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Supersymmetry and the spontaneous breakdown of $\mathcal{PT}$ symmetry

Patrick Dorey[1], Clare Dunning[2] and Roberto Tateo[3]

[1,3] *Dept. of Mathematical Sciences, University of Durham, Durham DH1 3LE, UK*

[2] *Dept. of Mathematics, University of York, York YO10 5DD, UK*

Abstract

The appearances of complex eigenvalues in the spectra of $\mathcal{PT}$-symmetric quantum-mechanical systems are usually associated with a spontaneous breaking of $\mathcal{PT}$. In this letter we discuss a family of models for which this phenomenon is also linked with an explicit breaking of supersymmetry. Exact level-crossings are located, and connections with $\mathcal{N}$-fold supersymmetry and quasi-exact solvability in certain special cases are pointed out.

[1] p.e.dorey@durham.ac.uk
[2] tcd1@york.ac.uk
[3] roberto.tateo@durham.ac.uk

1. In a recent paper [1], we discussed the spectra of the following family of $\mathcal{PT}$-symmetric eigenvalue problems:

$$\left[-\frac{d^2}{dx^2} - (ix)^{2M} - \alpha(ix)^{M-1} + \frac{l(l+1)}{x^2}\right]\psi(x) = \lambda\,\psi(x)\,, \quad \psi(x) \in L^2(\mathcal{C})\,. \qquad (1)$$

The wavefunction $\psi(x)$ is required to be square-integrable on the contour $\mathcal{C}$. For $M < 2$ this can be the real axis, while for $M \geq 2$, it should be distorted down into the complex x-plane, as in [2] for the special case $\alpha = l(l+1) = 0$.

The principal interest in $\mathcal{PT}$-symmetric problems lies in the fact that, despite not being in any obvious sense Hermitian, they often appear to have spectra which are entirely real [2–12]. In appendix B of [1], a *proof* of this property was given for the class of Hamiltonians (1), drawing on ideas from the so-called 'ODE/IM correspondence' [13–17] (see also [18]). More precisely, it was shown that, for M, α and l real and $M > 1$, the spectrum of (1) is

- $\quad$ *real* $\quad$ if $\quad \alpha < M + 1 + |2l+1|$; $\hspace{3cm}$ (2)
- $\quad$ *positive* if $\quad \alpha < M + 1 - |2l+1|$. $\hspace{3cm}$ (3)

Referring to figure 1, reality was proved for $(\alpha, l) \in B \cup C \cup D$, and positivity for $(\alpha, l) \in D$.
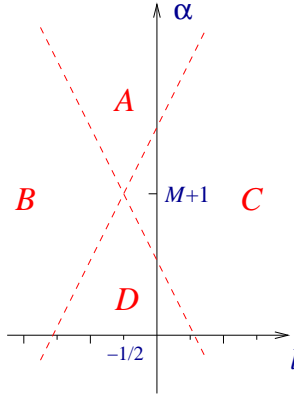

Figure 1: The initial 'phase diagram' at fixed M.

Special cases of this result were the subject of a number of previous conjectures: for $\alpha{=}l{=}0$, $M{=}3/2$ in [3]; for $\alpha{=}l{=}0$ in [2]; and for $\alpha{=}0$, l small in [15]. More unexpected was the possible loss of reality in the region A. The condition (2) arose for technical reasons in the proof in [1], though it was verified that at some (but not all) points inside A, reality did indeed break down. But the reason for this breakdown, and the physical significance – if any – of the boundaries of the region A, were left obscure.

In this note we show that the appearances of complex eigenvalues as the lines $\alpha = M + 1 + |2l+1|$ are crossed are directly related to the fact that on these lines the problem (1) can be reformulated in terms of a $\mathcal{PT}$-symmetric version of supersymmetric quantum mechanics, and exhibits level-crossing. Elsewhere in the phase diagram,

supersymmetry is explicitly broken and the level-crossing is lifted. Moving into the region A, this lifting occurs through the creation of a pair of complex-conjugate energy levels. Since the presence of complex energy levels indicates the spontaneous breaking of $\mathcal{PT}$ symmetry [2], we see that the model shows an interesting interplay between the two different ways in which a symmetry can be broken.

2. To streamline some formulae, we replace x by x/i in (1) by setting $\Phi(x) = \psi(x/i)$, so that the eigenproblem becomes

$$\left[-\frac{d^2}{dx^2} + x^{2M} + \alpha x^{M-1} + \frac{l(l+1)}{x^2}\right]\Phi(x) = -\lambda\,\Phi(x)\,, \quad \Phi(x) \in L^2(i\,\mathcal{C})\,. \qquad (4)$$

We now specify the quantization contour more precisely: $i\,\mathcal{C}$ starts and ends at $|x| = \infty$, joining the (Stokes) sectors $\mathcal{S}_{-1}$ and $\mathcal{S}_1$, where

$$\mathcal{S}_k = \left\{ x : \left|\arg(x) - \frac{2\pi k}{2M+2}\right| < \frac{\pi}{2M+2} \right\}\,. \qquad (5)$$
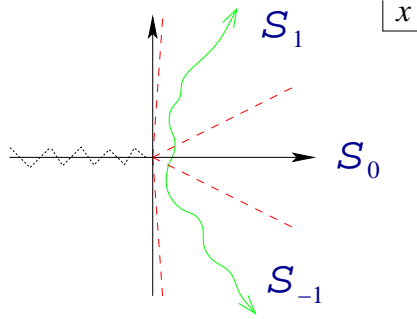
This is illustrated in figure 2.

Figure 2: The quantization contour $i\,\mathcal{C}$ (arrowed line).

It will also be convenient to adopt a new set of coordinates on the (α, l) plane, by setting

$$\alpha_\pm = \frac{1}{2M+2}\left[\alpha - M - 1 \pm (2l+1)\right] \qquad (6)$$

so that

$$\alpha = (M+1)(1 + \alpha_+ + \alpha_-)\,, \quad 2l + 1 = (M+1)(\alpha_+ - \alpha_-)\,. \qquad (7)$$

The domain boundaries – the dotted lines on figure 1 – are the lines $\alpha_\pm = 0$. On the line $\alpha_+ = 0$, $\alpha = M - 2l$ and the problem (4) factorises as

$$\mathcal{Q}_+\mathcal{Q}_-\Phi(x) = -\lambda\,\Phi(x)\,, \quad \text{with } \mathcal{Q}_\pm = \left[\pm\frac{d}{dx} + x^M - \frac{l}{x}\right]\,. \qquad (8)$$

Such a factorisation is usually taken to signal a relationship with supersymmetry [19,20]. Indeed, it is immediately seen that (8) has a $\lambda = 0$ eigenfunction in $L^2(i\mathcal{C})$:

$$\Psi(x) = x^{-l}\exp\left(\frac{1}{M+1}x^{M+1}\right)\,, \quad \mathcal{Q}_-\Psi(x) = 0\,, \qquad (9)$$

which can be interpreted as having unbroken supersymmetry. All other eigenfunctions are paired with those of the SUSY partner Hamiltonian $\widehat{\mathcal{H}} = \mathcal{Q}_-\mathcal{Q}_+$. This is found by replacing $(\alpha,l) = (M-2l,l)$ by $(\widehat{\alpha},\widehat{l}) = (-M-2l,l-1)$, or $(\alpha_+,\alpha_-) = (0,-\frac{2l+1}{M+1})$ by $(\widehat{\alpha}_+,\widehat{\alpha}_-) = (-1,-1-\frac{2l-1}{M+1}) = (-1,\alpha_--\frac{M-1}{M+1})$. (A rather different point of view on supersymmetry in $\mathcal{PT}$-symmetric quantum mechanics was taken in [7], where the action of the supersymmetry generators $\mathcal{Q}_\pm$ was supplemented by $\mathcal{T}$, resulting in a set-up with two states of zero energy, rather than one.) Replacing $-l$ by $l+1$ gives a $\lambda = 0$ eigenfunction on the line $\alpha_- = 0$. Thus the boundaries between the regions on figure 1 are picked out by the presence of a (supersymmetric) zero-energy state in the spectrum of the model. One would normally expect this to be the ground state, and indeed this is the case on the boundary of D. However, level-crossing means that $\Psi(x)$ is only the ground state on the boundary of A for $\alpha < M+3$. There is no contradiction with the usual theorems of supersymmetric quantum mechanics, since the problem under discussion is not Hermitian.

To verify that level-crossing does occur, we consider $T(0)$, where $T(-\lambda)$ is a spectral determinant which vanishes if and only if (4) has a solution, square-integrable on $i\mathcal{C}$, at that value of λ. In [1], an expression for $T(0)$ was found. In the 'light-cone' coordinates $\alpha_\pm$, this is

$$T(-\lambda,\alpha_+,\alpha_-)|_{\lambda=0} = \left(\tfrac{M+1}{2}\right)^{1+\alpha_++\alpha_-} \frac{2\pi}{\Gamma\left(-\alpha_+\right)\Gamma\left(-\alpha_-\right)}. \tag{10}$$

As expected from supersymmetry, $T|_{\lambda=0}$ is identically zero when either α_+ or α_- vanishes, on account of the presence of the state Ψ in the spectrum. Level-crossing will occur when a further level passes through zero, but the presence of Ψ makes this hard to detect from an examination of $T(-\lambda,\alpha_+,\alpha_-)|_{\lambda=0}$ alone. It is tempting suppose that the level-crossings happen at the double zeroes of $T|_{\lambda=0}$. Tempting, but wrong, for reasons which may become clearer when figures 3 and 4 below are examined. Better is to consider the SUSY partner potential to (8), which is isospectral to it save for the elimination of the state Ψ. Substituting the values of $\widehat{\alpha}_+$ and $\widehat{\alpha}_-$ into (10),

$$T(-\lambda,\widehat{\alpha}_+ = -1,\widehat{\alpha}_-)|_{\lambda=0} = \left(\tfrac{M+1}{2}\right)^{\widehat{\alpha}_-} \frac{2\pi}{\Gamma\left(-\widehat{\alpha}_-\right)} = \left(\tfrac{M+1}{2}\right)^{\widehat{\alpha}_-} \frac{2\pi}{\Gamma\left(\frac{M-1}{M+1}-\alpha_-\right)}. \tag{11}$$

Level-crossings with the state Ψ are indicated by simple zeroes of (11), and are at

$$(\alpha_+,\alpha_-) = (0,n+\tfrac{M-1}{M+1}), \qquad n = 0,1,\ldots. \tag{12}$$

(Had we looked instead for double zeroes of (10), we would have predicted – incorrectly – the values $\alpha_- = n$.) Swapping α_+ and α_- throughout gives the level-crossings on the line $\alpha_- = 0$. Note that these level-crossings are *exact* – the state Ψ is protected by supersymmetry, and cannot mix with any other state, even as the level-crossing value of l is approached. However, as soon as the supersymmetric line is left, the protection is lost and mixing does occur. This is the key point, showing why the boundaries of the

region A are of more than just technical significance. Since the model is $\mathcal{PT}$-symmetric, all energies are either real or occur in complex-conjugate pairs [2], and a real energy can only become complex if it first pairs off with another real energy. What we have just shown is that the hidden supersymmetry of the theory on the lines $\alpha_+ = 0$ and $\alpha_- = 0$ affords a mechanism for this pairing-off to occur, by permitting eigenvalues to become exactly-degenerate.

3. To verify this picture, we report some numerical data for $M = 3$. This value is chosen principally for convenience, since at $M = 3$ the fifth spectral equivalence discussed in [1] allows the 'lateral' eigenvalue problem (4), defined on the contour $i\mathcal{C}$, to be mapped onto a radial problem defined on the positive real axis. (Note that this mapping does not rule out the appearance of complex eigenvalues of (4), since at such points the corresponding radial problem has 'irregular' boundary conditions at the origin and is not self-adjoint.) The merit of the radial problem is that it is straightforward to treat numerically, using (for example) the MAPLE code in appendix A of [1].
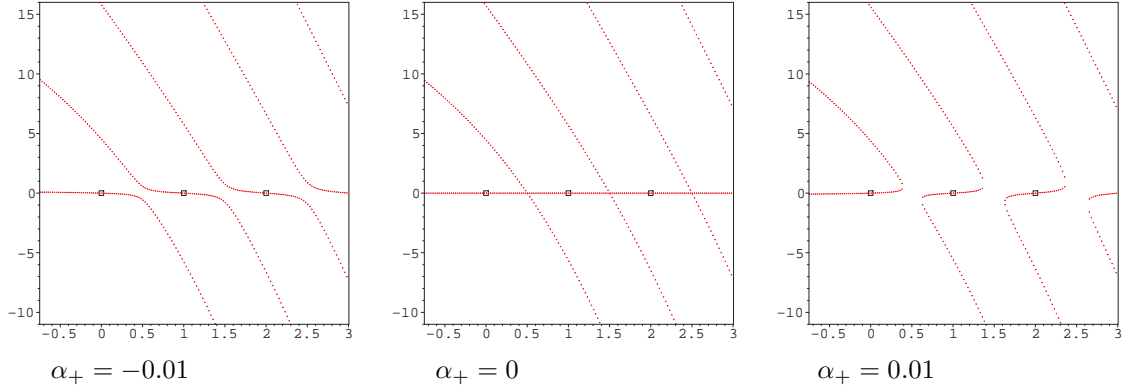

$$\alpha_+ = -0.01 \qquad\qquad \alpha_+ = 0 \qquad\qquad \alpha_+ = 0.01$$

Figure 3: The appearance of complex eigenvalues as the supersymmetric line $\alpha_+ = 0$ is crossed, for $M = 3$. The eigenvalues are plotted against α_-, with the points $\alpha_- \in \mathbb{Z}^+$, $E = 0$ indicated by boxes; for $\alpha_+ = 0$, these are double zeroes of $T(-\lambda, \alpha_+, \alpha_-)$.

In figure 3 we show the real energy levels along the supersymmetric line $\alpha_+ = 0$, and along two lines just either side of it. On the third of these plots – the only one partially inside the region A – pairs of eigenvalues join and become complex in the neighbourhoods of the level-crossing points. By contrast, on the first plot, which lies entirely inside regions covered by the proof of [1], the spectrum remains entirely real and the effect of supersymmetry-breaking is seen in the replacement of level-crossing by level-repulsion.

It might seem surprising that we are observing level-crossing in such a simple quantum-mechanical system. Again, the non-Hermiticity of the problem provides the explanation. As the level-crossing point is approached, not only the two eigenvalues but also their eigenfunctions become equal. (This is clear since eigenfunctions of (4) are uniquely characterised by the values of $\alpha_\pm$ and E, and their decay in the sector $\mathcal{S}_{-1}$.)

The Hamiltonian thus ceases to be completely diagonalisable, a situation which can be understood by considering the following 2×2 matrix, with eigenvalues $\pm\sqrt{\delta}$:

$$\mathcal{H}_\delta = \begin{pmatrix} 0 & 1+\delta \\ \delta & 0 \end{pmatrix} ; \qquad \mathcal{H}_\delta \begin{pmatrix} 1 \\ \pm\sqrt{\delta} \end{pmatrix} = \pm\sqrt{\delta} \begin{pmatrix} 1 \\ \pm\sqrt{\delta} \end{pmatrix} . \tag{13}$$

The value of δ corresponds to the distance from the level-crossing in the supersymmetry-breaking direction. (The addition of $\mathrm{diag}(-\eta, 0)$ to $\mathcal{H}_{\delta=0}$ would instead model a perturbation which preserves supersymmetry, leaving the theory on the line $\alpha_+ = 0$.) At $\delta = \eta = 0$ the two eigenvectors coincide. For this reason, the reference at the end of the last section to a *degeneracy* in the model at the level-crossing should be treated with caution – better maybe to say that the eigenproblem itself has become singular. The two eigenvalues become complex for δ negative, so the transition to complex eigenvalues can be traced to the singular nature of the eigenproblem at the level-crossing.

(To avoid confusion, we should stress that we are talking only of the 'bosonic' part of a fully supersymmetric problem here – the full SUSY QM system involves both $\mathcal{H}(\alpha_+, \alpha_-)$ and $\mathcal{H}(\widehat{\alpha}_+, \widehat{\alpha}_-)$, and the singularity of the eigenproblem at the level-crossing is reflected in the fact that the $\lambda = 0$ degeneracy jumps from 1 to 2 there, rather than from 1 to 3 as would naively have been expected.)

The lines $\alpha_\pm = 0$ are not the only ones along which there is a 'protected' zero-energy level in the spectrum of the system – from the formula (10), the same is true of all the lines $\alpha_\pm = n$, $n \in \mathbb{Z}^+$. At least for $M = 3$, we can understand this as being due to a hidden $\mathcal{N}$-fold supersymmetry in the model [1,21]. In figure 4 we illustrate how further complex levels are created as these lines are crossed.
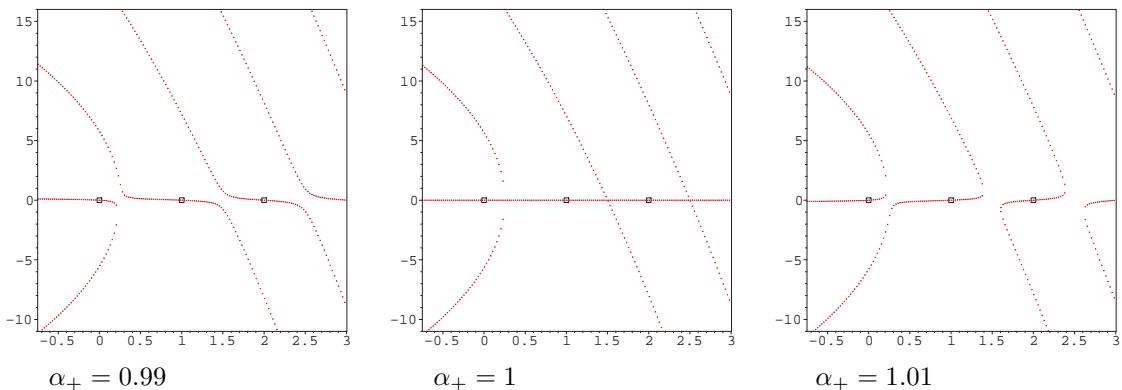

<center>$\alpha_+ = 0.99$ $\alpha_+ = 1$ $\alpha_+ = 1.01$</center>

Figure 4: Yet more complex eigenvalues for $M = 3$, this time near the line $\alpha_+ = 1$. Labelling as in figure 3.

Three of the low-lying levels in the middle plot of figure 4, two of which become complex for $\alpha_- > 1/4$, can in fact be found exactly: they are the roots of

$$P_3(E) = E^3 - 32[(5 - 3J)\alpha_- + J - 2]E = 0 \tag{14}$$

at $J = 3$. More generally for $M=3$, at $\alpha_+ = (J-1)/2$, $J = 1, 2, \ldots$, J levels of the $\mathcal{PT}$-symmetric problem (4) can be found exactly. This can be seen using the above-mentioned fifth spectral equivalence of [1]: on the lines $\alpha_+ = (J-1)/2$, (4) is mapped

<center>5</center>

to the radial problem at a point where it is quasi-exactly solvable [22]*. (Equivalently, it follows directly from the termination of the relevant Bender-Dunne [24] expansion.) For J odd, one of these QES levels is the exactly-zero level mentioned in the previous paragraph. This level (together with the other $J{-}1$ QES levels) can be eliminated by passing from $(\alpha_+, \alpha_-) = ((J{-}1)/2, \alpha_-)$ to $(\widehat{\alpha}_+, \widehat{\alpha}_-) = (-(J{+}1)/2, \alpha_- {-} J/2)$ [1], and the exact locations of the level-crossings can again be found: they are at $\alpha_- = J/2 + n$, $n = 0, 1, \ldots$. However, for the moment we do not know if a similar exact treatment can be given for other values of M, though we expect the qualitative features of the breaking of level-crossings to persist, at least for nearby values of M. (Recall that the 'protection' of the zero-energy state for $\alpha_\pm \in \mathbb{Z}^+$ holds for all M.)

To end this section, we show three further plots which should help the reader to understand how the levels reorganise passing between figures 3 and 4. The middle plot is another QES example. This middle plot illustrates an additional feature of the models on the QES lines $M{=}3$, $\alpha_\pm = (J{-}1)/2$: the levels which become complex along these lines always lie in the QES part of the spectrum. To prove this, we simply note that the 'dual' problems, with $(\alpha_\pm, \alpha_\mp) = (\widehat{\alpha}_\pm, \widehat{\alpha}_\mp) = (-(J{+}1)/2, \alpha_- {-} J/2)$, always lie in regions of the parameter space covered by the reality proof of [1]. As already mentioned, the spectrum of this dual problem is identical to that of the original problem, minus the QES levels. (Strictly speaking, the arguments of [1] applied to a radial problem, but the discussion of section 8 of that paper, based on intertwining operators, can be modified to cover the current, lateral, problem. One can alternatively argue via the 'fifth spectral equivalence' mentioned at the start of this section.) This demonstrates that the non-QES levels always remain real. A similar result was conjectured for quartic QES potentials in [25], but so far as we know remains unproven.
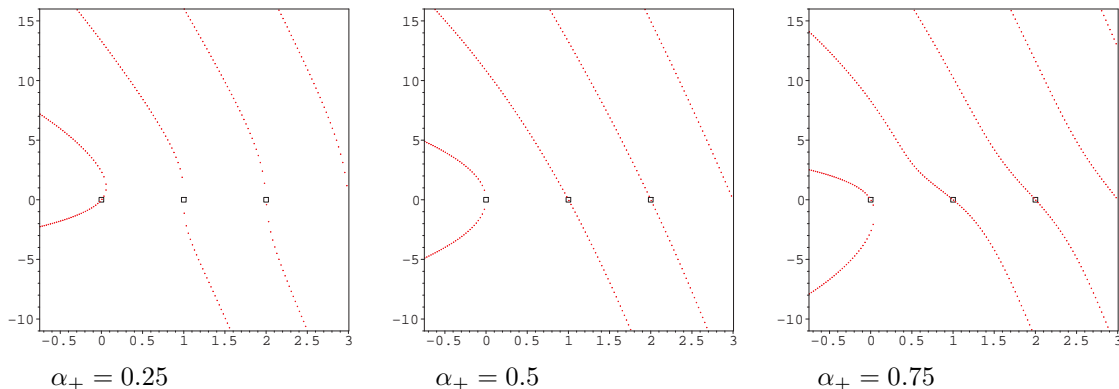

$\alpha_+ = 0.25$ $\alpha_+ = 0.5$ $\alpha_+ = 0.75$

Figure 5: The reorganisation of the levels passing from $\alpha_+ = 0$ to $\alpha_+ = 1$. Labelling as in figure 3.

4. The above discussion has confirmed that the domains where the spectrum of (4) has a complex component open out from the level-crossing points on the boundary of

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*We remark that a relationship between generalised supersymmetry and quasi-exact solvability, albeit for a different set of models, was originally discussed in [23].

the region A. These points are given exactly, for general M, by equation (12). It is interesting to see, at a qualitative level, how these 'domains of unreality' coalesce as one moves further into A. Figure 6 below exhibits some initial numerical results, found again for $M = 3$ using the MAPLE code of [1]. This should be viewed as a refinement of the initial 'phase diagram' of figure 1. The full domain of unreality is the interior of the curved line, a proper subset of A which only touches its boundary at the points $(\alpha_\pm, \alpha_\mp) = (0, n+1/2)$, which are the level-crossing points (12) for $M=3$. In the small, approximately-triangular region inside A but outside the curved line which abuts the point $\alpha_+ = \alpha_- = 0$, the spectrum is not only real but also entirely positive, despite the fact that it lies outside the domain D. This shows that, while the condition $\alpha < M + 1 - |2l+1|$ is sufficient for positivity of the spectrum, it is not necessary.

Perhaps the most striking feature of figure 6 is the pattern of cusps. In the absence of an analytical analysis, we do not yet know whether these are special to $M = 3$, or more generic. At least numerically, they lie exactly on the lines $\alpha_\pm = n$, along which the model possesses a protected zero-energy state (to guide the eye, segments of these lines have also been added on figure 6). Furthermore, for $M = 3$ the model is quasi-exactly solvable on these lines. As shown above, the levels which go complex then lie in the QES part of the spectrum, and this allows the cusps to be located exactly for M=3, assuming that they do indeed lie on the QES lines. For example, this places the two lowest pairs of cusps at $(\alpha, l) = (9, -1/2\pm3/2)$ and $(15-3/\sqrt{2}, -1/2\pm3/\sqrt{2})$.
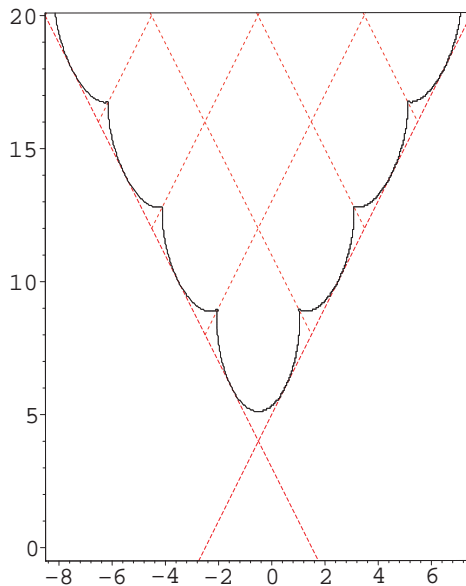

Figure 6: The domain of unreality for $M = 3$. Horizontal and vertical axes are l and α respectively, as in figure 1.

A detailed study of these properties must await future work; we expect that complex WKB techniques will be useful in this regard. These were recently employed in [9] for

a problem which can be mapped onto the $l = 0$, $M = 2$ case of (4), save for a different choice of Stokes sectors – $\mathcal{S}_{-1}$ and $\mathcal{S}_2$ – for the quantisation contour.

The intersections of the lines $\alpha_+ = (J_1 - 1)/2$ and $\alpha_- = (J_2 - 1)/2$ provoke one further thought, for $M = 3$. At these points the model is quasi-exactly solvable from two different points of view, with either J_1 or J_2 levels lying in the QES part of the spectrum. This is reflected in a curious factorisation property of the Bender-Dunne [24] polynomials $P_J(\alpha_\pm, E)$. (These polynomials encode the QES levels in their zeroes, equation (14) being one example.) Taking $J_1 < J_2$, at the point $(\alpha_+, \alpha_-) = ((J_1 - 1)/2, (J_2 - 1)/2)$ where the two QES lines intersect, $P_{J_1}(\alpha_-, E)$ is a factor of $P_{J_2}(\alpha_+, E)$. A direct proof of this result, using the Bender-Dunne three-term recursion relation [24], can also be given.

5. We conclude with some general comments. Our main purpose in this note has been to understand the physical reasons for the breakdown in the reality proof of [1]. We have shown that, on the lines along which the proof fails, the model has a hidden supersymmetry, and that once these lines are crossed, the reality property can, and at some points does, fail to hold. This implies that the conditions required by the proof are of more than technical significance. It would be interesting to extend the proof to cover the full domain for which the spectrum of (1) is real, but the complicated shape of the domain of unreality shown in figure 6 suggests that this will be a difficult task.

The model we have been discussing has turned out to have an unexpectedly rich structure, and should serve as a testing-ground for other aspects of $\mathcal{PT}$-symmetric quantum mechanics. More detailed studies should help us to gain a better understanding of the emergence of complex eigenvalues in general, as well as shedding light on the pattern of transitions revealed by figure 6. We should also remember that, via the ODE/IM correspondence [13–17], all of these results have potential implications in the field of integrable models in 1+1 dimensions.

Finally, we remark that similar appearances of complex levels preceded by level-crossings have been observed in a nonunitary model of quantum field theory – the boundary scaling Lee-Yang model [26]. In this case the level-crossings were protected by an identity between cylinder partition functions in models with differing boundary conditions, which followed from a set of functional equations called a T-system. It remains to be seen whether this phenomenon can also be understood on the basis of some hidden symmetry, as was the case for the level-crossings discussed in a simpler, quantum-mechanical context in this letter.

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