sqrt{\frac{2\pi}{k}}\bar{\psi}_{\hat{\imath}}^{(2\ell+1)}\eta^{\hat{\imath}}_{(2\ell+1)}, \quad f_2^{(2\ell)} = \sqrt{\frac{2\pi}{k}}\bar{\psi}_i^{(2\ell)}\eta^i_{(2\ell)}.
\end{aligned} \tag{4.12}$$

Note that $\mathcal{A}$ and $\hat{\mathcal{A}}$ are Grassmann even. Also $\bar{\eta}_{\hat{\imath}}^{(2\ell+1)}$, $\bar{\eta}_{i}^{(2\ell)}$, $\eta^{\hat{\imath}}_{(2\ell+1)}$ and $\eta^{i}_{(2\ell)}$ are Grassmann even, and so $\bar{F}_1$ and F_2 are Grassmann odd. To make $W_{1/2}$ SUSY invariant, we need [31]

$$\delta L_{1/2} = \mathcal{D}_\tau G \equiv \partial_\tau G + i[L_{1/2}, G], \tag{4.13}$$

for some Grassmann odd supermatrix

$$G = \begin{pmatrix} & \bar{G}_1 \\ G_2 & \end{pmatrix}. \tag{4.14}$$

Concretely, we need

$$\begin{aligned} \delta\mathcal{A} &= i(\bar{F}_1 G_2 - \bar{G}_1 F_2), \\ \delta\hat{\mathcal{A}} &= i(F_2 \bar{G}_1 - G_2 \bar{F}_1), \\ \delta\bar{F}_1 &= \mathcal{D}_\tau \bar{G}_1 \equiv \partial_\tau \bar{G}_1 + i\mathcal{A}\bar{G}_1 - i\bar{G}_1\hat{\mathcal{A}}, \\ \delta F_2 &= \mathcal{D}_\tau G_2 \equiv \partial_\tau G_2 + i\hat{\mathcal{A}} G_2 - iG_2\mathcal{A}. \end{aligned} \tag{4.15}$$

As in [13], we can use symmetry to guide the search for a 1/2 BPS Wilson loop. We break the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ R-symmetry to $\mathrm{U}(1)\times\mathrm{SU}(2)$ by writing $(i,\hat{\imath}) = (1,2,\hat{\imath})$. We wish to get a BPS Wilson loop with the SU(2) subgroup intact, and so we choose

$$\begin{aligned} &\bar{\eta}_i^{(2\ell)} = \bar{\eta}^{(2\ell)}\delta_i^1, && \eta^i_{(2\ell)} = \eta_{(2\ell)}\delta^i_1, && \bar{\eta}_{\hat{\imath}}^{(2\ell+1)} = \eta^{\hat{\imath}}_{(2\ell+1)} = 0, \\ &M^i{}_j = \mathrm{diag}(m_1, m_2), && M^{\hat{\imath}}{}_{\hat{\jmath}} = \mathrm{diag}(\hat{m}, \hat{m}), \\ &N_i{}^j = \mathrm{diag}(n_1, n_2), && N_{\hat{\imath}}{}^{\hat{\jmath}} = \mathrm{diag}(\hat{n}, \hat{n}). \end{aligned} \tag{4.16}$$

Then we need

$$\begin{aligned} \bar{G}_1 &= \mathrm{diag}(\bar{g}_1^{(0)}, \bar{g}_1^{(2)}, \cdots, \bar{g}_1^{(2n-2)}), \\ G_2 &= \mathrm{diag}(g_2^{(0)}, g_2^{(2)}, \cdots, g_2^{(2n-2)}). \end{aligned} \tag{4.17}$$

And then equations (4.15) become

$$\begin{aligned} \delta\mathcal{A}^{(2\ell+1)} &= i(\bar{f}_1^{(2\ell)} g_2^{(2\ell)} - \bar{g}_1^{(2\ell)} f_2^{(2\ell)}), \\ \delta\hat{\mathcal{A}}^{(2\ell)} &= i(f_2^{(2\ell)} \bar{g}_1^{(2\ell)} - g_2^{(2\ell)} \bar{f}_1^{(2\ell)}), \\ \delta\bar{f}_1^{(2\ell)} &= \mathcal{D}_\tau \bar{g}_1^{(2\ell)} \equiv \partial_\tau \bar{g}_1^{(2\ell)} + i\mathcal{A}^{(2\ell+1)}\bar{g}_1^{(2\ell)} - i\bar{g}_1^{(2\ell)}\hat{\mathcal{A}}^{(2\ell)}, \\ \delta f_2^{(2\ell)} &= \mathcal{D}_\tau g_2^{(2\ell)} \equiv \partial_\tau g_2^{(2\ell)} + i\hat{\mathcal{A}}^{(2\ell)} g_2^{(2\ell)} - ig_2^{(2\ell)}\mathcal{A}^{(2\ell+1)}. \end{aligned} \tag{4.18}$$

Without loss of generality, we suppose that

$$\gamma_0\theta^{1\hat{\imath}} = i\theta^{1\hat{\imath}}, \quad \hat{\imath} = \hat{1}, \hat{2}, \tag{4.19}$$

and then from (3.3) we have

$$\gamma_0\theta^{2\hat{\imath}} = -i\theta^{2\hat{\imath}}, \quad \bar{\theta}_{1\hat{\imath}}\gamma_0 = i\bar{\theta}_{1\hat{\imath}}, \quad \bar{\theta}_{2\hat{\imath}}\gamma_0 = -i\bar{\theta}_{2\hat{\imath}}. \tag{4.20}$$

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For ψ^2, $\psi^{\hat\imath}$ and $\bar\psi_2$, $\bar\psi_{\hat\imath}$ not appearing in $\delta\mathcal{A}^{(2\ell+1)}$ and $\delta\hat{\mathcal{A}}^{(2\ell)}$, we have to choose $m_1 = n_1 = -1$ and $m_2 = \hat m = n_2 = \hat n = 1$, and then we get

$$
\begin{aligned}
\delta\mathcal{A}^{(2\ell+1)} &= -\frac{8\pi i}{k}\left(\phi_{\hat\imath}^{(2\ell)}\bar\psi_1^{(2\ell)}\theta^{1\hat\imath} + \bar\theta_{1\hat\imath}\psi^1_{(2\ell)}\bar\phi^{\hat\imath}_{(2\ell)}\right), \\
\delta\hat{\mathcal{A}}^{(2\ell)} &= -\frac{8\pi i}{k}\left(\bar\psi_1^{(2\ell)}\phi_{\hat\imath}^{(2\ell)}\theta^{1\hat\imath} + \bar\theta_{1\hat\imath}\bar\phi^{\hat\imath}_{(2\ell)}\psi^1_{(2\ell)}\right).
\end{aligned} \tag{4.21}
$$

For $\delta\bar f_1^{(2\ell)}$ and $\delta f_2^{(2\ell)}$ satisfying the form of (4.18), we must choose

$$
\gamma_0\eta_{(2\ell)} = i\eta_{(2\ell)}, \quad \bar\eta^{(2\ell)}\gamma_0 = i\bar\eta^{(2\ell)}. \tag{4.22}
$$

Then we get

$$
\begin{aligned}
\delta\bar f_1^{(2\ell)} &= -i\sqrt{\frac{8\pi}{k}}\bar\eta^{(2\ell)}\theta^{1\hat\imath}\mathcal{D}_0\phi_{\hat\imath}^{(2\ell)}, \quad & \bar g_1^{(2\ell)} &= -i\sqrt{\frac{8\pi}{k}}\bar\eta^{(2\ell)}\theta^{1\hat\imath}\phi_{\hat\imath}^{(2\ell)}, \\
\delta f_2^{(2\ell)} &= i\sqrt{\frac{8\pi}{k}}\bar\theta_{1\hat\imath}\eta_{(2\ell)}\mathcal{D}_0\bar\phi^{\hat\imath}_{(2\ell)}, \quad & g_2^{(2\ell)} &= i\sqrt{\frac{8\pi}{k}}\bar\theta_{1\hat\imath}\eta_{(2\ell)}\bar\phi^{\hat\imath}_{(2\ell)}.
\end{aligned} \tag{4.23}
$$

One can show that, given[6]

$$
\eta_{(2\ell)}\bar\eta^{(2\ell)} = -i - \gamma_0, \tag{4.24}
$$

equations (4.18) are satisfied. It is similar for conformal SUSY transformation. Thus we get the 1/2 BPS Wilson loop along a timelike infinite straight line.

4.3 Relation between 1/4 and 1/2 BPS Wilson loops

We check

$$
W_{1/2} - W_{1/4} = QV, \tag{4.25}
$$

for some supercharge Q preserved by both $W_{1/4}$ and $W_{1/2}$ and some operator V. This is similar to the ABJM case in [13]. In the $\mathcal{N}=4$ SCSM theory we have the 1/4 and 1/2 BPS Wilson loops (4.8) and (4.9)

$$
\begin{aligned}
W_{1/4}(s,t) &= \mathcal{P}\exp\left(-i\int_t^s d\tau L_{1/4}(\tau)\right), \\
W_{1/2}(s,t) &= \mathcal{P}\exp\left(-i\int_t^s d\tau L_{1/2}(\tau)\right).
\end{aligned} \tag{4.26}
$$

In this subsection it is convenient to rearrange the rows and columns and rewrite

$$
\begin{aligned}
L_{1/4} &= \mathrm{diag}\left(L_{1/4}^{(0)}, L_{1/4}^{(1)}, \cdots, L_{1/4}^{(n-1)}\right), \\
L_{1/4}^{(\ell)} &= \begin{pmatrix} \mathcal{A}^{(2\ell+1)} & \\ & \hat{\mathcal{A}}^{(2\ell)} \end{pmatrix}, \\
L_{1/2} &= \mathrm{diag}\left(L_{1/2}^{(0)}, L_{1/2}^{(1)}, \cdots, L_{1/2}^{(n-1)}\right), \\
L_{1/2}^{(\ell)} &= \begin{pmatrix} \mathcal{A}^{(2\ell+1)} & \bar f_1^{(2\ell)} \\ f_2^{(2\ell)} & \hat{\mathcal{A}}^{(2\ell)} \end{pmatrix}.
\end{aligned} \tag{4.27}
$$

[6] We stress again that there are no summations of ℓ in this paper unless indicated explicitly.

Note that for $L_{1/4}$ there are $M^i{}_j = M^{\hat{\imath}}{}_{\hat{\jmath}} = N_i{}^j = N_{\hat{\imath}}{}^{\hat{\jmath}} = \mathrm{diag}(-1,1)$, and for $L_{1/2}$ there are $M^i{}_j = N_i{}^j = \mathrm{diag}(-1,1)$ and $M^{\hat{\imath}}{}_{\hat{\jmath}} = N_{\hat{\imath}}{}^{\hat{\jmath}} = \mathrm{diag}(1,1)$. Here $L_{1/2}^{(\ell)}$ with $\ell = 0, 1, \cdots, n-1$ can be thought as building blocks of the 1/2 BPS Wilson loop. Note that though $L_{1/2}^{(\ell)}$ with fixed ℓ only involves fermions $\psi^i_{(2\ell)}$ and $\bar{\psi}_i^{(2\ell)}$ which are in (anti-)bifundamental representation of $\mathrm{U}(N)_{(2\ell)} \times \mathrm{U}(N)_{(2\ell+1)}$, the involved scalar fields include not only $\phi_{\hat{\imath}}^{(2\ell)}, \bar{\phi}^{\hat{\imath}}_{(2\ell)}$, but also $\phi_i^{(2\ell\pm1)}, \bar{\phi}^i_{(2\ell\pm1)}$. In some sense, this construction is a kind of hybrid of 1/2 BPS Wilson loop in ABJM theory and 1/4 BPS Wilson loop in this $\mathcal{N}=4$ theory.

We also define

$$\begin{aligned} L_{1/2} - L_{1/4} = \tilde{L} &= \tilde{L}_B + \tilde{L}_F, \\ \tilde{L}_{1/2} &= L_{1/4} + \tilde{L}_B - \tilde{L}_F, \\ \tilde{W}_{1/2}(s,t) &= \mathcal{P}\exp\left(-i\int_t^s d\tau \tilde{L}_{1/2}(\tau)\right). \end{aligned} \tag{4.28}$$

Explicitly there are

$$\begin{aligned} \tilde{L}_B &= \mathrm{diag}\left(\tilde{L}_B^{(0)}, \tilde{L}_B^{(1)}, \cdots, \tilde{L}_B^{(n-1)}\right), \\ \tilde{L}_B^{(\ell)} &= \frac{4\pi}{k}\begin{pmatrix} \phi_{\hat{1}}^{(2\ell)}\bar{\phi}^{\hat{1}}_{(2\ell)} & \\ & \bar{\phi}^{\hat{1}}_{(2\ell)}\phi_{\hat{1}}^{(2\ell)} \end{pmatrix}, \\ \tilde{L}_F &= \mathrm{diag}\left(\tilde{L}_F^{(0)}, \tilde{L}_F^{(1)}, \cdots, \tilde{L}_F^{(n-1)}\right), \\ \tilde{L}_F^{(\ell)} &= \sqrt{\frac{2\pi}{k}}\begin{pmatrix} & \bar{\eta}^{(2\ell)}\psi^1_{(2\ell)} \\ \bar{\psi}_1^{(2\ell)}\eta_{(2\ell)} & \end{pmatrix}. \end{aligned} \tag{4.29}$$

Then we get

$$\begin{aligned} W_{1/2}(s,t) - W_{1/4}(s,t) &= -i\int_t^s d\tau\left[W_{1/4}(s,\tau)\tilde{L}(\tau)W_{1/2}(\tau,t)\right] \\ &= -i\int_t^s d\tau\left[W_{1/2}(s,\tau)\tilde{L}(\tau)W_{1/4}(\tau,t)\right]. \end{aligned} \tag{4.30}$$

We define

$$\begin{aligned} \Lambda &= \mathrm{diag}\left(\Lambda^{(0)}, \Lambda^{(1)}, \cdots, \Lambda^{(n-1)}\right), \\ \Lambda^{(\ell)} &= -\sqrt{\frac{2\pi}{k}}\begin{pmatrix} & \phi_{\hat{1}}^{(2\ell)} \\ \bar{\phi}^{\hat{1}}_{(2\ell)} & \end{pmatrix}, \end{aligned} \tag{4.31}$$

and a Grassmann odd operator

$$\begin{aligned} Q &= \mathrm{diag}\left(Q^{(0)}, Q^{(1)}, \cdots, Q^{(n-1)}\right), \\ Q^{(\ell)} &= \bar{\eta}^{(2\ell)}P^{1\hat{1}} + \bar{P}_{1\hat{1}}\eta_{(2\ell)}, \end{aligned} \tag{4.32}$$

with $P^{1\hat{1}}$ and $\bar{P}_{1\hat{1}}$ being Poncaré charges in (3.4). It can be checked that

$$Q\Lambda = \tilde{L}_F, \quad \alpha\Lambda^2 = \tilde{L}_B, \quad \alpha = 2. \tag{4.33}$$

Now we have the SUSY transformation

$$\begin{aligned}
&\delta W_{1/4}(s,t) = 0, \\
&\delta W_{1/2}(s,t) = -iG(s)W_{1/2}(s,t) + W_{1/2}(s,t)iG(t), \\
&\delta \tilde{W}_{1/2}(s,t) = iG(s)\tilde{W}_{1/2}(s,t) - \tilde{W}_{1/2}(s,t)iG(t),
\end{aligned} \tag{4.34}$$

with

$$\begin{aligned}
G &= \mathrm{diag}\left(G^{(0)}, G^{(1)}, \cdots, G^{(n-1)}\right), \\
G^{(\ell)} &= \begin{pmatrix} & \bar{g}_1^{(2\ell)} \\ g_2^{(2\ell)} & \end{pmatrix}.
\end{aligned} \tag{4.35}$$

Note that $\bar{g}_1^{(2\ell)}$ and $g_2^{(2\ell)}$ have been derived in (4.23). And then we have

$$\begin{aligned}
&QW_{1/4}(s,t) = 0, \\
&QW_{1/2}(s,t) = \alpha\Lambda(s)W_{1/2}(s,t) - \tilde{W}_{1/2}(s,t)\alpha\Lambda(t), \\
&Q\tilde{W}_{1/2}(s,t) = -\alpha\Lambda(s)\tilde{W}_{1/2}(s,t) + W_{1/2}(s,t)\alpha\Lambda(t).
\end{aligned} \tag{4.36}$$

Note that from (4.24) we have $\bar{\eta}^{(2\ell)}\eta_{(2\ell)} = -2i$ with no summation of ℓ.

We define

$$\begin{aligned}
S_1(s,t) &= -i\int_t^s d\tau \left[W_{1/4}(s,\tau)\Lambda(\tau)W_{1/2}(\tau,t)\right], \\
S_2(s,t) &= -i\int_t^s d\tau \left[W_{1/4}(s,\tau)\Lambda(\tau)\tilde{W}_{1/2}(\tau,t)\right], \\
S_3(s,t) &= -i\int_t^s d\tau \left[W_{1/2}(s,\tau)\Lambda(\tau)W_{1/4}(\tau,t)\right], \\
S_4(s,t) &= -i\int_t^s d\tau \left[\tilde{W}_{1/2}(s,\tau)\Lambda(\tau)W_{1/4}(\tau,t)\right].
\end{aligned} \tag{4.37}$$

We can show that

$$\begin{aligned}
&QS_1(s,t) = W_{1/2}(s,t) - W_{1/4}(s,t) - S_2(s,t)\alpha\Lambda(t), \\
&QS_4(s,t) = W_{1/2}(s,t) - W_{1/4}(s,t) - \alpha\Lambda(s)S_4(s,t).
\end{aligned} \tag{4.38}$$

For the infinite straight line, we have $s \to \infty$ and $t \to -\infty$, and we also assume $\Lambda(\pm\infty) = 0$. Then we get

$$W_{1/2} - W_{1/4} = QS_1 = QS_4. \tag{4.39}$$

Operator S_1 or S_4 is just the V we are looking for.

5 Straight line in Euclidean space

There are BPS Wilson loops along spacelike infinite straight lines in Euclidean space. Since they are similar to BPS Wilson loops along timelike infinite straight lines in Minkowski spacetime, and so it will be brief in this section.

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5.1 1/4 BPS Wilson loop

We use coordinates $x^\mu = (x^1, x^2, x^3)$ in Euclidean space. We have the 1/4 BPS Wilson loop along the infinite straight line $x^\mu = \tau\delta_1^\mu$ the same as (4.8) except that

$$M^i{}_j = M^{\hat{\imath}}{}_{\hat{\jmath}} = N_i{}^j = N_{\hat{\imath}}{}^{\hat{\jmath}} = \mathrm{diag}(i, -i). \tag{5.1}$$

The preserved Poncaré and conformal supersymmetries are

$$\begin{aligned} \gamma_1\theta^{1\hat{1}} &= \theta^{1\hat{1}}, \quad \gamma_1\theta^{2\hat{2}} = -\theta^{2\hat{2}}, \\ \gamma_1\vartheta^{1\hat{1}} &= \vartheta^{1\hat{1}}, \quad \gamma_1\vartheta^{2\hat{2}} = -\vartheta^{2\hat{2}}, \\ \theta^{1\hat{2}} &= \theta^{2\hat{1}} = \vartheta^{1\hat{2}} = \vartheta^{2\hat{1}} = 0. \end{aligned} \tag{5.2}$$

5.2 1/2 BPS Wilson loop

Also we have the 1/2 BPS Wilson loop along the infinite straight line $x^\mu = \tau\delta_1^\mu$ the same as (4.9) except that

$$\begin{aligned} &M^i{}_j = N_i{}^j = \mathrm{diag}(i, -i), \qquad M^{\hat{\imath}}{}_{\hat{\jmath}} = N_{\hat{\imath}}{}^{\hat{\jmath}} = \mathrm{diag}(-i, -i), \\ &\gamma_1\eta_{(2\ell)} = \eta_{(2\ell)}, \quad \bar{\eta}^{(2\ell)}\gamma_1 = \bar{\eta}^{(2\ell)}, \quad \eta_{(2\ell)}\bar{\eta}^{(2\ell)} = i(1+\gamma_1). \end{aligned} \tag{5.3}$$

The preserved supersymmetries are

$$\begin{aligned} \gamma_1\theta^{1\hat{\imath}} &= \theta^{1\hat{\imath}}, \quad \gamma_1\theta^{2\hat{\imath}} = -\theta^{2\hat{\imath}}, \\ \gamma_1\vartheta^{1\hat{\imath}} &= \vartheta^{1\hat{\imath}}, \quad \gamma_1\vartheta^{2\hat{\imath}} = -\vartheta^{2\hat{\imath}}, \end{aligned} \tag{5.4}$$

with $\hat{\imath} = \hat{1}, \hat{2}$.

5.3 Relation between 1/4 and 1/2 BPS Wilson loops

The check of $W_{1/2} - W_{1/4} = QV$ for a straight line in Euclidean space is similar to the case of a timelike straight line in Minkowski spacetime. The only differences are that

$$\begin{aligned} \tilde{L}_B^{(\ell)} &= -\frac{4\pi i}{k}\begin{pmatrix} \phi_{\hat{1}}^{(2\ell)}\bar{\phi}^{\hat{1}}_{(2\ell)} & \\ & \bar{\phi}^{\hat{1}}_{(2\ell)}\phi_{\hat{1}}^{(2\ell)} \end{pmatrix}, \\ \alpha &= -2i. \end{aligned} \tag{5.5}$$

6 Circle in Euclidean space

The $\mathcal{N} = 4$ SCSM theory is a superconformal theory, and a conformal transformation can change an infinite straight line to a circle. So there would be BPS circular Wilson loops if there exist BPS Wilson loops along infinite straight lines.

6.1 1/4 BPS Wilson loop

There is 1/4 circular BPS Wilson loop along $x^\mu = (\cos\tau, \sin\tau, 0)$ the same as (4.8) except that

$$M^i{}_j = M^{\hat{\imath}}{}_{\hat{\jmath}} = N_i{}^j = N_{\hat{\imath}}{}^{\hat{\jmath}} = \mathrm{diag}(i, -i). \tag{6.1}$$

The preserved Poncaré and conformal supersymmetries are

$$\begin{aligned} &\vartheta^{1\hat{1}} = i\gamma_3\theta^{1\hat{1}}, \quad \vartheta^{2\hat{2}} = -i\gamma_3\theta^{2\hat{2}}, \\ &\theta^{1\hat{2}} = \theta^{2\hat{1}} = \vartheta^{1\hat{2}} = \vartheta^{2\hat{1}} = 0. \end{aligned} \tag{6.2}$$

6.2 1/2 BPS Wilson loop

Also there is circular 1/2 BPS Wilson loop along $x^\mu = (\cos\tau, \sin\tau, 0)$ the same as (4.9) except that

$$\begin{aligned} M^i{}_j = N_i{}^j = \mathrm{diag}(i, -i), \qquad & M^{\hat{\imath}}{}_{\hat{\jmath}} = N_{\hat{\imath}}{}^{\hat{\jmath}} = \mathrm{diag}(-i, -i), \\ \bar{\eta}^{(2\ell)\alpha} = \bar{\beta}(e^{i\tau/2}, e^{-i\tau/2}), \qquad & \eta_{(2\ell)\alpha} = (e^{-i\tau/2}, e^{i\tau/2})\beta, \end{aligned} \tag{6.3}$$

with β, $\bar{\beta}$ being Grassmann even constants and satisfying $\beta\bar{\beta} = i$. Note that we have useful relations with no summations of ℓ

$$\eta_{(2\ell)}\bar{\eta}^{(2\ell)} = i\,(1 + \dot{x}^\mu\gamma_\mu)\,, \quad \bar{\eta}^{(2\ell)}\eta_{(2\ell)} = 2i. \tag{6.4}$$

Here we have antiperiodic boundary conditions

$$G(2\pi) = -G(0), \tag{6.5}$$

and so the gauge invariant Wilson loop is

$$\mathrm{Tr}W_{1/2}. \tag{6.6}$$

Now the preserved supersymmetries are

$$\vartheta^{1\hat{\imath}} = i\gamma_3\theta^{1\hat{\imath}}, \quad \vartheta^{2\hat{\imath}} = -i\gamma_3\theta^{2\hat{\imath}}, \tag{6.7}$$

with $\hat{\imath} = \hat{1}, \hat{2}$.

6.3 Relation between 1/4 and 1/2 BPS Wilson loops

The check of $W_{1/2} - W_{1/4} = QV$ for a circle in Euclidean space is different to the case of a straight line in Minkowski spacetime. Firstly we have the differences

$$\begin{aligned} \tilde{L}_B^{(\ell)} &= -\frac{4\pi i}{k}\begin{pmatrix} \phi_{\hat{1}}^{(2\ell)}\bar{\phi}^{\hat{1}}_{(2\ell)} & \\ & \bar{\phi}^{\hat{1}}_{(2\ell)}\phi_{\hat{1}}^{(2\ell)} \end{pmatrix}, \\ \Lambda^{(\ell)} &= -\sqrt{\frac{2\pi}{k}}e^{i\tau/2}\begin{pmatrix} & \phi_{\hat{1}}^{(2\ell)} \\ \bar{\phi}^{\hat{1}}_{(2\ell)} & \end{pmatrix}, \\ Q^{(\ell)} &= \bar{\zeta}^{(2\ell)}\left(P^{1\hat{1}} + i\gamma_3 S^{1\hat{1}}\right) + \left(\bar{P}_{1\hat{1}} + \bar{S}_{1\hat{1}}i\gamma_3\right)\zeta_{(2\ell)}, \\ \bar{\zeta}^{(2\ell)\alpha} &= \bar{\beta}(1,0), \quad \zeta_{(2\ell)\alpha} = (0,1)\beta, \quad \alpha = -2ie^{-i\tau}. \end{aligned} \tag{6.8}$$

Then, the construction of V is also different, since we have to treat the boundary terms carefully. The calculation is very involved, and so we collect them in appendix D.

7 Conclusion and discussion

In this paper, we have investigated the supersymmetric Wilson loops in $\mathcal{N} = 4$ SCSM theory. In Minkowski spacetime we have 1/2 BPS Wilson loops along null infinite straight lines, and 1/4 and 1/2 BPS Wilson loops along timelike infinite straight lines. In Euclidean space we have 1/4 and 1/2 Wilson loops along infinite straight lines, as well as circular 1/4 and 1/2 Wilson loops. We also gave a complete proof that the difference between 1/4 and 1/2 Wilson loops is Q-exact with Q being some supercharge that is preserved by both the 1/4 and 1/2 Wilson loops. On the gravity side, we also studied the probe M2-branes dual to half BPS circular Wilson loops in the fundamental representation, and give the holographic prediction of the VEV of this 1/2 BPS Wilson loops in the M-theory limit.

The VEV of the half-BPS circular Wilson loop in fundamental representation can be calculated using localization in the M-theory limit ($N \to \infty$ with k and n being fixed) based on results in [39]. It will be also interesting to compute the vacuum expectation values of these BPS Wilson loops beyond the M-theory limit. We can use the fermi gas approach [40, 41] to include all of the $1/N$ corrections. Similar to the ABJM case [42], it is interesting to study these Wilson loops in arbitrary representations. These results are in accordance with the gravity ones, and they will be presented in [43].

We think that the construction of DT type BPS Wilson loops here could be easily generalized to similar ones in $\mathcal{N} = 4$ theories obtained from orbifolding ABJ theory or $\mathcal{N} = 5$ theories in [29, 30]. It is an interesting question to study whether there exist DT type BPS Wilson loops in other $\mathcal{N} = 4$ theories [33, 34, 44–46][7] and $\mathcal{N} = 3$ theories [47–50]. As mentioned in the Introduction, in the latter case the DT type Wilson loops are believed to be at most 1/3 BPS [32].

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A Review of Wilson loops in ABJM theory

The ABJM theory is an $\mathcal{N} = 6$ SCSM theory, and it was constructed in [4]. ABJM theory has gauge group $\mathrm{U}(N) \times \mathrm{U}(N)$ and Chern-Simons levels $(k, -k)$, and the gauge fields are A_μ and $\hat{A}_\mu$ respectively. The complex scalar ϕ_I and Dirac spinor ψ_I are in $(N, \bar{N})$ bifundamental representation, and so $\bar{\phi}^I = \phi_I^\dagger$ and $\bar{\psi}_I = (\psi^I)^\dagger$ are in $(\bar{N}, N)$ representation. We adopt the convention of spinors in three-dimensional Minkowski spacetime and Euclidean

[7] This issue has been addressed recently in [28].

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space in [38]. We use $I, J, K, L, \cdots = 1, 2, 3, 4$ as indices of the SU(4) R-symmetry. A general SUSY transformation of ABJM theory is [30, 51–53]

$$
\begin{aligned}
\delta A_\mu &= \frac{4\pi}{k}\left(\phi_I \bar\psi_J \gamma_\mu \chi^{IJ} + \bar\chi_{IJ}\gamma_\mu \psi^J \bar\phi^I\right), \\
\delta \hat A_\mu &= \frac{4\pi}{k}\left(\bar\psi_J \gamma_\mu \phi_I \chi^{IJ} + \bar\chi_{IJ}\bar\phi^I \gamma_\mu \psi^J\right), \\
\delta\phi_I &= 2i\bar\chi_{IJ}\psi^J, \quad \delta\bar\phi^I = 2i\bar\psi_J\chi^{IJ}, \\
\delta\psi^I &= 2\gamma^\mu\chi^{IJ}D_\mu\phi_J + 2\vartheta^{IJ}\phi_J - \frac{4\pi}{k}\chi^{IJ}\left(\phi_J\bar\phi^K\phi_K - \phi_K\bar\phi^K\phi_J\right) - \frac{8\pi}{k}\chi^{KL}\phi_K\bar\phi^I\phi_L, \\
\delta\bar\psi_I &= -2\bar\chi_{IJ}\gamma^\mu D_\mu\bar\phi^J + 2\bar\vartheta_{IJ}\bar\phi^J + \frac{4\pi}{k}\bar\chi_{IJ}\left(\bar\phi^J\phi_K\bar\phi^K - \bar\phi^K\phi_K\bar\phi^J\right) + \frac{8\pi}{k}\bar\chi_{KL}\bar\phi^K\phi_I\bar\phi^L,
\end{aligned}
\tag{A.1}
$$

with $\chi^{IJ} = \theta^{IJ} + x^\mu\gamma_\mu\vartheta^{IJ}$ and $\bar\chi_{IJ} = \bar\theta_{IJ} - \bar\vartheta_{IJ}x^\mu\gamma_\mu$. The definitions of covariant derivatives are

$$
\begin{aligned}
D_\mu\phi_J &= \partial_\mu\phi_J + iA_\mu\phi_J - i\phi_J\hat A_\mu, \\
D_\mu\bar\phi^J &= \partial_\mu\bar\phi^J + i\hat A_\mu\bar\phi^J - i\bar\phi^J A_\mu.
\end{aligned}
\tag{A.2}
$$

Also θ^{IJ}, $\bar\theta_{IJ}$ and ϑ^{IJ}, $\bar\vartheta_{IJ}$ are Dirac spinors with constraint

$$
\begin{aligned}
\theta^{IJ} &= -\theta^{JI}, \quad (\theta^{IJ})^* = \bar\theta_{IJ}, \quad \bar\theta_{IJ} = \frac{1}{2}\epsilon_{IJKL}\theta^{KL}, \\
\vartheta^{IJ} &= -\vartheta^{JI}, \quad (\vartheta^{IJ})^* = \bar\vartheta_{IJ}, \quad \bar\vartheta_{IJ} = \frac{1}{2}\epsilon_{IJKL}\vartheta^{KL}.
\end{aligned}
\tag{A.3}
$$

Symbol ϵ_{IJKL} is totally antisymmetric with $\epsilon_{1234} = 1$. The θ, $\bar\theta$ terms denote Poncaré SUSY transformation, and ϑ, $\bar\vartheta$ terms denote conformal SUSY transformation. Note that we have $\delta A_\mu = \delta A_\mu^\dagger$, $\delta\hat A_\mu = \delta\hat A_\mu^\dagger$, $\delta\bar\phi^I = \delta\phi_I^\dagger$, and $\delta\bar\psi^I = \delta\psi_I^\dagger$.

For the Euclidean ABJM theory, the SUSY transformation is formally identical to (A.1), with $\chi^{IJ} = \theta^{IJ} + x^\mu\gamma_\mu\vartheta^{IJ}$ and $\bar\chi_{IJ} = \bar\theta_{IJ} - \bar\vartheta_{IJ}x^\mu\gamma_\mu$. But now equations (A.3) become

$$
\begin{aligned}
\theta^{IJ} &= -\theta^{JI}, \quad \bar\theta_{IJ} = \frac{1}{2}\epsilon_{IJKL}\theta^{KL}, \\
\vartheta^{IJ} &= -\vartheta^{JI}, \quad \bar\vartheta_{IJ} = \frac{1}{2}\epsilon_{IJKL}\vartheta^{KL}.
\end{aligned}
\tag{A.4}
$$

Note the twelve spinors θ^{IJ}, ϑ^{IJ} with $I, J = 1, 2, 3, 4$ are independent Dirac spinors.

In Minkowski spacetime, one has the 1/6 BPS Wilson loop along the timelike infinite straight line $x^\mu = \tau\delta_0^\mu$ [10–12]

$$
\begin{aligned}
W_{1/6} &= \mathcal{P}\exp\left(-i\int d\tau\mathcal{A}(\tau)\right), \\
\hat W_{1/6} &= \mathcal{P}\exp\left(-i\int d\tau\hat{\mathcal{A}}(\tau)\right), \\
\mathcal{A} &= A_\mu\dot x^\mu + \frac{2\pi}{k}M^I{}_J\phi_I\bar\phi^J|\dot x|, \\
\hat{\mathcal{A}} &= \hat A_\mu\dot x^\mu + \frac{2\pi}{k}N_I{}^J\bar\phi^I\phi_J|\dot x|, \\
M^I{}_J &= N_I{}^J = \mathrm{diag}(-1,-1,1,1).
\end{aligned}
\tag{A.5}
$$

Here $W_{1/6}$ and $\hat{W}_{1/6}$ can be combined to give the 1/6 BPS Wilson loop

$$\begin{aligned} W_{1/6} &= \mathcal{P}\exp\left(-i\int d\tau L_{1/6}(\tau)\right), \\ L_{1/6} &= \begin{pmatrix} \mathcal{A} & \\ & \hat{\mathcal{A}} \end{pmatrix}. \end{aligned} \tag{A.6}$$

The preserved Poncaré and conformal supersymmetries are

$$\begin{aligned} \gamma_0\theta^{12} = i\theta^{12}, &\quad \gamma_0\theta^{34} = -i\theta^{34}, \\ \theta^{13} = \theta^{14} &= \theta^{23} = \theta^{24} = 0, \\ \gamma_0\vartheta^{12} = i\vartheta^{12}, &\quad \gamma_0\vartheta^{34} = -i\vartheta^{34}, \\ \vartheta^{13} = \theta^{14} &= \vartheta^{23} = \vartheta^{24} = 0. \end{aligned} \tag{A.7}$$

Also one has the 1/2 BPS Wilson loop along the timelike infinite straight line $x^\mu = \tau\delta_0^\mu$ [13]

$$\begin{aligned} W_{1/2} &= \mathcal{P}\exp\left(-i\int d\tau L_{1/2}(\tau)\right), \\ L_{1/2} &= \begin{pmatrix} \mathcal{A} & \bar{f}_1 \\ f_2 & \hat{\mathcal{A}} \end{pmatrix}, \\ \mathcal{A} &= A_\mu \dot{x}^\mu + \frac{2\pi}{k} M^I{}_J \phi_I \bar{\phi}^J |\dot{x}|, \\ \hat{\mathcal{A}} &= \hat{A}_\mu \dot{x}^\mu + \frac{2\pi}{k} N_I{}^J \bar{\phi}^I \phi_J |\dot{x}|, \\ M^I{}_J &= N_I{}^J = \mathrm{diag}(-1,1,1,1), \\ \bar{f}_1 &= \sqrt{\frac{2\pi}{k}}\bar{\eta}_I \psi^I |\dot{x}|, \qquad f_2 = \sqrt{\frac{2\pi}{k}}\bar{\psi}_I \eta^I |\dot{x}|, \\ \bar{\eta}_I &= \bar{\eta}\delta_I^1, \quad \eta^I = \eta\delta_1^I, \\ \gamma_0\eta &= i\eta, \quad \bar{\eta}\gamma_0 = i\bar{\eta}, \quad \eta\bar{\eta} = -i - \gamma_0. \end{aligned} \tag{A.8}$$

The preserved Poncaré and conformal supersymmetries are

$$\begin{aligned} \gamma_0\theta^{1i} = i\theta^{1i}, &\quad \gamma_0\theta^{ij} = -i\theta^{ij}, \\ \gamma_0\vartheta^{1i} = i\vartheta^{1i}, &\quad \gamma_0\vartheta^{ij} = -i\vartheta^{ij}, \end{aligned} \tag{A.9}$$

with $i,j = 2,3,4$. One can use localization techniques to calculate the vacuum expectation value of the 1/6 BPS Wilson loop [19], and in order to generalize this to the 1/2 BPS Wilson loops [22, 23] one needs the relation between 1/6 and 1/2 BPS Wilson loops [13]

$$W_{1/2} - W_{1/6} = QV, \tag{A.10}$$

for some supercharge Q preserved by both $W_{1/6}$ and $W_{1/2}$ and some operator V. Operator V has been given in [13], and $W_{1/2} - W_{1/6} = QV$ can be checked for the first several orders.[8] Note that there is no spacelike BPS Wilson loop in Minkowski spacetime [38]. One has 1/2 BPS Wilson loops along null infinite straight lines.

In Euclidean space, we use coordinates $x^\mu = (x^1, x^2, x^3)$. One has the 1/6 BPS Wilson loop along the infinite straight line $x^\mu = \tau\delta_1^\mu$ the same as (A.6) except that

$$M^I{}_J = N_I{}^J = \mathrm{diag}(i, i, -i, -i). \tag{A.11}$$

The preserved Poncaré and conformal supersymmetries are

$$\begin{aligned} \gamma_1\theta^{12} = \theta^{12}, \quad \gamma_1\theta^{34} = -\theta^{34}, \\ \theta^{13} = \theta^{14} = \theta^{23} = \theta^{24} = 0, \\ \gamma_1\vartheta^{12} = \vartheta^{12}, \quad \gamma_1\vartheta^{34} = -\vartheta^{34}, \\ \vartheta^{13} = \theta^{14} = \vartheta^{23} = \vartheta^{24} = 0. \end{aligned} \tag{A.12}$$

Also one has the 1/2 BPS Wilson loop along the infinite straight line $x^\mu = \tau\delta_1^\mu$ the same as (A.8) except that

$$\begin{aligned} M^I{}_J = N_I{}^J = \mathrm{diag}(i, -i, -i, -i), \\ \gamma_1\eta = \eta, \quad \bar\eta\gamma_1 = \bar\eta, \quad \eta\bar\eta = i(1+\gamma_1). \end{aligned}$$

The preserved supersymmetries are

$$\begin{aligned} \gamma_1\theta^{1i} = \theta^{1i}, \quad \gamma_1\theta^{ij} = -\theta^{ij}, \\ \gamma_1\vartheta^{1i} = \vartheta^{1i}, \quad \gamma_1\vartheta^{ij} = -\vartheta^{ij}, \end{aligned}$$

with $i, j = 2, 3, 4$. The check of $W_{1/2} - W_{1/6} = QV$ for a straight line in Euclidean space is similar to the previous case.

Besides, in Euclidean space one has the circular 1/6 BPS Wilson loop along $x^\mu = (\cos\tau, \sin\tau, 0)$ the same as (A.6) except that

$$M^I{}_J = N_I{}^J = \mathrm{diag}(i, i, -i, -i). \tag{A.13}$$

The preserved Poncaré and conformal supersymmetries are

$$\begin{aligned} \vartheta^{12} = i\gamma_3\theta^{12}, \quad \vartheta^{34} = -i\gamma_3\theta^{34}, \\ \theta^{13} = \theta^{14} = \theta^{23} = \theta^{24} = 0, \\ \vartheta^{13} = \theta^{14} = \vartheta^{23} = \vartheta^{24} = 0. \end{aligned} \tag{A.14}$$

Also one has the circular 1/2 BPS Wilson loop along $x^\mu = (\cos\tau, \sin\tau, 0)$ the same as (A.8) except that

$$\begin{aligned} M^I{}_J = N_I{}^J = \mathrm{diag}(i, -i, -i, -i), \\ \bar\eta^\alpha = \bar\beta(e^{i\tau/2}, e^{-i\tau/2}), \quad \eta_\alpha = (e^{-i\tau/2}, e^{i\tau/2})\beta, \end{aligned} \tag{A.15}$$

[8] In fact, the localization is used to compute the BPS circular Wilson loops in the Euclidean space discussed in the following. The discussion on the relation of $W_{1/6}$ and $W_{1/2}$ here can be taken as a warm-up.

with β, $\bar{\beta}$ being constants and satisfying $\beta\bar{\beta} = i$. The preserved supersymmetries are

$$\vartheta^{1i} = i\gamma_3\theta^{1i}, \quad \vartheta^{ij} = -i\gamma_3\theta^{ij}, \tag{A.16}$$

with $i, j = 2, 3, 4$. There has been no general form of V in the check of $W_{1/2} - W_{1/6} = QV$, but there are V of the first several orders in [13].

B A simple proof of gauge covariance of Wilson lines

We have a general line in spacetime parameterized by $\tau \in [t, s]$. For gauge field $A(\tau)$ we define the Wilson line

$$W(s,t) = \mathcal{P}\exp\left(-i\int_t^s d\tau A(\tau)\right), \tag{B.1}$$

with $\mathcal{P}$ being path-ordering. For a general infinitesimal gauge transformation

$$\delta A \equiv D_\tau\Lambda = \partial_\tau\Lambda + i[A,\Lambda], \tag{B.2}$$

the Wilson line transforms as

$$\delta W(s,t) = -i\Lambda(s)W(s,t) + W(s,t)i\Lambda(t). \tag{B.3}$$

This gauge covariance of Wilson lines is well-known, one can see a complete proof in, for example, the textbook [54]. Here we give a simple proof using induction.

For $n \geq 0$ we define the symbols

$$\begin{aligned}
X_n(s,t) &= \int_t^s d\tau_1 \int_t^{\tau_1} d\tau_2 \cdots \int_t^{\tau_{n-1}} d\tau_n A_1 A_2 \cdots A_n, \\
Y_n(s,t) &= \int_t^s d\tau_1 \int_t^{\tau_1} d\tau_2 \cdots \int_t^{\tau_{n-1}} d\tau_n \left(\partial\Lambda_1 A_2 \cdots A_n + A_1 \partial\Lambda_2 A_3 \cdots A_n\right. \\
&\qquad\qquad \left. + \cdots + A_1 A_2 \cdots A_{n-1}\partial\Lambda_n\right), \\
Z_n(s,t) &= \int_t^s d\tau_1 \int_t^{\tau_1} d\tau_2 \cdots \int_t^{\tau_{n-1}} d\tau_n \left([A_1,\Lambda_1]A_2 \cdots A_n + A_1[A_2,\Lambda_2]A_3 \cdots A_n\right. \\
&\qquad\qquad \left. + \cdots + A_1 A_2 \cdots A_{n-1}[A_n,\Lambda_n]\right),
\end{aligned} \tag{B.4}$$

with the shorthand $A_i \equiv A(\tau_i)$, $\Lambda_i \equiv \Lambda(\tau_i)$ and $\partial\Lambda_i \equiv \partial_{\tau_i}\Lambda(\tau_i)$. Note that we have $X_0 = 1$ and $Y_0 = Z_0$=0. We have the relations

$$W = \sum_{n=0}^{+\infty}(-i)^n X_n, \quad \delta X_n = Y_n + iZ_n. \tag{B.5}$$

With the recursive relations for $n \geq 1$

$$\begin{aligned}
X_n(s,t) &= \int_t^s d\tau A(\tau) X_{n-1}(\tau,t), \\
Y_n(s,t) &= \int_t^s d\tau \left(\partial_\tau\Lambda(\tau)X_{n-1}(\tau,t) + A(\tau)Y_{n-1}(\tau,t)\right), \\
Z_n(s,t) &= \int_t^s d\tau \left([A(\tau),\Lambda(\tau)]X_{n-1}(\tau,t) + A(\tau)Z_{n-1}(\tau,t)\right),
\end{aligned} \tag{B.6}$$

we can use induction to prove

$$Y_{n+1}(s,t) - Z_n(s,t) = \Lambda(s) X_n(s,t) - X_n(s,t)\Lambda(t). \tag{B.7}$$

This leads to the infinitesimal version of the gauge covariance of the Wilson line (B.3).

We rewrite (B.3) as

$$\mathcal{P}\left(e^{-i\int_t^s d\tau A(\tau)} \int_t^s d\tau' D_{\tau'}\Lambda(\tau')\right) = \mathcal{P}\left(e^{-i\int_t^s d\tau A(\tau)}[\Lambda(s) - \Lambda(t)]\right). \tag{B.8}$$

Then it follows that more generally for $[t', s'] \subset [t, s]$ we can easily get

$$\mathcal{P}\left(e^{-i\int_t^s d\tau A(\tau)} \cdots \int_{t'}^{s'} d\tau' D_{\tau'}\Lambda(\tau') \cdots\right) = \mathcal{P}\left(e^{-i\int_t^s d\tau A(\tau)} \cdots [\Lambda(s') - \Lambda(t')] \cdots\right). \tag{B.9}$$

This is just the statement in [13] that one can integrate out the covariant derivative $D = d + iA$ term in the presence of path ordered $\exp(-i\int A)$.

C Alternative Wilson loops for a super connection

In this appendix we explore alternative definitions of Wilson loops for a super connection. The result is that we find no nontrivial ones.

A super connection L can be written as $L = B + F$ with Grassmann even part B being block diagonal and Grassmann odd part F being block off-diagonal

$$B = \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix}, \quad F = \begin{pmatrix} & F_1 \\ F_2 & \end{pmatrix}. \tag{C.1}$$

When defining the path-ordering for the Grassmann odd part F of the supermatrix L, we have ambiguities. We can do it as an ordinary matrix

$$\mathcal{P}F(\tau_1)F(\tau_2) = \begin{cases} F(\tau_1)F(\tau_2) & \tau_1 \geq \tau_2 \\ F(\tau_2)F(\tau_1) & \tau_1 < \tau_2 \end{cases}, \tag{C.2}$$

or we can define the super path-ordering as

$$\mathcal{SP}F(\tau_1)F(\tau_2) = \begin{cases} F(\tau_1)F(\tau_2) & \tau_1 \geq \tau_2 \\ -F(\tau_2)F(\tau_1) & \tau_1 < \tau_2 \end{cases}. \tag{C.3}$$

Note that when acting on two Grassmann even matrices, or one even matrix and one odd matrix, $\mathcal{SP}$ is no different with $\mathcal{P}$. Only when acting on two odd matrices, $\mathcal{SP}$ is different from $\mathcal{P}$ as shown above.

We can define the Wilson loop along a super connection L as that of an ordinary connection

$$W(s,t) = \mathcal{P}\exp\left(-i\int_t^s d\tau L(\tau)\right), \tag{C.4}$$

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and this is just what is done in main part of the paper. As shown in the last appendix, for a transformation

$$\delta L = \partial \Lambda + i[L, \Lambda], \tag{C.5}$$

with $\Lambda = \Sigma + \Xi$, Σ being Grassmann even and Ξ being odd, the Wilson loop transforms as

$$\delta W(s,t) = -i\Lambda(s)W(s,t) + W(s,t)i\Lambda(t). \tag{C.6}$$

This is the same as the case of an ordinary gauge filed. For super matrices there is also definition

$$[L, \Lambda\} = [B, \Sigma] + [B, \Xi] + [F, \Xi] + \{F, \Xi\}. \tag{C.7}$$

Note that for a transformation

$$\delta' L = \partial \Lambda + i[L, \Lambda\} = \delta L + 2i\Xi F, \tag{C.8}$$

the Wilson loop (C.4) transforms as

$$\delta' W(s,t) = -i\Lambda(s)W(s,t) + W(s,t)i\Lambda(t) + 2\mathcal{P}\left[\exp\left(-i\int_t^s d\tau L(\tau)\right)\int_t^s d\tau' \Xi(\tau')F(\tau')\right], \tag{C.9}$$

and so it is not covariant under $\delta' L$.

Alternatively, we can define the Wilson loop for a super connection L as

$$SW(s,t) = \mathcal{SP}\exp\left(-i\int_t^s d\tau L(\tau)\right), \tag{C.10}$$

with the super path-ordering defined in (C.3). We rewrite SW as

$$SW(s,t) = \mathcal{P}\left[\exp\left(-i\int_t^s d\tau B(\tau)\right)\mathcal{SP}\exp\left(-i\int_t^s d\tau F(\tau)\right)\right], \tag{C.11}$$

and then expand

$$\mathcal{SP}\exp\left(-i\int_t^s d\tau F(\tau)\right) = \sum_{n=0}^{+\infty}\frac{(-i)^n}{n!}T_n(s,t), \tag{C.12}$$

with

$$T_n(s,t) = \mathcal{SP}\int_t^s d\tau_1\int_t^{\tau_1} d\tau_2\cdots\int_t^{\tau_{n-1}} d\tau_n \sum_{\sigma\in S_n} F(\tau_{\sigma(1)})F(\tau_{\sigma(2)})\cdots F(\tau_{\sigma(n)}), \tag{C.13}$$

where S_n denotes order n permutation group. It is easy to see that $T_n = 0$ for $n \geq 2$. And then we get

$$SW(s,t) = \mathcal{P}\left[\exp\left(-i\int_t^s d\tau B(\tau)\right)\left(1 - i\int_t^s d\tau F(\tau)\right)\right]. \tag{C.14}$$

Thus the definition of SW (C.10) is trivial for several aspects.

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- For a loop, upon taken the trace Tr or super trace STr, there is no contribution from the block off-diagonal part,

$$\begin{aligned}\mathrm{Tr}SW &= \mathrm{Tr}\mathcal{P}\exp\left(-i\oint d\tau B(\tau)\right),\\ \mathrm{STr}SW &= \mathrm{STr}\mathcal{P}\exp\left(-i\oint d\tau B(\tau)\right).\end{aligned} \tag{C.15}$$

- Quantum mechanically, the Grassmann odd part will not contribute to the vacuum expectation value,

$$\langle SW\rangle = \left\langle \mathcal{P}\exp\left(-i\int d\tau B(\tau)\right)\right\rangle. \tag{C.16}$$

- Furthermore, it is not covariant under the transformation (C.5) or (C.8).

D A complete proof of 1/2 and 1/4 BPS Wilson loops differencing by a Q-exact term

The result here is general and applies not only to the $\mathcal{N}=4$ SCSM case, but also to the ABJM case. A complete proof of 1/2 and 1/6 BPS Wilson loops difference in ABJM theory being Q-exact is the same as what is presented here.

D.1 Some simplifications

First of all, let us repeat the problem that we are going to tackle and make some simplifications. We have a circle parameterized by $\tau\in[0,1]$ with $x^\mu(1)=x^\mu(0)$.[9] We have the 1/4 and 1/2 BPS Wilson loops

$$\begin{aligned}W_{1/4} &= \mathcal{P}\exp\left(-i\int d\tau L_{1/4}(\tau)\right),\\ W_{1/2} &= \mathcal{P}\exp\left(-i\int d\tau L_{1/2}(\tau)\right),\end{aligned} \tag{D.1}$$

with

$$L_{1/2} = L_{1/4}+\tilde{L} = L_{1/4}+\tilde{L}_B+\tilde{L}_F. \tag{D.2}$$

Here $L_{1/2}$ is a supermatrix, with $L_{1/4}+\tilde{L}_B$ being its Grassmann even block diagonal part and $\tilde{L}_F$ being its Grassmann odd block off-diagonal part. There is a Grassmann odd operator Q and a Grassmann even block off-diagonal matrix Λ. We have the relations

$$\begin{aligned}&Q\Lambda=\tilde{L}_F, \quad \kappa\Lambda^2=\tilde{L}_B, \qquad \kappa(1)=\kappa(0), \quad \Lambda(1)=-\Lambda(0),\\ &QL_{1/4}=0, \quad Q\tilde{L}_B=\{\tilde{L}_F,\kappa\Lambda\}, \quad Q\tilde{L}_F=\partial_\tau(i\kappa\Lambda)+i[L_{1/4},i\kappa\Lambda].\end{aligned} \tag{D.3}$$

Here the factor α has been redefined as κ. Note that $[\tilde{L}_B,\Lambda]=0$ has been used. We want to find some operators V and U that satisfy

$$QV(1,0) = W_{1/2}(1,0)-W_{1/4}(1,0)+i\kappa(1)\Lambda(1)U(1,0)+U(1,0)i\kappa(0)\Lambda(0). \tag{D.4}$$

[9] Note that this is different to what we have done before. For a circle we have used $\tau\in[0,2\pi]$.

Taking the trace we would have

$$QTrV(1,0) = TrW_{1/2}(1,0) - TrW_{1/4}(1,0). \tag{D.5}$$

We call this task I.

To avoid cluster of factors and indices, we make the following redefinitions

$$\begin{aligned} L_{1/4} \to iL_{1/4}, \quad L_{1/2} \to iL_{1/2}, \quad \tilde{L} \to iL, \quad \tilde{L}_B \to iB, \\ \tilde{L}_F \to iF, \quad \Lambda \to e^{i\pi/4}\Lambda, \quad Q \to e^{i\pi/4}Q. \end{aligned} \tag{D.6}$$

Then we have

$$\begin{aligned} W_{1/4} &= \mathcal{P}\exp\left(\int d\tau L_{1/4}(\tau)\right), \\ W_{1/2} &= \mathcal{P}\exp\left(\int d\tau L_{1/2}(\tau)\right), \end{aligned} \tag{D.7}$$

with

$$L_{1/2} = L_{1/4} + L = L_{1/4} + B + F. \tag{D.8}$$

We have the relations

$$\begin{aligned} Q\Lambda = F, \quad \kappa\Lambda^2 = B, \quad \kappa(1) = \kappa(0), \quad \Lambda(1) = -\Lambda(0), \\ QL_{1/4} = 0, \quad QB = \{F, \kappa\Lambda\}, \quad QF = \partial_\tau(\kappa\Lambda) - [L_{1/4}, \kappa\Lambda]. \end{aligned} \tag{D.9}$$

Our task is still to find some operators V and U that satisfy

$$QV(1,0) = W_{1/2}(1,0) - W_{1/4}(1,0) + \kappa(1)\Lambda(1)U(1,0) + U(1,0)\kappa(0)\Lambda(0). \tag{D.10}$$

We call this task II. Of course, it is equivalent to task I.

Furthermore, we may set $L_{1/4} = 0$ and redefine $W_{1/2} = W$ in task II. Now there are

$$\begin{aligned} W &= \mathcal{P}\exp\left(\int d\tau L(\tau)\right), \\ L &= B + F. \end{aligned} \tag{D.11}$$

We have the relations

$$\begin{aligned} Q\Lambda = F, \quad \kappa\Lambda^2 = B, \quad \kappa(1) = \kappa(0), \quad \Lambda(1) = -\Lambda(0), \\ QB = \{F, \kappa\Lambda\}, \quad QF = \partial_\tau(\kappa\Lambda). \end{aligned} \tag{D.12}$$

Our task is to find some operators V and U that satisfy

$$QV(1,0) = W(1,0) - 1 + \kappa(1)\Lambda(1)U(1,0) + U(1,0)\kappa(0)\Lambda(0). \tag{D.13}$$

We call it task III. It is a special case of task II, and so is easier.

D.2 Some definitions

Before tacking task III we make some formal definitions. We have a circle parameterized by $\tau \in [0,1]$ with $x^\mu(1) = x^\mu(0)$, and we will also use $s, t, \tau_1, \tau_2, \cdots$ to denote the parameter of the circle. We define two kinds of quantities on the circle.

- We call the first kind type 1, and a type 1 quantity has only one argument. Generally we denote them by lowercase Latin letters $a(\tau), b(\tau), c(\tau), \cdots$, or simply $a, b, c, \cdots$. Type 1 quantities below will include Λ, B, F, et al.

- The second kind is type 2, and a type 2 quantity has two arguments. We denote them generally by lowercase Greek letters $\alpha(s,t), \beta(s,t), \gamma(s,t), \cdots$, or simply $\alpha, \beta, \gamma, \cdots$. Note that $s \geq t$ is required. Type 2 quantities below will include W, W_n, V, V_n, U, U_n, Λ_{mn}, B_{mn}, F_{mn}, $S_{5,6,7,8}$, et al. We also define the identity type 2 quantity $I(s,t) = 1$.

We then define two kinds of operations $*$ and $\circ$. For two type 1 quantities a, b we define a type 2 quantity as

$$(a * b)(s,t) \equiv \int_t^s d\tau_1 \int_t^{\tau_1} d\tau_2 a(\tau_1) b(\tau_2) = \int_t^s d\tau_2 \int_{\tau_2}^s d\tau_1 a(\tau_1) b(\tau_2). \tag{D.14}$$

Note that $a * b \neq b * a$. For one type 1 quantity a and one type 2 quantity α we define type 2 quantities as

$$\begin{aligned}(\alpha * a)(s,t) &\equiv \int_t^s d\tau \alpha(s,\tau) a(\tau), \\ (a * \alpha)(s,t) &\equiv \int_t^s d\tau a(\tau) \alpha(\tau, t).\end{aligned} \tag{D.15}$$

Note that $\alpha * a \neq a * \alpha$. For two type 2 quantities α, β we do NOT define $\alpha * \beta$, and so it is illegal. For one type 1 quantity a and two type 2 quantity α, β, we define a type 2 quantity

$$(\alpha \circ a \circ \beta)(s,t) \equiv \int_t^s d\tau \alpha(s,\tau) a(\tau) \beta(\tau, t). \tag{D.16}$$

Note that symbol $\circ$ must appear in pair. We also define shorthand

$$(ab)(\tau) \equiv a(\tau) b(\tau), \quad (a\alpha)(s,t) \equiv a(s)\alpha(s,t), \quad (\alpha a)(s,t) \equiv \alpha(s,t) a(t). \tag{D.17}$$

Note that the shorthand is of the highest priority in calculation.

Under the above definitions there are some useful relations. There are

$$\begin{aligned}a * I = I * a, \quad (I * a) * b = a * b, \quad a * (I * b) = a * b, \\ I \circ a \circ \alpha = a * \alpha, \quad \alpha \circ a \circ I = \alpha * a.\end{aligned} \tag{D.18}$$

Note that $I * a \neq a$, because they are of different types. We can prove the associative relations

$$(a * b) * c = a * (b * c), \quad (a * \alpha) * b = a * (\alpha * b). \tag{D.19}$$

Also we know $a * (b * \alpha)$ and $(\alpha * a) * b$ are legal, and $(a * b) * \alpha$ and $\alpha * (a * b)$ are illegal. So we can write without ambiguity

$$a * b * c, \quad a * \alpha * b, \quad a * b * \alpha, \quad \alpha * a * b. \tag{D.20}$$

We can prove

$$a * (\alpha \circ b \circ \beta) = (a * \alpha) \circ b \circ \beta, \quad (\alpha \circ a \circ \beta) * b = \alpha \circ a \circ (\beta * b), \tag{D.21}$$

and so we can write directly

$$a * \alpha \circ b \circ \beta, \quad \alpha \circ a \circ \beta * b. \tag{D.22}$$

For the shorthand (D.17) we have useful relations

$$\begin{gathered} \alpha * ab = \alpha a * b, \quad ab * \alpha = a * b\alpha, \\ \alpha a \circ b \circ \beta = \alpha \circ ab \circ \beta = \alpha \circ a \circ b\beta. \end{gathered} \tag{D.23}$$

D.3 The main part

We tackle task III in this subsection and subsections D.4 and D.5. As done in [13], we expand (D.11) in powers of B and F. We set F to be of order one, and B to be of order two. And then we have

$$W = \sum_{n=0}^{+\infty} W_n, \tag{D.24}$$

with the first few orders being

$$\begin{aligned} & W_0 = I, \quad W_1 = I * F, \quad W_2 = I * B + F * F, \quad W_3 = B * F + F * B + F * F * F, \\ & W_4 = B * B + B * F * F + F * B * F + F * F * B + F * F * F * F. \end{aligned} \tag{D.25}$$

It can be seen that our definitions simplify these formulas significantly. There is recursive relation for $n \geq 2$

$$W_n = B * W_{n-2} + F * W_{n-1} = W_{n-2} * B + W_{n-1} * F. \tag{D.26}$$

Note that we can define $W_{-1} = 0$ to make the above equations apply to $n \geq 1$.

We also define

$$V = \sum_{n=1}^{+\infty} V_n, \quad U = \sum_{n=1}^{+\infty} U_n. \tag{D.27}$$

And then task III (D.13) becomes to find V_n and U_n for $n \geq 1$ to satisfy

$$QV_n = W_n + \kappa\Lambda U_n + U_n \kappa\Lambda. \tag{D.28}$$

We define

$$\tilde{W} = \mathcal{P} \exp\left[\int d\tau \, (B - F)\right], \tag{D.29}$$

and then we get

$$\tilde{W} = \sum_{n=0}^{+\infty} (-)^n W_n. \tag{D.30}$$

From (D.12) we have

$$QW = \kappa\Lambda W - \tilde{W}\kappa\Lambda, \quad Q\tilde{W} = -\kappa\Lambda\tilde{W} + W\kappa\Lambda, \tag{D.31}$$

and then for $n \geq 1$ we get

$$QW_n = \kappa\Lambda W_{n-1} + (-)^n W_{n-1}\kappa\Lambda. \tag{D.32}$$

Note that $W_0 = I$, $QW_0 = 0$, and we have set $W_{-1} = 0$. And then the above equation applies to $n \geq 0$.

For $m \geq -1$ and $n \geq -1$ we define

$$\Lambda_{mn} \equiv W_m \circ \Lambda \circ W_n, \quad B_{mn} \equiv W_m \circ B \circ W_n, \quad F_{mn} \equiv W_m \circ F \circ W_n. \tag{D.33}$$

Note that we have $\Lambda_{mn} = 0$ for $m = -1$ or $n = -1$, and it is similar to B_{mn} and F_{mn}. Because of (D.26), for $m \geq 1$ we have

$$\begin{aligned} B_{mn} &= B * B_{m-2,n} + F * B_{m-1,n}, \\ F_{mn} &= B * F_{m-2,n} + F * F_{m-1,n}, \end{aligned} \tag{D.34}$$

and for $n \geq 1$ we have

$$\begin{aligned} B_{mn} &= B_{m,n-2} * B + B_{m,n-1} * F, \\ F_{mn} &= F_{m,n-2} * B + F_{m,n-1} * F. \end{aligned} \tag{D.35}$$

And we can show

$$Q\Lambda_{mn} = (-)^m \left(B_{m-1,n} + B_{m,n-1} + F_{mn}\right) + \kappa\Lambda\Lambda_{m-1,n} + (-)^{m+n}\Lambda_{m,n-1}\kappa\Lambda. \tag{D.36}$$

Then we claim that the V_n and U_n with $n \geq 1$ we want in (D.28) are that for $k \geq 0$

$$\begin{aligned} V_{2k+1} &= \sum_{i=0}^{2k} \Lambda_{i,2k-i}, \quad U_{2k+1} = \sum_{i=0}^{2k-1} \Lambda_{i,2k-1-i}, \\ V_{2k+2} &= \frac{1}{2k+2} \sum_{i=0}^{2k+1} (-)^i \Lambda_{i,2k+1-i}, \\ U_{2k+2} &= -\frac{1}{2k+2} \sum_{i=0}^{2k} (-)^i \Lambda_{i,2k-i}. \end{aligned} \tag{D.37}$$

We will prove this claim using two different methods in subsections D.4 and D.5. The first few orders are

$$\begin{aligned}
&V_1 = I * \Lambda, \quad U_1 = 0, \quad V_2 = \frac{1}{2}(\Lambda * F - F * \Lambda), \quad U_2 = -\frac{1}{2} I * \Lambda, \\
&V_3 = \Lambda * B + B * \Lambda + \Lambda * F * F + F * \Lambda * F + F * F * \Lambda, \quad U_3 = \Lambda * F + F * \Lambda, \\
&V_4 = \frac{1}{4}(\Lambda * B * F - F * B * \Lambda + B * \Lambda * F - B * F * \Lambda + \Lambda * F * B - F * \Lambda * B \\
&\qquad + \Lambda * F * F * F - F * \Lambda * F * F + F * F * \Lambda * F - F * F * F * \Lambda), \\
&U_4 = -\frac{1}{4}(\Lambda * B + B * \Lambda + \Lambda * F * F - F * \Lambda * F + F * F * \Lambda).
\end{aligned} \tag{D.38}$$

Note that $V_{1,2,3}$ are just the ones given in [13]. Here we have gone further, and give a general expression of V_n and U_n for all integers $n \geq 1$. Besides, we will give a general proof of (D.28). There are two methods of doing so, and then we split the main part to two branches.

D.4 The first branch

In the first branch we use (D.36) and get for $k \geq 0$

$$\begin{aligned}
QV_{2k+1} &= \sum_{i=0}^{2k} (-)^i F_{i,2k-i} + \kappa \Lambda U_{2k+1} + U_{2k+1} \kappa \Lambda, \\
QV_{2k+2} &= \frac{1}{2k+2} \left(2 \sum_{i=0}^{2k} B_{i,2k-i} + \sum_{i=0}^{2k+1} F_{i,2k+1-i} \right) + \kappa \Lambda U_{2k+2} + U_{2k+2} \kappa \Lambda.
\end{aligned} \tag{D.39}$$

Thus to prove (D.28) we need for $k \geq 0$

$$\begin{aligned}
\sum_{i=0}^{2k} (-)^i F_{i,2k-i} &= W_{2k+1}, \\
2 \sum_{i=0}^{2k} B_{i,2k-i} + \sum_{i=0}^{2k+1} F_{i,2k+1-i} &= (2k+2) W_{2k+2},
\end{aligned} \tag{D.40}$$

the first few orders of which can be verified easily. Furthermore, for $k \geq 0$ we can prove the above two equations and the following two equations

$$\begin{aligned}
\sum_{i=0}^{2k+1} (-)^i F_{i,2k+1-i} &= 0, \\
2 \sum_{i=0}^{2k+1} B_{i,2k+1-i} + \sum_{i=0}^{2k+2} F_{i,2k+2-i} &= (2k+3) W_{2k+3},
\end{aligned} \tag{D.41}$$

using induction. In the process (D.26) and (D.34) are used. Thus the proof of (D.28) is done.

D.5 The second branch

In the second branch, we firstly define

$$\begin{aligned} S_5 &= W \circ \Lambda \circ W, \quad S_6 = \tilde{W} \circ \Lambda \circ \tilde{W}, \\ S_7 &= \tilde{W} \circ \Lambda \circ W, \quad S_8 = W \circ \Lambda \circ \tilde{W}. \end{aligned} \tag{D.42}$$

Then we can show

$$\begin{aligned} S_5 + S_6 &= 2\sum_{k=0}^{+\infty} V_{2k+1}, & S_5 - S_6 &= 2\sum_{k=0}^{+\infty} U_{2k+1}, \\ S_7 + S_8 &= -2\sum_{k=0}^{+\infty}(2k+2)U_{2k+2}, & S_7 - S_8 &= 2\sum_{k=0}^{+\infty}(2k+2)V_{2k+2}. \end{aligned} \tag{D.43}$$

Note that $U_1 = 0$ has been used. From (D.31) we get

$$\begin{aligned} QS_5 &= \tilde{W} \circ F \circ W + \kappa\Lambda S_5 - S_6\kappa\Lambda, \\ QS_6 &= W \circ F \circ \tilde{W} - \kappa\Lambda S_6 + S_5\kappa\Lambda, \\ QS_7 &= W \circ (2B + F) \circ W - \kappa\Lambda S_7 - S_8\kappa\Lambda, \\ QS_8 &= \tilde{W} \circ (-2B + F) \circ \tilde{W} + \kappa\Lambda S_8 + S_7\kappa\Lambda. \end{aligned} \tag{D.44}$$

And then using (D.24) and (D.30), as well the results (D.40) in the first branch, we can get

$$\begin{aligned} Q(S_5 + S_6) &= 2\sum_{k=0}^{+\infty} W_{2k+1} + \kappa\Lambda(S_5 - S_6) + (S_5 - S_6)\kappa\Lambda, \\ Q(S_7 - S_8) &= 2\sum_{k=0}^{+\infty}(2k+2)W_{2k+2} - \kappa\Lambda(S_7 + S_8) - (S_7 + S_8)\kappa\Lambda. \end{aligned} \tag{D.45}$$

Thus for $k \geq 0$ we get

$$\begin{aligned} QV_{2k+1} &= W_{2k+1} + \kappa\Lambda U_{2k+1} + U_{2k+1}\kappa\Lambda, \\ QV_{2k+2} &= W_{2k+2} + \kappa\Lambda U_{2k+2} + U_{2k+2}\kappa\Lambda. \end{aligned} \tag{D.46}$$

This is indeed just (D.28). However, we note that neither V or U can be written directly as combinations of $S_{5,6,7,8}$.

D.6 Back to the main part

Now we turn back to the main part, with task III being completed. From task III to task II we need to use (B.9) and just make some simple changes in the proof of task III. All the type 1 quantities do not change, and every type 2 quantity changes as

$$\alpha(s,t) = \mathcal{P}\left[W_{1/4}(s,t)\underline{\alpha}(s,t)\right], \tag{D.47}$$

with $\underline{\alpha}$ being the old type 2 quantity and α being the new one. Especially, the "identity" type 2 quantity becomes

$$I(s,t) \to W_{1/4}(s,t). \tag{D.48}$$

For type 1 quantities $a, b, c, \cdots$ and the old type 2 quantities $\underline{\alpha}, \underline{\beta}, \underline{\gamma}, \cdots$, there are still $*$ and $\circ$ operations defined as before. For type 1 quantities $a, b, c, \cdots$ and the new type 2 quantities $\alpha, \beta, \gamma, \cdots$ we define $\circledast$ and $\circledcirc$ operations as

$$\begin{aligned}(a \circledast b)(s,t) &\equiv \mathcal{P}\left[W_{1/4}(s,t)(a * b)(s,t)\right], \\ (a \circledast \alpha)(s,t) &\equiv \mathcal{P}\left[W_{1/4}(s,t)(a * \underline{\alpha})(s,t)\right], \\ (\alpha \circledast a)(s,t) &\equiv \mathcal{P}\left[W_{1/4}(s,t)(\underline{\alpha} * a)(s,t)\right], \\ (\alpha \circledcirc a \circledcirc \beta)(s,t) &\equiv \mathcal{P}\left[W_{1/4}(s,t)(\underline{\alpha} \circ a \circ \underline{\beta})(s,t)\right].\end{aligned} \tag{D.49}$$

Now we have shorthand

$$\begin{aligned}(a\alpha)(s,t) &\equiv \mathcal{P}\left[W_{1/4}(s,t)a(s)\underline{\alpha}(s,t)\right], \\ (\alpha a)(s,t) &\equiv \mathcal{P}\left[W_{1/4}(s,t)\underline{\alpha}(s,t)a(t)\right].\end{aligned} \tag{D.50}$$

Keeping the above changes in mind, we can tackle task II with few efforts. For example, we need to change (D.24) to

$$W_{1/2} - W_{1/4} = \sum_{n=1}^{+\infty} W_n, \tag{D.51}$$

with W_n being changed as (D.47). And (D.29) is changed to

$$\tilde{W}_{1/2} = \mathcal{P}\exp\left[\int d\tau \left(L_{1/4} + B - F\right)\right]. \tag{D.52}$$

In the second branch equations (D.42) are changed to

$$\begin{aligned}S_5 &= W_{1/2} \circledcirc \Lambda \circledcirc W_{1/2}, \quad S_6 = \tilde{W}_{1/2} \circledcirc \Lambda \circledcirc \tilde{W}_{1/2}, \\ S_7 &= \tilde{W}_{1/2} \circledcirc \Lambda \circledcirc W_{1/2}, \quad S_8 = W_{1/2} \circledcirc \Lambda \circledcirc \tilde{W}_{1/2}.\end{aligned} \tag{D.53}$$

Thus task II is completed following task III and (B.9). Since task I is equivalent to task II, task I is competed too.

In summary we have given a complete proof that the difference of circular 1/2 and 1/4 BPS Wilson loops in this $\mathcal{N} = 4$ SCSM theory is Q-exact, with Q being some supercharge that is preserved by the both the 1/2 and 1/4 BPS Wilson loops. As we have stated, this proof also applies to the 1/2 and 1/6 BPS Wilson loops in ABJM theory.

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