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SUPERSYMMETRY ON CURVED SPACES AND HOLOGRAPHY

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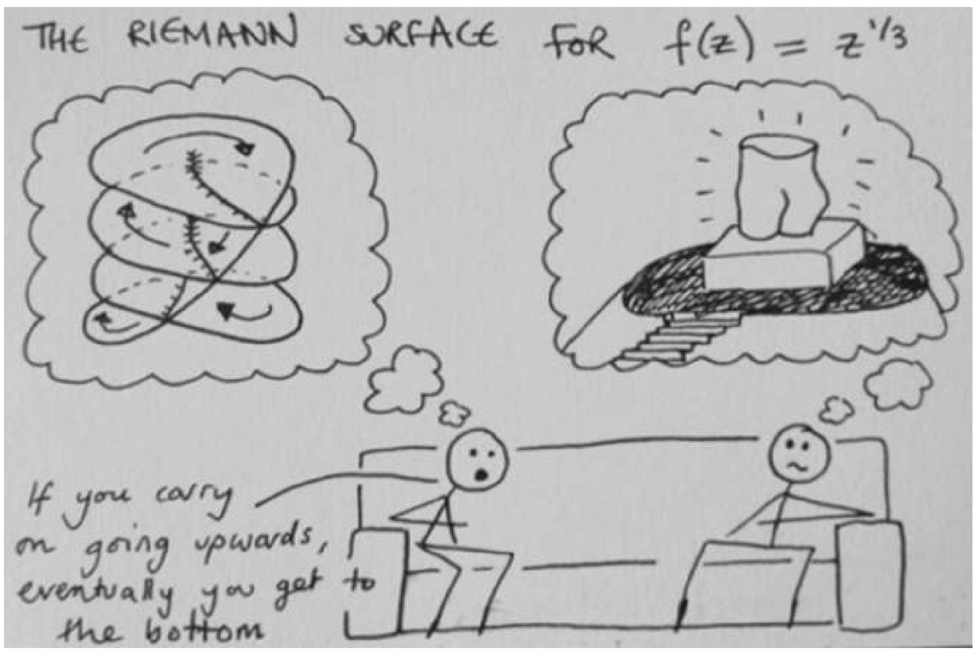


Discussion with the supervisor. Or: It's a long way to the top. © P.J. Dawson

Abstract

This thesis deals with superconformal and supersymmetric field theories on curved spaces with a view toward applications to holography and localisation. It contains two fairly different parts.

In the first (and main) part we classify Euclidean and Lorentzian four-manifolds with some preserved $\mathcal{N} = 1, 2$ supersymmetry, and Euclidean three-manifolds with some preserved $\mathcal{N} = 2$ supersymmetry. We take a holographic approach, starting with manifolds that preserve superconformal symmetries. Preserved supersymmetry for asymptotically locally AdS solutions implies the existence of a certain (generalised) "conformal Killing spinor" on the boundary. In the non-conformal case a closely related spinor exists, which also will be discussed. In this thesis we classify the manifolds in three and four dimensions that admit such spinors. In particular we find for the case with four supercharges that supersymmetry can be preserved in four dimensions on every Euclidean complex manifold and on any Lorentzian space-time with a null conformal Killing vector. In three Euclidean dimensions we find a condition very similar to complexity in four dimensions. When the field theory has eight supercharges, supersymmetry can generically be preserved on manifolds with time-like conformal Killing vectors; there are singular cases depending on the signature, in the Lorentzian there is a degenerate case reducing to the $\mathcal{N} = 1$ analysis, in the Euclidean there is a degenerate case corresponding to the topological twist. The supersymmetric curvature couplings are systematically understood in the rigid limit of supergravity. We give explicit formulae for the background fields that one needs to turn on in order to preserve some supersymmetry. This first part of the thesis is based on the papers [1–3].

Putting supersymmetric field theories on curved manifolds has led to interesting results over the past years. In the second part of the thesis we analyse a matrix model for the partition function of three dimensional field theories on S^3, which was obtained by supersymmetric localisation. In the large N limit one can evaluate the matrix model, allowing us to perform a non-classical and non-perturbative check of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence and Seiberg duality. In particular, we compute the large-N free energy of various three dimensional quiver gauge theories with arbitrary R-charges, which are dual to M-theory on Sasaki-Einstein seven-manifolds. In particular, we check that the free energy functional depending on the R-charges is minimised for the *exact* R-symmetry, an extremisation that is dual to the volume-minimisation of the Sasaki-Einstein manifold in the gravity sector. The second part of the thesis is based on [4].

Contents

Introduction and Motivations

The curvature of space-time is usually either treated as a dynamical object in the context of gravity, or it is simply neglected in the context of common (rigid) field theories. The past years have seen a change away from this snobbish attitude, at least in the case of supersymmetric field theories. In fact, there has been quite a boom of activity involving rigid supersymmetric field theories on various curved manifolds. Before we get to discuss the issues and difficulties of the very definition of supersymmetry on curved space, we want to dedicate this section of introduction & motivations to review some of this recent activity, providing a physics context for the rather formal questions we will pursue in the rest of this thesis.

Much of the progress seen in the last years in the subject of supersymmetric field theory on curved spaces is tightly connected to the beautiful concept of supersymmetric localisation. Introduced by Witten in the context of topological field theory [5], it turns out a powerful tool applicable in many other scenarios, too. Localisation relies on the existence of a nilpotent symmetry Q, for us this will be supersymmetry. One can argue that, when the Lagrangian is deformed by any Q-exact term, the path integral does not depend on the size of the coupling of such a deformation. We can thus evaluate the path integral at any coupling of the deformation we want, the answers should be equivalent. If we take the coupling to infinity only the zero locus of the deformation term contributes to a meaningful path integral and that can lead to drastic simplifications. Often, the path integral reduces to a matrix model, which is an integral over a finite number of variables. Crucially, in order to have a finite and well defined matrix model, one typically needs to consider a compact space-time manifold.

The most prominent example of such a matrix model obtained by supersymmetric localisation on a curved space was found by Pestun [6], who considered $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric field theories on the four-sphere. Let us report on one specific discovery that has emerged from Pestun's work (yet there are many others). Quite remarkably, the authors of [7] (AGT) observed that a particular S^4 matrix model can be identified with four-point correlators in two-dimensional Liouville theory on the sphere. Roughly speaking, the formula for the four-point correlators is composed of three-point correlators, which can be identified with the perturbative piece of the matrix model, and of conformal blocks which are identical to the non-perturbative part of the matrix model, being Nekrasov's instanton sum [8]. In [7] the comparison is generalised to n-point correlators and an entire class of Gaiotto theories [9]. Note also that one way of refining the correspondence, which is tightly related with the subject of supersymmetry on curved manifolds, is obtained by considering the partition function on a four dimensional ellipsoid [10]. The squashing parameter of the ellipsoid corresponds to varying the central charge in Liouville theory. In fact in section 7 we will come back to this example. Over the years, the whole business has now established a much unexpected collection of dualities between field theories in different dimensions.

Much work has been done also in three dimensions. The authors of [11] have generalised Pestun's work to $\mathcal{N} \geq 3$ ABJM-like theories on the three-sphere, again yielding a matrix model for the path integral. This matrix model has been evaluated at large N and the puzzling $N^{3/2}$ scaling of the free energy, which had been predicted long time ago [12] for theories with a gravity dual, has indeed successfully been observed [13]. The analogous matrix model for field theories with $\mathcal{N} = 2$ supersymmetry has been found in [14, 15]. In that case the matrix model depends explicitly on the R-charges of the fields as supersymmetry does not fix them to canonical values.

Jafferis [14] conjectured that in three dimensions the 'free energy' $F = -\ln Z$, where Z is the partition function on S^3, is extremised for the values of the R-charge that correspond to the *exact* R-symmetry at the superconformal fixed point. This solved a longstanding problem in conformal field theory, as it had been predicted from gravity [16] that, at least for SCFTs with a gravity dual, some analogue principle to the a-maximisation in $4d$ [17] should exist also in $3d$. Since there is no anomaly in three dimensions, it had not been clear what other object could play the role of a, though. In a sense, now F came to a rescue there. Moreover, it was also suggested [18] that F obeys a *c-theorem* [19] in three dimensions, decreasing monotonically along the RG flow.

This thesis contains holography in the title and that is for two reasons. In part I, chapter 1 we will learn about one way how holography enters the story of supersymmetry on curved space, which in fact will be the main focus. Here we want to review another one. As mentioned, a/F-maximisation for superconformal field theories has a beautiful counterpart on the gravity side [16, 20, 21]. The field theoretical 'c-function' corresponds to a 'Z-function' for the dual geometry, which is a functional depending on the choice of a certain vector field. Being able to identify these two functions amounts to a very non-trivial check of the AdS/CFT correspondence.

The whole three dimensional story has been a very active field of research, in part II of this thesis we will in fact report on our small contribution to it, much of it in the context of $\mathrm{AdS}_4/\mathrm{CFT}_3$. This will also include a more complete list of references and a more detailed review of the recent activity.

Supersymmetric partition functions have also proved useful for checking Seiberg duality. As the resulting formulae often are exact, non-classical and non-perturbative, comparing them for two different phases represents some of the most profound accessible checks of these dualities. The partition function defined on $S^1 \times S^3$ computes what is known as the supersymmetric index [22, 23] of the theory. For the physicist after the mid-90ies an almost trivial thing, identifying the index for two Seiberg dual phases of the same theory turns out to be at the edge of knowledge for the mathematicians [24, 25]. In fact, it involves recently discovered identities on elliptic hypergeometric integrals [26, 27]. A similar story holds in $3d$. The S^3 partition function for $\mathcal{N} = 2$ theories involves hyperbolic gamma functions and the most simple Seiberg duality for the field theory translates to highly complicated and only recently studied [28] integral identities amongst them [29].

A remarkable exact result extracted from a rigid field theory on a curved manifold has also been obtained in two dimensions. In [30, 31] the partition function for $\mathcal{N} = (2, 2)$ theories on the two-spere has been calculated by supersymmetric localisation. Subsequently it was conjectured in [32] that, when the CFT_2 is related to a GLSM with Calabi-Yau target space X_3, the localised path integral computes exactly the quantum Kähler moduli space of X_3, see [33] for a proof that this indeed is true. It is worth pointing out that this computation does not involve any usage of mirror symmetry. In fact, in [32] the authors have computed new Gromov-Witten invariants for Calabi-Yau manifolds without known mirror.

Finally we want to briefly report about the progress in higher dimensions. Most mysterious is the infamous $(2, 0)$ theory in six dimensions, whose Lagrangian – if existent at all – is not known. Every ever so tiny grain towards its understanding is celebrated like a treasure by the community. There exist intriguing conjectures, that maximally supersymmetric Yang-Mills in $5d$, naively power-counting non-renormalisable as it is, might already itself be UV complete, if one includes the non-perturbative sector in the spectrum [34, 35]. It is hence of double interest to look at the partition function of five dimensional field theories on S^5, this has been done in [36, 37]. These results are not yet entirely complete, as there is also an instantonic contribution to the matrix model, which so far no one has been able to compute. The subject is very active field of research and various related results have been obtained in the recent past, we will briefly mention only a few. Neglecting the contribution from instantons, the matrix model shows nevertheless a N^3 scaling behaviour for the free energy [38], which is the sought-after $6d$ prediction from gravity [12]. Another interesting idea is to extend the S^5 partition function to the six-dimensional index, which is discussed in [39], also here a N^3 behaviour could be observed. Less immediately related to the $(2,0)$ theory, there are even localisation results directly in 6 dimensions [40].

It is a good motivation to our work to keep these manifold applications in the back of our minds. Let us now give an overview about the contents of this thesis.

In part I we will take one step back from the applications and address a more basic question. We have seen that a compact manifold seems fundamental for the recent developments and in part I of this work will address the question when and how it is possible to put a supersymmetric field theory on a general curved space-time, preserving some of the fermionic symmetries. We take a holographic point of view starting with superconformal theories in chapter 2, where we show that superconformality on curved spaces is governed by (sometimes generalised) conformal Killing spinors. In chapters 3 - 7 we then explicitly classify the geometry of $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry in various dimensions. We will also show how our results extend to the non-conformal case. In chapter 1 we have outlined the general strategy pursued in part I, including a summary of the classification results. Here the reader not interested in technical details can find an overview over the information contained in the first part of the thesis.

Part II considers the aforementioned matrix model for $\mathcal{N}=2$ superconformal field theories on the three-sphere. In chapter 8 we give an introduction to the whole subject and summarise our general findings. We present some necessary material about the field theories in question and the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence in chapter 9. In chapter 10 we review a large N saddle point ansatz for the matrix model of the theory on S^3. We then solve this ansatz for various examples and match the result with the gravity duals. In chapter 11 we show how, for theories with multiple gauge groups, the saddle point equations are invariant under Seiberg duality. We conclude with a more speculative chapter 12, about a curious observation of an alternative 'geometric' formula for the free energy.

Part I

Supersymmetry on curved spaces

Chapter 1

Introduction, Outline and Summary of the Results

Statement of the problem

Field theories that are supersymmetric in flat space generically loose all their fermionic symmetries if we naively put them on a curved manifold without adding extra couplings. This is most easily seen by considering the extra contribution within the covariant derivatives stemming from the non-vanishing spin connection.

Nevertheless, supersymmetric theories on curved spaces have been constructed since a long time, see e.g. [41, 42] for early accounts of AdS spaces. One has to add extra terms to both the flat-space action and the flat-space supersymmetry variations, in order to render the theory supersymmetric. This can be done, most straightforwardly, by a perturbative construction in $1/r$, where r is the scale of the curved space. A priori it is not clear, however, for which manifolds such a construction can be performed successfully and the problem needs to be studied case-by-case. Moreover, it is a cumbersome exercise to determine the deformed Lagrangian and the supersymmetry variations in this way. Anyhow, the problem can also be studied more systematically:

A clever solution

We can understand the field theory on curved space as arising in the rigid limit of supergravity [43][1]. The bosonic fields of the gravity multiplet are frozen to a supersymmetric configuration and treated as background fields for the matter sector. In this way, they

[1] Similar but less systematic ideas have appeared earlier in the literature, see e.g. [44] and [45].

naturally generate the additional curvature couplings that ensure preserved supersymmetry. Thus one gets, in a sense, the Lagrangian and the supersymmetry variations "for free", as the supergravities are often well known. Note that the metric is part of the supergravity multiplet, its background value can be considered as an external input. It is worthwhile to point out that the gravity is treated non-dynamically, in the sense that we do not impose the equations of motion for the background fields. In particular, we keep the auxiliary fields and freeze the full off-shell multiplet to supersymmetric values. The problem of determining the manifolds supporting a supersymmetric theory can now be solved very systematically by determining the class of spaces on which one can find such a supersymmetric configuration of the gravity multiplet. As we will show in this thesis, in many cases this boils down to the existence of a (charged) *conformal Killing spinor* (CKS)

$$P^A_\mu \cdot \epsilon = \big(\nabla^A_\mu - \frac{1}{d}\gamma_\mu D^A\big)\epsilon = 0 \tag{1.1}$$

where ^{A} denotes twisting with the R-symmetry gauge field, $\nabla^A_\mu = \nabla_\mu - iA_\mu$, and D is the Dirac operator $D = \gamma^\mu \nabla_\mu$. When the flat space theory has more then 4 supercharges, the condition for unbroken supersymmetry amounts to a generalised version of this conformal Killing spinor, including various extra background fields.

Hello, holography!

If the field theory is superconformal, there is yet a different approach to supersymmetry on curved spaces. In that case we can study the problem using holography. Now, the curved space-time arises as boundary information of the dual gravity in the bulk. Clearly, the supergravity in the bulk has a proper physical interpretation as the holographic description of the field theory in one more dimension. Nevertheless, we argue that there is a tight connection between this gravity and the "auxiliary" gravity in one dimension less that we described in the last paragraph. Before we do that, we want to give a few more details about the nature of these two theories. The natural auxiliary supergravity that couples to a super*conformal* field theory is conformal supergravity[2]. Recall that in conformal supergravity the entire superconformal group is gauged, as opposed to Poincaré supergravity. In particular, there are gauge fields for the R-symmetry, for special conformal and for Weyl transformations. On the other hand, in the bulk, we will assume minimal gauged supergravity. That basically contains the metric $g_{\mu\nu}$, the gravitino ψ_μ and a graviphoton A_μ, which are the appropriate gauge fields dual to the three symmetry

[2] See [46] for a nice review of conformal supergravity. Some original papers are [47–53].

currents of a generic SCFT, being the energy-momentum tensor $T_{\mu\nu}$, the supersymmetry current $\mathcal{J}_\mu$ and the R-symmetry current J_μ.

Conformal Killing spinors from the bulk

Remarkably, it turns out that this minimal gauged supergravity in an asymptotically, locally AdS space-time reduces to conformal supergravity on the boundary [1, 2, 54, 55]. We will explicitly perform this computation in the next chapter, here we report only few details. If one assumes the metric of asymptotically, locally AdS$_{d+1}$,

$$ds^2_{d+1} = \frac{dr^2}{r^2} + (r^2 ds^2_{M_d} + O(r))$$

where M_d is any curved d-dimensional manifold, one can expand the supersymmetry variations of minimal gauged supergravity in $1/r$,

$$\delta_\xi \psi_M \quad \xrightarrow{r\to\infty} \quad \nabla^A_\mu \epsilon - \gamma_\mu \eta$$

where the indices $_M$ and $_\mu$ run in ALAdS$_{d+1}$ and M_d, respectively, and the bulk supersymmetry parameter ξ has the asymptotic expansion $\xi = r^{\frac{1}{2}}\epsilon + r^{-\frac{1}{2}}\eta$. Upon identifying the boundary profile of the graviphoton appearing in ∇^A with an R-symmetry gauge field, and the asymptotic spinors ϵ and η with Q- and S- supersymmetry parameters respectively, we uncover the supersymmetry variation of conformal supergravity in d dimensions. Setting to zero this variation actually implies the existence of a charged conformal Killing spinor on the boundary manifold as in (1.1), which can be seen by gamma-tracing $\nabla^A_\mu \epsilon = \gamma_\mu \eta$ and plugging back the expression obtained for η. The treatment is generalised appropriately in the case of extended supersymmetry, where other background fields and also the dilatino variation need to be included. In this outline section we will focus on the simplest case with four supercharges, the generalised CKS is discussed in detail in the chapters 6 and 7.

Classifying geometries with conformal Killing spinors, a.k.a. SUSY

We now have established that manifolds which admit a charged conformal Killing spinor (1.1), preserve some supersymmetry[3]. In this thesis we want to classify such manifolds

[3] Again, if the flat space theory has *extended* supersymmetry, the manifolds need to admit a *generalised* version of CKS.

in various dimensions. The strategy which was applied in [1–3][4] is based on the following idea. By looking at all possible spinor bilinears one can determine the geometry that is defined by a generic spinor. Restricting on the bilinears made from conformal Killing spinors then gives certain constraints on this general geometry. We want to make some of the ideas outlined in this chapter more explicit by (repeatedly) discussing a concrete example.

> **Ex.:** Let us consider $\mathcal{N} = 1$ in four Euclidean dimensions. The complete analysis of this example will be presented in chapter 3 (see also [43]), here we will be very sketchy. A chiral spinor in four dimensions defines a real two-form j and a complex two-form ω [5]
>
> $$j_{mn} = \epsilon_+^\dagger \gamma_{mn} \epsilon_+ \qquad\qquad \omega_{mn} = \epsilon_+^T \gamma_{mn} \epsilon_+ \,.$$
>
> They fulfil $\omega^2 = 0$ and $\omega \wedge \bar{\omega} = 2j^2$. Note that if ϵ_+ is charged under a $U(1)$ gauge field, so will be ω. In the language of G-structures, a charged chiral spinor in four dimensions thus defines a $U(2)$-structure.

Next we want to get a handle on the geometry. To this end one can parameterise the derivatives of the forms which are defined by the spinors, in some natural base. The coefficients are often called *intrinsic torsions*.

> **Ex.:** In our example, they look like
>
> $$dj = W^4 \wedge j \qquad\qquad d\omega = W^5 \wedge \omega + W^3 \wedge \bar{\omega} \tag{1.2}$$
>
> where the one-forms W^i are the intrinsic torsions. As we see, the natural base for us is given by the forms themselves.

The last step is to impose that the spinor defining the forms is conformal Killing. This will give certain geometrical constraints on the manifold. Moreover, in all cases studied in this thesis we will also solve for the values of the background fields in terms of the geometrical data. That allows us not only to determine the class of manifolds which preserve some

[4] There are several related works [40, 56–69] in some of which a similar strategy has been applied and some of which significantly overlap with the results presented in this thesis. Supersymmetric backgrounds have been classified in [1, 2, 56, 57, 62] for $\mathcal{N} = 1$ theories in $4d$, in [1, 58, 59] for $\mathcal{N} = 2$ theories in $3d$ and in [3, 69] for $\mathcal{N} = 2$ theories in $4d$. We will point out the precise overlap in the respective chapters.

[5] **Note**: In order to make most clear the underlying structure, we omit here and in other formulae of this overview section various factors. All formulae are given in their full blossom in the detailed sections.

supersymmetry, but it also tells us *how* we can preserve it, i.e. which background fields we need to turn on in order to generate the curvature couplings. An important observation is that in the Euclidean we allow these background fields to take *complex* values. This is justified as we do not consider their physical fluctuations. In consequence, however, the resulting Lagrangians for the field theory possibly are not reflection-positive. On the contrary in the Lorentzian analogue, a non-unitary theory clearly would not be acceptable and we require the background fields to be real.

> **Ex.:** More explicitly, in our example, we need to impose the differential constraint $P_m^A \cdot \epsilon_+ = 0$, where the conformal Killing operator has been defined in (1.1). Here, $d = 4$ and A_m is a $U(1)$ gauge field, which being in the Euclidean we allow for taking complex values. It is slightly technical to translate the spinorial CKS condition (1.1) into a differential condition on the forms. The result is roughly[6]
>
> $$W^3 = 0 \qquad\qquad A_m \approx W_m^5 + W_m^4\,. \tag{1.3}$$
>
> The first equation is a geometrical constraint. Looking at (1.2) shows that it enforces M_4 to be *complex*. The second equation does not impose additional conditions on the geometry, it simply determines the value of the gauge field A_m in terms of the torsions W^4 and W^5, which are unconstrained geometrical data. As a word of caution we should mention here that this analysis is local, we assumed that the spinor ϵ_+ has no zeros. In order to obtain a global solution one needs to analytically continue the results into the vanishing locus.

This example, though the simplest one and with all technicalities omitted, yet captures the main conceptual features of the analysis performed in part I of the thesis. We have identified the class of Euclidean four dimensional manifolds that preserve some supersymmetry and we have given the values of the background fields, which generate the necessary curvature couplings in the field theory.

The curved space Lagrangian

Conformal supergravity has the important feature that the matter sector does not appear in the supersymmetry variations of the gravity multiplet. Hence a configuration of $(g_{\mu\nu}, A_\mu, \psi_\mu)$ is supersymmetric independently from arbitrary matter couplings. In the

[6] see (3.25) for the exact formulae for A_m, including a contribution from the norm of the spinor ϵ_+.

matter sector, the frozen background values of the gravity multiplet generate extra couplings which are not present in the flat space theory, as we have discussed above. On the linearised level one has for instance

$$-\frac{1}{2}g_{\mu\nu}T^{\mu\nu} + A_\mu J^\mu + \bar{\psi}_\mu \mathcal{J}^\mu \tag{1.4}$$

where $T_{\mu\nu}$ is the energy momentum tensor and J_μ and $\mathcal{J}_\mu^\alpha$ are respectively the R-symmetry and the supersymmetry current in the boundary field theory. The non-linear level works conceptually the same and the matter coupled Lagrangian for conformal supergravity freezes to the curved space Lagrangian of the rigid theory.

Ex.: Coming back to our case study of $\mathcal{N} = 1$ theories in four dimensions, one simple example for a complex manifold is the Hopf surface $S^1 \times S^3$ [7]. This manifold has been prominently used to study superconformal indices, see *e.g.* [22–24]. If we choose the metric $ds^2 = dt^2 + ds^2_{S^3}$, we can compute the torsions in (1.2) as $dj = -2dt \wedge dj$ and $d\omega = -2dt \wedge \omega$. One confirms that indeed $W^3 = 0$. Moreover, for a t-independent spinor ϵ_+, we find from (1.3) a purely imaginary gauge field with a leg along the S^1 circle,

$$A = -\frac{i}{2}dt\,.$$

Via couplings as in (1.4) or inside the covariant derivative, this value for A_m indeed generates the extra terms in the Lagrangian and the supersymmetry variations of the field theory on $S^3 \times S^1$, see e.g. [22].

We have thus seen how we can start with a geometry, extract the background fields from that, and then determine the curved space Lagrangian and supersymmetry transformations.

From CFT to QFT

So far, formally we only discussed *conformal* field theories. We coupled them to conformal supergravity, and gave a physical interpretation of this coupling as boundary residue of the holographic bulk dual. It turns out that many of the results extend to the case where the rigid field theory is just super*symmetric*, without necessarily being conformal. In that case, one would naturally couple the theory to a suitable version of off-shell Poincaré supergravity. Recall that all known off-shell formulations of Poincaré supergravity can be

[7] The complex structure works by pairing the Hopf fibre in S^3 with the S^1 factor.

obtained from conformal supergravity via the old-established formalism of superconformal tensor calculus. There, the conformal symmetry is removed by introducing “compensator multiplets”, upon which the unwanted gauge freedom is fixed [70, 71] (see also [46] for a review). Some components of these compensator multiplets remain unfixed, they correspond to additional background fields. Different choices for the compensator lead to the various off-shell formulations. We found that often the same additional background fields can be introduced by identifying them with a particular combination of the intrinsic torsions (1.2), without loosing any degrees of freedom. In this sense, their value also is determined by the geometry; and together with the CKS in that geometry they constitute a solution to the corresponding Poincaré supersymmetry condition.

Ex.: In the case in which the Euclidean $\mathcal{N} = 1$ field theory preserves R-symmetry, it is natural to couple it to ”new minimal supergravity” [72], which is a Poincaré supergravity with a $U(1)$ gauge field. In the formalism of superconformal tensor calculus it is obtained by choosing a tensor multiplet as conformal compensator. Fixing the conformal symmetry introduces a tensor b_{mn} as an additional background field. It turns out that this freedom exists also in the conformal supergravity. To see this, note that we can rewrite the CKS equation (1.1) by expanding $D^A\epsilon$ in the basis $\gamma^m\epsilon$ for chiral spinors,

$$D^A\epsilon \equiv 2i\not{v}\epsilon$$

where at this point the $2iv_m$ are arbitrary expansion coefficients. In the form language, this translates[8] to $2iv = W^4_{0,1} - W^4_{1,0} = -i * dj$, where $_{0,1}$ and $_{1,0}$ denotes the (anti-)holomorphic part with respect to the complex structure of M_4. It is straightforward to see that the CKS equation becomes (we stress that this is formally just a re-writing)

$$\nabla_m\epsilon = -i\left(\frac{1}{2}v^n\gamma_{nm} + (v-a)_m\right)\epsilon \ , \qquad a \equiv A + \frac{3}{2}v \ . \tag{1.5}$$

When v_m is the dualised tensor, $v_m = (*db)_m$, this is precisely the supersymmetry condition of new minimal supergravity. Since this is the case for our v, the conditions for some preserved supersymmetry in a theory with an R-symmetry is indeed in a one-to-one correspondence with the condition for supersymmetry in a superconformal theory.

[8] The translation is no equivalence, though. There is an ambiguity in the definition of v_m, the discussion of which we postpone to sec. 3. Also, our discussion needs to be refined wherever ϵ has a zero.

In summary, if we can identify a particular combination of the intrinsic torsions with the background fields specific for an off-shell version of Poincaré supergravity (and we often can), the geometrical conditions for unbroken supersymmetry are exactly the same whether the theory is superconformal or just supersymmetric.

Manifolds that allow supersymmetry - Results

We have obtained a classification of manifolds that admit supersymmetry in three and four dimensions with varying signature and for different amounts of flat-space supersymmetry. In this paragraph we will summarise our results.

$\mathcal{N} = 1$ **in Euclidean** $4d$ This situation has been worked out in [1], see also [56] which has a big overlap, and [57, 62]. It has been the case study of the outline section so far and we will only state the final result.

On any Euclidean four manifold which is *complex* one can define an $\mathcal{N} = 1$ conformal field theory that preserves some of its four flat-space supercharges. The same is true for supersymmetric but not superconformal theories whenever the theory has an $U(1)$ R-symmetry.

$\mathcal{N} = 1$ **in Lorentzian** $4d$ This situation has been worked out in [2], similar results have also appeared in [63]. A chiral spinor in four Lorentzian dimensions defines a real null vector z and a complex two-form $\omega = z \wedge w$, where w is a complex one-form. Up to a shift symmetry by a complex parameter, this set of forms is equivalent to a (null) vierbein with line element $ds^2 = ze^- + w\bar{w}$. Supersymmetry requires that the chiral spinor is CKS, which in turn implies that the null vector z must be conformal Killing, i.e. $\nabla_{(\mu} z_{\nu)} = \lambda g_{\mu\nu}$. That is the only geometrical constraint. In addition, the supersymmetric values of the background fields are determined by the geometry. In the case when z becomes Killing[9] we can choose a general set of coordinates. We give explicit expressions for the background fields in these coordinates.

To sum up, on any Lorentzian four manifold with a *null conformal Killing vector* one can define an $\mathcal{N} = 1$ conformal field theory that preserves some of its four flat-space supercharges. If the theory is only supersymmetric (with an R-symmetry) and not superconformal, the vector needs to be Killing instead of conformal Killing.

[9] We can always achieve this by a conformal rescaling. As discussed in section 4 this actually becomes an additional constraint when the theory is supersymmetric but not conformal.

$\mathcal{N}=2$ **in Euclidean** $3d$ This situation has been worked out in [1], a broad analysis has also appeared in [58]. Any (charged) spinor in three dimensions defines a dreibein (up to a phase). Supersymmetry requires again that the spinor is charged CKS, which gives a condition on the geometry looking very similar to complexity in $4d$, namely $do = W \wedge o$ where $o = e^1 + ie^2$ and W some one-form.[10] The CKS condition also determines the value of the background gauge field in terms of the geometrical data. The supersymmetry condition for an $\mathcal{N}=2$ non-conformal theory with an R-symmetry can be obtained from the four dimensional $\mathcal{N}=1$ case by dimensional reduction. As in $4d$, the analysis is very similar to the conformal one and the resulting condition on the geometry is the same.

To sum up, on any Euclidean three-manifold with a dreibein $e^3, o, \bar{o}$ which fulfils $do = W \wedge o$ with W some one-form, one can define a $\mathcal{N}=2$ theory which preserves some of its flat-space supersymmetry.

$\mathcal{N}=2$ **theories in** $4d$ This situation has been worked in [3], similar results had also appeared in [69]. If the flat-space theory has 8 supercharges, the analysis is a little bit more complicated. That is because the gravity multiplet is richer and so are the supersymmetry conditions. Albeit major technical differences, the analysis for Euclidean and Lorentzian signature runs fairly parallel and in this sketchy summary we will not discuss the two cases separately. A chiral spinor doublet of the $U(2)$ R-symmetry defines, amongst other things, a tetrad e^α_μ, where $\alpha = 0, 1, 2, 3$. The core of the supersymmetry condition remains the existence of certain generalised conformal Killing spinors, involving other background fields in the definition and also additional differential constraints. With one of the spinors vanishing identically, we get degenerate solutions, including the topological twist in the Euclidean. In the generic case the differential equations don't constrain the space-time and the sole geometrical constraint is that e^0_μ is a time-like conformal Killing vector. The set of remaining equations fixes again the values of the background fields, leaving three unconstrained degrees of freedom in the gauge field. Upon a special choice of coordinates we could explicitly solve the differential equations.

To sum up, on any four dimensional manifold with a time-like *conformal Killing vector*, one can define a $\mathcal{N}=2$ theory which preserves some of its flat-space supersymmetry. As somewhat degenerate cases arise also the possibilities of topological twisting on any manifold in the Euclidean and manifolds with a *null* CKV in the Lorentzian (which is basically the reduction to the $\mathcal{N}=1$ case).

[10] The geometries are known in the math literature as transversely holomorphic foliations with a transversely hermitian metric, see the discussion and maths references in [73].

From bulk to boundary

Sometimes the solutions to our bulk theory, i.e. minimal gauged supergravity, have been classified in the literature, see *e.g.* [74, 75]. To check whether the holographic understanding of the curved manifolds is more than a curious observation, it is useful to expand the results of such a bulk-classification in asymptotically locally AdS space and compare against the classification done on the boundary, reported about in the previous paragraphs. We have in fact done such a comparison for five dimensional minimal gauged supergravity (in Lorentzian signature), whose solutions have been classified in [74]. There the authors have found that all solutions fall into two classes, which have a time-like or a null Killing vector, respectively. We showed that on the boundary of ALAdS both classes reduce to solutions with a *null conformal* Killing vector, in agreement with our four dimensional findings. Furthermore, all other differential and algebraic constraints of the 5d classification reduce to the set of equations which we found in the purely four dimensional analysis.

Organisation of the rest of part I

The rest of the first part of this thesis is organised very straightforwardly. In chapter 2 we discuss locally asymptotically AdS space-times and superconformal theories on the curved boundary. In particular, we show how conformal Killing spinors emerge from the bulk. In the chapters 3 – 7 we classify geometries with (generalised) conformal Killing spinors in various dimensions and for different amount of flat-space supersymmetry, according to the chapter's titles. In each case we discuss issues, particularities and examples that are specific for that dimension/supersymmetry.

Chapter 2

The Holographic Perspective – Conformal Killing Spinors from the Bulk

In this chapter, we review how supersymmetry in the bulk implies the existence of a "conformal Killing spinor" on the boundary [54]; a very similar version of this computation has also appeared in [76, App. E]. We will focus on the case with 8 bulk supercharges, which correspond to boundaries with $\mathcal{N} = 1$ supersymmetry in four, and $\mathcal{N} = 2$ in three dimensions. The analogous case of extended supersymmetry, where the bulk-to-boundary analysis gives rise to a generalised version of the CKS, together with an extra supersymmetry equation, is discussed in detail in [55] for asymptotically AdS, the generalisation to asymptotically *locally* AdS is straightforward and we will not review it here. We will describe the emergence of a CKS from five-dimensional gravity in section 2.1, and in section 2.2 for four-dimensional gravity. In section 2.3, we will interpret the result in terms of the superconformal theory at the boundary.

The material presented in this chapter has appeared originally in the papers [1, 2]

2.1 From five-dimensional gravity to CFT$_4$'s

We will mainly discuss the Lorentzian signature case here, eventually discussing the Wick rotation to the Euclidean. Our starting point is $\mathcal{N} = 2$ gauged supergravity with an AdS$_4$ vacuum corresponding to the dual of a four-dimensional conformal field theory on flat space. According to the holographic dictionary, other solutions of the bulk theory which are asymptotically AdS describe deformations (or different vacua) of the CFT. We are interested in studying the CFT on a curved manifold M_4 and therefore we look for solutions of the bulk theory with a conformal boundary M_4. This can be implemented by assuming for the asymptotic bulk metric the Fefferman–Graham form

$$ds_5^2 = \frac{dr^2}{r^2} + (r^2 ds_{M_4}^2 + O(r)) \ . \tag{2.1}$$

This describes a *locally asymptotically* AdS_5 space-time. For our further analysis, it will be useful to compute the asymptotic fünfbein

$$\hat{e}^\alpha \ = \ r\, e^\alpha + \mathcal{O}(1)\,, \qquad \hat{e}^5 \ = \ \frac{dr}{r} \ , \tag{2.2}$$

where $\alpha = 0, \ldots, 3$ are flat four-dimensional indices and $e^\alpha = e^\alpha_\mu(x) dx^\mu$ is a vierbein for M_4. We denote with a hat five-dimensional quantities that might be confused with four-dimensional ones. Similarly, the associated spin connection to leading order in r is

$$\hat{\omega}^{\alpha\beta} = \omega^{\alpha\beta}\,, \qquad \hat{\omega}^{\alpha 5} = r e^\alpha\,. \tag{2.3}$$

In general, in order to define the theory on the curved manifold in a supersymmetric way, we will need to turn on a non trivial background for the R symmetry current. This corresponds to a relevant deformation $A_\mu J^\mu$ of the CFT and we expect a non trivial profile of the graviphoton field in the bulk. On the other hand, we do not want to include explicit deformations induced by scalar operators so we can safely assume that all the scalars in the bulk vanish at the boundary.

It is hence natural to look at minimal gauged supergravity in the bulk. In Lorentzian mostly plus signature the bosonic part of the action is[1]

$$S \ = \ \frac{1}{4\pi G} \int \left(\left(\frac{1}{4}\hat{R} + \frac{3}{\ell^2} \right) * 1 - \frac{1}{2}\hat{F} \wedge * \hat{F} - \frac{2}{3\sqrt{3}} \hat{F} \wedge \hat{F} \wedge \hat{A} \right) \ , \tag{2.4}$$

where $\hat{F} = d\hat{A}$ and $\ell \neq 0$ is a real constant.

The Killing spinor equation corresponding to a vanishing gravitino variation is

$$\left[\hat{\nabla}_A + \frac{i}{4\sqrt{3}} \left(\gamma_A{}^{BC} - 4\delta_A^B \gamma^C \right) \hat{F}_{BC} \right] \epsilon^I + \frac{1}{2} \epsilon^{IJ} \left(i\gamma_A + 2\sqrt{3}\hat{A}_A \right) \epsilon^J \ = \ 0 \ , \tag{2.5}$$

[1]We use notations adapted from [74], to which we refer for details. To compare with [74], one has to identify $\chi = 2\sqrt{3}$. Moreover, one needs to switch between mostly plus and mostly minus signature, which means flipping the sign of the metric and taking $\gamma^A_{\text{here}} = -i\,\gamma^A_{\text{there}}$. Also, for ease of reading we set $\ell = 1$ in most other formulae.

where we are using flat A, B, C five-dimensional space-time indices. Our conventions for the spinors, which are symplectic-Majorana, can be found in Appendix A.1.

We want to expand this in the asymptotic metric (2.1). For the bulk gauge field we assume the asymptotic behaviour

$$\hat{A}_\mu(x,r) = -\frac{1}{\sqrt{3}}A_\mu(x) + \mathcal{O}(r^{-1}) \ , \qquad \hat{A}_r(x,r) = 0 \ , \tag{2.6}$$

which is compatible with the equations of motion. Here and in the following, $\mu, \nu, \ldots$ are curved Lorentzian indices on M_4. From the point of view of the gravity solution, this corresponds to the non-normalisable mode for A, which indeed is interpreted in AdS/CFT as the deformation of the theory induced by a background field for the R-symmetry. We can also turn on the bulk fields corresponding to global symmetries of the CFT but, allowing for a redefinition in A, this will not change the form of the supersymmetry transformation. It follows that $\hat{F}_{\mu\nu} = \mathcal{O}(1)$ and $\hat{F}_{\mu r} = \mathcal{O}(r^{-2})$.

At leading order in the asymptotic expansion in r, the radial part of the Killing spinor equation (2.5) gives rise to

$$\partial_r \epsilon^I + \frac{i}{2r}\gamma_5\, \epsilon^{IJ}\epsilon^J \ = \ 0\,, \tag{2.7}$$

where the index on γ_5 is flat. Note that the contribution of the gauge field strength obtained from (2.6) is sub-leading and therefore drops out. Eq. (2.7) implies that the two symplectic-Majorana spinors take the asymptotic form

$$\begin{aligned} \epsilon^1 \ &= \ r^{1/2}\epsilon + r^{-1/2}\eta \ , \\ \epsilon^2 \ &= \ i\gamma_5(r^{1/2}\epsilon - r^{-1/2}\eta) \ \ , \end{aligned} \tag{2.8}$$

where ϵ and η are independent of r. Plugging these expressions back into the remaining components of (2.5), one finds that at leading order the spinors obey the following equation

$$\Big(\nabla_\mu - iA_\mu\gamma_5\Big)\epsilon + \gamma_\mu\gamma_5\eta \ = \ 0\,. \tag{2.9}$$

Here, a term from the covariant derivative relative to the metric ds_5^2 has combined with the term linear in γ_A in (2.5). In the gamma matrix representation we adopted (see appendix A.1), the symplectic-Majorana condition in five dimensions implies that the four-dimensional spinors obey $\epsilon^* = \epsilon$ and $\eta^* = -\eta$. We can also use γ_5 to define the chirality for the boundary spinors,

$$\epsilon = \epsilon_+ + \epsilon_- \,, \qquad \eta = \eta_+ + \eta_- \,, \tag{2.10}$$

where $\gamma_5\epsilon_\pm = \pm\epsilon$ and $\gamma_5\eta_\pm = \pm\eta$. Taking the trace of (2.9) allows us to solve for η:

$$\eta = -\frac{1}{4}(\gamma_5\nabla_\mu + iA_\mu)\gamma^\mu\epsilon \ . \tag{2.11}$$

Finally, inserting this back into (2.9), we find

$$\nabla^A_\mu\epsilon_+ = \frac{1}{4}\gamma_\mu D^A\epsilon_+ \ , \tag{2.12}$$

where $\nabla^A_\mu = \nabla_\mu - iA_\mu$ and $D^A = \gamma^\mu\nabla^A_\mu$. This is the equation for a (charged) *conformal Killing spinor*, and will be the starting point of our subsequent analysis. In the math literature such a spinor is also known as *twistor spinor*. We will review the mathematics behind it and classify its solutions in section 3.1 for Euclidean and in section 4.1 for Lorentzian signature. Note that in the Lorentzian a similar equation is given for ϵ_- by complex conjugation.

Let us comment on the Euclidean case. Upon Wick rotation we drop the Majorana condition $\epsilon^* = \epsilon$. This implies that ϵ_+ and ϵ_- are not charge conjugate any longer and we have two independent equations,

$$\nabla^A_m\epsilon_\pm = \frac{1}{4}\gamma_m D^A\epsilon_\pm \ ,$$

where $\nabla^A_m\epsilon_\pm = (\nabla_m \mp iA_m)\epsilon_\pm$ and we allow for complex values of A_m. $m, n, \ldots$ will denote curved indices for Euclidean M_4. As usual, in the Euclidean the spinors have been doubled.

2.1.1 Generalisation to extended supersymmetry

The analogous case with $\mathcal{N} = 2$ supersymmetry on the boundary is slightly more involved. Minimal gauged $\mathcal{N} = 4$ supergravity in five dimensions [77] contains more fields, and so does $\mathcal{N} = 2$ conformal supergravity in four [78, 79]. In particular, there is an additional spinor, but also a tensor and a scalar in the off-shell multiplet. Nevertheless, it is still true that the latter can be obtained from the former in an asymptotical AdS background, see e.g. [55]. We will not review this here. The structure and supersymmetry conditions for $\mathcal{N} = 2$ conformal supergravity will be discussed in detail in section 6.1 for Lorentzian signature and in section 7.1 for the Wick rotation to the Euclidean. Here we just want to briefly report on the fact that also for extended supersymmetry, superconformality is ensured by the existence of a somewhat 'generalised' conformal Killing spinor. More explicitly, the vanishing of the gravitino variation gives

$$\nabla^A_\mu\epsilon^i_+ + \frac{1}{4}T^+_{\mu\nu}\gamma^\nu\epsilon^i_- = \frac{1}{4}\gamma_\mu D^A\epsilon^i_+ \ , \tag{2.13}$$

where $\nabla^A_\mu \epsilon^i_+ = \nabla_\mu \epsilon^i_+ - iA_\mu{}^i{}_j \epsilon^j_+$ is twisted with the $U(2)$ R-symmetry gauge field, $i = 1, 2$ is a $U(2)$ index and $T^+_{\mu\nu}$ is the tensor field mentioned above. We recognise the similarity to the CKS equation (2.12) found above. There is another equation (see (6.4b)) coming from the vanishing of the aforementioned 'additional' spinor in the gravity multiplet, which a priori gives extra conditions for the generalised Killing spinor reported here. We will discuss these conditions in detail in the later chapters.

2.2 From four-dimensional gravity to CFT$_3$'s

The analysis of $\mathcal{N} = 2$ gauged supergravity in four dimensions is very similar and we will be brief here. In one dimension less, the asymptotic bulk metric we will be using is, analogously to (2.1),

$$ds^2_4 = \frac{dr^2}{r^2} + \left(r^2 ds^2_{M_3} + O(r)\right) , \tag{2.14}$$

where M_3 is the Euclidean[2] three dimensional boundary manifold we are interested in. The asymptotic behaviour of the gauge field parallels that of the four dimensional one and we find again that to leading order in $\frac{1}{r}$ the curvature of the gauge field drops out of the Killing spinor equation on the boundary. In frame indices $(a, 4)$, $a = 1, 2, 3$, we get for the leading order asymptotic Killing spinor equation

$$\left(\partial_4 + \frac{1}{2}\gamma_4\right)\epsilon = 0 \ , \qquad \left(\nabla^A_a + \frac{r}{2}\gamma_a(1+\gamma_4)\right)\epsilon = 0 \ , \tag{2.15}$$

where $\nabla^A_a \equiv \nabla_a - iA_a$ is now the covariant derivative with respect to the metric $ds^2_{M_3}$.

Since γ_4 squares to one, we can divide spinors into eigenspaces of eigenvalue ± 1, $\gamma_4 \epsilon_\pm = \pm\epsilon_\pm$. The first equation in (2.15) then gives

$$\epsilon = r^{\frac{1}{2}}\epsilon_- + r^{-\frac{1}{2}}\epsilon_+ \ . \tag{2.16}$$

Plugging this into the second equation of (2.15) gives, at leading order,

$$\nabla^A_a \epsilon_- + \gamma_a \epsilon_+ = 0 \ . \tag{2.17}$$

We can use γ^4 to reduce spinors from four dimensions to three. In a basis where

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \sigma^a & 0 \end{pmatrix} , \qquad \gamma^4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} , \tag{2.18}$$

[2] For a discussion of supersymmetry on Lorentzian three-manifolds, see [59]

the spinors $\epsilon_\pm$ can be rewritten as $\epsilon_\pm = \left(\begin{smallmatrix} \pm i\chi_\pm \\ \chi_\pm \end{smallmatrix}\right)$, where $\chi_\pm$ are three-dimensional spinors. This gives

$$\nabla_a^A \chi_- = -i\sigma_a \chi_+ \ . \tag{2.19}$$

We can get an expression for χ_+ by taking the trace:

$$\chi_+ = \frac{i}{3} D^A \chi_- \ , \tag{2.20}$$

where $D^A \equiv \sigma^a \nabla_a$ is the Dirac operator. We have obtained again

$$\left(\nabla_a^A - \frac{1}{3}\sigma_a D^A \right) \chi_- = 0 \ . \tag{2.21}$$

Which we recognise as three dimensional version of the (charged) *conformal Killing spinor*. We will classify the spaces that admit a solution to this equation in section 5.1.

2.3 Conformal Killing spinors and superconformal theories

The appearance of the equation (2.21) and (2.12) for a conformal Killing spinor at the conformal boundary of the gravity solution can be easily explained.

As we reviewed in the outline section, if we want to define a supersymmetric theory on a curved manifold M, an efficient strategy [43] consists in coupling the theory to supergravity and then freeze the fields of the gravitational multiplet. The value of the auxiliary fields determines the coupling of the theory to the curved background.

For a superconformal theory, one can proceed similarly and couple the theory to the fields of conformal supergravity g_{mn}, ψ_m and A_m. At the linearised level, these fields couple to the superconformal currents:

$$-\frac{1}{2} g_{mn} T^{mn} + A_m J^m + \bar{\psi}_m \mathcal{J}^m \ , \tag{2.22}$$

where J^m is the R-symmetry and $\mathcal{J}^m$ is the supersymmetry current. For us, the fields of conformal gravity will play the role of background fields; since we work in Euclidean signature, we will allow the auxiliary field A_m to be complex.

In order to preserve some supersymmetry, the gravitino variation must vanish. For simplicity, we write the variation for a four-dimensional theory where they read (with obvious redefinitions) [46, 50]

$$\delta\psi_m = (\nabla_m - iA_m\gamma_5)\,\epsilon + \gamma_m\gamma_5\eta \tag{2.23}$$

where ϵ is the parameter for the supersymmetries Q and η for the superconformal transformations S. It is crucial for our arguments that, as stressed many times in the old days[3], the algebra of the superconformal transformations of g_{mn}, ψ_m, A_m closes off shell. Therefore the variation (2.23) depends only on the background field A_m and is not modified by the coupling to matter. Moreover, the supergravity action for the fields g_{mn}, ψ_m, A_m is separately invariant and can be safely omitted without spoiling the superconformal invariance of the matter part.

The vanishing of the gravitino variation constrains the manifolds where we can have supersymmetry. As expected, equation (2.23) is identical to (2.9) which, in turn, is equivalent to the conformal Killing equation. Notice that ϵ and η in our bulk computation appear in the asymptotic expansion (2.8) with a different power of r corresponding precisely to the conformal dimension of the supercharges Q and S.

[3]See for instance [80] for a very simple example of the logic we will be using in this thesis.

Chapter 3

The geometry of $\mathcal{N} = 1$ in four Euclidean dimensions

In this chapter, we focus on the four dimensional case with one of the four $\mathcal{N} = 1$ supercharges preserved in Euclidean signature. We saw in section 2.1 that starting from very mild assumptions on the structure of the gravity dual, the boundary manifold will have a *conformal Killing spinor*, namely a solution of the conformally invariant equation $\left(\nabla^A_m - \frac{1}{d}\gamma_m D^A\right)\epsilon = 0$, where $D = \gamma^m\nabla_m$ is the Dirac operator, and ^{A} denotes twisting by a gauge field, in general complex. Following the same logic as in [43], applied this time to conformal supergravity [47–53], the existence of a conformal Killing spinor is exactly the condition one needs in order to preserve one supercharge for a superconformal theory[1].

We show that a conformal Killing spinor exists on any complex manifold (Kähler or not)[2]. Thus, a superconformal field theory on any complex manifold preserves at least one supersymmetry, if we turn on a background field (in general complex) for the R-symmetry. In fact, conformal supergravity gives rise to ordinary supergravity once one breaks conformal invariance by giving expectation value to fields in auxiliary compensator multiplets (as for example in [70]; for a review see [46]). In particular, applying this idea to a tensor multiplet gives rise to "new minimal supergravity" [72]. This suggests that one might extend our results to any supersymmetric theory with an R-symmetry (not necessarily conformal); and we indeed show, using again the method in [43], that a

[1] The relevance of conformal Killing spinors was also realised for superconformal σ models in [81, 82].

[2] There exist many complex manifolds which are not Kähler. The most famous example is perhaps the Hopf surface, which is diffeomorphic to $S^3 \times S^1$, or more generally all "class VII" Kodaira surfaces. Primary and secondary Kodaira surfaces in the Enriques–Kodaira classification are also non-Kähler (see for example [83, Ch. VI]).

supersymmetric theory preserves at least one supercharge on any complex manifold. A very similar analysis has been presented for the "old minimal supergravity" in [57, 62] [3].

Our analysis can also be used to identify concretely the Lagrangian of the theory dual to a given supergravity background. Suppose we have a supergravity theory whose AdS solution is dual to a given CFT in flat space. If one has another solution of the same supergravity theory which is asymptotically locally AdS, it is possible to read off the value of the boundary metric and of the background field for the R-symmetry, and to write the Lagrangian of the CFT on the resulting curved space using our discussion in section 2.3 (and in particular (2.22)), in agreement with the standard AdS/CFT dictionary. Moreover, from (3.30) one can also identify the background field appearing in new minimal supergravity, which is crucial to write the Lagrangian for any supersymmetric but non-conformal deformation of the CFT.

This chapter is organised as follows. We did already review how conformal Killing spinors arise from holography in section 2.1, now in section 3.1 we will study the geometry of such spinors; in the charged case, we find that any complex manifold admits one. In section 3.2, we show that any supersymmetric field theory preserves one supercharge on a complex manifold.

The material presented in this chapter is based on the paper [1]. Similar results have appeared in [56].

3.1 Geometry of conformal Killing spinors

We have seen in the previous chapter that preserved supersymmetry implies the existence of a charged conformal Killing spinor. In this section, we will review some geometry behind the defining equation

$$P^A_m \epsilon \equiv \left(\nabla^A_m - \frac{1}{d}\gamma_m D^A \right) \epsilon = 0\,, \tag{3.1}$$

and classify its solutions. Notice that the conformal Killing operator P^A_m is covariant under Weyl rescaling. The operator $\bar{P}^A_m$ of the rescaled metric $\bar{g} = e^f g$ is indeed given by

$$\bar{P}^A_m e^{f/4} = e^{f/4} P^A_m \ . \tag{3.2}$$

[3] See also [61].

3.1.1 The $A = 0$ case

In the uncharged case ($A = 0$), the conformal Killing spinor equation reads

$$\left(\nabla_m - \frac{1}{d}\gamma_m D\right)\epsilon \equiv P_m\epsilon = 0 \ . \tag{3.3}$$

One way to think of the operator P_m is the following. The covariant operator ∇_m goes from the bundle of spinors Σ to the bundle $T \otimes \Sigma$ of vector-spinors. The sections of the latter are a reducible representation of the orthogonal (or Lorentz) group. It can be written as the direct sum of two representations: a "trace", defined by taking a section ψ_m and multiplying it by γ^m, and the traceless part.[4] The orthogonal projector on this second irreducible representation can be written as $\delta_m^n - \frac{1}{d}\gamma_m\gamma^n$. Now, projecting ∇_m on the trace representation gives the Dirac operator D, while projecting on the traceless part gives

$$\left(\delta_m^n - \frac{1}{d}\gamma_m\gamma^n\right)\nabla_n = \nabla_m - \frac{1}{d}\gamma_m D = P_m \ . \tag{3.4}$$

So in a sense P_m is the "complement" of the Dirac operator. Some of the properties of P_m (and of its zero modes, the conformal Killing spinors) have been studied by mathematicians; see for example [84] in the Euclidean case and [85] in the Lorentzian case. In particular, some of these results can be used to classify completely the manifolds on which a conformal Killing spinor can exist, as we will now review.

Consider a conformal Killing spinor ϵ. One can show that $D^2\epsilon \propto R\,\epsilon$, where R is the scalar curvature. Using the solution to the Yamabe problem [86, 87], one can make R constant by a conformal rescaling of the metric. ϵ is then an eigenspinor for D^2; namely,

$$(D^2 - \mu^2)\epsilon = (D - \mu)(D + \mu)\epsilon = 0 \ , \tag{3.5}$$

for some μ. The spinors

$$\psi \equiv \epsilon + \frac{1}{\mu}D\epsilon \ , \qquad \tilde{\psi} \equiv \epsilon - \frac{1}{\mu}D\epsilon \ , \tag{3.6}$$

are then eigenspinors of D. A theorem by Hijazi [88] now tells us that any eigenspinor of D is also a *Killing spinor*, namely a spinor ϵ that satisfies

$$\nabla_m\epsilon = \mu\gamma_m\epsilon \ . \tag{3.7}$$

[4] This is familiar from the NS⊗R sector of the NSR superstring, which decomposes into a dilatino (the trace) and the gravitino (the traceless part).

Such spinors are familiar from the supergravity literature; for example, one can find explicit expression for Killing spinors on the sphere S^n in [89]. One can readily check that every Killing spinor is a conformal Killing spinor; thus, a priori (3.7) would seem to be more restrictive than (3.3). However, as we have just described, existence of a solution to (3.3) is in fact equivalent to existence of a solution to (3.7) (with a Weyl rescaled metric).

In fact, manifolds which admit Killing spinors have been classified. Notice first that the usual compatibility between different components of (3.7) gives $R_{mn} = -2\mu^2 g_{mn}$. This implies that μ should be either real or purely imaginary. The real case can be shown [90] to be realised only on non-compact manifolds, which are in fact a warped product of $\mathbb{R}$ with any manifold M, with metric $dr^2 + e^{-4\mu r} ds^2_M$. When μ is purely imaginary, one can observe [91] that the existence of a Killing spinor on M_d implies the existence of a covariantly constant spinor on the cone $C(M_d)$. Such manifolds are in turn classified using their restricted holonomy.

For example, in dimension four, the cone $C(M_4)$ would be a five-dimensional manifold with restricted holonomy, which can only be $\mathbb{R}^5$. This tells us that S^4 is the only four-manifold with Killing spinors, and thus the only four-manifold with conformal Killing spinors (up to Weyl rescaling).

The case of S^4 is also instructive in other respects. It is known that there is no almost complex structure on this manifold. A chiral spinor defines at each point an almost complex structure; thus, there can be no chiral spinor without zeros on S^4. On the other hand, a Killing spinor has no zeros, because (3.7) implies that the norm of ϵ is constant. There is no contradiction: a Killing spinor is never chiral; so $\epsilon = \epsilon_+ + \epsilon_-$, where $\epsilon_\pm$ are chiral. Both ϵ_+ and ϵ_- have one zero, which explains why there is no almost complex structure on S^4, but the norm of ϵ is still constant. In fact, [84, Th. 7] shows that, in any dimension, the sphere S^d is the only manifold on which a conformal Killing spinor can have a zero.

In dimension three the situation is similar. Since $\mathbb{R}^4$ is the only four-manifold with restricted holonomy, S^3 (or quotients thereof) is the only compact three-manifold with Killing spinors. Unlike in the four dimensional case, the conformal Killing spinors on S^3 never vanish. In higher dimensions, we have a larger class of possibilities. The existence of Killing spinors identifies Sasaki-Einstein manifolds in five dimensions and nearly-Kähler manifolds in six [91]. The corresponding cones with restricted holonomy are Calabi-Yau three-folds and G_2 manifolds, respectively. Only in the case of S^6 the conformal Killing spinor is allowed to have a zero.

Let us summarise the results we have reviewed in this subsection. If a manifold admits a conformal Killing spinor (uncharged, namely with $A = 0$), it also admits a closely related Killing spinor. Manifolds with Killing spinors, in turn, are also completely classified; they are either warped products of a manifold with $\mathbb{R}$, or bases of cones with covariantly constant spinors.

3.1.2 The $A \neq 0$ case in four dimensions

We will now turn to the case with $A \neq 0$:

$$\left(\nabla^A_m - \frac{1}{d}\gamma_m D^A\right)\epsilon = P^A_m \epsilon = 0 \ , \qquad m = 1, \ldots, 4 \ . \tag{3.8}$$

In general, A will be a connection on a bundle. We will actually take ReA to be a connection on a U(1) bundle $\mathcal{U}$, and ImA to be a one-form. Accordingly, ϵ will be not quite a spinor, but a "charged", or Spin^c, spinor; namely, a section of

$$\mathcal{U} \otimes \Sigma \ , \tag{3.9}$$

where Σ is the spinor bundle.

(3.8) has also been considered by mathematicians (see e.g. [92, Part III]), but in this case it is no longer true that existence of its solutions is equivalent to the existence of charged Killing spinor (which have been studied for example in [93]). Thus a complete classification of the solutions to (3.8) is currently not available.

We will thus study (3.8) here. In this section, we will deal with the four-dimensional case. Since (3.8) does not mix different chiralities (unlike (3.7)), we can consider its chiral solutions separately. For simplicity, we will assume that $\epsilon = \epsilon_+$ is a spinor of positive chirality.

Intrinsic torsions

We can borrow some of the tools that have been successfully used in the analysis of supersymmetric solutions in supergravity. The first idea is to parameterise the covariant derivatives of ϵ_+ in terms of a basis of spinors. This strategy has been used for a long time (for example [94, (2.2)]); in the case of a four-dimensional Euclidean manifold, this was used recently in [62, 95]. In the case at hand, a basis in the space of spinors with positive chirality is given by

$$\epsilon_+ \ , \qquad \epsilon^C_+ \equiv C\epsilon^* \ , \tag{3.10}$$

where C is the intertwiner such that $\gamma_m^* = C^{-1}\gamma_m C$. A basis for the space of spinors of negative chirality is given by either

$$\gamma_m \epsilon_+ \tag{3.11}$$

or by

$$\gamma_m \epsilon_+^C \ ; \tag{3.12}$$

as we will see shortly, this second choice is related to (3.11). Using the basis (3.10), we can expand

$$\nabla_m \epsilon_+ = p_m \epsilon_+ + q_m \epsilon_+^C \ . \tag{3.13}$$

p_m, q_m are locally complex one-forms. Globally speaking, Imp is a connection on $\mathcal{U}$, Rep is a one-form, and q is a section of $\mathcal{U}^2 \otimes T^*$.

An alternative, perhaps more transparently geometrical, point of view, consists in noticing that ϵ_+ defines an U(2) structure on M_4. We can express it in terms of forms by considering the bispinors

$$\epsilon_+ \otimes \epsilon_+^\dagger = \frac{1}{4} e^B e^{-ij} \ , \qquad \epsilon_+ \otimes \overline{\epsilon_+} = \frac{1}{4} e^B \omega \ , \tag{3.14}$$

where $\bar\epsilon \equiv \epsilon^T C^{-1}$, $e^B \equiv ||\epsilon_+||^2$, and j is a real two-form. ω is locally a complex two-form; globally, it is actually a section of

$$\mathcal{U}^2 \otimes \Omega^{2,0} \ , \tag{3.15}$$

where recall $\mathcal{U}$ is the U(1) bundle for which A is a connection. j and ω satisfy

$$\omega^2 = 0 \ , \qquad \omega \wedge \bar\omega = 2 j^2 \ , \tag{3.16}$$

or, more symmetrically,

$$\begin{aligned} &j \wedge \mathrm{Re}\omega = \mathrm{Re}\omega \wedge \mathrm{Im}\omega = \mathrm{Im}\omega \wedge j = 0 \ , \\ &j^2 = (\mathrm{Re}\omega)^2 = (\mathrm{Im}\omega)^2 \ . \end{aligned} \tag{3.17}$$

One can use these forms to relate the two choices for a basis of spinors of negative chirality, (3.11) and (3.12):

$$\gamma_m \epsilon_+ = \frac{1}{2} \omega_{mn} \gamma^n \epsilon_+^C \ . \tag{3.18}$$

Notice that this also implies that ϵ_+ and ϵ_+^C are annihilated by half of the gamma matrices[5]:

$$\bar\Pi_m{}^n \gamma_n \epsilon_+ = 0 \ , \qquad \Pi_m{}^n \gamma_n \epsilon_+^C = 0 \ , \tag{3.19}$$

[5]In other words, they are *pure*; it is indeed well-known that any spinor in even dimension ≤ 6 has this property [96, Rem. 9.12].

where $\Pi_m{}^n \equiv \frac{1}{2}(\delta_m^n - iI_m{}^n) = \frac{1}{4}\omega_{mp}\bar{\omega}^{np}$ is the holomorphic projector (relative to the almost complex structure $I_m{}^n \equiv j_{mp}g^{pn}$).

Strictly speaking, the previous discussion should be taken with a grain of salt in the case where ϵ_+ has zeros. In general j and ω will not be well-defined on any zero z_i, and will define a U(2) structure only on $M_4 - \{z_i\}$.

Similarly to (3.13), it is easy to parameterise the derivatives of j and ω, by decomposing dj and $d\omega$ in SU(2) representations. A three-form α_3 can always be written as $\alpha_3 = \alpha_1 \wedge j$, where α_1 is a one-form; or, it can be decomposed into its $(2,1)$ part and its $(1,2)$ part, which can in turn be re-expressed as $\omega \wedge \beta_{0,1}$ and $\bar{\omega} \wedge \tilde{\beta}_{1,0}$. These two possibilities can be exchanged with one another by using (3.18).

We follow both strategies to write

$$dj = w^4 \wedge j \ , \qquad d\omega = w^5 \wedge \omega + w^3 \wedge \bar{\omega} \ . \tag{3.20}$$

w^4 is a real one-form. w^5 and w^3 are locally complex one-forms; globally, $\mathrm{Im} w^5$ is a connection on $\mathcal{U}^2$, $\mathrm{Re} w^5$ is a one-form, and w^3 is a section of $\mathcal{U}^4 \otimes T^*$.

The w^i are collectively called "intrinsic torsion". Our choice to write dj using j and $d\omega$ using ω and $\bar{\omega}$, and our names for the one-forms w^i, might seem mysterious. We made these choices to be as close as possible to a notation commonly used for U(3) structures on six-manifolds, where the intrinsic torsion consists of forms $W_1, \ldots, W_5$ of various degrees (a notation which is also not particularly suggestive, but which has become traditional; see [94, 97]). Notice that w_5 can be assumed to have $(0,1)$ part only, and w_3 to have only $(1,0)$ part.

Our parameterisation of $\nabla\epsilon_+$ in (3.13) is nothing but a spinorial counterpart of the intrinsic torsions w_i in (3.20). In fact, we can easily compute a relation between the two, using the definitions (3.14) of j and ω. We get:

$$w^4 = -2\mathrm{Re}(\bar{q} \llcorner \omega) \ , \qquad w^5_{0,1} = 2i(\mathrm{Im} p)_{0,1} - \frac{1}{2} q \llcorner \bar{\omega} \ , \qquad w^3_{1,0} = \frac{1}{2} q \llcorner \omega \ . \tag{3.21}$$

As a byproduct, we also obtain a relation on B (which was defined earlier as $e^B \equiv ||\epsilon_+||^2$):

$$dB = 2\mathrm{Re} p \ . \tag{3.22}$$

General solution

In supergravity applications, it is usually straightforward to compute dj and $d\omega$ directly from the spinorial equations imposed by supersymmetry. In this case, it is more convenient to compute first the torsions p and q in (3.13) from the conformal Killing spinor equation (3.8). The computation involves the action of γ_{mn}:

$$\begin{aligned}\gamma_{mn}\epsilon_+ &= ij_{mn}\epsilon_+ - \omega_{mn}\epsilon_+^C \ ,\\ \gamma_{mn}\epsilon_+^C &= -ij_{mn}\epsilon_+^C + \bar\omega_{mn}\epsilon_+ \ .\end{aligned} \tag{3.23}$$

This allows us to rewrite (3.8) as[6]

$$p^A_{1,0} = 0 \ , \qquad 2p^A_{0,1} + q_{1,0}\llcorner\bar\omega = 0 \ , \qquad q_{0,1} = 0 \ . \tag{3.24}$$

where $p^A_m \equiv p_m - iA_m$.

(3.24) can also be translated into equations for the intrinsic torsions w_i defined in (3.20):

$$w^3 = 0 \ , \tag{3.25a}$$

$$iA_{1,0} = -\frac{1}{2}\overline{w^5_{0,1}} + \frac{1}{4}w^4_{1,0} + \frac{1}{2}\partial B \ , \tag{3.25b}$$

$$iA_{0,1} = +\frac{1}{2}w^5_{0,1} - \frac{3}{4}w^4_{0,1} + \frac{1}{2}\bar\partial B \ . \tag{3.25c}$$

We see that (3.25b) and (3.25c) simply determine A and do not impose any constraints on the geometry. On the other hand, $w^3 = 0$ has a geometrical meaning: namely,

$$(d\omega)_{1,2} = 0 \ . \tag{3.26}$$

When ϵ has no zeros anywhere, this is just a way of saying that the manifold M_4 should be *complex*.

Let us briefly review why[7]. From its definition as a bispinor in (3.14), we know that the two-form ω is decomposable, i.e. it can locally be written as a wedge of two one-forms:

$$\omega = e^1 \wedge e^2 \ . \tag{3.27}$$

These one-forms e^i can be taken as generators of the holomorphic tangent bundle $T^{1,0}$; this defines an almost complex structure I_ω. Clearly, if I_ω is integrable, $d\omega$ is a $(2,1)$-form, and hence (3.26) holds. To see that the converse is also true, observe that (3.26) can only be true if d of a $(1,0)$ form never contains a $(0,2)$ part; or, by conjugation, if

$$(de^{\bar i})_{(2,0)} = 0 \ , \tag{3.28}$$

[6]Writing (3.8) as $E^1_m\epsilon_+ + E^2_m\epsilon_+^C = 0$, one would expect two vector equations; decomposing each into $(1,0)$ and $(0,1)$ parts would give four equations. However, $E^1_{0,1}$ and $E^2_{1,0}$ can be shown to be equivalent. This can also be seen from the fact that multiplying (3.8) by γ^m is automatically zero, and from (3.18).

[7]This idea is usually attributed to Andreotti.

where $i = 1, 2$ is a holomorphic index. Consider now any two $(1,0)$ vectors E_i, E_j. We have the following chain of equalities:

$$([E_j, E_k]_{\rm Lie})\llcorner e^{\bar{i}} = [\{d, E_j\llcorner\}, E_k\llcorner]\llcorner e^{\bar{i}} = -E_k\llcorner\{d, E_j\llcorner\}e^{\bar{i}} = -E_k\llcorner E_j\llcorner de^{\bar{i}} = 0 \ . \qquad (3.29)$$

In the first step, we have used Cartan's magic formulas relating d, Lie derivatives and vector contractions. (3.29) means that the Lie bracket of any two $(1,0)$ vectors is still $(1,0)$, which is the definition of integrability. So I_ω is a complex structure, and the manifold M_4 is complex.

Conversely, if M_4 is complex, there exists a solution of (3.8). Given a complex structure I, let ω_I be a section of its canonical bundle $K \equiv \Lambda T^*_{1,0}$. I defines a $\mathrm{Gl}(2,\mathbb{C})$ structure on M_4; but $\mathrm{Gl}(2,\mathbb{C})$ is homotopy equivalent to $\mathrm{U}(2)$, and for this reason there is actually a $\mathrm{U}(2)$ structure on M_4. This means that there always exists a two-form j compatible with ω, in the sense that $j \wedge \omega = 0$ (as in (3.16)), or in other words that j is a $(1,1)$ for I; this also implies that j and I define together a metric via $g = Ij$.[8] The volume form of this metric is just $-\frac{1}{2}j^2$; by choosing an appropriate function B, we can now define a normalised $\omega = e^{-B}\omega_I$ so that $\omega \wedge \bar{\omega} = 2j^2$ is also true (again as in (3.16)). We can now define the w^i from (3.20); since I is complex, $w^3 = 0$. Finally, as remarked earlier, (3.25b) and (3.25c) simply determine A in terms of the w^i and B; it can be checked that it transforms as a connection.

If ϵ has zeros z_i, only $M_4 - \{z_i\}$ will be complex, and not the whole of M_4. This is for example the case for S^4. As we discussed at the end of section 3.1.1, in this case a chiral conformal Killing spinor ϵ_+ has a zero at one point; the complement of that point is conformally equivalent to $\mathbb{R}^4$, which obviously admits a complex structure. Conversely, if one finds a complex structure on $M_4 - \{z_i\}$, one can determine A through (3.25b), (3.25c), and one should then check whether it extends smoothly to the entire M_4.

To summarise, the charged version of the conformal Killing spinor equation, (3.8), is much less restrictive than the uncharged version studied in section 3.1.1. We found that the only requirement on the geometry is (3.26), which can be solved for example by requiring that the manifold is complex. Moreover, A is determined in terms of the geometry by (3.25b), (3.25c).

[8] One can alternatively reason as follows. Given any metric $\tilde{g}$ on M_4, the 'projected' metric $g_{mn} \equiv (\Pi_m{}^p \bar{\Pi}_n{}^q + \bar{\Pi}_m{}^p \Pi_n{}^q)\tilde{g}_{pq} = \frac{1}{2}(\tilde{g}_{mn} + I_m{}^p I_n{}^q \tilde{g}_{pq})$ is hermitian with respect to I. One can then define a two-form via $J_{mn} \equiv I_m{}^p g_{pn}$, which is indeed antisymmetric, as one can easily check.

3.2 Supersymmetric theories on curved spaces from new minimal supergravity

In the previous section, we have studied the constraints imposed by the presence of at least one supercharge in a superconformal theory, by coupling the theory to conformal supergravity. We will now show that those results can be interpreted very naturally also as the coupling of a supersymmetric theory to "new minimal supergravity" [72, 98]. In particular, we show that every solution of the new minimal equations is a conformal Killing spinor. Vice versa, every conformal Killing spinor (without zeros[9]) gives rise to a solution of the new minimal equations. We can then use the results in section 3.1 to understand when we can consistently define a supersymmetric, but not necessarily conformal, theory with an R-symmetry on a curved manifold.

3.2.1 Equivalence with conformal Killing spinor equation

We start with a solution of the conformal Killing spinor equation (3.8) without zeros, charged under a connection A.

As a first step, notice that $D^A\epsilon_+$ is a negative chirality spinor, and as such can be expanded in the basis (3.11):

$$D^A\epsilon_+ \equiv 2iv\epsilon_+ \tag{3.30}$$

where $v = v^m\gamma_m$ and v^m is a vector.[10] Since ϵ has no zeros, v^m is defined everywhere. An easy computation now shows that (3.8) can be rewritten as[11]

$$\nabla_m\epsilon_+ = -i\left(\frac{1}{2}v^n\gamma_{nm} + (v-a)_m\right)\epsilon_+ \ , \qquad a \equiv A + \frac{3}{2}v \ . \tag{3.31}$$

This is exactly the condition for the existence of at least one unbroken supersymmetry in new minimal supergravity [72, 98]. When this condition has a solution, we can consistently define supersymmetric theories on the four-manifold M_4 using the strategy in [43].

Actually, v starts its life as the auxiliary field of a tensor multiplet; so one should impose that it can be dualised back:

$$d * v = 0 \ . \tag{3.32}$$

[9]We thank the authors of [56] for a useful comment about this point.

[10]Its $(1,0)$ part is immaterial because of (3.19); its $(0,1)$ part can be written in terms of the intrinsic torsions in (3.13) as $v_{0,1} = -\frac{i}{2}\left(p^A_{0,1} - \frac{1}{2}q\llcorner\bar{\omega}\right)$.

[11]We use lower-case letters a and v for the auxiliary fields of new minimal supergravity, in order to avoid confusion with the A of conformal supergravity we have been using until now.

We can use the ambiguity in the definition of v to arrange this condition. Since v is defined only up to its $(1,0)$ part, we have two arbitrary complex parameters that can be used to enforce (3.32). An alternative geometrical perspective is the following. We can first choose v imaginary and then perform a conformal rescaling of the metric

$$g_{mn} \to e^{2f} g_{mn} \;\Rightarrow\; D \to \left(e^{-f} D + \frac{d-1}{2} \partial_m f \gamma^m \right) . \tag{3.33}$$

In $d = 4$, this transforms $v \to v - i\frac{3}{4} df$, and one can use the freedom in choosing f to arrange so that (3.32) is satisfied.

Hence we have shown that one can take the charged conformal Killing spinor equation (3.8) to the condition of unbroken supersymmetry in new minimal supergravity (3.31). The fact that one can bring (3.8) to (3.31) was to be expected because of the formalism of conformal compensators (for a review see [46]). In that formalism, one obtains new minimal supergravity by coupling a tensor multiplet to conformal supergravity, and by then giving an expectation value to the tensor multiplet.

By reversing the previous argument, it is clear that every solution of the new minimal equation (3.31) is also a solution of the conformal Killing equation (3.8).

We have now two ways of defining a conformal field theory on a curved background, either by coupling to conformal supergravity or by coupling to new minimal supergravity. The resulting theory is however the same. The coupling to new minimal supergravity will add linear and quadratic terms in the auxiliary fields a, v, as discussed in [43] At the linear level in the auxiliary fields the bosonic action contains the coupling to the supercurrent multiplets [43, 72]

$$-\frac{1}{2} g_{mn} T^{mn} + \left(a_m - \frac{3}{2} v_m \right) J^m + \bar{\psi}_m \mathcal{J}^m - \frac{1}{2} b^{mn} t_{mn} , \tag{3.34}$$

where J^m and $\mathcal{J}^m$ are the R-symmetry and the supersymmetry current, respectively. In the non conformal case, the multiplet of currents also contains a conserved t_{mn} ($\nabla^m t_{mn} = 0$) which measures precisely the failure of the theory at being conformally invariant. t_{mn} couples to the dual of the auxiliary field v: $v^m = \epsilon^{mnpr} \partial_n b_{pr}$. In the conformal case $t_{mn} = 0$ and the linear coupling to v_m vanishes. The remaining terms reproduce the couplings to the conformal supergravity 2.22 since $A = a - \frac{3}{2} v$. The quadratic terms work similarly.

3.2.2 One supercharge

Even though we have already analysed the geometrical content of the conformal Killing spinor equation (3.8) in section 3.1.2, it is instructive to repeat the analysis starting directly from (3.31).

We again introduce j and ω as in (3.14). This time it is most convenient to calculate directly dj and $d\omega$ from (3.31). First of all we compute

$$\begin{aligned} d(\epsilon_+\epsilon_+^\dagger) &= (-2\text{Im}a\wedge +i\text{Re}v\llcorner)\epsilon_+\epsilon_+^\dagger + \frac{1}{2}e^B(\text{Im}v - i*\text{Re}v) \ , \\ d(\epsilon_+\overline{\epsilon_+}) &= 2ia\wedge\epsilon_+\overline{\epsilon_+} \ . \end{aligned} \tag{3.35}$$

Using (3.14), we get expressions for dj and $d\omega$, which in turn give us

$$v_{0,1} = -\frac{i}{2}w^4_{0,1} \ , \tag{3.36a}$$

$$a_{0,1} = -\frac{i}{2}(\bar\partial B + w^5_{0,1}) \ , \tag{3.36b}$$

$$\left(a - \frac{3}{2}v\right)_{1,0} = -\frac{i}{4}w^4_{1,0} + \frac{i}{2}\overline{w^5_{0,1}} - \frac{i}{2}\partial B \ , \tag{3.36c}$$

as well as

$$(d\omega)_{1,2} = 0 \ . \tag{3.37}$$

Not surprisingly, these relations are consistent with (3.25), which we found by directly analysing the conformal Killing spinor equation (3.8). In particular, we have found again that the vector $A = \left(a - \frac{3}{2}v\right)$ is completely determined in terms of the geometry and B, and that the constraint on the geometry can be solved by taking the manifold complex, by following the steps described in section 3.1.2.

The vector v must satisfy $d{*}v = 0$. We can actually solve this condition for v explicitly: although $A_{1,0} = \left(a - \frac{3}{2}v\right)_{1,0}$ is fixed by (3.36c), $a_{1,0}$ and $v_{1,0}$ are not. By choosing $a_{1,0}$ and $v_{1,0}$ in a convenient way we can impose (3.32). There is a particularly simple choice that always works. By choosing $a_{1,0} = -\frac{i}{2}\partial B + \frac{i}{2}(w^4_{1,0} + \overline{w^5_{0,1}})$, we get $v_{1,0} = \frac{i}{2}w^4_{1,0}$, which together with (3.36a) gives

$$v = -\frac{1}{2} * dj \ . \tag{3.38}$$

This obviously satisfies (3.32).

Due to the ambiguity in choosing $a_{1,0}$ and $v_{1,0}$, we can have different pairs (a, v) that solve all constraints for supersymmetry. For particular manifolds, for example $\mathbb{R} \times M_3$, there can be different and more natural choices for v, as discussed below.

To summarise, using new minimal supergravity and the strategy in [43], a supersymmetric theory with an R-symmetry on any complex manifold M_4 preserves at least one supercharge. This is in agreement with our result in section 3.1 for superconformal theories.

We will now comment in particular on the important subcase where M_4 is Kähler.

Kähler manifolds

A very simple case is $v = 0$ and a real. The new minimal condition 3.31 reduces to the equation for a covariantly constant charged spinor

$$(\nabla_m - ia_m)\,\epsilon_+ = 0 \ , \tag{3.39}$$

which is well known to characterise Kähler manifolds. In our formalism this can be seen easily from equations (3.35). Using (3.14) we learn that B is constant and

$$dj = 0\,, \qquad d\omega = 2ia \wedge \omega \ . \tag{3.40}$$

The second condition implies, as already stressed, that M_4 is complex and the first that it is a Kähler manifold.

It is interesting to consider the case of conical Kähler metrics

$$ds_4^2 = dr^2 + r^2 ds_{M_3}^2 \ . \tag{3.41}$$

The three dimensional manifold M_3 is, by definition, a Sasaki manifold. The cone is conformally equivalent to the direct product $\mathbb{R} \times M_3$ through the Weyl rescaling $ds_4^2 \to \frac{1}{r^2} ds_4^2$. $\mathbb{R} \times M_3$ will also support supersymmetry but with different a, v. If we keep the norm of the spinor fixed, the new minimal conditions (3.36) for a Weyl rescaled metric $e^{2f} ds_4^2$ will be satisfied with the replacement

$$v \to v - idf \qquad a \to a - idf \ . \tag{3.42}$$

In the case of $\mathbb{R} \times M_3$, we see that a and v have acquired an imaginary contribution idt in terms of the natural variable $r = e^t$ parameterising $\mathbb{R}$. Notice that v is not of the form (3.38) but that it nevertheless satisfies $d * v = 0$.

The theory on $\mathbb{R} \times M_3$ can be reduced to give a three dimensional supersymmetric theory on M_3. We will return to the study of three dimensional theories in chapter 5.

3.2.3 Two supercharges

It is interesting to consider the case where we have two supercharges $\epsilon_\pm$ of opposite chirality. The Euclidean spinors should satisfy the new minimal equations [43] which in the Euclidean read

$$\begin{aligned} \nabla_m \epsilon_+ &= -i\left(\frac{1}{2} v^n \gamma_{nm} + (v-a)_m\right)\epsilon_+ \ , \\ \nabla_m \epsilon_- &= +i\left(\frac{1}{2} v^n \gamma_{nm} + (v-a)_m\right)\epsilon_- \ . \end{aligned} \tag{3.43}$$

With two spinors, in addition to (3.14), we can construct the odd bispinors

$$\epsilon_+ \otimes \epsilon_-^\dagger = \frac{1}{4} e^B (z_2 + *z_2) \ , \qquad \epsilon_+ \otimes \overline{\epsilon_-} = \frac{1}{4} e^B (z_1 + *z_1) \ , \tag{3.44}$$

where $\{z^i\}$ is a holomorphic vielbein, in terms of which $\omega = z^1 \wedge z^2$ and $j = \frac{i}{2}\left(z^1 \wedge \overline{z^1} + z^2 \wedge \overline{z^2}\right)$. It is easy to show that $z^1_m = \overline{\epsilon_-}\gamma_m \epsilon_+$ is a Killing vector. In fact

$$\nabla_{\{m} z^1_{n\}} = 0 \, , \qquad \nabla_{[m} z^1_{n]} = -i \epsilon_{mn}{}^{pr} z^1_p v_r \ . \tag{3.45}$$

We thus learn that we always have two isometries when there are two supercharges of opposite chirality.

The commutator of the two supersymmetries closes on the isometry generated by z^1. For example, if we take the transformation rules for a chiral multiplet [43, 62]:

$$\delta\phi = -\overline{\epsilon_+}\psi_+ \ , \qquad \delta\bar{\phi} = -\overline{\epsilon_-}\psi_- \ ; \tag{3.46a}$$

$$\delta\psi_+ = F\epsilon_+ + \nabla^a_m \phi \gamma^m \epsilon_- \ , \qquad \delta\psi_- = \bar{F}\epsilon_+ - \nabla^a_m \bar{\phi}\gamma^m \epsilon_+ \ ; \tag{3.46b}$$

$$\delta F = \overline{\epsilon_-}\gamma^m \left(\nabla^a_m - \frac{i}{2} v_m\right) \psi_+ \ , \qquad \delta\bar{F} = \overline{\epsilon_+}\gamma^m \left(\nabla^a_m + \frac{i}{2} v_m\right) \psi_- \ ; \tag{3.46c}$$

a straightforward use of Fierz identities shows that

$$[\delta_{\epsilon_+}, \delta_{\epsilon_-}]\mathcal{F} = \mathcal{L}_{z_1}\mathcal{F} \ , \tag{3.47}$$

where $\mathcal{F}$ is any field in the multiplet $(\phi, \psi_\pm, F)$ and the Lie derivative $\mathcal{L}$ is covariantised with respect to a.

Chapter 4

The geometry of $\mathcal{N} = 1$ in four Lorentzian dimensions

From the point of view of rigid supersymmetry, the Lorentzian signature case has so far been less studied; one exception is anti-de Sitter, which has been considered for a long time, since e.g. [41, 42, 99], and more recently in e.g. [45, 61] (which also contain a more complete list of references). In this chapter we consider the same question as in the last one, namely under what conditions a $\mathcal{N} = 1$ supersymmetric field theory can preserve any supersymmetry on a curved space, in Lorentzian signature.[1] We begin by considering superconformal theories. In section 2.1 we found very generally that the boundary M_4 needs to admit a *conformal Killing spinor* (CKS) ϵ, possibly charged under a gauge field A_μ. The smallest amount of supersymmetry corresponds to ϵ being chiral; since the equation is linear, whenever ϵ is a conformal Killing spinor $i\epsilon$ is one too. So the minimal amount of supersymmetry is two supercharges; we focus on this case. As we will show, the condition on the geometry of M_4 for this to happen is very different from the Euclidean case. Namely, M_4 has a conformal Killing spinor if and only if it has a *null conformal Killing vector* z. The gauge field A_μ can then be determined purely from data of the metric on M_4.

As usual, one can also study supersymmetric theories on curved spaces using the method proposed in [43]. We recall that this consists in coupling the theory to supergravity, and then freezing its fields to background values. For a superconformal theory, the appropriate gravity theory is conformal supergravity the result of this procedure is again that M_4 should admit a conformal Killing spinor. For a supersymmetric theory with an

[1] One might also attack this question using superspace, as in [100, Chap. 6].

R-symmetry which is not superconformal, it is natural to use new minimal supergravity [72], where the off-shell gravity multiplet contains $g_{\mu\nu}$ and two vectors a_μ, v_μ (the former coupling to the R-symmetry current). For the theory obtained by this procedure to be supersymmetric on a curved M_4, one should then solve an equation for ϵ which is (locally) equivalent to the CKS equation, with a suitable map of a_μ, v_μ with A_μ and some data of the geometry. This map in general produces a v_μ which is complex, which in Lorentzian signature is not acceptable; imposing that it should be real turns out to require that the conformal Killing vector z is now actually a *Killing vector*. As we will see, this stronger condition arises automatically from the bulk perspective when a certain natural choice of coordinates is used.

After having determined that supersymmetry leads to clear geometrical requirements, one naturally wonders how this is related to the geometry in the bulk. The geometry of supersymmetric solutions of (Lorentzian) five-dimensional minimal gauged supergravity was considered in [74], and it is interesting to compare our result to their classification. Indeed, one of the conditions found in [74] in the bulk was the existence of a Killing vector V, which may be time-like or null. We will show that this vector always becomes null at the boundary, and reduces to the conformal Killing vector z. We will also check that the other conditions from the bulk become redundant at the boundary, in agreement with our results.

The rest of this chapter is organised as follows. Recall that in section 2.1 we have shown that supersymmetric asymptotically locally AdS solutions in the bulk imply the existence of a charged conformal Killing spinor on the boundary M_4. Here in section 3.1 we will show that such a spinor can exist if and only if M_4 has a null conformal Killing vector, and thus that this is the condition for a superconformal theory on M_4 to preserve some supersymmetry. In section 4.2 we extend our analysis to theories which are not necessarily superconformal, but simply supersymmetric with an R-symmetry; we show that the condition on M_4 is now that it admits a null Killing vector. In sections 4.3 and 4.4 we compare our results on M_4 with the bulk analysis of supersymmetric solutions of gauged minimal supergravity performed in [74], and find agreement.

This chapter is based on the paper [2].

4.1 Geometry of conformal Killing spinors in Lorentzian signature

In this section we will analyse the geometrical content of conformal Killing spinors (2.12), charged under a gauge field A, in Lorentzian signature. This equation is also known as *twistor equation*, and it is well-studied in conformally flat spaces [101]. The case where $A = 0$ has already been analysed in [102]: all possible spaces on which a conformal Killing spinor exists were classified. It turns out that they fall in two classes: Fefferman metrics, and pp-wave space-times. We will review these two as particular cases (with $A = 0$) of our more general classification in section 4.2.4. As stated in the introduction, we will find that a charged conformal Killing spinor exists if and only if there exists a null conformal Killing vector. To explain the computations that lead to this result, we need to review first some geometrical aspects of four-dimensional spinors in Lorentzian signature.

4.1.1 Geometry defined by a spinor

In this section we review the geometry associated with a Weyl[2] spinor ϵ_+ in Lorentzian signature. As in section 2.1, we work in the signature $(-,+,+,+)$ and with real gamma matrices. We start with a spinor of positive chirality ϵ_+ and its complex conjugate $\epsilon_- \equiv (\epsilon_+)^*$. We can use ϵ_+ and $\gamma_\mu\epsilon_-$ to form a basis for the spinor of positive chirality and ϵ_- and $\gamma_\mu\epsilon_+$ for those of negative chirality. A convenient way of choosing the basis is obtained as follows. At every point where ϵ_+ is not vanishing, it defines a real null vector z and a complex two form ω. We can express this fact in terms of bispinors [3]

$$\epsilon_+ \otimes \bar{\epsilon}_+ = z + i * z \ , \qquad \epsilon_+ \otimes \bar{\epsilon}_- \equiv \omega \ , \tag{4.1}$$

where, as usual, $\bar{\epsilon} = \epsilon^\dagger\gamma^0$. Equivalently, as spinor bilinears the forms read

$$z_\mu = \frac{1}{4}\bar{\epsilon}_+\gamma_\mu\epsilon_+ \, , \qquad \omega_{\mu\nu} = -\frac{1}{4}\bar{\epsilon}_-\gamma_{\mu\nu}\epsilon_+ \, . \tag{4.2}$$

It can be shown easily that $z \wedge \omega = 0$, which implies that we can write

$$\omega = z \wedge w \tag{4.3}$$

for some complex one-form w. The form ω looks very similar to the holomorphic top-form of an almost complex structure; in section 4.2.2 we will make this similarity more precise

[2] We could also use a Majorana spinor.

[3] We are using conventions where $*\alpha \wedge \alpha = ||\alpha||^2 \mathrm{Vol}_4$ and $\mathrm{Vol}_4 = e^{0123} = \frac{1}{2}e^{+-23}$.

by introducing the concept of *CR-structure*. One can then show that the spinor ϵ_+ is annihilated by z and w, namely

$$z \cdot \epsilon_+ = w \cdot \epsilon_+ = 0 \,, \tag{4.4}$$

and

$$z^2 = z \cdot w = w^2 = 0 \ , \qquad w \cdot \bar{w} = 2 \ . \tag{4.5}$$

We can think of z and w as elements of a local frame: $z = e^+$, $w = e_2 - ie_3$. In order to complete the frame we can introduce another real one-form e^- such that

$$(e^-)^2 = 0 \ , \qquad e^- \cdot z = 2 \ , \qquad e^- \cdot w = 0 \ . \tag{4.6}$$

The four-dimensional metric then takes the form

$$ds^2 = z\, e^- + w\, \bar{w} \ . \tag{4.7}$$

Notice that the pair (z, ω) is uniquely defined by the spinor, while the frame $\{z, e^-, w, \bar{w}\}$ is not. This is because, given a w that satisfies (4.3), any other one-form of the form $w+\alpha z$ still satisfies it. After having fixed w, e^- is uniquely determined by the conditions (4.6). Alternatively, one can pick any null e^- such that $e^- \cdot z = 2$; a complex w orthogonal to e^- and z and such that $w^2 = 0$, $w \cdot \bar{w} = 2$ is then uniquely determined.

In summary, the vielbein $\{z, e^-, w, \bar{w}\}$ is not uniquely determined by ϵ_+; rather, it is determined up to the ambiguity

$$w \to w + \alpha z \ , \qquad e^- \to e^- - \bar{\alpha} w - \alpha \bar{w} - |\alpha|^2 z \ . \tag{4.8}$$

The complex function α has to do with the fact that ϵ_+ by itself describes an $\mathbb{R}^2$ structure[4], rather than the identity structure that would be described by the vielbein $\{z, e^-, w, \bar{w}\}$.

4.1.2 Intrinsic torsions

We are now ready to define a basis of spinors. For the positive chirality we can take

$$\epsilon_+ \ , \qquad e^- \cdot \epsilon_- \tag{4.9}$$

and for negative chirality

$$\epsilon_- \ , \qquad e^- \cdot \epsilon_+ \ . \tag{4.10}$$

[4]The stabiliser of the light-like vector z is $\mathrm{SO}(2) \ltimes \mathbb{R}^2$; w breaks the $\mathrm{SO}(2)$ to the identity. For more details see [103]. See also section 4 of [104] for a similar discussion in six dimensions.

It follows from (4.4) and the previous definitions that

$$\gamma^\mu \epsilon_+ = -w^\mu \epsilon_- + \frac{1}{2} z^\mu e^- \cdot \epsilon_+ \ . \tag{4.11}$$

Using the basis (4.9), we can expand

$$\nabla_\mu \epsilon_+ = p_\mu \epsilon_+ + q_\mu e^- \cdot \epsilon_- \ . \tag{4.12}$$

p_μ, q_μ are (locally) complex one-forms. They can be interpreted as intrinsic torsions for the $\mathbb{R}^2$ structure defined by ϵ_+.[5] It is also possible to express p and q in terms of exterior differentials. In order to do so, we can use the auxiliary piece of data e^-, which allows to define a vielbein $\{z, e^-, w, \bar{w}\}$, as described in section 4.1.1. This vielbein is an identity structure. The intrinsic torsion of an identity structure e^a is expressed by the "anholonomy coefficients" $c^a{}_{bc}$ defined by $de^a = c^a{}_{bc} e^b \wedge e^c$. As shown in Appendix B, we can parameterise the de^a as

$$dz = 2\mathrm{Re}\, p \wedge z + 4\mathrm{Re}(q \wedge \bar{w}) \ , \tag{4.13a}$$

$$dw = -2\rho \wedge z + 2i\mathrm{Im} p \wedge w - 2q \wedge e^- \ , \tag{4.13b}$$

$$de^- = 4\mathrm{Re}(\rho \wedge \bar{w}) - 2\mathrm{Re} p \wedge e^- \ . \tag{4.13c}$$

Here p and q are precisely the one-forms appearing in the covariant derivative of the spinor (4.12), while ρ is a new one-form which is an intrinsic torsion for the identity structure $\{z, e^-, w, \bar{w}\}$ but not for the $\mathbb{R}^2$ structure defined by ϵ_+. In four dimensions, we have $4 \times 6 = 24$ real anholonomy coefficients, which we can identify with three complex one-forms p, q and ρ.

Alternatively, we can extract p and q from the forms z and ω defined in (4.1). One can indeed derive the following differential constraints

$$dz = 2\mathrm{Re}\, p \wedge z + 4\mathrm{Re}(q \wedge \bar{w}) \, , \tag{4.14a}$$

$$d\omega = 2p \wedge \omega - 2q \wedge (z \wedge e^- + w \wedge \bar{w}) \, , \tag{4.14b}$$

$$(e^- \cdot \nabla)\, \omega = 2(p \cdot e^-)\omega - 2(q \cdot e^-)(z \wedge e^- + w \wedge \bar{w}) \, , \tag{4.14c}$$

which allow to determine p and q from the geometry. Notice that dz and $d\omega$ alone would not be enough to determine p and q.

[5] For a G-structure, one decomposes $\Lambda^2 T = g \oplus k$ (where g is the Lie algebra of G); the intrinsic torsion is then given by $k \otimes T$. In our case, $G = \mathbb{R}^2$, so k is 4-dimensional, and $k \otimes T$ is 16-dimensional. These are precisely the eight complex components of the complex one-forms p and q.

4.1.3 Conformal Killing spinors are equivalent to conformal Killing vectors

In this section we study the geometrical constraints imposed by the existence of a charged conformal Killing spinor ϵ_+. On the resulting curved backgrounds one can define a superconformal field theory preserving some supersymmetry, see section 2.3. We will show that existence of a charged conformal Killing spinor is equivalent to the existence of a conformal Killing vector. In turn, this allows to introduce local coordinates, in which the metric (or equivalently the frame) takes a canonical form, generalising the one discussed in [102], corresponding to $A = 0$. We shall present this metric in section 4.2, in the case when the conformal Killing vector becomes a Killing vector. The two metrics are simply related by a Weyl rescaling.

First of all, notice that not only does a spinor ϵ_+ determine a null vector z (via (4.1) or (4.2)), but also that in a sense the opposite is true. Indeed, let us study the map $\epsilon_+ \mapsto z$. The space of spinors with fixed $\epsilon_+^t \epsilon_+$ is an S^3 in the four-dimensional space of all spinors. This is mapped by (4.2) into the space of all null vectors z with fixed z^0, which is an S^2 (the so-called "celestial sphere"). This is the Hopf fibration map, whose fibre is an S^1. So, to any null vector z one can associate a U(1) worth of possible spinors ϵ_+ whose bilinear is z.

Let us now move on to differential constraints. We consider the equation defining a *charged* conformal Killing spinor, or twistor-spinor,

$$\nabla_\mu^A \epsilon_+ = \frac{1}{4}\gamma_\mu D^A \epsilon_+ \ , \tag{4.15}$$

where $\nabla_\mu^A = \nabla_\mu - iA_\mu$ and D^A is the covariantised Dirac operator $D^A = \gamma^\mu \nabla_\mu^A$. Note that the equation does not mix chiralities, and we consider the case of a positive chiral spinor. A is a real connection and ϵ_+ is a section of the U(1) Hopf fibration described in the previous paragraph.

We can expand equation (4.15) in the basis (4.9). Using (4.12), we obtain a set of linear equations for p, q and the gauge field A. Since the gamma-trace of equation (4.15) is trivial we find a total of six complex constraints

$$\begin{aligned} q\cdot z &= 0 \ , \qquad p^A\cdot e^- = 0 \ , \qquad p^A\cdot z = 2q\cdot\bar{w} \ , \\ q\cdot w &= 0 \ , \qquad p^A\cdot \bar{w} = 0 \ , \qquad p^A\cdot w = -2q\cdot e^- \ , \end{aligned} \tag{4.16}$$

where $p_\mu^A = p_\mu - iA_\mu$.

Two of these conditions will determine the real gauge field A. The remaining eight real conditions are constraints to be imposed on the geometry. We now show that these constraints are equivalent to the existence of a conformal Killing vector.

A short computation shows that

$$\nabla_\mu z_\nu = 2\mathrm{Re}\left(p_\mu z_\nu + 2\,\bar{q}_\mu w_\nu\right) \ . \tag{4.17}$$

Taking the anti-symmetric part of this equation we reproduce the first equation in (4.13). Taking the symmetric part and imposing that z is a conformal Killing vector,

$$(\mathcal{L}_z g)_{\mu\nu} = 2\nabla_{(\mu} z_{\nu)} = \lambda g_{\mu\nu} \ , \tag{4.18}$$

we obtain the conditions

$$\begin{aligned} \mathrm{Re}\, p \cdot e^- &= 0 \ , \qquad & q \cdot z = q \cdot w &= 0 \ , \\ \mathrm{Re}\, p \cdot z &= 2\mathrm{Re}(q \cdot \bar{w}) \ , \qquad & \mathrm{Re}\, p \cdot w &= -q \cdot e^- \ , \end{aligned} \tag{4.19}$$

with

$$\lambda \equiv 4\,\mathrm{Re}(q \cdot \bar{w}) \ . \tag{4.20}$$

This set of eight real conditions is precisely the subset of the constraints (4.16) not involving A, as previously stated.

To summarise, we showed that on any manifold M_4 with a null conformal Killing vector we can find a charged conformal Killing spinor. In section 4.1.5 we will give an expression for the gauge field A under which the conformal Killing spinor is charged.

Notice that the presence of a conformal Killing spinor also implies

$$d\omega = i\Big(2A - 3 * (q \wedge e^- \wedge \bar{w})\Big) \wedge \omega \ . \tag{4.21}$$

In the Euclidean signature case, we found a very similar condition (3.26), $d\omega = W \wedge \omega$, where ω is the $(2,0)$ form of a complex structure. While in that case that condition turns out to be necessary and sufficient for the existence of a charged conformal Killing spinor, in the present case of Lorentzian signature this condition alone is not sufficient to imply supersymmetry.

4.1.4 Conformal Killing spinors are equivalent to conformal Killing–Yano forms

Our two-form ω satisfies an interesting property, namely it is a *charged* conformal Killing–Yano form (CKF). In general, a p-form φ on a d-dimensional space(-time) (M, g) is conformal Killing–Yano (or simply conformal Killing) if it satisfies the equation

$$\nabla_\rho \varphi_{\mu_1 \ldots \mu_p} = \nabla_{[\rho} \varphi_{\mu_1 \ldots \mu_p]} + \tfrac{p}{d-p+1}\, g_{\rho[\mu_1} \nabla^\sigma \varphi_{|\sigma|\mu_2 \ldots \mu_p]} \ . \tag{4.22}$$

This is a conformally invariant equation: if φ is a conformal Killing form on (M,g) and the metric is rescaled as $g \to \tilde{g} = e^{2f} g$, then the rescaled form $\tilde{\varphi} = e^{(p+1)f}\varphi$ is conformal Killing on $(M,\tilde{g})$.

In the uncharged case ($A=0$), it is known that the bilinears of conformal Killing spinors are conformal Killing forms, see e.g. [105]. We already saw that this is true for z, since a conformal Killing one-form is just the dual of a conformal Killing vector. For a two-form in four dimensions, this is easiest to check in the two-component formalism for spinors. Because of its definition in (4.1), ω can be written as $\omega^{\alpha\beta} = \epsilon^\alpha \epsilon^\beta$, and the CKF equation (4.22) reads

$$D_{\dot\alpha}^{(\beta}\omega^{\gamma\delta)} = 0 \ . \tag{4.23}$$

The CKS equation reads in this formalism $D_{\dot\alpha}^{(\beta}\epsilon^{\gamma)} = 0$, which implies obviously (4.23). Since for us the CKS is actually charged under A, we obtain that $(\nabla^A)_{\dot\alpha}^{(\beta}\omega^{\gamma\delta)} = 0$, where $\nabla^A_\mu = \nabla_\mu - 2iA_\mu$; or, going back to four-component language,

$$\nabla^A_\rho \omega_{\mu\nu} \;=\; \nabla^A_{[\rho}\omega_{\mu\nu]} - \tfrac{2}{3}\, g_{\rho[\mu}\nabla^{A\,\sigma}\omega_{\nu]\sigma} \ , \tag{4.24}$$

which is a charged version of the standard conformal Killing form equation.

It is interesting to ask to what extent this property can be used to characterise our space-time, similarly to what we saw in section 4.1.3. First of all, we should ask when a two-form ω can be written as a spinor bilinear as in (4.1). One possible answer is that the form should define an $\mathbb{R}^2$ structure; namely, that the stabiliser of ω in SO(3,1) should be $\mathbb{R}^2$. We can also give an alternative, more concrete characterisation by using again the two-component formalism for spinors. The two-form ω should be imaginary self-dual, which means it is in the $(1,0)$ representation of SO(3,1); the corresponding bispinor then is a symmetric matrix $\omega^{\alpha\beta}$. As a 2×2 matrix, this can be factorised as $\epsilon^\alpha\epsilon^\beta$ if and only if it has rank 1, which is equivalent to $\det(\omega) = \frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}\omega^{\alpha\gamma}\omega^{\beta\delta} = 0$; in the original form language, $\omega_{\mu\nu}\omega^{\mu\nu} = 0$. So we have obtained that a two-form ω can be written as a bispinor as in (4.1) if and only if it is imaginary self-dual and null:

$$\omega = \epsilon_+ \otimes \overline{\epsilon_-} \qquad \Longleftrightarrow \qquad \begin{cases} *\omega = i\omega \ , \\ \omega^{\mu\nu}\omega_{\mu\nu} = 0 \ . \end{cases} \tag{4.25}$$

Remarkably, it turns out that the content of the equation (4.24) for a CKF is exactly the same as the content of the system (4.15) for a CKS. Indeed, if one uses (4.12) in (4.24) (or in (4.23)), the system one finds is exactly (4.16). Thus, we can conclude that a choice of metric and gauge field A admits a charged CKS if and only if it admits a null, imaginary self-dual charged CKF.

This reformulation is slightly less interesting than the one involving a null CKV in section 4.1.3. Although CKF's do have physical applications (such as helping in finding first integrals of the geodesic equation, see for example [106]), their geometrical meaning is less compelling than that of a CKV. Moreover, one needs both the data of the geometry and of the gauge field A to check the condition (4.24), whereas in the previous section we saw that the presence of a null CKV tells us that a geometry can admit a charged CKS for some A (without having to guess its form, which will actually be determined in section 4.1.5). Last but not least, the CKV condition is computationally easier to check than the CKF condition (4.24).

4.1.5 Determining the gauge field

The gauge field A can be determined by the four equations in (4.16) involving A. One possible expression is

$$A = \mathrm{Im}\big(p + i * (q \wedge e^- \wedge \bar{w})\big) \ . \tag{4.26}$$

Here, p and q are intrinsic torsion forms that can be computed for example from (4.13). The fact that (4.26) involves e^- might look puzzling, since, as we stressed in section 4.1.1, e^- is an auxiliary degree of freedom, not one determined by the spinor ϵ_+. More precisely, the vielbein $\{z, e^-, w, \bar{w}\}$ is only defined up to the freedom (4.8). From the definition of p and q in (4.12), using (4.11) we see that under (4.8)

$$p_\mu \to p_\mu - 2\bar{\alpha} q_\mu \ , \qquad q_\mu \to q_\mu \ . \tag{4.27}$$

Using this and (4.26), we can show that A is invariant under (4.8). This means it is independent on the choice of e^-, and is in fact determined by ϵ_+ alone.

We now show that the gauge curvature is invariant under the action of the vector field z, namely that

$$\mathcal{L}_z F = 0 \, . \tag{4.28}$$

We first compute the Lie derivative of a set of vielbein with respect to the vector z. Using equation (4.13) and the constraints (4.16) imposed by the conformal Killing spinor equation we find

$$\begin{aligned}
\mathcal{L}_z z &= \iota_z dz = \lambda z \, , \\
\mathcal{L}_z w &= \iota_z dw = -2(\mathrm{Re} p \cdot w + \rho \cdot z) z + \left(\frac{\lambda}{2} + i z \cdot A + 3i \mathrm{Im}(q \cdot \bar{w})\right) w \, , \\
\mathcal{L}_z e^- &= \iota_z de^- = 2(\mathrm{Re} p \cdot \bar{w} + \bar{\rho} \cdot z) w + 2(\mathrm{Re} p \cdot w + \rho \cdot z)\bar{w} \ ,
\end{aligned} \tag{4.29}$$

where λ has been defined in (4.20). In order to simplify these expressions we can make use of the freedom in the choice of a basis (4.8) to set

$$\mathrm{Re}\, p \cdot \bar{w} + \bar{\rho} \cdot z = 0 \tag{4.30}$$

and the gauge invariance to impose

$$z \cdot A \;=\; -3\,\mathrm{Im}(q \cdot \bar{w})\,. \tag{4.31}$$

At this point, the Lie derivative of the vielbein simply becomes[6]

$$\mathcal{L}_z z = \lambda z \;, \qquad \mathcal{L}_z w = \frac{\lambda}{2} w \;, \qquad \mathcal{L}_z e^- = 0 \;, \tag{4.32}$$

which is consistent with (4.18). We can also take the Lie derivative of (4.13) to compute the Lie derivatives of the torsions

$$\begin{aligned} \mathcal{L}_z p \;&=\; \frac{1}{4}(d\lambda \cdot z)e^- + \frac{1}{4}(d\lambda \cdot w)\bar{w} \;, \\ \mathcal{L}_z q \;&=\; \frac{\lambda}{2} q - \frac{1}{8}(d\lambda \cdot z)w - \frac{1}{8}(d\lambda \cdot w)z\,, \\ \mathcal{L}_z \rho \;&=\; -\frac{\lambda}{2}\rho - \frac{1}{8}(d\lambda \cdot w)e^- + \frac{1}{8}(d\lambda \cdot e^-)w\,. \end{aligned} \tag{4.33}$$

It is then straightforward to check from equation (4.26) that in our gauge $\mathcal{L}_z A = 0$. It follows that

$$\iota_z F \;=\; \mathcal{L}_z A - d(z \cdot A) \;=\; d(3\mathrm{Im}(q \cdot \bar{w}))\;. \tag{4.34}$$

Notice that this expression is independent of the choice of gauge and frame (due to (4.16)) and it is valid in general. It follows from (4.34) that F is invariant, $\mathcal{L}_z F = 0$.

4.2 Supersymmetric theories with an R-symmetry

In this section we will discuss an alternative supersymmetry equation, that arises as the rigid limit of new minimal supergravity [72, 98]. This formulation is particularly well suited to describe supersymmetric field theories with an Abelian R-symmetry, and it may be thought as a special case of the CKS equation.

[6] As we will see in section 4.2, in the new minimal case $\lambda = 0$, hence z is a Killing vector and the Lie derivative of the vielbein vanishes, $\mathcal{L}_z z = \mathcal{L}_z w = \mathcal{L}_z e^- = 0$. In the notation of section 4.2 the gauge condition (4.31) reads $a \cdot z = 0$.

4.2.1 New minimal supersymmetry equation

Solutions of the conformal Killing spinor equation (4.15) are closely related to solutions of the supersymmetry equation

$$\nabla_\mu \epsilon_+ = -i \left(\frac{1}{2} v^\nu \gamma_{\nu\mu} + (v-a)_\mu \right) \epsilon_+ \ , \tag{4.35}$$

arising from the rigid limit of new minimal supergravity[7] [72, 98]. Here a and v are real vectors and v is required to satisfy $d * v = 0$. When this condition has a solution, we can consistently define supersymmetric field theories on the four-manifold M_4, with background fields v and a, using the strategy of [43].

It is simple to see that a solution of (4.35) is a conformal Killing spinor associated with the gauge field $A = a - \frac{3}{2} v$. It follows from our analysis in section 4.1.3 that there should exist a null conformal Killing vector. It is in fact straightforward to see with a direct computation that equation (4.35) implies that $z_\mu = \frac{1}{4} \bar{\epsilon}_+ \gamma_\mu \epsilon_+$ is not only conformal Killing, but actually even Killing.

Vice versa, if we start with a solution of the conformal Killing spinor equation (4.15) without zeros, charged under a connection A, we can define a complex vector v through

$$D^A \epsilon_+ \equiv 2i \, v \cdot \epsilon_+ \ . \tag{4.36}$$

Every spinor of negative chirality can indeed be written as a linear combination of gamma matrices acting on ϵ_+. If ϵ_+ has no zeros, v is defined everywhere[8]. Using (4.12) we can express some components of v in terms of q

$$w \cdot v \ = \ 2i \, q \cdot e^- \ , \qquad\qquad z \cdot v \ = \ -2i \, q \cdot \bar{w} \ . \tag{4.37}$$

All other components of v are immaterial and v itself is not uniquely determined, since we can always add to it a term along z and w (recall that $z \cdot \epsilon_+ = w \cdot \epsilon_+ = 0$). We can use this freedom to make v real, except for an imaginary part given by

$$\mathrm{Im}\, v \ = \ -\frac{\lambda}{4} e^- \, , \tag{4.38}$$

where λ was defined in (4.20). This rewriting of q in terms of v will be useful in section 4.3, where we will perform a comparison between bulk and boundary solutions. It is now easy to show that (4.15) can be rewritten as equation (4.35) with $a = A + \frac{3}{2} v$.

[7] We use lower-case letters a and v for the auxiliary fields of new minimal supergravity, in order to avoid confusion with the A of conformal supergravity we have been using until now.

[8] Conformal Killing spinors with zeros do exist; see for example [107] for a characterisation.

So far, all we have done is rewriting the equation for conformal Killing spinors (4.15) as (4.35); v, however, is potentially still complex, with imaginary part given by (4.38), and in general $d * v \neq 0$. We will now show that v can be made real by an appropriate Weyl rescaling $g_{\mu\nu} \to e^{2f} g_{\mu\nu}$. To see this, remember that z is a conformal Killing vector, namely a vector satisfying (4.18). However, a null conformal Killing vector can always be made a null Killing vector by a Weyl transformation. In particular,

$$\mathcal{L}_z g_{\mu\nu} = \lambda g_{\mu\nu} \;\Rightarrow\; \mathcal{L}_z (e^{2f} g_{\mu\nu}) = (\lambda + 2z \cdot df) g_{\mu\nu}. \tag{4.39}$$

In coordinates where $z = \frac{\partial}{\partial y}$, it is then enough to solve $2\frac{\partial f}{\partial y} = -\lambda$. This is possible as long as there are no closed time-like curves. In the rescaled metric $e^{2f} g_{\mu\nu}$, z is now a Killing vector, which implies that $\lambda = 0$; from (4.38), we then see that v is real. Moreover, a similar argument shows that we can use the remaining ambiguity in shifting the z component of v to arrange for

$$d * v = 0\ . \tag{4.40}$$

Hence we have shown that, by a conformal rescaling of the metric, one can take the charged conformal Killing spinor equation (4.15) to the condition of unbroken supersymmetry in new minimal supergravity (4.35). The fact that one can bring (4.15) to (4.35) was to be expected because of the formalism of conformal compensators (for a review see [46]). In that formalism, one obtains new minimal supergravity by coupling a tensor multiplet to conformal supergravity, and by then giving an expectation value to the tensor multiplet.

To summarise, the geometrical constraints imposed by the new minimal equation just amounts to the existence of a null Killing vector z. As a check, we can count components. The new minimal equation (4.35) brings 16 real constraints and the existence of a Killing vector brings 9 real conditions. The remaining 7 real constraints can be used to determine the components of the gauge fields: 4 for a and 3 for v. a and v can now be computed as follows. v can be computed from (4.37), and from (4.40), while $a = A + \frac{3}{2}v$, where A was given in (4.26). Recall that the intrinsic torsion p and q can be computed for example from (4.13).

Finally, we observe that a solution ϵ_+ of the new minimal equations is defined up to a multiplication by a complex number. We can form two independent Majorana spinors $\epsilon_1 = \epsilon_+ + \epsilon_-$ and $\epsilon_2 = i(\epsilon_+ - \epsilon_-)$ corresponding to two independent real supercharges. The commutator of these two supersymmetries closes on the isometry generated by z:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\Phi = \mathcal{L}^a_z \Phi\ , \tag{4.41}$$

where Φ is any field in the Lagrangian and the Lie derivative $\mathcal{L}^a$ is covariantised with respect to a. The superalgebra can be easily extracted from the transformation rules of matter fields in the new minimal supergravity or from the algebra of local supergravity transformations [72, 98].

4.2.2 Introducing coordinates

We can obtain more explicit expressions for v and A after introducing a local set of coordinates, as promised earlier. First, notice that, using (4.37), (4.38) and the fact that $\lambda = 0$, equations (4.13a) and (4.21) simplify considerably:

$$dz = -2\iota_v * z \ , \tag{4.42a}$$

$$d\omega = 2ia \wedge \omega \ . \tag{4.42b}$$

As noticed at the end of section 4.1.3, the second of these equations is similar to the equation that in Euclidean signature implies that the manifold is complex. (4.42) can be used to compute all the components of a and v not along z. In particular, (4.42a) can be inverted to give

$$v^\perp \equiv v - \tfrac{1}{2}(e^- \cdot v)z = -\frac{1}{4}\iota_{e^-} * dz \, . \tag{4.43}$$

As discussed before, the component of v along z is ambiguous and is determined by requiring (4.40).

Given a null Killing vector, there exists a set of natural coordinates adapted to this. We will follow the discussion in [104]. We can introduce a coordinate y such that as vector field

$$z = \frac{\partial}{\partial y} \ , \tag{4.44}$$

and then the vector field dual to the one-form e^- introduced earlier can be parameterised as

$$e^- = 2H\left(\frac{\partial}{\partial u} - \mathcal{F}\frac{\partial}{\partial y}\right) , \tag{4.45}$$

for some H and $\mathcal{F}$. Taking as coordinates on the four-dimensional space (y, u, x^m), the functions H and $\mathcal{F}$ do not depend on y, and are otherwise arbitrary functions of u and x^m. In these coordinates, the four-dimensional metric can be written as

$$ds^2 = 2H^{-1}(du + \beta)\Big(dy + \varrho + \mathcal{F}(du + \beta)\Big) + Hh_{mn}dx^m dx^n , \tag{4.46}$$

where h_{mn} is a two-dimensional metric, while $\beta = \beta_m dx^m$ and $\varrho = \varrho_m dx^m$ are one-forms. Everything depends on u and x^m, but not on y. Therefore, as one-forms,

$$z \; = \; H^{-1}(du + \beta)\,, \qquad e^- \; = \; 2\,(dy + \varrho + \mathcal{F} H z)\;. \tag{4.47}$$

The remaining elements of the vielbein can be complexified as $w = e^2 - ie^3$.

Our four-dimensional manifold M_4 can be seen as an $\mathbb{R}$ fibration (with coordinate y) over a three-dimensional manifold M_3 (spanned by $\{z, w\}$). The latter admits a *CR structure*: namely, a one-dimensional complex subbundle $T_{1,0} \subset TM_3$, such that $T_{1,0} \cap \overline{T_{1,0}} = \{0\}$. Roughly speaking, this can be thought of as a complex structure on two of the three dimensions of M_3. For us, $T_{1,0}$ is spanned by the vector dual to the one-form $\bar{w}$. From a dual point of view, the subbundle of T^*M_3 spanned by one-forms which are orthogonal to $\overline{T_{1,0}}$ has dimension two, and it is spanned by the one-forms we have been calling z and w; so its volume form is $z \wedge w$, which is the form we have been calling ω, and which in a sense can be used to characterise the CR structure. The role of $z \wedge w$ actually becomes clearer in higher odd dimension $2n+1$; the bundle $T_{1,0}$ is now an n-dimensional bundle which should be closed under Lie bracket (just like for a complex manifold). The subbundle of T^*M_3 orthogonal to $\overline{T_{1,0}}$ now has dimension $n+1$, and it is spanned by forms z, $w_1, \ldots, w_n$. Integrability of $T_{1,0}$ is equivalent to the statement that $d(z \wedge w_1 \wedge \ldots \wedge w_n) = a \wedge (z \wedge w_1 \wedge \ldots \wedge w_n)$ for some one-form a. Summing up, on our three-dimensional manifold M_3 the form $z \wedge w$ is the analogue of a holomorphic volume form for a CR-structure, and can be used to characterise it.

Let us now present expressions for v and A in these coordinates. Evaluating (4.43) we find

$$v^\perp = \frac{1}{4} H^{-2} \left[*_2(\beta \wedge \partial_u \beta - d_2 \beta) \right] e^- + \frac{1}{2} H *_2 \left[\partial_u (H^{-1}\beta) - d_2(H^{-1}) \right]\,, \tag{4.48}$$

where we defined $d_2 = dx^m \partial_m$ and $*_2$ is the Hodge star operator with respect to the metric h_{mn}. Inserting $a \equiv a^\perp + \frac{1}{2}(a \cdot e^-) z$ into (4.42b), we determine $a^\perp$ as

$$a^\perp \; = \; \frac{1}{4} *_2 \left[d_2(H^{-1}\bar{w}) - \partial_u (H^{-1}\beta \wedge \bar{w}) \right] w + \text{c.c.}\,, \tag{4.49}$$

where c.c. denotes the complex conjugate[9]. The remaining component of the gauge field is given by $a \cdot e^- = A \cdot e^- + \frac{3}{2} v \cdot e^-$. As already noticed the component $v \cdot e^-$ is ambiguous; $A \cdot e^-$ can be extracted for example from the second and third equations in (B.7) and

[9] Comparing with section 4.1.5 we see that $a \cdot z = 0$ is a consequence of the Lie derivative $\mathcal{L}_z$ of our vielbein being zero, and corresponds to the gauge condition (4.31) (see footnote 6).

reads

$$A \cdot e^{-} \;=\; \frac{1}{2} H^{-1} *_2 \left[d_2\varrho + \mathcal{F} d_2\beta + (\partial_u \varrho + \mathcal{F}\partial_u \beta) \wedge \beta + H \,\mathrm{Re}(\bar{w} \wedge \partial_u w) \right] \;. \tag{4.50}$$

Perhaps it is worth emphasising that the data entering in the metric (4.46), gauge field A and v $(H, \mathcal{F}, \beta, \varrho, h_{mn})$ are completely arbitrary. This is in stark contrast with the typical situation in supergravity, where e.g. the Bianchi identities and equations of motion impose more stringent constraints on the geometry.

4.2.3 Non-twisting geometries

In the special case that $z \wedge dz = 0$ everywhere, z is hypersurface orthogonal, in the sense that the distribution defined by vectors orthogonal to z is integrable (by Frobenius theorem). As we will show in section 4.4, this corresponds to the case where the Killing vector in the bulk is *null*. Since z is hypersurface orthogonal, there exist preferred functions H and u such that

$$z = H^{-1} du \;. \tag{4.51}$$

Comparing with equation (4.47), we see that in these particular coordinates $\beta = 0$. After performing a further local change of coordinates to eliminate ϱ, the metric can be brought to the pp-wave form, namely

$$ds^2 \;=\; 2H^{-1} du \,(dy + \mathcal{F} du) + H h_{mn} dx^m dx^n \;. \tag{4.52}$$

In addition we have

$$v^{\perp} \;=\; -\frac{1}{2} H *_2 d_2 (H^{-1}) \tag{4.53}$$

$$a^{\perp} \;=\; \frac{1}{4} \left[*_2 d_2 (H^{-1} \bar{w}) \right] \, w + \text{c.c.} \tag{4.54}$$

$$A \cdot e^{-} \;=\; \frac{1}{2} *_2 \left[\mathrm{Re}(\bar{w} \wedge \partial_u w) \right] \tag{4.55}$$

where in particular notice $v \cdot z = 0$.

4.2.4 The case $A = 0$

It is interesting to study what happens in the particular case $A = 0$. Actually, as we showed in this section, every solution to the new minimal equation (4.35) is also a solution to the CKS equation (4.15), and hence must be included in the classification of uncharged conformal Killing spinors obtained in [102]. We will first consider the case $z \wedge dz \neq 0$, and then the case $z \wedge dz = 0$.

When $z \wedge dz \neq 0$, z is a *contact form* on the three-dimensional manifold M_3, spanned by $\{z, w\}$. It follows from (4.42a) that $\frac{1}{2} z \cdot v \equiv v_- \neq 0$; using (4.8), we can then make $w \cdot v = 0$, so that[10]

$$v = v_- e^- + v_z z \ . \tag{4.56}$$

Since $A = 0$, we have that $a = A + \frac{3}{2} v = \frac{3}{2}(v_- e^- + v_z z)$. Moreover, from (4.34) and (4.37), we see that v_- is actually constant. Our (4.42) now become

$$dz = 2iv_- w \wedge \bar{w} \ , \tag{4.57a}$$

$$d\omega = 3iv_- e^- \wedge \omega \ . \tag{4.57b}$$

Moreover, from (B.7) and the fact that $\mathrm{Re}(\sigma \cdot \bar{w}) = 0$ as a consequence of (4.57a), we also have

$$e^- \wedge dw \wedge \bar{w} = \frac{1}{8}(\iota_w \iota_{\bar{w}} de^-) e^- \wedge z \wedge w \wedge \bar{w} \, . \tag{4.58}$$

A metric of the form (4.7), such that (4.57) and (4.58) hold, is called a *Fefferman metric* [108]. It has the property that, if one rescales the one-form $z \to \tilde{z} = e^{2\lambda} z$, where λ is a function on the CR manifold M_3, and one computes new $\tilde{w}$, $\tilde{e}^-$ so that (4.57) and (4.58) are still satisfied, the new metric $\tilde{z}\tilde{e}^- + \tilde{w}\overline{\tilde{w}}$ is equal to $e^{2\lambda}(ze^- + w\bar{w})$.[11] Notice that (4.57) are very similar to the conditions for Sasaki–Einstein manifolds (which have Euclidean signature rather than Lorentzian, and odd dimension rather than even). The fact that we found a Fefferman metric in the $A = 0$, $z \wedge dz \neq 0$ case is in agreement with the classification in [102].

Let us now consider the case $z \wedge dz = 0$. Using (4.53) and (4.54) it follows that $A^\perp = 0$ implies

$$d_2(\sqrt{H} w) = 0 \ . \tag{4.59}$$

Hence, we can choose a complex coordinate ζ and a function α so that locally $\sqrt{H} w = d\zeta + \alpha du$. We can then rearrange (4.52) as $ds^2 = H^{-1}[du(2dy + d\zeta\bar{\alpha} + d\bar{\zeta}\alpha + \mathcal{F} du) + d\zeta d\bar{\zeta}]$, after suitably redefining $\mathcal{F}$. Moreover, from (4.55) we learn that the component of $d\alpha$ along w is real. This implies in turn that the one-form $d\zeta\bar{\alpha} + d\bar{\zeta}\alpha$ is closed, up to terms

[10] As pointed out after (4.37), the component v_z is immaterial, and can be used to set $d * v = 0$.

[11] Such a characterisation of Fefferman metric can be found in [109]. The metric is defined there by [109, (3.7)]. The term L_θ in that equation is defined composing (4.57a) with the complex structure associated to $T_{1,0}$, the one-dimensional complex subbundle of TM_3 defining the CR structure on M_3, and corresponds to our term $w\bar{w}$ in (4.7). Moreover, the term $2\theta\sigma$ in [109, (3.7)] is identified with the term $e^- z$ in our (4.7), once we compare our (4.57b) with [109, (3.5)]. Finally, [109, (3.6)] is (4.58).

$du \wedge (\ldots)$; locally, we can then write $d\zeta\bar{\alpha} + d\bar{\zeta}\alpha = df + gdu$, for some functions f and g. We can now further redefine y and $\mathcal{F}$ to obtain

$$ds^2 = H^{-1}[du\,(2dy + \mathcal{F}du) + d\zeta d\bar{\zeta}] \ , \tag{4.60}$$

which agrees (locally) with the classification in [102, Eq. (41)].

4.3 Boundary geometry from the bulk

The general analysis of the supersymmetry conditions in the minimal gauged supergravity in five dimensions was performed in [74]. Here we would like to asymptotically expand these results, to extract a set of conditions on a four-dimensional boundary geometry. Not surprisingly, at leading order we find agreement with the conditions that we derived from the CKS equation on the boundary in sections 4.1 and 4.2.

4.3.1 Asymptotic expansion of the bilinears

The analysis in [74] uses the following set of five-dimensional bilinears:

$$\begin{aligned} f\epsilon^{IJ} &= i\bar{\epsilon}^I\epsilon^J \ , \\ V_A\epsilon^{IJ} &= \bar{\epsilon}^I\gamma_A\epsilon^J \ , \\ X^{(1)}_{AB} + iX^{(2)}_{AB} &= -i\bar{\epsilon}^1\gamma_{AB}\epsilon^1 \ = \ -(i\bar{\epsilon}^2\gamma_{AB}\epsilon^2)^* \ , \\ X^{(3)}_{AB} &= \bar{\epsilon}^1\gamma_{AB}\epsilon^2 \ = \ \bar{\epsilon}^2\gamma_{AB}\epsilon^1 \, . \end{aligned} \tag{4.61}$$

Here, f is a real scalar, V is a real one-form, and $X^{(i)}$, $i = 1, 2, 3$, are real two-forms[12]. We will also define $\Omega = X^{(2)} - iX^{(3)}$.

We can expand the bulk bilinears (4.61) near the boundary using (2.8). In order to facilitate the comparison with the boundary results, let us again define a complex one-form v via the covariant derivative of ϵ as in (4.36), which when plugged into (2.11) yields

$$\eta = i\gamma^\mu \mathrm{Re}(v_\mu\epsilon_+) = \frac{1}{2}\gamma^\mu\left(i\mathrm{Re}v_\mu\epsilon - \mathrm{Im}v_\mu\gamma_5\epsilon\right) , \tag{4.62}$$

where we have used that the Majorana condition on ϵ implies that $\epsilon_+^* = \epsilon_-$. Recall that v has a complex part given by (4.38). Recall also the definition of the boundary bilinears (4.2), which together with the Hodge dual of z correspond to the four-dimensional bilinears

[12] They satisfy a set of algebraic relations that can be found in equations (2.8)–(2.12) of [74] (changing the sign of the metric in order to take into account the opposite choice of signature).

defined by a single chiral spinor ϵ_+, and determine an $\mathbb{R}^2$ structure. Note also that using the properties of the 4d gamma matrices, one can check that

$$* \, \omega \; = \; i \, \omega \tag{4.63}$$

where the Hodge star is four-dimensional and the metric is the boundary metric $g_{\mu\nu}$.

With these definitions, it is straightforward to compute the asymptotic expansion of the bulk bilinears (4.61) at leading order in r, namely

$$\begin{aligned}
f/8 \quad &\sim \quad -\mathrm{Re} v \cdot z \ , && (4.64a)\\
\ell^{-1} V/8 \quad &\sim \quad r^2 z + r^{-1} \mathrm{Im} v \cdot z \; dr \ , && (4.64b)\\
\ell^{-2} X^{(1)}/8 \quad &\sim \quad r dr \wedge z - r^2 \left(\mathrm{Re} \, \iota_v * z + \mathrm{Im} v \wedge z\right) \ , && (4.64c)\\
\ell^{-2} \, \overline{\Omega}/8 \quad &\sim \quad i r^3 \omega + dr \wedge \iota_v \omega \ . && (4.64d)
\end{aligned}$$

Using (4.63), as well as the identity $* \iota_v \omega = i v \wedge \omega$, one also finds that at leading order

$$\ell^{-3} \, \hat{*} \, \overline{\Omega}/8 \quad \sim \quad -i r^3 v \wedge \omega - r^2 dr \wedge \omega \ , \tag{4.65}$$

where $\hat{*}$ denotes the five-dimensional Hodge star.

4.3.2 Differential conditions from the bulk

The conditions for the existence of supersymmetric solutions in the bulk can be written in terms of a set of differential conditions on the bilinears [74]. We will now expand these conditions near the boundary where the metric is given by (2.1) and the gauge field by (2.6). They read

$$\begin{aligned}
df \quad &= \quad -\frac{2}{3} i_V F && (4.66a)\\
\hat{\nabla}_A V_B \quad &= \quad \ell^{-1} X^{(1)}_{AB} + \cdots && (4.66b)\\
\hat{\nabla}_A X^{(1)}_{BC} \quad &= \quad 2 \ell^{-1} \, \eta_{A[B} V_{C]} + \cdots && (4.66c)\\
\hat{\nabla}_A \overline{\Omega}_{BC} \quad &= \quad -i \ell^{-1} \left(2\sqrt{3} \, \hat{A}_A \overline{\Omega}_{BC} + (* \overline{\Omega})_{ABC} \right) + \cdots && (4.66d)
\end{aligned}$$

where we omitted terms containing F, whenever they are manifestly sub-leading in r.[13]

We can now further expand the bulk differential conditions (4.66) near the boundary. In the computation, we will need the expressions of the Christoffel symbols for the five-dimensional metric with expansion given in (2.1). We have the following identities

$$\hat{\Gamma}^{\ \mu}_{rr} = \hat{\Gamma}^{\ r}_{\mu r} = 0 \ , \qquad \hat{\Gamma}^{\ r}_{rr} = -\frac{1}{r} \ , \tag{4.67}$$

[13] Full expressions can be found in eqs. (2.15), (2.17), (2.18), (2.19) in [74].

as well as the expansions

$$\hat{\Gamma}^{\ \rho}_{\mu\nu} \ = \ \Gamma^{\ \rho}_{\mu\nu} + \mathcal{O}(r^{-1}) \ , \qquad \hat{\Gamma}^{\ \nu}_{\mu r} \ = \ \frac{1}{r}\delta^{\nu}_{\mu} + \mathcal{O}(r^{-1}) \ , \qquad \hat{\Gamma}^{\ r}_{\mu\nu} \ = \ -r^3 g_{\mu\nu} + \mathcal{O}(r^2) \ , \tag{4.68}$$

where $\Gamma^{\ \rho}_{\mu\nu}$ denotes the Christoffel symbols of the four-dimensional metric $g_{\mu\nu}$. Let us start with equation (4.66b). Its symmetric part is simply

$$\hat{\nabla}_{(A}V_{B)} \ = \ 0 \ , \tag{4.69}$$

which states that V^A is a Killing vector in the bulk. It is an easy check to see that, at leading order in r, this just says that z^{μ} is a conformal Killing vector on the boundary. To this end, note that the equations having components along r do not give rise to any conditions on the boundary, while the ones without leg along r imply

$$\nabla_{(\mu}z_{\nu)} \ = \ -g_{\mu\nu}\mathrm{Im}v \cdot z \ \equiv \ \frac{\lambda}{2}\, g_{\mu\nu} \, . \tag{4.70}$$

This is the same condition that we found from the purely four-dimensional analysis in (4.18) and (4.38).

Having reproduced the existence of a boundary conformal Killing vector from the gravity analysis, let us now consider the other differential conditions. We have reformulated the conditions on the boundary geometry in (B.8) and (B.9) in such a way to make the comparison with the bulk analysis of this section most straightforward. Plugging (4.64b) and (4.64c) into the anti-symmetric part of (4.66b), we find that again the only non trivial information at leading order in r comes from the four-dimensional part. We get the condition

$$dz = -2(\mathrm{Re}\,\iota_v * z + \mathrm{Im}\, v \wedge z) \ , \tag{4.71}$$

which is just the anti-symmetric part of (B.8). Next on the list are equations (4.66c) and (4.66d). At leading order, (4.66c) and the $(\mu\nu 5)$-part of (4.66d) do not give any new information. On the other hand, upon using (4.65), the four-dimensional part of equation (4.66d) yields

$$(\nabla_{\rho} - 2iA_{\rho})\,\omega_{\mu\nu} \ = \ i(v\wedge\omega)_{\mu\nu\rho} + i\,(g_{\rho\nu}\omega_{\mu\sigma} - g_{\rho\mu}\omega_{\nu\sigma})\, v^{\sigma} \, , \tag{4.72}$$

which is precisely the equation (B.9).

The final equation (4.66a) would seem to be more problematic. It involves the scalar bilinear in the bulk which has no correspondence in the boundary and involves the curvature of the gauge field which we usually neglected because sub-leading in r. However, equation (4.66a) expands to

$$df = -\frac{16}{3}\iota_z F \, , \tag{4.73}$$

a relation that we also found on the boundary. It corresponds indeed to (4.34), as we can see by using (4.37) and (4.64a).

We have thus shown that, as expected, all the conditions for supersymmetry in the bulk reduce to conditions that can be derived from the CKS equation on the boundary. In other words, any supersymmetric bulk solution that can be written asymptotically in the Fefferman–Graham form (2.1) and with a gauge field A satisfying (2.6) reduces to the boundary to a metric with a null conformal Killing vector. This vector is associated with a conformal Killing spinor ϵ_+ charged under A. Vice versa, any Lorentzian metric with a null conformal Killing vector gives rise to a bulk metric (2.1) that solves, at leading order, the supersymmetry conditions of gravity. In this regard, we expect to be able to find a supersymmetric bulk solution with a given boundary condition order by order in r, in the spirit of the Fefferman–Graham construction. It is then a very hard problem to determine which boundary metrics give rise to regular solutions in the bulk. Few examples are known in the literature and they will be reviewed in the next section.

4.4 Time-like and null solutions in the bulk

In this section we will analyse in more detail the classification of supersymmetric solutions of minimal five-dimensional gauged supergravity given in [74]. We will demonstrate how to extract the boundary data from a bulk solution and we will discuss how the examples found in [74] fit in the general discussion of supersymmetric boundary geometries.

Let us analyse some general features of the bulk solutions that can be written asymptotically in the Fefferman–Graham form (2.1) and with a gauge field A satisfying (2.6). As discussed in the previous section, the five-dimensional vector V is Killing and its asymptotic expansion (written here as the dual one-form),

$$V \sim r^2 z + r^{-1} \mathrm{Im} v \cdot z \, dr + \cdots \,, \qquad \mathrm{Im} v \cdot z = -\frac{1}{2}\lambda \,, \tag{4.74}$$

gives rise to a null conformal Killing vector z on the boundary. As in [74], we can introduce a coordinate y such that

$$V = \frac{\partial}{\partial y} \,. \tag{4.75}$$

In this (particularly natural) coordinate system the metric is independent of y and so will be the boundary metric. This means that z is actually Killing and we can identify the bulk coordinate y here with the coordinate y introduced in section 4.2.2. We also learn that the term $\mathrm{Im} v \cdot z = -\frac{1}{2}\lambda$, which controls the failure of z at being Killing (recall

(4.38)), must vanish and

$$V \sim r^2 z + \cdots \ . \tag{4.76}$$

There is no loss of generality here. As discussed in section 4.2, one can always make z Killing by a Weyl rescaling of the boundary metric. But Weyl rescalings in the boundary are part of diffeomorphisms in the bulk and they can be arranged with a suitable choice of coordinates. In a different coordinate system, for example with an (unnatural) choice of radial coordinate depending on y, we would find that z restricts to a conformal Killing vector on the boundary.

The boundary data can be easily extracted from the bulk metric. It follows from our discussion that the natural framework where to discuss the boundary supersymmetry is that of the new minimal equation. The boundary metric and gauge field A can be read off from equations (2.1) and (2.6). To have full information about the supersymmetry realised on the boundary we also need the vector v. This is real and satisfies equation (4.42a). It can easily be computed starting from z, using for example (4.43); the component of v along z is ambiguous and is determined by requiring (4.40). We will see explicit examples of this procedure in the following.

Note that while z is always null with respect to the boundary metric, the five-dimensional Killing vector V can be null or time-like [74]. This follows from the algebraic constraint (equation (2.8) in [74])

$$V^2 = -f^2 \ . \tag{4.77}$$

For $f \neq 0$, V is time-like while, for $f = 0$, V is null. The time-like and null solutions have different properties and, following [74], we will discuss them separately. Recall from (4.64a) that

$$f \sim -8 v \cdot z \ , \tag{4.78}$$

so the time-like and null bulk solutions correspond to $v \cdot z \neq 0$ and $v \cdot z = 0$, respectively. As already noticed in section 4.2.3, these correspond to $z \wedge dz \neq 0$ and $z \wedge dz = 0$, respectively. It then follows that the null bulk case corresponds to the non-twisting geometries discussed in section 4.2.3. In the following, we consider these cases in turn, also discussing two explicit examples as an illustration of our general results.

4.4.1 Time-like case

In the time-like case, the bulk metric can be written as a time-like fibration over a four-dimensional base B_4, as

$$ds^2 = -f^2 (dy + \tau)^2 + f^{-1} ds^2(B_4) \ , \tag{4.79}$$

where f, τ and $ds^2(B_4)$ do not depend on y. As a one-form, V reads

$$V = -f^2(dy + \tau)\ . \tag{4.80}$$

Supersymmetry in the bulk requires the base B_4 to be Kähler [74]; in particular, as shown in [74], this is equivalent to the equations[14]

$$dX^{(1)} \ = \ 0\,, \qquad d\Omega \ = \ i\ell^{-1}(2\sqrt{3}\hat{A} - 3f^{-1}V) \wedge \Omega\,. \tag{4.81}$$

We are interested in metrics that can be written in the Fefferman–Graham form (2.1). As already discussed, such metrics have the properties that at large r, f is independent of r and $V \sim r^2 z$. It then follows that $\tau = O(r^2)$. The boundary supersymmetry is determined by the background fields a and v. v is extracted from (4.42a), while $a = A + \frac{3}{2}v$, where A can be read off from (2.6). Once again, we can check that v is real in such solutions. Indeed, a non-vanishing term $r^{-1}\mathrm{Im}v \cdot z\, dr$ in (4.74) would contradict the mutual consistency of the two metrics (4.79) and (2.1) by introducing $dydr$ terms.

It is interesting to write explicitly the asymptotic Kähler structure $(X^{(1)}, \Omega)$ on the base manifold B_4. Using the freedom to take $v = v_- e^- + v_z z$, combining (4.64) with (4.81) we get

$$\ell^{-2}X^{(1)} \ \sim \ rdr \wedge z - 2v_- r^2 w \wedge \bar{w} = \tfrac{1}{2} d\left(r^2 z\right)\,, \tag{4.82}$$

$$\ell^{-2}\Omega \ \sim \ (2v_- dr - ir^3 z) \wedge \bar{w}\,, \tag{4.83}$$

and correspondingly the Kähler metric reads

$$\ell^{-2}ds^2(B_4) \ \sim \ 2v_- \left(\frac{dr^2}{r^2} + r^2 w\bar{w}\right) + \frac{1}{2v_-} r^4 z^2\,. \tag{4.84}$$

Eq. (4.82) characterises Kähler cones, however the asymptotic metric is not homogeneous in r, and this is reflected by the $(2,0)$-form Ω. Equations (4.82)–(4.84) may be thought of as boundary conditions that a Kähler base B_4 should satisfy. We also note that on surfaces of constant r, $\overline{\Omega}$ pulls back to a form proportional to $z \wedge w$, which characterises the CR structure on M_3, as we saw in section 4.2.2.

In [74, 110], an explicit time-like solution was presented, in which AdS_5 is deformed by a gauge field, and two supercharges are preserved. The Kähler base of the five-dimensional space-time is the Bergmann space, which is an analytic continuation of $\mathbb{CP}^2$. The five-dimensional metric takes the asymptotic form (2.1), with boundary metric

$$ds^2 = -\frac{1}{\ell}\left(dt + \mu\ell^2\sigma_1\right)\sigma_3 + \frac{1}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)\,, \tag{4.85}$$

[14] These equation are valid in the gauge $\iota_V \hat{A} = -\frac{\sqrt{3}}{2} f$, used in [74]. (4.81) are therefore equations on the base B_4.

where the σ's are right-invariant one-forms on S^3:

$$\begin{aligned}\sigma_1 &= \sin\phi d\theta - \sin\theta\cos\phi d\psi\,,\\ \sigma_2 &= \cos\phi d\theta + \sin\theta\sin\phi d\psi\,,\\ \sigma_3 &= d\phi + \cos\theta d\psi\;.\end{aligned} \tag{4.86}$$

This is a non-Einstein, non-conformally flat metric on $\mathbb{R}\times S^3$. In our conventions, the gauge field at the boundary reads

$$A = \frac{3}{2\ell}(dt + \mu\ell^2\sigma_1)\,. \tag{4.87}$$

Here, μ is a parameter of the solution. When $\mu = 0$, the gauge field is trivial, the boundary metric becomes the standard one on $\mathbb{R}\times S^3$ (after a coordinate transformation), and the bulk space-time is just AdS_5.

Identifying the frame as

$$e^+ = \frac{\sigma_3}{2}\,, \qquad e^- = -\frac{2}{\ell}(dt + \mu\ell^2\sigma_1) + \frac{\sigma_3}{2}\,, \qquad w = \frac{1}{2}(\sigma_1 - i\sigma_2)\,, \tag{4.88}$$

we see that (4.85) agrees with our general description of section 4.2.2, with the coordinate identification $\{y,\, u,\, x^1,\, x^2\} = \{-t/\ell,\, \phi,\, \theta,\, \psi\}$. We also need to identify

$$H = 2\,, \qquad \mathcal{F} = \frac{1}{4}\,, \qquad \beta = \cos\theta d\psi\,, \qquad \varrho = -\mu\ell\sigma_1\,, \tag{4.89}$$

and the metric h_{mn} with the round metric on S^2. One can also check that the gauge field in (4.87) is consistent with our general formulae in section 4.2.2. Using (4.48), we find that $v^\perp = \frac{1}{2}e^-$ and this can be completed by choosing $e^-\cdot v = -1$, so that

$$v \;=\; \frac{1}{2}(e^- - e^+) \;=\; -e^0 \tag{4.90}$$

satisfies $d * v = 0$. Finally, $a = A + \frac{3}{2}v = 0$ is consistent with (4.49) and (4.50).

We checked that with these values of v and a, the new minimal equation (4.35) is solved by a constant spinor ϵ_+ satisfying the projection $\gamma^0\gamma^1\epsilon_+ = \epsilon_+$. This shows that the background preserves precisely two supercharges[15]. Finally, we note that from the point of view of the boundary geometry we could deform the metric on S^3 in various ways. However, which deformations can be completed to a non-singular solution in the bulk is a very hard question to address.

[15] Note that in order to map the frame chosen in [110] into the five-dimensional frame used here, one needs to perform an r-dependent Lorentz transformation. Acting on the spinors, this transforms the spinors in [110], which are independent of r, into r-dependent spinors, with asymptotic form given in (2.8). Note also that the t-dependence of the spinors in [110] arises as a consequence of a different gauge for A. In particular, in [110]: $A\cdot z = 0$.

4.4.2 Null case

In the null case, $f = 0$ and the bulk metric can be written as [74]

$$ds^2 \;=\; -2\hat{H}^{-1}du\left(dy + \tfrac{1}{2}\mathcal{F}du\right) + \hat{H}^2\gamma_{mn}dx^m dx^n\,, \tag{4.91}$$

where $\hat{H}$, γ_{mn} and $\mathcal{F}$ depend only on u and x^m, $m = 1, 2, 3$, but not on y. Here

$$V = \hat{H}^{-1}du \tag{4.92}$$

and by comparison with (4.76) we see that $\hat{H}^{-1} = r^2H^{-1} + \ldots$, and $z = H^{-1}du$, in agreement with the results of section 4.2.3.

Explicit asymptotically locally AdS solutions in the null case are also discussed in [74]. These are the magnetic string solutions of [111, 112]. The boundary is $\mathbb{R}^{1,1} \times M_2$, with metric (after some obvious rescaling)

$$ds^2 = 2\,dudy + ds^2(M_2)\;, \tag{4.93}$$

and the gauge field is

$$F = -\frac{k}{2}\mathrm{vol}(M_2)\;. \tag{4.94}$$

Here, M_2 is S^2 if $k > 0$ (with radius $k^{-1/2}$), $\mathbb{T}^2$ if $k = 0$, or the hyperbolic space $\mathbb{H}^2$ if $k < 0$ (with radius $(-k)^{-1/2}$). The bulk space-time has a regular horizon when $k < 0$, while it has a naked singularity when $k > 0$. Setting $H = 1$, $\mathcal{F} = 0$, we find that the formulae in our section 4.2.3 are consistent with $v = 0$ and $F = da$.

Notice that these bulk solutions can be easily Wick-rotated to Euclidean signature, giving boundary metrics on $\mathbb{R}^2 \times M_2$, or $\mathbb{T}^2 \times M_2$. In the case $M_2 = \mathbb{H}^2$, the Wick-rotated bulk solution is non-singular, and interpolates between Euclidean AdS$_5$ asymptotically and $\mathbb{H}^3 \times \mathbb{H}^2$ in the interior. Indeed in [62] these four-dimensional geometries were shown to preserve supersymmetry in Euclidean signature.

Chapter 5

The geometry of $\mathcal{N} = 2$ in three Euclidean dimensions

A big part of the excitement that recent years have seen about exact results in supersymmetric field theories has been due to the ability of computing partition functions of various theories on three dimensional curved spaces, see the discussing in the introduction & motivations chapter. One in the context of holography particularly interesting result are the gravity duals [113–116] to field theories on various squashed three-spheres [117, 118] and Lens spaces . These are amongst the first examples of AdS/CFT dualities with a conformally non-flat boundary.

In this chapter we want to address the same question as in the last two chapters, namely on which general curved spaces we can define a supersymmetric field theory with some residue fermionic symmetry. Our interest will be in Euclidean space-times and three dimensional $\mathcal{N} = 2$ theories. The analysis will parallel closely the one of chapters 3 and 4 and we will hence be brief with the introduction.

Again we will start with the superconformal case. We have seen in section 2.2 that also in three dimensions, holography tells us that supersymmetry is equivalent with the existence of a charged conformal Killing spinor (2.21). Alternatively, following the strategy of [43] we can couple the theory directly on the boundary to three dimensional conformal supergravity [119]. In consistency with the holographic approach, the condition for unbroken supersymmetry is again the existence of a conformal Killing spinor. We find that such a spinor exists on every manifold with dreibein e^i, that fulfils the condition $do = W \wedge o$ where $o = e^1 + ie^2$ and W some one-form. The condition we find is very similar to the one for complexity in four dimensions. As was discussed in [73], it is equivalent to what is known in the mathematic literature as transversely holomorphic foliation with transversely hermitian metric.

We can also get the three dimensional analogue of new minimal supergravity by dimensional reduction of (3.31) and study the condition for unbroken supersymmetry without necessary conformal invariance.[1] As it turns out the geometrical condition that one finds is exactly the same as for the conformal case. We also determine the values of the background fields in terms of the geometrical data. As an application for our formulae we discuss various examples, such as round and squashed spheres, which had been so popular in the context of supersymmetric partition functions.

The rest of the chapter is as follows. In section 5.1 we classify manifolds with charged conformal Killing spinors in $3d$. In section 5.2 we discuss the reduction of new minimal supergravity to three dimensions and solve again the supersymmetry equations. To illustrate our findings we provide various examples.

This chapter is based on the paper [1], a detailed analysis with some overlap has also appeared in [58].

5.1 Geometry of conformal Killing spinors

In this section, we will deal with equation (3.8) in $d = 3$. The arguments are very similar to those in $d = 4$, and we will be brief.

Given a spinor χ, we can complete it to a basis with its complex conjugate:

$$\chi \ , \qquad \chi^C \equiv C\chi^* \ , \tag{5.1}$$

where $C^{-1}\sigma_m C = -\sigma_m^T$. Any nowhere-vanishing χ defines an identity structure. If it is charged this is true up to a phase. We can indeed construct the bispinors

$$\chi \otimes \chi^\dagger = \frac{1}{2} e^B \left(e_3 - i\mathrm{vol}_3\right) \ , \qquad \chi \otimes \overline{\chi} = -\frac{i}{2} e^B \, o \qquad (o \equiv e_1 + ie_2) \ , \tag{5.2}$$

where e_a are a vielbein for the metric on M_3. We defined $\overline{\chi} = \chi^T C^{-1}$. Notice that in odd dimensions the map between bispinors and forms is not bijective; a bispinor can be identified both with an even or an odd differential form. In writing (5.2) we opted for odd forms. In terms of this vielbein, one can also show

$$\sigma^m \chi = e_3^m \chi - i o^m \chi^C \ , \qquad m = 1, 2, 3 \ . \tag{5.3}$$

[1] This problem has been studied very detailed also in [58].

We can now define "intrinsic torsions" by expanding $\nabla_m \chi$ in the basis (5.1):

$$\nabla_m \chi \equiv p_m \chi + q_m \chi^C \ . \tag{5.4}$$

Alternatively, we can simply use the "anholonomy coefficients" $c^a{}_{bc}$ defined by $de^a = c^a{}_{bc} e^b \wedge e^c$. It is more convenient to work with e^3 and $o = e^1 + ie^2$, and to organise the $c^a{}_{bc}$ as

$$\begin{aligned} de_3 &\equiv \mathrm{Re}(w^1 e_3 \wedge o) + iw^2\, o \wedge \bar{o} \ , \\ do &\equiv w^3 e_3 \wedge \bar{o} + w^4\, o \wedge \bar{o} + w^5 e_3 \wedge o \ . \end{aligned} \tag{5.5}$$

Here, w^2 is real, while all the other w^i are complex, which gives a total of nine (which is the correct number for the $c^a{}_{bc}$). Together with dB, these are in one-to-one correspondence with the p and q in (5.4):

$$\begin{aligned} dB &= 2\mathrm{Re}p \ , \qquad & w^1 &= -2i\bar{q} \cdot e_3 \ , \qquad & w^2 &= \mathrm{Re}(q \cdot \bar{o}) \\ w^3 &= iq \cdot o \ , \qquad & w^4 &= -i\mathrm{Im}p \cdot o \ , \qquad & w^5 &= iq \cdot \bar{o} + 2i\mathrm{Im}p \cdot e_3 \ . \end{aligned} \tag{5.6}$$

We are now ready to impose (3.8). Using (5.4) and (5.3), we get

$$2p^A \cdot e_3 = iq \cdot \bar{o} \ , \qquad p^A \cdot o = -2iq \cdot e_3 \ , \qquad p^A \cdot \bar{o} = 0 = q \cdot o \ . \tag{5.7}$$

The first three simply determine A. The last can be written as $w^3 = 0$, which means that the sole geometrical constraint is that

$$do = w \wedge o \tag{5.8}$$

for some w, in analogy to (3.26).

5.2 New Minimal Supergravity reduced to Three Dimensions

It is also of some interest to reduce the condition of supersymmetry to three dimensions, where partition functions of Euclidean supersymmetric theories have been recently studied and computed using localisation.

In this section we will study the solutions of the dimensionally reduced new minimal condition

$$\nabla_m \chi = -i\left(v^n \sigma_{nm} + (v-a)_m\right)\chi + \frac{v_4}{2}\sigma_m \chi \ , \qquad m, n = 1, 2, 3 \ , \tag{5.9}$$

where χ is a two-component three-dimensional spinor on the manifold M_3, v and a are vectors and v_4 is a scalar. Similarly to four dimensions, v is subject to the constraint $d(*v) = 0$. A discussion of the off-shell $\mathcal{N} = 2$ new minimal supergravity in three dimensions can be found in [120, 121].

Every solution of (5.9) can be uplifted to a solution of the four-dimensional new minimal condition (3.31) on a manifold with metric

$$ds^2 = e^{-2b} ds^2_{M_3} + e^{2b} \left(d\phi + \mu\right)^2 \ , \tag{5.10}$$

with connection μ determined by

$$v + i\, db = \frac{i}{4} e^b * d\mu \tag{5.11}$$

and background fields $v_{4d} \equiv (v_4, v)$, $a_{4d} \equiv (v_4, a)$ satisfying $d(*v_{4d}) = 0$. We split all four-dimensional vectors in a component along $e^4 = e^b(d\phi + \mu)$ and a vector on M_3. We used the basis (2.18) and wrote the chiral spinor ϵ_+ as $\epsilon_+ = \left(\begin{smallmatrix}\chi \\ 0\end{smallmatrix}\right)$. Notice that we identified $a_4 = v_4$.

More general reductions from four to three may exist and a more general analysis can be performed, but (5.9) will be sufficient to illustrate various examples.

To characterise the geometry, we can use the bispinors we defined in (5.2). The equations they satisfy follow readily from the new minimal condition (5.9):

$$de_3 = -(dB + 2\,\mathrm{Im} a) \wedge e_3 + 4 * \mathrm{Re} v + i\, \mathrm{Im} v_4\, o \wedge \bar{o} \ , \tag{5.12}$$

$$do = (2\, v_4 e_3 + 2i\, a - dB) \wedge o \ , \tag{5.13}$$

$$dB = 2\,\mathrm{Im}(v - a) + i\, \mathrm{Re} v \llcorner (o \wedge \bar{o}) + \mathrm{Re} v_4 e_3 \ . \tag{5.14}$$

As in four dimensions, the problem of finding solutions of the new minimal condition (5.9) is closely related to the problem of finding solutions of the conformal Killing equation (3.8). In fact, any solution of equation (3.8) without zeros in $3d$ is also solution of (5.9) with the scalar and vector (v_4, v) defined by $D^A \chi \equiv 3 \left(i v^m \sigma_m + \frac{1}{2} v_4\right) \chi$ and $a = A + 2v$. It should come as no surprise that all results obtained in section 5.1 are consistent and equivalent to the set of equations (5.12-5.14). It is also obvious from this discussion that not all components of the auxiliary fields (v_4, v) are independent and there is some redundancy in their use.

We will now discuss some simple examples.

5.2.1 Spheres, round and squashed

Supersymmetric theories on the round sphere have been considered in [11, 14, 15]. Localisation leads to a matrix model for the partition function, whose applications have been a very active subject of research in the past years. In part II of this thesis we will discuss some aspects of this.

On S^3 we have Killing spinors satisfying $\nabla_m \epsilon = \pm\frac{1}{2}\sigma_m \epsilon$ and we can satisfy the new minimal condition (5.9) with $a = v = B = 0$ and $v_4 = \pm i$. It is easy to see how this translates in terms of forms. We can define left- and right-invariant vielbeine:

$$ds^2 = \sum_{a=1}^{3} l_a^2 = \sum_{a=1}^{3} r_a^2 \ . \tag{5.15}$$

They satisfy $dl_a = \epsilon_{abc} l_b \wedge l_c$ and $dr_a = -\epsilon_{abc} r_b \wedge r_c$. The equations (5.12) simplify to

$$de_3 = 2\mathrm{Im} v_4 \, e_1 \wedge e_2 \qquad do = 2 v_4 \, e_3 \wedge o \ , \tag{5.16}$$

which can be solved by taking e_a to be a permutation of the l_a for $\mathrm{Im} v_4 = +1$ or a permutation of the r_a for $\mathrm{Im} v_4 = -1$. It is worthwhile to notice that, if we define a superconformal theory on S^3, there will be no couplings linear in v_4 in the Lagrangian, and the theory will be invariant under all the supersymmetries with $\nabla_m \epsilon = \pm\frac{1}{2}\sigma_m \epsilon$; they obviously close to the superconformal algebra on S^3. If we instead consider a generic supersymmetric theory, v_4 will appear explicitly in the Lagrangian and we can only keep half of the supersymmetries.

One can also consider the squashed three-sphere

$$ds^2 = l_1^2 + l_2^2 + \frac{1}{s^2} l_3^2 \ . \tag{5.17}$$

Several different supersymmetric theories have been constructed on the squashed three-sphere [117, 118] and have attracted some attention in the context of localisation and the AGT correspondence [7]. For the interested reader, we quote the corresponding background fields. The simplest theory [117] is based on a deformation of the left invariant vielbein

$$e_3 = \frac{l_3}{s} \ , \qquad e_1 = l_1 \ , \qquad e_2 = l_2 \ , \tag{5.18}$$

which corresponds to the background fields $v_4 = \frac{i}{s}, a = \left(1 - \frac{1}{s^2}\right) l_3$ and $v = B = 0$. A different theory has been constructed in [118] and it is based instead on a deformation of the right invariant vielbein

$$e_a = \cos\theta r_a + \sin\theta \epsilon_{abc} n_b r_c \tag{5.19}$$

where n_a is a unit vector on the sphere and $e^{i\theta} = \frac{1+i\sqrt{1-s^2}}{s}$. The background fields are $v_4 = -\frac{i}{2s}, a = v = \frac{i}{2}\frac{\sqrt{1-s^2}}{s} l_3$ and $B = 0$. The squashed three-sphere lifts to a four-dimensional bundle (5.10) with connection proportional to v, as also discussed in [118]. The gravity dual of the theory in [118] have been identified in [114], where also an analytical continuation of the theory, $(\theta, v, a) \to i(\theta, v, a)$, has been considered. One can explicitly check that the asymptotic behavior of the spinors in the gravity dual [114] is consistent with our general discussion in section 2.2.

Using the v and a computed for these examples to couple to the reduction of new minimal supergravity, one can check that one gets the same Lagrangians as in [15, 118].

5.2.2 Sasaki Manifolds

Another very general class of solutions is provided by Sasaki three-manifolds M_S. The spinorial characterisation [93] of a Sasaki manifold is the existence of a solution of the charged Killing equation

$$(\nabla_m - ia_m)\,\chi = \frac{i}{2}\sigma_m \chi \tag{5.20}$$

with real a. The new minimal condition (5.9) provides a characterisation in terms of a vielbein $\{e_a\}$:

$$de_3 = 2e_1 \wedge e_2\,, \qquad\qquad do = 2i\alpha \wedge o\,. \tag{5.21}$$

where $a = \alpha - e_3$. Notice that e_3 has the property that $e_3 \wedge de_3$ is nowhere zero, which makes it a *contact form* on M_S.

An equivalent characterisation of a Sasaki manifold is the fact that the cone

$$ds_4^2 = dr^2 + r^2 ds^2_{M_S} \tag{5.22}$$

is Kähler. Let us briefly review why the two characterisations are equivalent. First of all, it is easy to check that the spinorial equation (5.20) lifts to the condition for a charged parallel spinor on the cone (equation (3.39)) whose existence is equivalent to the Kähler condition, as discussed in section 3.2.2. Alternatively, using the differential conditions (5.21), we can construct a Kähler form $j = \frac{1}{2}d(r^2 e_3)$ and a complex two form $\omega = r(dr + ire_3) \wedge o$ with $d\omega = w^5 \wedge \omega$.

We can generalise this example and include squashed Sasaki metrics [122]

$$ds^2 = e_3^2 + \frac{1}{h^2} o_S \bar{o}_S \tag{5.23}$$

where e_3 and o_S satisfy (5.21) and h is a function with no component along the contact form ($e_3 \lrcorner dh = 0$). We can easily solve the new minimal conditions (5.9) with background fields $v_4 = ih^2$ and

$$a = \alpha - h^2 e_3 + \frac{i}{2h} \left(\partial_{o_S} h - \partial_{\bar{o}_S} h\right) \tag{5.24}$$

where $\partial_{o_S} h, \partial_{\bar{o}_S} h$ are the components of dh along o_S and $\bar{o}_S$, respectively. This class of manifolds is quite general and include for example Seifert manifolds, which are U(1) bundles over Riemann surfaces. On all these spaces we can easily define a supersymmetry field theory with at least a supercharge.

Notice that the new minimal condition here reads

$$\left(\nabla_m - i a_m\right) \chi = i \frac{h^2}{2} \sigma_m \chi \ . \tag{5.25}$$

This kind of generalised Killing equations with a non-trivial functions on the right hand-side have solution only in dimensions less than or equal to three [122].

Chapter 6

The geometry of $\mathcal{N} = 2$ in four Lorentzian dimensions

In this chapter we analyse the $\mathcal{N} = 2$ case in four Lorentzian dimensions and we determine the general couplings to auxiliary backgrounds fields that preserve some of the extended supersymmetry. The case where part of the superconformal invariance is gauge fixed by compensators has already been analysed and completely solved in [69] for applications to black hole entropy[1]. Here we generalise the result to the full CKS equation.

For theories with higher supersymmetry, the analysis done so far is more involved since the conformal gravity multiplet contains another dynamical fermion in addition to the gravitino. Its supersymmetry variation leads to additional, differential equations which should be added to a generalised CKS equation involving various background fields.

Although some of the constraints are differential, we show that they can always be solved by choosing appropriate local coordinates. We give very explicit expressions for the auxiliary fields that we need to turn on to preserve some supersymmetry. In general, the auxiliary fields are not unique and there is some arbitrariness in their choice.

Our strategy is as in the last chapters to couple the theory to some background supergravity [43]. We mainly focus on superconformal theories, where it is natural to couple to $\mathcal{N} = 2$ conformal supergravity [78, 79]. As usual, this can be understood also from the holographic dual. As discussed in section 2, minimal supergravity in asymptotically locally AdS spaces reduces to conformal supergravity on the (non-trivial) boundary.

The condition for preserving some supersymmetry for an $\mathcal{N} = 2$ theory in four di-

[1] We thank Sameer Murthy for pointing out to us their results, which have substantial overlap with this chapter.

mensions is equivalent to a 'generalised' CKS, involving additional background fields and differential constraints, see (6.4). We show that these equations are equivalent to the existence of a conformal Killing vector (CKV). Of course, it is simple to see that the existence of a (charged) CKS implies the existence of a CKV. The converse is also true: the geometric constraints following from supersymmetry just amount to the existence of a CKV, all other constraints determine the background auxiliary fields of the gravity multiplet. Analogously to three dimensions [59], the CKV can be null or time-like, the null case being related to the existence of an $\mathcal{N} = 1$ subalgebra, the case discussed in detail in chapter 4.

In general, Poincaré supergravities arise from conformal supergravity through the coupling to compensator multiplets which gauge fix the redundant symmetry. Some of our results can be used also to define general supersymmetric field theory on curved space, although we do not discuss this issue in this chapter. The case where the spinor η associated with the conformal supersymmetry vanishes has been analysed thoroughly by [69] in the context of the study of black hole entropy. When $\eta = 0$, the conformal Killing vector actually becomes Killing.

The chapter is organised as follows. In section 6.1 we review the field content of $\mathcal{N} = 2$ conformal supergravity and we write the supersymmetry variations which include a *generalised* conformal Killing spinor equation. In section 6.2 we then discuss the geometrical structure induced by a pair of chiral spinors and the technical tools we will be using in this chapter. Section 6.3 contains the main results for this chapter: the proof that any manifold with a conformal Killing vector supports some supersymmetry and the explicit expressions for the background auxiliary fields.

This chapter is based on [3], it overlaps at various points with the paper [69].

6.1 The multiplet of conformal supergravity

The structure of conformal supergravity in 4 dimensions is nicely reviewed in [123][2]. The independent field content of the gravity (Weyl) multiplet is

$$g_{\mu\nu} \qquad \psi^i_\mu \qquad T^+_{\mu\nu} \qquad \tilde{d} \qquad \chi^i \qquad A_{\mu 0} \qquad A_\mu{}^i{}_j \tag{6.1}$$

[2] In our discussion of conformal supergravity in 4 dimensions, we mostly follow the conventions of [123] with few changes that are discussed in the Appendix. We have also redefined the gauge field and the scalar.

where $A_{\mu 0}$ and $A_\mu{}^i{}_j = A_{\mu x}\sigma^{x\,i}{}_j$ are the gauge fields of the $U(1)$ and $SU(2)$ R-symmetry, respectively (with $\vec{\sigma}$ the usual Pauli matrices), T^+ is a (complex) self-dual tensor, $\tilde{d}$ a scalar field and χ^i the dilatino. The fermions are chiral spinors in the **2** of $U(2)$. The fermionic part of the supersymmetry variations is

$$\begin{aligned}
\delta\psi^i_\mu &= \nabla^A_\mu \epsilon^i_+ + \frac{1}{4} T^+_{\mu\nu}\gamma^\nu \epsilon^i_- - \gamma_\mu \eta^i_- \\
\delta\chi^i &= \frac{1}{6}\nabla^A_\mu T^{+\,\mu}{}_\nu \gamma^\nu \epsilon^i_- - \frac{i}{3} R^{SU(2)\,i}{}_j \cdot \epsilon^j_+ + \frac{2i}{3} R^{U(1)} \cdot \epsilon^i_+ + \frac{\tilde{d}}{2}\epsilon^i_+ + \frac{1}{12} T^+_{\mu\nu}\gamma^{\mu\nu}\eta^i_+
\end{aligned} \tag{6.2}$$

with

$$\begin{aligned}
\nabla^A_\mu \epsilon^i_+ &= \nabla_\mu \epsilon^i_+ - i A_{\mu\alpha}\sigma^{\alpha i}{}_j \epsilon^j_+ \\
\nabla^A_\mu T^+ &= (\nabla_\mu - 2iA_{\mu 0})\, T^+ \\
R^{SU(2)\,i}{}_j &= (\partial_\mu A_{\nu x} + A_{\mu y} A_{\nu z}\epsilon^{yz}{}_x)\sigma^{x\,i}{}_j \gamma^{\mu\nu} \\
R^{U(1)} &= \partial_\mu A_{\nu 0}\gamma^{\mu\nu}
\end{aligned}$$

where we have introduced

$$\sigma^{\alpha\,i}{}_j = (\mathbb{1}, \vec{\sigma})^i{}_j\,. \tag{6.3}$$

The spinor doublets ϵ^i and η^i are the Q- and S- supersymmetry parameters, respectively. Our conventions are summarised in appendix A.2, in particular $\alpha, \beta, \dots = 0, 1, 2, 3$ and $x, y, \dots = 1, 2, 3$.

To preserve some supersymmetry on a manifold with metric $g_{\mu\nu}$ we need to find a configuration of auxiliary fields of the Weyl multiplet that solves (6.2). Obviously, this is not always possible and we now want to analyse in which cases it can be done. The special case where $\eta^i = 0$ has already been analysed in [69], in the following we extend this result to the general case.

We can eliminate η^i by taking the γ-trace of the first equation $\eta^i_- = \frac{1}{4} D^A \epsilon^i_+$, with $D^A = \gamma^\mu \nabla^A_\mu$. Note that T drops out of the computation since $\gamma_\mu T^+ \gamma^\mu = 0$. The supersymmetry condition can then be re-written as[3]

$$\nabla^A_\mu \epsilon^i_+ + \frac{1}{4} T^+_{\mu\nu}\gamma^\nu \epsilon^i_- - \frac{1}{4}\gamma_\mu D^A \epsilon^i_+ = 0 \tag{6.4a}$$

$$\nabla^A_\mu T^{+\,\mu}{}_\nu \gamma^\nu \epsilon^i_- + D^A D^A \epsilon^i_+ + 4i\nabla_\mu A_{\nu 0}\gamma^{\mu\nu}\epsilon^i_+ + 2d\,\epsilon^i_+ = 0 \tag{6.4b}$$

where we have used the first equation and we have redefined $d \equiv \tilde{d} + \frac{1}{6}R$, with R being the curvature scalar.

[3] See also [10] for a similar presentation.

The gravitino equation (6.4a) can be seen as a generalisation of the charged conformal Killing spinor equation found in the $\mathcal{N}=1$ case of chapter 3. For $T^+=0$ we simply obtain a non-abelian version of the CKS equation. In general, the situation is more involved due to the presence of the tensor T^+, however the equation (6.4a) shares many similarities with the CKS equation, in particular, it is conformally covariant. If the doublet ϵ^i_+ is a solution to the equation with metric $g_{\mu\nu}$, the rescaled doublet $e^{\lambda/2}\epsilon^i_+$ is a solution to the equation with rescaled metric $e^{2\lambda}g_{\mu\nu}$. In particular, the tensor has conformal weight $+1$

$$T^+_{\mu\nu} \to e^{\lambda}T^+_{\mu\nu}\,.$$

A further complication that affects all extended supergravities is the presence of the dilatino equation. As a difference with the generalised CKS equation (6.4a) which can be analysed in terms of a set of algebraic constraints for the geometric quantities involved, as in the previous chapters, the dilatino equation contains derivatives of the auxiliary fields and it seems to be more complicated to analyse. However, we will show how to extract the relevant information from it. It turns out that it is only the gravitino equation that restricts the geometry of the space-time, while the conditions coming from the dilatino equation (6.4b) merely fix some of the background field values.

We stress that the supersymmetry variations of conformal supergravity do not depend on the explicit matter content of the field theory. Our result will therefore be valid for any conformal theory with rigid supersymmetry. We also notice that all Poincaré supergravities arise from conformal supergravity through the coupling to compensator multiplets which gauge fix the redundant symmetry. Some of our results can also be used to define general supersymmetric field theories on curved space, although we do not discuss that issue further in this thesis. The close relation between $\mathcal{N}=1$ CKS and new minimal supergravity spinors is described in detail in chapters 3 and 4. For the $\mathcal{N}=2$ case we just observe that we can always set $\eta^i=0$ after a partial gauge fixing of the superconformal symmetry using a hypermultiplet compensator. The results in [69], where this gauge fixing is done, can therefore be used to define general supersymmetric field theories on curved space. As can be seen in [69], or by specialising the results in section 6.3 to the case $\eta^i=0$, the conditions of supersymmetry now requires that the conformal Killing vector is actually Killing. This is similar to what happens in the $\mathcal{N}=1$ case of chapter 4.

The supersymmetry transformations and the most general Lagrangian for matter couplings to conformal supergravity can be found in [123][4]. For completeness, here we give

[4] An analysis of supersymmetry Lagrangians using conformal supergravity has also appeared in [124]

the Lagrangian for the vector multiplet $(\phi, W_\mu, \psi_{+i}, Y^{ij})$,

$$\begin{aligned}\mathcal{L}^{\text{Lorentz}} &= d\,\phi\bar{\phi} + \nabla^A_\mu\phi\nabla^{A\mu}\bar{\phi} + \frac{1}{8}Y^{ij}Y_{ij} - g\left[\bar{\phi},\phi\right]^2 + \frac{1}{8}F_{\mu\nu}F^{\mu\nu} \\ &\quad + \left\{\frac{1}{4}\bar{\psi}^i_- D^A\psi_{+i} + \frac{1}{2}g\bar{\psi}^i_-\left[\phi,\psi_{-i}\right] - \frac{1}{4}\phi F_{\mu\nu}T^{+\mu\nu} - \frac{1}{16}\phi^2 T^+_{\mu\nu}T^{+\mu\nu} + \text{h.c.}\right\}\end{aligned} \tag{6.5}$$

where $Y_{ij} = (Y^{ij})^* = \epsilon_{ik}\epsilon_{jl}Y^{kl}$ is a triplet of real $SU(2)$ scalars.

In the next sections we will classify the geometries in which one can solve (6.4). The background fields that we will determine in terms of the geometry can be coupled to arbitrary vector and hypermultiplets as in (6.5), giving rise to supersymmetric field theories on curved space.

6.2 The geometry of spinors

In our analysis we follow a similar formalism as in the previous chapters and [59, 69]. We want to analyse the geometry that is defined by a chiral spinor doublet of $U(2)$. To this end, let us look at the two spinor bilinears

$$\begin{aligned} s &= \frac{1}{2}\epsilon_{ij}\,\bar{\epsilon}^i_-\epsilon^j_+ \\ z_\mu &= \frac{1}{2}\bar{\epsilon}_{+i}\gamma_\mu\epsilon^i_+ \end{aligned} \tag{6.6}$$

with $z_\mu z^\mu = -\|s\|^2$. It is a simple consequence of the generalised CKS equation that z is a conformal Killing vector. To proceed, we need to distinguish two situations.

When $s = 0$, the two spinors in the doublet are linearly proportional $\epsilon^1_+ \propto \epsilon^2_+$ and z_μ is null. We essentially fall in the $\mathcal{N} = 1$ case that has been studied in chapter 4. In fact, by a gauge transformation we can always set one of the spinors to zero. By further restricting the gauge fields to a suitable abelian subgroup and by setting the tensor T^+ to zero, we obtain the charge CKS equation discussed and solved in chapter 4. In that reference it is shown that any manifold with a null CKV supports supersymmetry. It is enough to turn on a background abelian gauge field whose explicit form is given in chapter 4[5].

We therefore refer to chapter4 for the case with $s = 0$ and from now on we will discuss the non-degenerate situation where $s \equiv e^{B+i\beta} \neq 0$. Apart from the complex scalar s, a

where various explicit solutions have been discussed.

[5]It is not excluded that *more* general solutions with non abelian gauge fields and a non-vanishing tensor exist but we will not discuss this case in this thesis.

generic chiral doublet of spinors in four dimensions defines four real vectors e^n_μ and three self-dual two-forms $\eta^x_{\mu\nu}$

$$\begin{aligned} e^B \sum_\alpha e^\alpha_\mu \, \sigma^{\alpha i}{}_j &= \bar\epsilon_{+j}\gamma_\mu \epsilon^i_+ \\ 2e^{B+i\beta}\eta^x_{\mu\nu}\, \sigma^{xij} &= \bar\epsilon^i_- \gamma_{\mu\nu}\epsilon^j_+ \end{aligned} \tag{6.7}$$

where $\sigma^{\alpha ij} = \sigma^{\alpha i}{}_k \epsilon^{kj}$ and $e^0_\mu = e^{-B} z_\mu$. The self-duality condition reads

$$\sqrt{|g|}\, \epsilon_{\mu\nu}{}^{\rho\sigma}\eta^x_{\rho\sigma} = 2i\, \eta^x_{\mu\nu}$$

where g is the determinant of the metric.

Using the Fierz identities one can show that e^α_μ is a tetrad, $e^\alpha \cdot e^\beta = \eta^{\alpha\beta}$. One can also show that in this frame the η_x's have components

$$\eta^{\alpha\beta}_x = e^\alpha_\mu e^\beta_\nu \eta^{\mu\nu}_x = \delta^\alpha_0 \delta^\beta_x - \delta^\beta_0 \delta^\alpha_x - i\epsilon_x{}^{\alpha\beta} \, . \tag{6.8}$$

We give some details in the appendix.

Note that the information contained in the spinor doublet can also be written in bispinor language[6]

$$\begin{aligned} \epsilon^i_+ \bar\epsilon_{+j} &= \frac{e^B}{4} \sum_\alpha \left(e^\alpha + i * e^\alpha \right) \sigma^{\alpha i}{}_j \\ \epsilon^i_+ \epsilon^j_- &= \frac{e^{B+i\beta}}{4} \eta_\alpha \sigma^{\alpha ij} \end{aligned} \tag{6.9}$$

where $\eta_0 = -\,(1+\gamma)$, $\eta_x = \frac{1}{2}\eta_{x\mu\nu}\gamma^{\mu\nu}$ and $e^\alpha = e^\alpha_\mu \gamma^\mu$. Here, $i * e^\alpha = \gamma e^\alpha$.

From the definition in (6.7) we see that e^x and η^x transform as vectors under the action of the $SU(2)$ R-symmetry. From (6.7) it also follows that we can gauge away β by a $U(1)$ R-symmetry transformation. Similarly, since our equations are conformally covariant, we can – at least locally – also set B to zero by an appropriate Weyl rescaling.

We will find useful to work in the frame defined by e^n where the action of the gamma matrices on the spinors takes a very simple form. For example it is easy to show that

$$\begin{aligned} \gamma^\alpha \epsilon^i_- &= e^{-i\beta}\sigma^{\alpha i}{}_j \epsilon^j_+ \\ \gamma^{\alpha\beta}\epsilon^i_+ &= \eta^{\alpha\beta}_x \sigma^{xi}{}_j \epsilon^j_+ \, . \end{aligned} \tag{6.10}$$

From (6.10) it readily follows that, in the frame defined by e^n, the spinors are constant, up to an overall norm factor.

[6] See [103], in particular Appendix A, for a nice review of the formalism.

Before we attack the supersymmetry conditions, note that it is useful to choose a covariant basis for the space of chiral spinor doublets[7]

$$\sigma^{\alpha i}{}_j \epsilon^j_+ \,. \tag{6.11}$$

We can expand in particular the covariant derivative of a spinor in this basis

$$\nabla_\mu \epsilon^i_+ = P_{\mu\alpha} \sigma^{\alpha i}{}_j \epsilon^j_+ \tag{6.12}$$

where we call the coefficients P *intrinsic torsions*. More explicitly, in our frame with constant spinors one has

$$P_{\mu x} = \frac{1}{4} \omega_\mu{}^{\alpha\beta} \eta_{x\alpha\beta} \qquad\qquad P_{\mu 0} = \frac{1}{2} \partial_\mu (B + i\beta) \tag{6.13}$$

where $\eta_{x\alpha\beta}$ projects on the self-dual part of the spin connection. A similar parameterisation has appeared in [125].

6.3 Solving the supersymmetry conditions

In this section we show that all manifolds with a timelike conformal Killing vector (CKV) admit a solution to the equations (6.4).

Let us analyse the two conditions for unbroken supersymmetry (6.4) separately. We define the symmetric traceless part of the torsion[8]

$$p_{\alpha\beta} \equiv P_{(\alpha\beta)} - \frac{\eta_{\alpha\beta}}{4} P^\gamma_\gamma$$

and the gravitino equation (6.4a) is readily solved by requiring

$$\mathrm{Re}(p_{\alpha\beta}) = 0 \tag{6.14a}$$

$$\mathrm{Im}\,(p_{\alpha\beta}) = A_{(\alpha\beta)} - \frac{\eta_{\alpha\beta}}{4} A^\gamma_\gamma \tag{6.14b}$$

$$T^+_{\alpha\beta} = -4 e^{i\beta} \left(P^+_{\alpha\beta} - i A^+_{\alpha\beta} \right) \,. \tag{6.14c}$$

The first line is a constraint on the geometry of the manifold, while the second and the third line are merely fixing some of the background fields in terms of this geometry. Let us discuss this more explicitly.

[7] Another obvious choice of basis would be $\gamma_\mu \epsilon^i_-$. The two bases are related by (6.10).

[8] It will be convenient to consider $P_{(\alpha\beta)}$ and $A_{(\alpha\beta)}$ as four-by four matrices where the indices are raised and lowered with $\eta_{\alpha\beta}$. All our formulae will be valid in the frame define by the e^α.

The geometrical interpretation of (6.14a) is very simple, it is equivalent to $z = e^B e^0$ being a conformal Killing vector, i.e. fulfilling

$$\nabla_{(\alpha} z_{\beta)} = \lambda \eta_{\alpha\beta}\,. \tag{6.15}$$

To see this, note that (6.15) written in our frame reads

$$\lambda = -e^B \partial_0 B \qquad\qquad \omega_{(x}{}^0{}_{y)} = \delta_{xy}\partial_0 B \qquad\qquad \omega_0{}^0{}_x = \partial_x B \tag{6.16}$$

where we defined $\partial_n B \equiv e_n^\mu \partial_\mu B$ etc. Then, using (6.13), it is easy to see that this is equivalent to $\mathrm{Re}\,(p_{\alpha\beta}) = 0$.

The other two equations determine the "symmetric traceless" part of the gauge field and the value of the tensor field, respectively. In our frame, (6.14b) reads

$$A_{(\alpha\beta)} - \frac{\eta_{\alpha\beta}}{4} A_\gamma^\gamma = -\frac{1}{4}\Big(\epsilon_{uv(\alpha}\omega_{\beta)}{}^{uv} + \frac{1}{2}\delta^0_{(\alpha}\partial_{\beta)}\beta - \frac{\eta_{\alpha\beta}}{4}(\epsilon_{yz}{}^x\omega_x{}^{yz} + \frac{1}{2}\partial_0\beta)\Big) \tag{6.17}$$

while (6.14c) fixes T^+ in terms of the "antisymmetric" part of the gauge field, which we will determine in the next paragraph.

It actually turns out that z being a CKV is the only geometrical constraint for unbroken supersymmetry. The dilatino equation (6.4b) gives no extra conditions for the manifold, in fact it has exactly the right amount of degrees of freedom to fix the background fields which are yet undetermined. There are 8 components and we still have to determine the values of $A_{[\alpha\beta]}$, A_γ^γ and d.

To this end it is useful to parameterise the gauge fields satisfying (6.17) as

$$\begin{aligned}
A_{x0} &\equiv -b_x + \frac{1}{2}\partial_x\beta \\
A_{0x} &\equiv b_x - \frac{1}{4}\epsilon_{xuv}\omega_0{}^{uv} \\
A_{00} &\equiv -\frac{1}{4}\alpha + \frac{1}{2}\partial_0\beta \\
A_{xy} &\equiv \epsilon_{xy}{}^z a_z + \frac{1}{4}\delta_{xy}\alpha - \frac{1}{4}\epsilon_{uvy}\omega_x{}^{uv}\,.
\end{aligned} \tag{6.18}$$

The new quantities α, a_x and b_x parameterise the "trace" and the "antisymmetric" part of the gauge field $A_{\alpha\beta}$ [9]. Then, the background value for the complex tensor T^+ becomes

$$\frac{1}{2}e^{-i\beta}T^{+}{}_{0x} = i(b_x - \frac{1}{4}\epsilon_x{}^{yz}\omega_y{}^0{}_z) + (a_x + \frac{1}{2}\partial_x B)\,. \tag{6.20}$$

The spatial components are given via self-duality $T^+_{xy} = i\epsilon_{xy}{}^z T^+_{0z}$.

In this language the dilatino equation can be re-written in a particularly simple form. This re-writing is a bit lengthy but straightforward. As a result we find eight real equations, coming from the real and imaginary part of (6.4b) in the basis (6.11)

$$\begin{aligned} \partial_0 a_x + (\omega_x{}^y{}_0 - \omega_0{}^y{}_x)a_y &= -\frac{1}{2}e^{-B}\partial_x(e^B\partial_0 B) \\ \partial_0 b_x + (\omega_x{}^y{}_0 - \omega_0{}^y{}_x)b_y &= 0 \\ \partial_0(e^B\alpha) &= 0 \end{aligned} \tag{6.21}$$

$$\begin{aligned} d = 2(\partial_x + \omega_u{}^u{}_x + a_x)a^x - (\epsilon_x{}^{yz}\omega_y{}^0{}_z + 2b_x)b^x + \frac{1}{4}\alpha^2 \\ + \frac{1}{4}(\omega_x{}^{0y})^2 + \partial_0\partial_0 B - \frac{1}{2}(\partial_x B)^2 + \frac{5}{4}(\partial_0 B)^2 \end{aligned} \tag{6.22}$$

where expressions like $\partial_0\partial_0 B$ are to be understood as $e_0^\mu\partial_\mu(e_0^\nu\partial_\nu B)$ etc. The seven equations (6.21) determine the missing parts of the gauge field in terms of the geometry, while equation (6.22) fixes the scalar d.

Note that we can solve these equations, at least locally, by choosing a particular set of coordinates. So far, (6.21) is valid for any frame in which z is conformal Killing. We can choose a Weyl representative of the metric such that it becomes *Killing* instead. Then, one can choose coordinates such that $z = \partial/\partial t$ and the metric can locally be written as

$$ds^2 = -e^{2B}(dt + 2\mathcal{F})^2 + \mathcal{H}_{ij}dx^i dx^j \tag{6.23}$$

where B, $\mathcal{F}$ and $\mathcal{H}$ do not depend on t. $\mathcal{F}$ is a one-form on the spatial part transverse to z. As a one-form, we have

$$z = e^{2B}(dt + 2\mathcal{F})\,. \tag{6.24}$$

In such coordinates we have additional symmetries of the spin connection. If we also choose a t-independent frame for the spatial dimensions, we have $e^0 \cdot de^x = 0$, which

[9] Note that a_x, b_x correspond precisely to the imaginary antisymmetric part of the ("twisted") intrinsic torsions

$$\begin{aligned} P^A_{x0} &= -P^A_{0x} = ib_x + \frac{1}{2}\partial_x B \\ P^A_{xy} &= -i\big(\epsilon_{xy}{}^z a_z + \frac{1}{4}\delta_{xy}\alpha\big) - \frac{1}{2}\omega_x{}^0{}_y \end{aligned} \tag{6.19}$$

where the superscript A denotes twisting with the $U(2)$ gauge field $P^A_{\alpha\beta} = P_{\alpha\beta} - iA_{\alpha\beta}$.

implies

$$\omega_0{}^x{}_\alpha + \omega_\alpha{}^{x0} = 0 \tag{6.25}$$

and we gain more explicit expressions for the spin connection

$$\omega^0{}_x = \partial_x B\, e^0 + e^B (d\mathcal{F}_{yx})\, e^y \qquad\qquad \omega^x{}_y = \tilde{\omega}^x{}_y - e^B (d\mathcal{F})_{xy}\, e^0 \tag{6.26}$$

where $\tilde{\omega}^{xy} = \omega_z{}^{xy} e^z$ is the spin connection on the three-dimensional space transverse to z.

Taking into account the symmetry (6.25), the differential constraints (6.21) boil down to

$$\partial_t \alpha = \partial_t a_x = \partial_t b_x = 0\,. \tag{6.27}$$

We see that we have the freedom to choose arbitrary values for a_x, b_x and α as long as they do not depend on the isometry coordinate t. One simple solution can be obtained for example by requiring T^+ to vanish, yielding

$$\begin{aligned}
T^+_{\alpha\beta} &= 0\\
A_{00} &= -\frac{\alpha}{4}\\
A_{x0} &= \frac{1}{2}\big(\partial_x \beta - e^B (\tilde{*}\, d\mathcal{F})_x\big)\\
A_{0x} &= e^B (\tilde{*}\, d\mathcal{F})_x\\
A_{xy} &= -\frac{1}{4}\big(\epsilon_{uvy}\omega_x{}^{uv} + 2\epsilon_{xy}{}^z \partial_z B - \delta_{xy}\alpha\big)
\end{aligned} \tag{6.28}$$

where $\tilde{*}$ is the three-dimensional Hodge dual, acting on forms living on the spatial part. The value for the scalar field follows immediately from (6.22).

We can explicitly check that, by restricting to the case where $\eta^i = 0$, we reproduce the results found in [69]. In this particular case the conformal Killing vector becomes Killing.

To summarise, we can preserve some supersymmetry on any manifold with a time-like Killing vector. To this end, we have to turn on the "symmetric traceless" part of the background gauge field as determined in (6.17), the background tensor field as in (6.20), and the background scalar as in (6.22). Upon picking special coordinates (6.23), we are free to choose a t-independent "trace" and "antisymmetric" part of the gauge field.

6.4 Examples, comments and possible extensions to higher dimensions

We have seen that any metric with a non-vanishing timelike Killing vector (and all conformally equivalent metrics) supports some supersymmetry. The general form of such metrics, up to Weyl rescaling, is given in (6.23).

In the particular case where the manifold is the direct product $\mathbb{R} \times M_3$ with M_3 being an Euclidean three-manifold we can just use the $SU(2)$ gauge group and set $A_{\mu 0} = 0$ and $T^+ = 0$. This is obvious from our solution (6.28), which for a direct product space collapses to $A_{\mu x} = -\frac{1}{4}\epsilon_{uvx}\tilde{\omega}_\mu^{uv}$, with all other background fields vanishing. We have taken the spinor to be constant. The $SU(2)$ R-symmetry background field is identified with the $SO(3)$ spin connection on M_3, making the spinor covariantly constant. This is a particular instance of the Euclidean Witten twist applied to the three-manifold M_3. These kind of solutions have an interesting application in holography as boundary theories of supersymmetric non-abelian black holes in AdS_5 [126].

We should mention that we have assumed up to now that the norm of the Killing vector were nowhere vanishing. In the cases where it becomes null on some sub-manifold more attention should be paid to the global properties of the solution. Examples of this kind are discussed explicitly in the three-dimensional case in [59].

We can also make some speculations about extended supersymmetry in higher dimensions. Curiously, a counting of degrees of freedom in the Weyl multiplet of conformal supergravity suggests the possibility that theories with 8 supercharges generally preserve some of their supersymmetry precisely on manifolds with a conformal Killing vector. It is easy to check that in $4, 5$ and 6 dimensions the number of conditions coming from the vanishing of the gravitino and dilation variations, i.e. the d-dimensional analogue of (6.2), is exactly the same as the number of conditions corresponding to the existence of a conformal Killing vector plus the number of components of the bosonic auxiliary fields in the Weyl multiplet.

Lat us discuss for example the 5d case. The generalised CKS equation of the gravitino imposes 4×8 conditions[10] and the dilatino brings another set of 8, making a total of 40 constraints coming from supersymmetry. This is to be confronted with the auxiliary bosonic background fields, which have a total of 26 components [123]. The $SU(2)$ gauge field has 15 components, the scalar 1 and the tensor $T_{\mu\nu}$ additional 10. The remaining 14 degrees of freedom can be stored in a traceless symmetric degree 2 tensor, corresponding

[10] Note that in $5d$, we have symplectic Majorana spinors.

to the CKV condition , $\nabla_{(\mu}z_{\nu)} = \lambda g_{\mu\nu}$. The analysis is analogous in 6d. In some sense, the CKV condition takes the role of the degrees of freedom in the graviton, the tracelessness being related to the Weyl invariance of the equation. The previous counting can be then reformulated as the equality of the fermionic and bosonic off-shell degrees of freedom in the conformal gravity multiplet. The off-shell closure of the algebra is actually true only modulo gauge transformations and the previous argument should be taken as an analogy, although it probably can be made more precise.

The previous argument suggests that extended supersymmetry can be preserved on spaces with a conformal Killing vector also in 5 and 6 dimensions.

Chapter 7

The geometry of $\mathcal{N} = 2$ in four Euclidean dimensions

Supersymmetric theories on curved Euclidean manifolds have attracted much interest in the last years. This is mainly due to the possibility of calculating the partition function on some of these spaces, using supersymmetric localisation techniques.

A word of caution should be spend for the case of $\mathcal{N} = 2$ in four Euclidean dimensions. Of course, it is well known that we can define a consistent $\mathcal{N} = 2$ theory on *any* Euclidean four manifold by a topological twist [5]. The twist was analysed in the language of this thesis long time ago [44]. However this is not the only way of preserving supersymmetry. For example, we can define a SCFT on any conformally flat curved space just by a conformal mapping from flat space. This is the case of the theories studied in [6], for example. In between these two extreme situations there is a full spectrum of possibilities with different auxiliary fields that is investigated in this chapter.

The different ways of putting a theory on curved space have lead to very different results. For instance in the topological twist the energy-momentum tensor is Q-exact and the quantum field theory is topological, i.e. the partition function and correlators are independent of the metric. In fact, the correlation functions compute the Donaldson polynomials of the four-manifold, which are topological invariants. Instead in [6] the partition function was used to find a matrix model description for certain Wilson loops in the $\mathcal{N} = 2$ theory. Curiously, [7] observed that the basic matrix model in [6] can be identified with four point functions in two dimensional Liouville theory. The construction of Pestun has been generalised in [10] where the same theories were put on a squashed four-sphere, preserving some supersymmetry. The generalisation allowed for a more detailed comparison with the 2-dimensional duals[1]. In this more general example background gauge and tensor fields take non-trivial values.

In this chapter we want to systematically discuss the Euclidean geometries preserving $\mathcal{N} = 2$ supersymmetry and the possible background fields we need to turn on for this. The general result turns out to be very similar to the one we found for Lorentzian signature in chapter 6, with the exception of few degenerate cases.

The conditions for supersymmetry involve two symplectic Majorana Weyl spinors of opposite chirality. A degenerate case arises where we preserve supersymmetry by using spinors of one chirality. The conditions of supersymmetry collapse to a non-abelian version of the CKS equation. The topological twist [5] falls in this class; the spinor is made covariantly constant by identifying the $SU(2)$ R-symmetry of the theory with the spin connection of the four manifold. This works for any four-manifold. In this thesis we analyse in detail the general case where supersymmetry is preserved using spinors of both chirality. In this case, as in Lorentzian signature, the condition for preserving some supersymmetry is equivalent to the *existence of a conformal Killing vector (CKV)*, all other constraints determining the background auxiliary fields for which we provide general expressions.

It is worth mentioning that in the case of extended supersymmetry the results turn out to be almost independent of the space-time signature. In fact, the Euclidean results are actually similar to the ones for Lorentzian theories with four supercharges in three and four dimensions, see chapter 4 and [59], and somehow different from the Euclidean results with four supercharges where the geometric constraints require the manifold to be complex in four dimensions (see chapter 3 and [56]) and to possess a suitably constrained contact structure in three, see chapter 5 and [58].

The chapter is organised as follows. In section 7.1 we discuss the field content of the Euclidean version of $\mathcal{N} = 2$ conformal supergravity and we write the supersymmetry variations which include a *generalised* conformal Killing spinor equation and a Lagrangian for vector multiplets coupled to gravity. In section 7.2 we discuss the geometrical structure induced by a pair of chiral spinors and the technical tools we will be using in this chapter. Section 7.3 contains the main results of this chapter: the proof that any manifold with a conformal Killing vector supports some supersymmetry and the explicit expressions for the background auxiliary fields. Finally in section 7.4 we discuss some examples, ranging from various topological twists to the supersymmetry on round and squashed spheres are then discussed.

[1] See also [127] where further details in this context have been studied.

This chapter is based on [3].

7.1 Wick rotation of the conformal supergravity

In order to obtain the Euclidean supersymmetry conditions we have to Wick rotate (6.2). Our strategy is to double the equations and then impose a symplectic Majorana-Weyl condition on the spinors

$$\left(\epsilon^i_+\right)^c = i\epsilon_{ij}\epsilon^j_+ \qquad\qquad (\epsilon_{-i})^c = -i\epsilon^{ij}\epsilon_{-j} \tag{7.1}$$

where $(\epsilon)^c \equiv B^{-1}\epsilon^*$. $\pm$ still denotes chirality. See the appendix A.2.2 for more details. We get two real equations for the gravitino

$$\begin{aligned} &\nabla^A_m\epsilon^i_+ + \frac{i}{4}T^+_{mn}\gamma^n\epsilon^i_- - \frac{1}{4}\gamma_m D^A\epsilon^i_+ = 0 \\ &\nabla^A_m\epsilon^i_- + \frac{i}{4}T^-_{mn}\gamma^n\epsilon^i_+ - \frac{1}{4}\gamma_m D^A\epsilon^i_- = 0\,. \end{aligned} \tag{7.2}$$

Note that we have redefined A_{ma} and $T^{\pm}_{mn}$ which are now both real. The vanishing of the dilatino gives the two conditions

$$\begin{aligned} &i\nabla^A_m T^{+m}{}_n\gamma^n\epsilon^i_- + D^A D^A\epsilon^i_+ - 2\nabla_m A_{n4}\gamma^{mn}\epsilon^i_+ + 2d\,\epsilon^i_+ = 0 \\ &i\nabla^A_m T^{-m}{}_n\gamma^n\epsilon^i_+ + D^A D^A\epsilon^i_- + 2\nabla_m A_{n4}\gamma^{mn}\epsilon^i_- + 2d\,\epsilon^i_- = 0\,. \end{aligned} \tag{7.3}$$

We have

$$\begin{aligned} &\nabla^A_m\epsilon^i_+ = \nabla_m\epsilon^i_+ + \frac{i}{2}A_{ma}\bar\sigma^{ai}{}_j\epsilon^j_+ \qquad\qquad &&\nabla^A_m T^+_{mn} = (\nabla_m + A_{m4})T^+_{mn} \\ &\nabla^A_m\epsilon^i_- = \nabla_m\epsilon^i_- + \frac{i}{2}A_{ma}\sigma^{ai}{}_j\epsilon^j_- &&\nabla^A_m T^-_{mn} = (\nabla_m - A_{m4})T^-_{mn} \end{aligned}$$

where we have defined

$$\sigma^{ai}{}_j = (\vec\sigma, i)^i{}_j \qquad\qquad \bar\sigma^{ai}{}_j = (\vec\sigma, -i)^i{}_j\,. \tag{7.4}$$

After Wick rotation, A_{m4} becomes an $SO(1,1)$ gauge field, the total R-symmetry being $SO(1,1)\times SU(2)$. This is consistent with the R-symmetry group coming from compactifying $\mathbb{R}^{1,9}$ on $\mathbb{R}^6$ to four Euclidean dimensions, see *e.g.* [6] and with the known classification of Euclidean superconformal algebras. The non-compactness of A_{m4} is also necessary to make equations (7.2) and (7.3) consistent with the symplectic Majorana condition.

Similarly, we can Wick rotate the Lagrangian and the supersymmetry transformations of arbitrary vector- and hypermultiplets coupled to the Weyl multiplet above. Here we note the form of the Lagrangian for a vector multiplet

$$\begin{aligned}\mathcal{L}^{\text{Euclid}} = d\,\phi\bar{\phi} + \nabla^A_m\phi\nabla^{Am}\bar{\phi} + \frac{1}{8}Y^i{}_jY^k{}_i - g\left[\bar{\phi},\phi\right]^2 + \frac{1}{8}F_{\mu\nu}F^{\mu\nu} \\ + \frac{1}{4}\big(\bar{\psi}^i_- D^A\psi_{+i} + \bar{\psi}_{-i}D^A\psi^i_-\big) + \frac{i}{2}g\big(\bar{\psi}^i_-\left[\phi,\psi_{-i}\right] - \bar{\psi}_{+i}\left[\bar{\phi},\psi^i_+\right]\big) \\ - \frac{1}{4}\big(\phi F_{mn}T^{+mn} + \bar{\phi}F_{mn}T^{-mn}\big) - \frac{1}{16}\big(\phi^2T^+_{mn}T^{+mn} + \bar{\phi}^2T^-_{mn}T^{-mn}\big) \quad (7.5)\end{aligned}$$

where the Euclidean $SU(2)$ triplet satisfies $\left(Y^i{}_j\right)^* = -Y_i^j \equiv -\epsilon^{jk}Y_{ik}$. We will comment on the supersymmetry transformations below.

In the following we want to discuss on which manifolds we can solve the equations (7.2) and (7.3). There is a degenerate situation where the spinor of one chirality is set to zero. In this case the equations collapse to a non-abelian version of the CKS equation for the remaining spinor, which we can always solve by a twist, as discussed later[2]. From now on we will focus on the general case where supersymmetry is preserved using spinors of both chiralities. In this case the result is very similar to the Lorentzian one: the only geometrical constraint imposed by supersymmetry is the existence of a CKV. We will solve explicitly the condition of supersymmetry and determine the auxiliary fields for any manifold with a CKV.

7.2 The geometry of spinors

It turns out that –albeit the technical details are quite different – many of the equations that we have seen in the Lorentzian describing the geometry defined by the spinors have a very close analogue in the Euclidean. In fact, two chiral Majorana-Weyl spinors define two scalars, two sets of real (anti-)selfdual two-forms η^x_{mn} and $\bar{\eta}^x_{mn}$ and again a tetrad e^a_n.

[2]We can not exclude that more general solutions may exist.

To see this, note that we can construct the following spinor bilinears

$$\begin{aligned}
e^A &= \frac{1}{2}\epsilon^\dagger_{+i}\epsilon^i_+ \\
e^B &= \frac{1}{2}\epsilon^{\dagger\,i}_-\epsilon_{-i} \\
e^a_m &= -\frac{e^{-b}}{2}\epsilon^\dagger_{-j}\gamma_m\epsilon^i_+\bar\sigma^{aj}{}_i \\
\bar\eta^x_{mn} &= -i\frac{e^{-A}}{2}\epsilon^\dagger_{+j}\gamma_{mn}\epsilon^i_+\sigma^{xj}{}_i \\
\eta^x_{mn} &= i\frac{e^{-B}}{2}\epsilon^\dagger_{-j}\gamma_{mn}\epsilon^i_-\sigma^{xj}{}_i
\end{aligned} \tag{7.6}$$

where $b = (A+B)/2$. As opposed to the Lorentzian, in the Euclidean all the forms are real. The η^x and $\bar\eta^x$ are (anti-)selfdual, respectively

$$\sqrt{|g|}\,\epsilon_{mn}{}^{pq}\bar\eta^x_{pq} = -2\bar\eta^x_{mn} \qquad\qquad \sqrt{|g|}\,\epsilon_{mn}{}^{pq}\eta^x_{pq} = 2\eta^x_{mn}\,.$$

Applying the Fierz identities, one can show that, similarly to the Lorentzian case, the e^a_n's form a tetrad,

$$e^a\cdot e^b = \delta^{ab}\,.$$

In this frame, the two-forms have components

$$\bar\eta^x_{ab} = \delta^4_a\delta^x_b - \delta^x_a\delta^4_b + \epsilon^x{}_{ab} \qquad\qquad \eta^x_{ab} = -\delta^4_a\delta^x_b + \delta^x_a\delta^4_b + \epsilon^x{}_{ab} \tag{7.7}$$

and we give more details in the appendix. For completeness, let us mention how the forms defined by the spinors can be stored elegantly into the bispinors

$$\begin{aligned}
\epsilon^i_+\epsilon^\dagger_{-j} &= -\frac{e^b}{4}\sum_a\left(e^a + *e^a\right)\sigma^{ai}{}_j \\
\epsilon^i_+\epsilon^\dagger_{+j} &= \frac{ie^A}{4}\eta_a\sigma^{ai}{}_j \\
\epsilon^i_-\epsilon^\dagger_{-j} &= \frac{ie^B}{4}\bar\eta_a\sigma^{ai}{}_j
\end{aligned} \tag{7.8}$$

where we have defined $\eta^4 = -i(\gamma+1)$ and $\bar\eta^4 = i(\gamma-1)$. In this language, $e^a = e^a_m\gamma^m$, $\bar\eta_x = \frac{1}{2}\bar\eta_{amn}\gamma^{mn}$ and $\eta_x = \frac{1}{2}\eta_{amn}\gamma^{mn}$. The hodge dual of a one-form is $*e^a = \gamma e^a$.

As in the Lorentzian case, we will mainly work in the frame defined by the e^a's. Again, the action of the flat gamma matrices can then be translated into multiplication with Pauli matrices,

$$\begin{aligned}
\gamma^a\epsilon^i_+ &= -e^{\Delta}\sigma^{ai}{}_j\epsilon^j_- \qquad\qquad & \gamma^{ab}\epsilon^i_+ &= -i\bar\eta^{ab}_x\sigma^{xi}{}_j\epsilon^j_+ \\
\gamma^a\epsilon^i_- &= -e^{-\Delta}\bar\sigma^{ai}{}_j\epsilon^j_+ & \gamma^{ab}\epsilon^i_- &= -i\eta^{ab}_x\sigma^{xi}{}_j\epsilon^j_-
\end{aligned} \tag{7.9}$$

where $\Delta = (A - B)/2$. This can be used to show that in this frame the spinors are constant (up to an overall norm factor).

Before we discuss the supersymmetry conditions in the next section, note that a convenient base for spinor doublets of positive and negative chirality is given by, respectively,

$$\bar{\sigma}^{ai}{}_{j}\epsilon_{+}^{j} \qquad\qquad \sigma^{ai}{}_{j}\epsilon_{-}^{j}\,. \tag{7.10}$$

Since the spinors are Majorana-Weyl, each of the two bases contains 4 real components.

We define the intrinsic torsions

$$\nabla_m\epsilon_+^i = -\frac{i}{2}\bar{P}_{ma}\bar{\sigma}^{ai}{}_{j}\epsilon_-^j \qquad\qquad \nabla_m\epsilon_-^i = -\frac{i}{2}P_{ma}\sigma^{ai}{}_{j}\epsilon_+^i \tag{7.11}$$

where P_{ma} and $\bar{P}_{ma}$ are independent real objects. In the frame defined by the spinors, they are, as in the Lorentzian case, composed of the spin connection and the norms of the spinors

$$\begin{aligned} \bar{P}_{m4} &= -\partial_m A & \qquad \bar{P}_{mx} &= \frac{1}{2}\omega_m{}^{ab}\bar{\eta}_{xab} \\ P_{m4} &= \partial_m B & \qquad P_{mx} &= \frac{1}{2}\omega_m{}^{ab}\eta_{xab} \end{aligned} \tag{7.12}$$

Here η_{xab} and $\bar{\eta}_{xab}$ project on the (anti-)selfdual part of the spin connection, respectively.

Our choice of frame degenerates at the points where the spinors vanish. A more detailed analysis will be necessary at the vanishing locus of the spinors to ensure that the results we will obtain are globally defined.

We are now ready to classify the solutions to the equations (7.2) and (7.3).

7.3 Solving the supersymmetry conditions

The analysis will be very similar to the Lorentzian one in section 6.3. Again we define the symmetric traceless part of the torsions

$$p_{ab} \equiv P_{(ab)} - \frac{1}{4}\delta_{ab}P_c^c \qquad\qquad \bar{p}_{ab} \equiv \bar{P}_{(ab)} - \frac{1}{4}\delta_{ab}\bar{P}_c^c$$

and it is an easy exercise to check that the gravitino equation (7.2) is solved by

$$\bar{p}_{ab} = p_{ab} \tag{7.13a}$$

$$p_{ab} = A_{(ab)} - \frac{\delta_{ab}}{4}A_c^c \tag{7.13b}$$

$$T_{ab}^{+} = -2e^{\Delta}\left(\bar{P}_{ab}^{+} - A_{ab}^{+}\right) \tag{7.13c}$$

$$T_{ab}^{-} = -2e^{-\Delta}\left(P_{ab}^{-} - A_{ab}^{-}\right)\,. \tag{7.13d}$$

The first equation tells us that $z = e^b e^4$ is a conformal Killing vector, while the other three equations fix some parts of the background fields. More explicitly, the conformal Killing condition $\nabla_{(m} z_{n)} = \lambda g_{mn}$ implies

$$\lambda = e^b \partial_4 b \qquad \omega_{(x}{}^4{}_{y)} = -\delta_{xy}\partial_4 b \qquad \omega_4{}^4{}_x = \partial_x b \tag{7.14}$$

where we have denoted $\partial_4 = e_4^m \partial_m$ etc. It is an easy task to see that these conditions are equivalent to (7.13a), using the explicit formulae given in (7.12). The other three equations in (7.13) determine the "symmetric traceless" part of the gauge field and the value of the tensor field, respectively. As in the Lorentzian case, it still remains to determine the "antisymmetric" part of the gauge field, its "trace" and the scalar d. In close analogy, this is done by the dilatino equation, which does not impose any new restrictions on the geometry.

Similarly to the Lorentzian case, we introduce a re-definition of the gauge field

$$\begin{aligned}
A_{x4} &\equiv -b_x - \partial_x(b+\Delta) \\
A_{4x} &\equiv b_x + \frac{1}{2}\epsilon_{xyz}\omega_4{}^{yz} + \partial_x b \\
A_{xy} &\equiv \epsilon_{xy}{}^z a_z + \frac{1}{4}\delta_{xy}\alpha + \frac{1}{2}\epsilon_{uvy}\omega_x{}^{uv} + \omega_{[x}{}^4{}_{y]} \\
A_{44} &\equiv \frac{1}{4}\alpha - \partial_4\Delta\,.
\end{aligned} \tag{7.15}$$

Note that a_x and b_x correspond to the anti-symmetric part of the twisted torsion $\bar{P}$

$$\bar{P}_{[4x]} - A_{[4x]} = -b_x \qquad \bar{P}_{[xy]} - A_{[xy]} = -\epsilon_{xy}{}^z a_z \tag{7.16}$$

and, in terms of this re-definition, the tensor field in (7.13) can be written as

$$\begin{aligned}
e^{-\Delta} T^+_{4x} &= b_x + a_x \\
e^{\Delta} T^-_{4x} &= b_x - a_x + 2\partial_x b - \epsilon_x{}^{yz}\omega_y{}^4{}_z\,.
\end{aligned} \tag{7.17}$$

Note that $2\partial_x b - \epsilon_x{}^{yz}\omega_y{}^4{}_z = -4\left(\nabla_{[4} z_{x]}\right)^- = \eta_x{}^{ab}\nabla_a z_b$. The remaining components of T^+ and T^- are fixed according to self-duality.

The analysis of the dilatino equation (7.3) is similar to the Lorentzian case. It can be re-written in a way that it fixes the seven missing pieces of the gauge field

$$\begin{aligned}
\partial_4 a_x + (\omega_x{}^y{}_4 - \omega_4{}^y{}_x)a_y &= e^{-b}\partial_x(e^b\partial_4 b) \\
\partial_4 b_x + (\omega_x{}^y{}_4 - \omega_4{}^y{}_x)b_y &= -e^{-b}\partial_x(e^b\partial_4 b) \\
(\partial_4 + \partial_4 b)\alpha &= 0
\end{aligned} \tag{7.18}$$

and the scalar field d

$$
\begin{aligned}
d = & -\partial_4\partial_4 b - 2\,(\partial_4 b)^2 + \frac{1}{16}\alpha^2 \\
& - (\partial_x + \omega_z{}^z{}_x - \frac{1}{2}\epsilon_x{}^{yz}\omega_y{}^4{}_z - \frac{1}{2}a_x)a^x + \frac{1}{2}(\epsilon_x{}^{yz}\omega_y{}^4{}_z + 2\partial_x b + b_x)b^x
\end{aligned}
\tag{7.19}
$$

In these equations, expressions like $\partial_x\partial_y b$ are to be understood as $e_x^m\partial_m(e_y^n\partial_n b)$ etc.

We can again pick a set of local coordinates that automatically solve (7.18) and leave us with an arbitrariness in the "anti-symmetric" and the "trace"-part of the gauge field. After a Weyl rescaling and a choice of coordinates such that $z = \partial/\partial\xi$, the metric can locally be written as

$$
ds^2 = e^{2b}(d\xi + \mathcal{F})^2 + \mathcal{H}_{ij}dx^i dx^j \tag{7.20}
$$

where b, $\mathcal{F}$ and $\mathcal{H}$ do not depend on ξ. $\mathcal{F}$ is a one-form on the "spatial" part transverse to z. As a one-form, we have

$$
z = e^{2b}(d\xi + \mathcal{F})\,. \tag{7.21}
$$

In such coordinates we can have additional symmetries of the spin connection. By choosing the three-dimensional frame e^x such that $e^4 \cdot de^x = 0$, one gets

$$
\omega_4{}^y{}_x = \omega_x{}^y{}_4 \tag{7.22}
$$

and (7.18) boils down to

$$
\partial_\xi\alpha = \partial_\xi a_x = \partial_\xi b_x = 0\,. \tag{7.23}
$$

We see that also in the Euclidean case we have the freedom to choose arbitrary values for a_x, b_x and α as long as they do not depend on the isometry ξ.

For example we can always use the freedom in the gauge field to locally impose $T^+_{mn} = T^-_{mn} = 0$. From (7.17) we see that this condition is satisfied for

$$
a_x = -b_x = \partial_x b - \frac{1}{2}\epsilon_x{}^{yz}\omega_y{}^4{}_z
$$

and it is a quick check that these a_x and b_x solve (7.18), where one has to use the Bianchi identities. In the coordinates introduced in (7.20), we find for the gauge and scalar fields

$$
\begin{aligned}
A_{44} &= \frac{\alpha}{4} \\
A_{x4} &= -\big(\partial_x\Delta + e^b\,(\tilde{*}\,d\mathcal{F})_x\big) \\
A_{4x} &= 2e^b\,(\tilde{*}\,d\mathcal{F})_x \\
A_{xy} &= \epsilon_{xy}{}^z\partial_z b + \frac{1}{2}\epsilon_{uvy}\omega_x{}^{uv} + \frac{\delta_{xy}}{4}\alpha
\end{aligned}
\tag{7.24}
$$

where $\tilde{*}$ denotes the three dimensional Hodge dual. The value for the scalar d follows from (7.19).

To summarise, in complete analogy to the study of the Lorentzian case, the only constraint imposed by supersymmetry is the existence of a conformal Killing vector. Given such a vector $z \cdot z = e^{2b}$, we can preserve some supersymmetry by turning on background values for the $SO(1,1) \times SU(2)$ gauge field as determined in (7.15) and for the tensor field as in (7.17), where a_x, b_x, α and Δ are free parameters of the solution subjected to (7.18). One also has to turn on a background scalar field as in (7.19).

The previous expressions may become singular at the points where the spinors vanish, in particular where the conformal Killing vector degenerates. A more careful analysis is required near the zeros of the spinors in order to ensure that the solution is regular. The large arbitrariness in the choice of auxiliary fields should usually allow to find globally defined solutions.

7.4 Examples

In this section we present some examples for the formalism we have introduced above.

Round and squashed spheres

Round and squashed spheres have been a main focus in the study of supersymmetry on curved spaces and exact results in quantum field theory. In fact, the work of Pestun [6], who considered $\mathcal{N} = 2$ theories on the round S^4, has in a sense triggered the recent activity in this field. This work has been generalised in [10], where $\mathcal{N} = 2$ theories on a squashed sphere, or ellipsoid, were considered. The squashing preserves a $U(1) \times U(1)$ isometry and the manifold can be described by the equation

$$\frac{x_1^2 + x_2^2}{\ell} + \frac{x_3^2 + x_4^2}{\tilde{\ell}} + \frac{x_5^2}{r^2} = 1$$

where ℓ and $\bar{\ell}$ are the squashing parameters and r is the radius of the sphere. A metric for this space is given by

$$(g^2 + h^2)d\rho^2 + 2fh \sin\rho \, d\theta \, d\rho + \sin^2\rho \, (f^2 d\theta^2 + \ell^2 \cos^2\theta \, d\phi^2 + \tilde{\ell}^2 \sin^2\theta \, d\chi^2) \qquad (7.25)$$

where the functions f, g and h are defined in [10]

$$\begin{aligned} f &= \sqrt{\ell^2 \sin^2\theta + \tilde{\ell}^2 \cos^2\theta} \\ h &= (\tilde{\ell}^2 - \ell^2) f^{-1} \cos\theta \sin\theta \cos\rho \\ g &= \sqrt{\ell^2 \tilde{\ell}^2 f^{-2} \cos^2\rho + r^2 \sin^2\rho} \end{aligned} \tag{7.26}$$

On the ellipsoid there is a Killing vector (here identified with the dual one-form)

$$z = \frac{1}{\ell}\partial_\phi + \frac{1}{\tilde{\ell}}\partial_\chi \hat{=} \sin^2\rho \, (\ell \cos^2\theta \, d\phi + \tilde{\ell} \sin^2\theta \, d\chi) \, .$$

with $z \cdot z = e^{2b} = \sin^2\rho$. In order to apply the formulae discussed in this thesis, we choose a frame with $e^4 = e^{-b} z$,

$$\begin{aligned} e^1 &= f \sin\rho \, d\theta + h d\rho & \qquad e^3 &= \sin\rho \, \cos\theta \sin\theta (\ell d\phi - \tilde{\ell} d\chi) \\ e^2 &= g d\rho & e^4 &= \sin^{-1}\rho \; z \end{aligned} \tag{7.27}$$

which is related to the vierbein in [10] by an $SO(4)$ rotation. Let us discuss the round and the squashed sphere separately.

The round sphere result of Pestun, with all background fields but the scalar vanishing, is recovered in the limit $\ell = \tilde{\ell} = r$. In particular, it should be reproduced by our special solution with vanishing tensor field described around (7.24). Plugging the explicit frame (7.27) into (7.24), we find that the vector field is pure gauge, $A^i{}_j = -(d\phi + d\chi)\sigma^{2i}{}_j$, when $\Delta = \ln \cot \frac{\rho}{2}$ and $\alpha = 0$. Note that $SU(2)$ gauge transformations correspond to local Lorentz transformations in the three "spatial" coordinates, as follows from the definition of the vierbein in (7.8). In fact, if we rotate the frame (7.27) by $(\phi + \chi)$ around e^2, we find that the gauge field vanishes identically. The scalar field (7.19) is constant, $d = 2/r^2 = R/6$ [3].

Now let us discuss the case of the ellipsoid. The solution found in [10] has non-vanishing tensor fields $T^{\pm}$ and an $SU(2)$ gauge field $A_{\mu x}$ whose explicit expressions can be found in equations (3.28) and (3.29) of that paper. Of course, the same result follows from our formalism. After computing the spin connection from the vierbein (7.27), all background fields are fixed in terms of this geometrical data, plus the eight free parameters in our solutions. This is to be contrasted with the result in [10], which is a 3-parameter family instead. The reason for the mismatch is that the authors of [10] work with explicit spinors and have switched off the $SO(1,1)$ gauge field. We checked that choosing Δ

[3] This value corresponds to a vanishing scalar $\tilde{d} = 0$ in the original Weyl multiplet (6.1).

appropriately and tuning the $SO(1,1)$ vector to zero indeed leads to their 3-parameter solution. More explicitly, setting $A_{m4} = 0$ eliminates four of the parameters,

$$\alpha = 0 \qquad\qquad b_x = \delta_{x2}\frac{1}{g}\tan\frac{\rho}{2} \tag{7.28}$$

where we set the value of $e^{\Delta} = \cot\frac{\rho}{2}$. Our scalar (7.19), the gauge field (7.15) and the tensor (7.17) are then identical[4] to the ones in [10], when we identify our parameters a_x with their c_1, c_2 and c_3,

$$a_x = \delta_{x1}\left(4c_2 - \frac{h}{fg}\tan\frac{\rho}{2}\right) + \delta_{x2}\left(4c_1 - \frac{1}{f}\tan\frac{\rho}{2}\right) - \delta_{x3}4c_3\,.$$

The three parameters c_i can be chosen as in [10] in order to have a regular solution on S^4 which reduces at the North and South pole of the squashed sphere to the Ω-background [128] which plays an important role in reducing the computation of the partition function to the Nekrasov partition functions for instantons. The local form of this solution and the Ω-background are discussed in the next section.

Twisting the theory

In this subsection we want to discuss the special case $\Delta = -b$. This choice is is actually related to a general solution discussed in [128]. There the authors have noticed that it is possible to preserve extended supersymmetry on every manifold with a Killing vector z by generalising the Witten twist for $\mathcal{N} = 2$ theories [5].

Let us first discuss the Witten twist in our formalism. The twist corresponds to the degenerate situation where one uses only one chiral spinor, say ϵ_+, while the spinor of the other chirality vanishes and we cannot immediately use our previous formulae. Since only the self-dual part of the spin-connection acts on ϵ_+, corresponding to an $SU(2)$ subgroup of the tangent group $SO(4)$, the spinor can be made covariantly constant by cancelling the spin-connection with the $SU(2)$-gauge field in the covariant derivative. More explicitly, a constant spinor $\partial_m \epsilon^i_+ = 0$ will be covariantly constant if we turn on the $SU(2)$-gauge field

$$A_{mx} = \frac{1}{2}\omega_m{}^{ab}\bar{\eta}_{xab}\,. \tag{7.29}$$

The generalised CKS equation is then satisfied with vanishing tensor fields. One can check that the dilatino equation is also satisfied with a value for the scalar field $\tilde{d} = -R/6$. This computation was first done long time ago in [44].

[4] Note that there is a reshuffling in the order of $SU(2)$ indices. Furthermore, the two gauge fields differ by a gauge transformation.

Let us now consider a more general case, where the gauge field is still as in (7.29) but the negative chirality spinor is different from zero. As suggested in [128], the pair $(\epsilon^i_+, \epsilon^i_- = i\gamma^a z_a \epsilon^i_+)$ preserves supersymmetry on any manifold with Killing vector z. We can easily see this in the picture of coupling to conformal supergravity discussed in this thesis. As it is clear from equation (7.9), the relation $\epsilon^i_- = i\gamma^a z_a \epsilon^i_+$ is true for our spinors exactly when the norm of ϵ_+ vanishes, or

$$\Delta = -b\,.$$

One can then check that for the choice of background fields

$$d = 0 \qquad\qquad T^+_{ab} = 0 \qquad\qquad T^-_{ab} = -4e^b\left(\nabla_{[a}z_{b]}\right)^- \tag{7.30}$$

both ϵ_+ and $\epsilon_- = iz\epsilon_+$ are generalised conformal Killing spinors fulfilling (7.2) and (7.3). There is obviously an analogous solution with a covariantly constant spinor ϵ^i_- and a self-dual tensor T^+.

For completeness, let us derive it from our general expressions above. For the choice $\Delta = -b$, our gauge field takes the value

$$A_{m4} = b_m \qquad\qquad A_{mx} = \frac{1}{2}\omega_m{}^{ab}\bar{\eta}_{xab} + \delta^y_m \epsilon_{yx}{}^z a_z$$

where we have called $b_4 \equiv \alpha + 4\partial_4 b$. Recall that the covariant derivative of ϵ_+ is

$$\nabla^A_a \epsilon^i_+ = \frac{i}{2}\bar{P}^A_{ab}\gamma^b\epsilon^i_-\,. \tag{7.31}$$

If we set $b_4 = 0$ by an appropriate choice of α, the torsion $\bar{P}$ becomes anti-symmetric, $\bar{P}_{mn} = \bar{P}_{[mn]}$, and is given by linear expressions in a_x and b_x as in (7.16). So we see that for vanishing a_x and b_x the spinor ϵ_+ is covariantly constant and the gauge field cancels the self-dual part of the spin-connection as in (7.29). Hence, the class of our solutions with $\Delta = -b$ and $b_m = a_x = 0$ reduces to the generalised twist solution described above. The value of the scalar (7.19) and the tensor field (7.17) become as in (7.30).

The Ω-background

One notable example of the Nekrasov-Okounkov twist discussed in the previous section is the Ω-background on flat $\mathbb{R}^4$. It uses the Killing vector

$$z = \epsilon_1\left(x_1\partial_{x_2} - x_2\partial_{x_1}\right) + \epsilon_2\left(x_3\partial_{x_4} - x_4\partial_{x_3}\right)$$

An $\mathcal{N} = 2$ theory will still preserve some supersymmetry on flat space if the tensor field

$$T^{-} = -2(\epsilon_1 + \epsilon_2)(dx_1 dx_2 + dx_3 dx_4)$$

is turned on. The corresponding field theory has a prepotential determined in terms of the Nekrasov instanton partition function. Alternatively, in the analogue solution with a covariantly constant negative chirality spinor, supersymmetry requires turning on

$$T^{+} = 2(\epsilon_1 - \epsilon_2)(dx_1 dx_2 - dx_3 dx_4)\,.$$

This example is discussed at length in [10] where the background fields on the squashed sphere have been chosen in order to reduce to the Ω-background near the poles, with tensors of opposite chirality at the North and South pole.

Part II

Some exact results in supersymmetric field theory and the $\mathbf{AdS_4/CFT_3}$ correspondence

Chapter 8

Introduction and Outline

So far in this thesis we have studied the very formal question of supersymmetry on curved spaces. In this second part we now want to become more practical, if at a certain point not even a bit technical. Recall the brief tour through some of the most exciting, recent results obtained considering supersymmetric field theories on curved spaces, which we had given in the introduction to the thesis. We had mentioned that there is a matrix model for the $\mathcal{N}=2$ partition function of field theories on S^3, obtained in [11, 14, 15] following the seminal four dimensional work of [6].

This matrix model will play a crucial role in our further analysis. Before we will be able to appreciate solving it for some explicit examples, we should however briefly review the field theories we are interested in.

$\mathcal{N}=2$ Chern-Simons gauge theories and $\mathrm{AdS}_4/\mathrm{CFT}_3$

We are interested in superconformal field theories with a holographic dual. Consider a stack of N M2 branes sitting at the tip of a conical Calabi-Yau four-manifold $X_4 = C(Y/\mathbb{Z}_k)$, where Y is a Sasaki-Einstein 7-manifold. Such a set-up has two dual descriptions [129–132]. On one side of the duality we have M-theory on $\mathrm{AdS}_4 \times Y_7/\mathbb{Z}_k$, including fluxes. On the other side of the duality we have generically a three dimensional $\mathcal{N}=2$ Chern-Simons gauge theory, with some product gauge group $\prod_a U(N)_{k_a}$ and bifundamental matter. Here k_a denotes the (integer) Chern-Simons level of the ath gauge group and $\gcd(\{k_a\}) = k$. For $Y = S^7$ this is just the ABJM model [129] with $\mathcal{N}=6$ supersymmetry. One fundamental check of the correspondence is that the abelian vacuum moduli space of the field theory equals the transverse Calabi-Yau geometry. The field theory is weakly coupled for $k \gg N$, while $N \gg k^5$ corresponds to the M-theory limit of the gravity dual.

Questions

We want to present a list of issues that one might want to address in this context.

From the holographic perspective, one can wonder if it is possible to go beyond matching the moduli spaces when comparing the two descriptions. This matching gives beautiful evidence for a large class of duals but it is clearly desirable to perform a deeper check of the correspondence.

The brane picture brings a puzzling prediction for the scaling of the free energy. From the weekly coupled point of view we expect the partition function to factorise such that each gluon contributes the same amount to the free energy $F = -\ln Z$. In the large N limit of a $U(N)$ theory this implies a scaling as $F \propto N^2$, growing with the number of gluons. However, at strong coupling the gravity description for M2 branes predicts F to scale like $N^{3/2}$ [12]. It has been a longstanding puzzle to observe such a counterintuitive scaling in any field theory with a weekly coupled description.

Even from the purely CFT point of view we find some interesting issues with these field theories. Recall the notion of *exact R-symmetry* that is selected by the superconformal algebra of a $\mathcal{N} = 2$ theory. It is the R-symmetry which sits in the current multiplet and gives $R = \Delta$, with Δ being the scaling dimension of a chiral multiplet, via saturation of a unitary bound. This feature makes it useful to know the exact R when constructing a superconformal theory, yet it is non trivial to determine it. Formally any symmetry that does not commute with the supersymmetry is a good R-symmetry, in fact there is an infinite family. It is desirable to have a mechanism that selects from this infinite family the one in the superconformal algebra. The four dimensional problem was solved in the beautiful work [17], where the authors showed that the anomaly coefficient a is maximised when the R symmetry is chosen to be the exact one. In AdS/CFT there exist a dual geometrical mechanism [16] whose arguments do actually not depend on the dimension of the set-up. This suggests that a similar process to [17] should also exist in $3d$, in absence of anomalies it is though not obvious what that should be. To find an appropriate $3d$ analogue of a-maximisation has been a longstanding puzzle.

Related to the last point is also the quest for a 'c-function' in three dimensions. Following the pioneering c-theorem in two dimensions [19], it is desirable to have a function that decreases along the RG flow, similar to the central charge c in a two dimensional field theory. It has been argued that in $4d$ the role of c is played again by the anomaly coefficient a [133–136] and it has not been obvious for a long time if any such function could exist also in three dimensions.

Answers

It turned out that the free energy $F = -\ln Z$, where Z is the partition function of the field theory on S^3, seems to give answers to most of the questions raised in the last paragraph.

AdS/CFT ties together what will be presented in the rest of the second part of this thesis and the remarkable success with which everything fits together gives a significant backing of the conjecture. We will come back to this several times.

Let us first mention again that in [13] the peculiar $N^{3/2}$ scaling has indeed been observed to arise at strong coupling for the partition function for the ABJM model.

As for the analogue of a-maximisation, Jafferis proposed that F itself is the right function which determines the exact R-symmetry via extremisation [14][1], and in [137] it has been argued that it is actually *maximised*. In this picture, recall that the matrix model for $\mathcal{N} = 2$ theories depends on the choice of the R charges for the matter fields. Moreover, it has been suggested that F obeys an *F-theorem*, providing a monotonically decreasing function along the RG flow [18].

These proposals have passed many tests [4, 18, 138–146]. In chapter 10 we will see how the F-maximisation works in the context of AdS/CFT. There we will also verify that F decreased when flowing from one theory to another by giving a vev to some chiral multiplet.

Outline of part II

Let us elaborate on the AdS/CFT picture of F-maximisation. As we said earlier, R-symmetry is not uniquely defined. Formally any symmetry that does not commute, by the amount of some given normalisation, with the supersymmetry is a good R-symmetry. We can parameterise this ambiguity by fixing a reference R_0 and mixing it with all possible flavour symmetries F_i,

$$R[a_i] = R_0 + \sum_i a_i F_i \ , \tag{8.1}$$

where the a_i's are parameters keeping track of the mixing. The sum runs over the set of abelian flavour groups. For IR fixed points, we can think about the charges under the

[1] This is actually not that endlessly far off from the even dimensional analogous statement one might first think. In a very brought sense one could say that it is always the finite part of F that plays the role we are after. Note that in odd dimensions F has only power-like divergences, which all can be absorbed in a local counterterm, hence F itself can be given a finite meaning. In even dimensions, however, there is also a logarithmic divergence for which this cannot be true. Nevertheless, the coefficient of this logarithmic divergence is proportional to a, which makes this very brought picture close.

reference R_0 as the canonical UV values Δ of the free theory, and the mixing as happening along an RG flow, but we don't have to do that.

Now we come to the main point. The statement about the $N^{3/2}$ scaling can actually be made precise at the level of coefficients, also for theories with less supersymmetry [147, 148],

$$F[a_i] = N^{3/2} \sqrt{\frac{2\pi^6}{27\,\mathrm{Vol}_b\,(Y)}}\,, \tag{8.2}$$

where Y is the Sasaki-Einstein 7-manifold at the base of the non-compact Calabi-Yau moduli space cone. The volume $\mathrm{Vol}_b(Y)$ depends on a certain vector b [16] and in order to match both sides of the equation we need to give a map between this b and the R-charges in $F[a_i]$, parameterised as in (8.1). Recall that the computation of F depends on a choice for the R-symmetry.

The authors of [16] have defined a 'Z-functional' for any odd-dimensional Sasaki-Einstein manifold Y, depending on a trial vector b. This functional is extremised when b is the *Reeb* vector of Y, then Z becomes $\mathrm{Vol}(Y)$. Abusing some language, we should hence view $\mathrm{Vol}_b(Y)$ as a trial, or 'off-shell' volume for Y, which is minimised at the Sasaki-Einstein value. In $4d$, this is the aforementioned geometrical dual of 'a-maximisation' (which predicted the existence of something like F-maximisation in $3d$).

The beautiful formula (8.2) relates a very non-trivial non-perturbative result in field theory for F to the geometrical object $\mathrm{Vol}(Y)$. One should appreciate that the actual computation of F is quite involved and a certain technical challenge. Nevertheless, (8.2) has been checked to hold in many examples [4, 18, 138, 139, 149]. We will discuss this formula to more detail in chapter 10. In particular we will suggest a very natural map between $R[a_i]$ and b for theories where X_4 is toric. In that case we can use the systematic technology of brane tilings [150, 151] extended to M2 branes [152, 153] to compute the volume as a function of the R-charges. The subject is rather technical and we refer the reader to chapter 9 and section 10.2 for details. Before we go on, it is worth mentioning that (8.2) holds even before extremisation. This is curious, it is an off-shell statement. The analogue fact for a in four dimensions has been shown to be generally true [20, 21], in three dimensions so far this holds only at the level of observation.

The whole picture has one big caveat. No-one has been able to evaluate the matrix model of [14, 139] at large N for 'chiral' theories. A theory is called chiral when the matter fields do not come in charge conjugate pairs (and the four dimensional YM theory would be chiral). In all these cases, the free energy does not even show the correct $N^{3/2}$ scaling behaviour. Therefore, many interesting models are excluded from the analysis presented so far. We will briefly take up this issue in section 10.3.5, where we also report on some attempts to overcome the problem.

For 'vector-like' theories where the fields do come in charge conjugate pairs, and the large N formalism gives us well-behaved formulae, we also discuss Seiberg duality with multiple gauge groups. In chapter 11 we observe that the large N free energy is preserved even before extremisation, as in the $\mathcal{N} = 3$ case [148] under the rules derived in [154–156]. A similar discussion has also appeared in [157].

The last topic that we discuss is related to the construction of a purely field theoretical quantity from the geometrical data, along the lines of [20]. Indeed in that paper it was shown that the central charge a can be obtained directly from the information of the dual geometry. In chapter 12 we show that the generalisation does not follow straightforwardly. By exploiting the symmetries of toric Calabi-Yau four-folds, we give a procedure to generalise the cubic formula of [20, 158] to three-dimensional field theories. We apply our general discussion to many examples and find, quite surprisingly, a formula that is quartic in the R charges and reproduces the field theory computations by only using the geometrical data and without any reference to localisation. The results in this chapter have been extended in the work [149].

The rest of part II is organised as follows. In chapter 9 we review the computation of the moduli space of toric $\mathcal{N} = 2$ gauge theories and review the methods to extract the geometrical data from the field theory. In chapter 10 we compute the free energy in vector-like models and compare with the geometric dual predictions. We also comment on chiral-like theories. The Seiberg dual phases of a large class of vector-like theories with multiple gauge groups are discussed in chapter 11 and the duality is verified at the level of the large N free energy. We conclude be commenting on a different formulation of the extremisation problem in field theory and on its relation to the volume minimisation in chapter 12.

Part II of this thesis is based on the paper [4].

Chapter 9

$\mathcal{N} = 2$ quiver gauge theories in $3d$ with a toric moduli space

9.1 The field theory description

In this section we briefly review the main aspects of the gauge theories that we study in the rest of this chapter. They are three dimensional $\mathcal{N} = 2$ supersymmetric quiver gauge theories which are believed to describe the low energy dynamics of a stack of M2 branes probing a conical CY$_4$ singularity [129–132]. Generically they have a product gauge group $\prod U(N)_{k_a}$ with Chern-Simons (CS) kinetic terms at level k_a and bifundamental matter X_{ab}, transforming in the fundamental of $U(N)_{k_a}$ and the anti-fundamental of $U(N)_{k_b}$. The field content is conveniently represented by a 'quiver diagram', see the first part of figures 10.1, 10.2, 10.3 and 10.4 for examples. In $\mathcal{N} = 2$ language, the Lagrangian reads

$$\begin{aligned}\mathcal{L} \; &= \; \sum_a \frac{k_a}{2\pi} \int d^4\theta \int_0^1 dt\, V_a \overline{D}^\alpha (e^{-tV_a} D_\alpha e^{tV_a}) + \int d^4\theta \sum_{X_{ab}} X_{ab}^\dagger e^{V_a} X_{ab} e^{-V_b} \\ &+ \; \int d^2\theta\, W(X) + c.c. \end{aligned} \tag{9.1}$$

The first term is the CS Lagrangian at level k_a for the gauge superfield V_a associated to the gauge group $U(N)_{k_a}$. The second term is the usual minimal coupling between matter and gauge fields, and W is the superpotential. The three dimensional vector superfield is (in WZ gauge)

$$V = i\theta\bar{\theta}\,\sigma + \theta\gamma^\mu\bar{\theta}\,A_\mu - \theta^2\bar{\theta}\,\bar{\lambda} - \bar{\theta}^2\theta\,\lambda + \theta^2\bar{\theta}^2\,D\,. \tag{9.2}$$

The field σ is an extra scalar with respect to the four dimensional vector multiplet, coming from the fourth component of the gauge field upon dimensional reduction. In terms of component fields, the Chern-Simons Lagrangian reads

$$S_{CS} = \sum_a \frac{k_a}{4\pi} \int \mathrm{Tr}\left(A_a \wedge \mathrm{d}A_a + \frac{2}{3} A_a \wedge A_a \wedge A_a - \overline{\lambda}_a \lambda_a + 2 D_a \sigma_a \right) \qquad (9.3)$$

The abelian moduli space $\mathcal{M}$ is a Calabi-Yau four-fold that corresponds to the transverse space of the M2-branes. We will briefly review how it can be obtained [131, 132]. Roughly speaking, it is the solution to the F-term equations $\partial_{X_{ab}} W = 0$, with fixed complexified gauge freedom. The solution to the F-terms is a $(g+2)$ dimensional toric variety sometimes called 'master space' $\mathcal{F}$ [159], here g denotes the number of gauge groups. We obtain

$$\mathcal{M} = \mathcal{F}/G^{\mathbb{C}} \ , \qquad (9.4)$$

where $G^{\mathbb{C}}$ is a complexified gauge group. The action of $G^{\mathbb{C}}$ in (9.4) is a bit subtle for $3d$ Chern-Simons theories, it is not the full gauge group $\prod_{a=1}^{g} U(1)_{k_a}$ that acts here. One can show that from the g gauge groups, only the $(g-2)$ combinations sitting in the kernel of

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ k_1 & k_2 & \dots & \dots & k_{G-1} & k_G \end{pmatrix} . \qquad (9.5)$$

are imposed in (9.4) [131, 132].[1] It includes also a discrete part, see footnote 1 for a very brief discussion and references. As a result, we find that

$$\mathcal{M} = CY_4/\mathbb{Z}_k \ .$$

In this thesis we will be interested in 'toric field theories'. We call a theory toric when the moduli space is a *toric* Calabi-Yau. This will be satisfied when every matter field appears exactly twice in the superpotential, once in a term with a positive sign and once in a term with a negative sign. In that case there is a developed algorithm for solving (9.4) by an ambient space construction. In chapters 9.2 and 9.3 we will review this formalism in more detail.

9.2 The toric description

In the case of toric quiver gauge theories, the information about the moduli space of the field theory is encoded in a set of combinatorial data which is represented in the so-called toric diagram. The toric diagram can be extracted from the field theory in various

[1] By taking linear combinations, one can see that only $(g-2)$ non-trivial groups have a vanishing D-term, which can be ditched by imposing complexified gauge invariance. The diagonal gauge group $\sum_a U(1)_{k_a}$ is trivial, no field is charged under it. The remaining group, which can be taken along $\sum_a k_a U(1)_{k_a}$, is broken to a discrete $\mathbb{Z}_k$ subgroup [129, 131, 132, 160, 161]. Here, $k = \gcd(\{k_a\})$.

(equivalent), purely algorithmic ways. This is true for four as for three dimensional field theories, see *e.g.* [150–153, 162, 163] and references therein. The particular algorithm we will use relies on the results presented in [152, 162, 163].

In order to obtain the toric diagram from the field theory data, we construct, in two steps, the so-called perfect matching matrix. Due to the toric condition, there is an even number of superpotential terms, half of them come with a positive sign, and the other half with a negative sign. Moreover, every field X appears exactly once in each set of terms, say in the i-th term of the positive set and in the j-th term of the negative set. As a first step, we construct a matrix by adding the fields X to the (i, j)-th entries. Let us denote its determinant by K and the number of terms in K by c. As a second step we construct another matrix, this time the (I, α) entry is 1 if the I-th field is in the α-th term of K, and 0 otherwise. This is is the perfect matching matrix, which we denote P. We can decompose the fields as

$$X_I = \prod_{\alpha=1}^{c} p_\alpha^{P_{I\alpha}} \ , \tag{9.6}$$

where the p_α's are called perfect matchings and I labels the set of chiral matter fields. The whole point of the procedure is that this decomposition, which is completely algorithmic, by construction solves the F-term equations.

We then define the incidence matrix d of the quiver. Each row corresponds to a gauge group, and each column to a field. The (i, j) entry is 1 if the j-th field transforms in the fundamental representation of the i-th gauge group, -1 if the field transforms according to the anti-fundamental representation, and 0 otherwise. By using the incidence matrix and the perfect matching matrix we can define a new matrix Q by

$$d = Q \cdot P^T \tag{9.7}$$

It is the charge matrix of the associated GLSM, and gives the D-terms when modded out by the gauge symmetry. Similarly, the perfect matching matrix P, extracted from the superpotential alone, gives the F-terms. Putting this together, the toric diagram for the complex four-dimensional Calabi-Yau cone is given by

$$G_T = \mathrm{Ker} \begin{pmatrix} Q_F \\ Q_D \end{pmatrix}^T \equiv \mathrm{Ker} \begin{pmatrix} \mathrm{Ker}(P)^T \\ \mathrm{Ker}(C) \cdot Q \end{pmatrix}^T \tag{9.8}$$

where C is the matrix in (9.5). G_T is a matrix with four rows and the n columns are the n four-vectors generating the fan for the four-dimensional toric Calabi-Yau cone. See [164] for an introduction to toric geometry. Every column of this matrix is in one-to-one correspondence with the perfect matchings represented as the c columns of P or the c terms in K. By a $SL(4,\mathbb{Z})$ transformation we can rotate all the vectors such that each last component is 1, viz. $v_i = (w_i, 1)$, $i = 1, \ldots, n$. This is special for Calabi-Yau manifolds. The convex hull of the w's is called toric diagram, by some abuse of notation we will sometimes also call the entire matrix G_T toric diagram as the incorporated information is equivalent. See figures 10.1, 10.2, 10.3 and 10.4 for examples of toric diagrams and their dual quiver graphs.

The toric diagram encodes all the relevant data of the toric Calabi-Yau cone $\mathcal{M} = C(Y)$. In particular, we can compute the volume of the seven-dimensional Sasaki-Einstein base manifold Y and of its five-cycles without any knowledge of the metric just by looking at the vectors in G_T [16]. Each independent compact five-cycle is in correspondence with an external point of the toric diagram and its volume can be determined as a function of a certain vector b. Let $v_i = (w_i, 1)$ be the vector in G_T, corresponding to the external point w_i in the toric diagram and consider the counter-clockwise ordered sequence of vectors $w_k, k = 1, \ldots, n_i$, that are adjacent to v_i. We can compute the volume of a 5-cycle Σ_i as [16, 152]

$$\mathrm{Vol}(\Sigma_i) = \sum_{k=2}^{n_i-1} \frac{(v_i, w_{k-1}, w_k, w_{k+1})(v_i, w_k, w_1, w_{n_i})}{(v_i, b, w_k, w_{k+1})(v_i, b, w_{k-1}, w_k)(v_i, b, w_1, w_{n_i})} \tag{9.9}$$

where (v_1, v_2, v_3, v_4) denotes the determinant of four vectors $v_{1,2,3,4}$. Note that this is a functional of the vector b. Let us define the sum of these volumes as

$$Z = \sum_i \mathrm{Vol}(\Sigma_i) \ . \tag{9.10}$$

The central statement in [16] is that this functional is minimised when b is the *Reeb* vector and that then Z is proportional to the volume of Y. We can thus view b in (9.9) as a 'trial Reeb vector'. Note that the Calabi-Yau condition $v_4 = 1$, implies setting the fourth component of the Reeb vector $b_4 = 4$.

A five-brane wrapped on a given five-cycle Σ_i corresponds to an operator with dimension [165]

$$\Delta_i = \frac{2\mathrm{Vol}(\Sigma_i)}{Z} \tag{9.11}$$

In the next section we present a different way to compute the volume functional of the underlying moduli space geometry, directly from the field theory data but having a very natural interpretation in the toric language.

9.3 The Hilbert series

A convenient way to extract the volume of the moduli space, which does not require the geometrical technologies involving the Reeb vector and individual 5-cycles, is related to counting the mesonic operators of the field theory. The counting can be performed by the Hilbert series, which is the partition function for the mesons on the M2 moduli space, see e.g. [152, 159, 166].

As a crucial fact, the pole of the series gives the volume of the Sasaki-Einstein space, while keeping track of the dependency on the global symmetries [167]. We want to elaborate a bit on how exactly this track-keeping works in our context.

In the toric case the counting becomes particularly easy, since we we can systematically solve the F-terms through perfect matchings, which results in the quotient description of the moduli space

$$\mathcal{M} = \mathbb{C}^d / \left(\mathbb{C}^*\right)^{d-4} . \tag{9.12}$$

Here, d denotes the number of perfect matchings assigned to external points of the toric diagram and the charge matrix of the quotient is given by the linear relations amongst the corresponding vectors in the fan. This quotient construction makes manifest the dependency on all of the global symmetries, which the moduli space inherits from the natural isometries of the ambient space $\mathbb{C}^d$. Generically, there are more external perfect matchings than global symmetries. This is because we had to introduce extra fields together with spurious symmetries, which are not seen by the physical fields. It is the prize we have to pay when solving the F-terms via perfect matchings. Upon parameterising the symmetries by the perfect matchings we might encounter a redundancy.

Nevertheless, it is a prize we are happy to pay; we will find it convenient to parameterise the set of global symmetries of the field theory in terms of coordinates on $\mathbb{C}^d$, as this will make the comparison with the geometry most straightforward.

The Hilbert series for the flat ambient space reduces to the geometrical series Hil $\sim 1/(1-t)^d$, and the quotient can be realised by projecting on the singlets under the $\left(\mathbb{C}^*\right)^{d-4}$ action,

$$\mathrm{Hil}(t_i; \mathcal{M}) = \oint \prod_{k=1}^{d-4} \frac{\mathrm{d}z_k}{2\pi i z_k} \frac{1}{\prod_{i=1}^{d} (1 - t_i Z_i)} , \tag{9.13}$$

where $Z_i = Z_i(z_k)$ is the monomial weight of the i-th homogeneous coordinate under the $\left(\mathbb{C}^*\right)^{d-4}$ action in (9.12). If we further set $t_i = e^{-2\epsilon a_i}$ and take the $\epsilon \to 0$ limit, we have [152, 167]

$$\mathrm{Hil}(t_i; \mathcal{M}) \sim \frac{\mathrm{Vol}_{a_i}(Y)}{\epsilon^4} + \dots , \tag{9.14}$$

which gives us an expression for the volume of the base Y in terms of the parameters a_i. The equation (9.14) is a geometric formula that knows about the flavour symmetries of the field theory, precisely through the a_i's which appear on the CFT side via the perfect matchings. We will see in section 10.2 how we can reproduce this geometric formula from a purely field theoretical computation.

Chapter 10

The free energy in the large N limit and holography

10.1 The large N free energy of vector-like quivers

In this section we discuss the computation of the leading order term of the free energy of vector-like field theories in a large N expansion. Localisation leads to a matrix model for the partition function on the three-sphere [11, 14, 15],

$$Z[\Delta] = \int \prod_{i,a} d\lambda_i^{(a)} \exp\Big(- F\big(\lambda_i^{(a)}, \Delta\big)\Big) \tag{10.1}$$

where the integral is over the Cartan of the gauge group $\prod_{a=1}^{g} U(N)_{k_a}$, parameterised by the $g \times N$ many variables $\lambda_i^{(a)}$, $i = 1 \ldots N$ and

$$F\big(\lambda_i^{(a)}, \Delta\big) = \ln \prod_a \left[e^{\frac{i \sum k_a \lambda_i^{(a)2}}{4\pi} - \sum \Delta_m^{(a)} \lambda_i^{(a)}} \prod_{i<j} \sinh^2 \left(\frac{\lambda_i^{(a)} - \lambda_j^{(a)}}{2} \right) \prod_\rho e^{l\left(1 - \Delta + i\rho\left(\frac{\lambda}{2\pi}\right)\right)} \right] \tag{10.2}$$

Furthermore, $\Delta_m^{(a)}$ is a monopole charge associated with the ath gauge group [14] and $\ell(z)$ is the one loop determinant of the matter fields computed in [14, 15]

$$\ell(z) = -z \ln\left(1 - e^{2\pi i z}\right) + \frac{i}{2}\left(\pi z^2 + \frac{1}{\pi} Li_2 e^{2\pi i z}\right) - \frac{i\pi}{12} \tag{10.3}$$

with derivative

$$\ell'(z) = -\pi z \cot\left(\pi z\right) \ . \tag{10.4}$$

Finally, ρ refers to the weights of the representation of each matter field with R-charge Δ, the product is over all chiral fields. Note that the full matrix model is a functional of the R-charges.

We want to solve this matrix model in a large N approximation, following closely the procedure of [18]. The integral at large N and finite k_a is dominated by the minimum of the free energy $F\big(\lambda_i^{(a)}, \Delta\big)$ and can be approximated by configurations that obey the saddle point equations $\partial_{\lambda_i^{(a)}} F = 0$. For vector-like theories, it turns out that a sensible ansatz for the eigenvalues λ is given by

$$\lambda_i^{(a)} = N^{1/2} x_i + i y_i^{(a)} \tag{10.5}$$

For large enough N, one can replace the discrete set (10.5) with g continuous variables. The real part of the eigenvalues becomes a dense set with density $\rho(x) = ds/dx$ and the imaginary parts $y_i^{(a)}$ become functions $y_a(x)$. The free energy can be split in two parts, a piece from the Chern-Simons and monopole terms

$$F_{CS} = \frac{N^{3/2}}{2\pi} \int \mathrm{d}x\, \rho(x)\, x \sum_a \big(k_a y_a + 2\pi \Delta_m^{(a)}\big) \tag{10.6}$$

and a second piece from the one loop determinant of the vector and the matter fields. For a pair of bifundamental and anti-bifundamental fields (X_{ab}, X_{ba}) in a vector-like theory, charged under the ath and the bth gauge group, with R-charges $(\Delta_{ab}, \Delta_{ba})$, we have

$$F_{1-loop} = -N^{3/2} \frac{(2-\Delta_{ab}^+)}{2} \int \mathrm{d}x \rho(x)^2 \left(\delta y_{ab}^2 - \frac{\pi^2}{3} \Delta_{ab}^+ (4 - \Delta_{ab}^+)\right) \tag{10.7}$$

where $\delta y_{ab} \equiv y_a(x) - y_b(x) + \pi \Delta_{ab}^-$ and $\Delta_{ab}^{\pm} = \Delta_{ab} \pm \Delta_{ba}$. For an adjoint field we use (10.7) with $a = b$ and divide by a factor two. Equation (10.7) is only valid in the range $|\delta y_{ab}| \leq \pi \Delta_{ab}^+$, which will indeed be satisfied by all our solutions.

In the continuum limit, the resulting free energy is extremised as a functional of ρ and the y's at the saddle point. The eigenvalue density is subjected to the consistency constraints

$$\int \mathrm{d}x\, \rho(x) = 1 \qquad \rho(x) \geq 0 \ \text{ point} - \text{wise}$$

One can impose the first constraint through a Lagrange multiplier μ. This set of rules is enough to compute the free energy of the vector like theories as a function of the R charges, $F(\Delta)$ [18].

As observed in [18] the expressions (10.6) and (10.7) possess flat directions which parameterise the symmetries on the eigenvalues and on the R charges. By defining the real parameters $\eta^{(a)}$ they are

$$\begin{aligned} y_a &\to y_a - 2\pi \eta^{(a)} \\ \Delta_{ab} &\to \Delta_{ab} + \eta^{(a)} - \eta^{(b)} \\ \Delta_m^{(a)} &\to \Delta_m^{(a)} + k_a \eta^{(a)} \end{aligned} \tag{10.8}$$

10.2 Relation with the geometry

We want to put forward the immediate coincidence of the mesonic expression for the volume of the Sasaki-Einstein space Y given in equation (9.14),

$$\mathrm{Hil}(t_i;\mathcal{M}) \sim \frac{\mathrm{Vol}_{a_i}(Y)}{\epsilon^4} + \dots \ , \tag{10.9}$$

with the free energy of the field theory evaluated at large N.[1] The free energy is a function of the R-charges Δ_I of the fields X_I and we can identify

$$F\left(\Delta_I\right) = N^{3/2}\sqrt{\frac{2\pi^6}{27\,\mathrm{Vol}_{a_i}\left(Y\right)}}\ , \tag{10.10}$$

where we have the identification

$$\Delta_I = \sum_i P_{Ii}\, a_i \ , \tag{10.11}$$

with i running over the external perfect matchings and P_{Ii} being the matrix introduced around equation (9.6). The flat directions (10.8) of F identify $g-2$ baryonic symmetries which do not contribute to the free energy functional. A similar decoupling is well-known from the four-dimensional case [20]. The flat directions of the free energy are analogue to the invariance of the Hilbert series under $(g-2)$ combination of gauge symmetries, recall that we projected on the mesonic singlets as in equation (9.4). In the explicit construction of (9.4) via (9.12), and correspondingly upon (10.11), we also encounter spurious extra flat directions via relations amongst the perfect matchings. This will be discussed for some concrete examples in 10.3.

We identify several advantages when inferring the volumes from the Hilbert series. First, (10.9) provides a fast and more direct way than (9.10) to obtain the geometrical informations of the volumes, without the need of mapping the R-charges of the PM with the volumes of the 5-cycles as in (9.11). Moreover we can compute the Hilbert series even in non-toric models, where we cannot use the simple formulas (9.10) and (9.13) anymore, opening the way for a more general analysis as in [21].

In the next section we want to make all these words more concrete and present some examples. In the appendix C we also discuss the matching of the field theory free energy with the geometrical Z-function at arbitrary trial Reeb vector.

[1] Related discussions appeared in [168, 169].

10.3 Examples

We now apply the discussion above to compute the free energy of some models. Our aim is to compare the field theoretical quantity $F(\Delta)$ with the volume of the dual Sasaki-Einstein manifold and verify equation (10.10). In all our examples we find that the result from the Hilbert Series and the large N free energy coincide even before extremisation. Our results do not rely on the underlying symmetries enjoyed by the quiver gauge theories at the infrared fixed point and generalise some of the results in [138].

We first study the vector-like theories $\mathbb{C}\times\mathcal{C}$, $\widetilde{\mathrm{SPP}}$ and $\widetilde{\mathcal{C}/\mathbb{Z}_2}$. Here we use the notation of [152], where the tilded names are inherited from the four-dimensional theories with the same quiver but Yang-Mills instead of Chern-Simons gauge interactions. The mesonic moduli space and the Hilbert Series of the first two and various other models have been studied in [152]. The three theories we will first study are connected by an RG flow, which on the field theory side corresponds to giving a vev to one of the scalar fields and then integrating it out. This is reproduced on the gravity side by a partial resolution of the singularity, which can be conveniently represented as removing an external point in the toric diagram. The geometric RG flow is equivalent to blowing up a singularity, which in turn implies that the volume of the manifold increases. Hence, once established the relation $F^2 \sim 1/\mathrm{Vol}$, the decreasing of F follows immediately in these cases, in agreement with the conjectured F-theorem [18]. We will comment on this in slightly more detail in section 10.3.3.

In section 10.3.5 we will also report on some attempts to solve the matrix model for *chiral* theories.

10.3.1 $\mathbb{C}\times\mathcal{C}$

Consider a theory with gauge group $U(N)_k \times U(N)_{-k}$, two adjoints ϕ_i and two pairs A_i, B_i, $i = 1,2$ of bifundamental fields in the $(\mathbf{N},\bar{\mathbf{N}})$ and $(\bar{\mathbf{N}},\mathbf{N})$, respectively, as depicted in the quiver of figure 10.1. The superpotential is

$$W = \mathrm{Tr}\left(\phi_1(A_1B_2 - A_2B_1) + \phi_2(B_1A_2 - B_2A_1)\right) \tag{10.12}$$

and the moduli space is $\mathbb{C}\times\mathcal{C}$, where $\mathcal{C}$ is the conifold. Finding the exact superconformal R symmetry requires, a priori, an arbitrary choice of combining the $g+2=4$ abelian symmetries (subjected to $R_{\mathrm{trial}}[W]=2$) to parameterise the dimensions Δ and eventually finding the exact choice of R by extreme zing F. We want to keep an eye on the correspondence and parameterise the dimensions in a way that allows for natural comparison

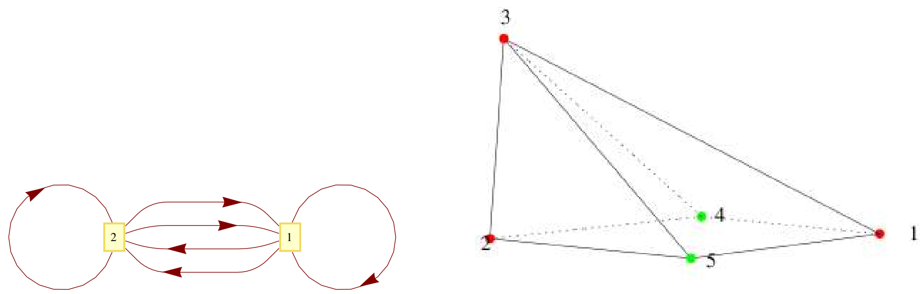


Figure 10.1: Quiver and toric diagram for $\mathbb{C} \times \mathcal{C}$.

with the geometry even before extremisation. To this end, we assign to each external perfect matching p_i the charge a_i, where we deliberately over-count global symmetries by the number of relations between the external p_i's. Since perfect matchings correspond to points in the toric diagram, given in figure 10.1, we can then directly incorporate the toric data of the moduli space.

The perfect matching matrix suggests the charge assignment

$$\begin{aligned} \Delta_{A_1} &= a_1 + a_4 \qquad & \Delta_{A_2} &= a_2 + a_4 \qquad & \Delta_{\phi_1} &= a_3 \\ \Delta_{B_1} &= a_1 + a_5 \qquad & \Delta_{B_2} &= a_2 + a_5 \qquad & \Delta_{\phi_2} &= a_3 \end{aligned} \tag{10.13}$$

where the marginality condition on the superpotential $\Delta(W) = 2$ is reflected by $\sum_i a_i = 2$. Following the rules outlined in the previous section, we get the free energy functional

$$\frac{F[\rho, u, \mu]}{N^{3/2}} = \int \mathrm{d}x \left(\frac{k}{2\pi}\, \rho\, x\, u + \rho^2 \left[P_{a_i} - a_3(\pi(a_4 - a_5) + u)^2 \right] \right) - \frac{\mu}{2\pi} \left(\int \rho - 1 \right)$$

where

$$P_{a_i} = -a_3 \pi^2 \left((a_1 - a_2)^2 - (a_3 - 2)^2 \right) \tag{10.14}$$

and we defined $u = u(x) \equiv y_1(x) - y_2(x)$. Note that we have included the monopole charge Δ_m not via a topological term in the free energy functional, but via $a_{4/5} \to a_{4/5} \pm 2\Delta_m$, corresponding to the direction in the abelian gauge space which is broken to $\mathbb{Z}_k$. This can be done by shifting $y^a \to y^a - \Delta_m / k^a$. The free energy functional is extremised for

$$\rho = \begin{cases} \frac{\mu + 2\, k\, \pi\, x\, A_2}{8\pi^3 (A_1 - A_2)\phi(A_2 + B_2)} & , \quad -\frac{\mu}{2k\pi A_2} < x < -\frac{\mu}{2k\pi A_1} \\ \frac{\mu/\pi + k\, x (B_1 - B_2)}{4 P_{a_i}} & , \quad -\frac{\mu}{2k\pi A_1} < x < \frac{\mu}{2k\pi B_1} \\ \frac{\mu - 2k\, \pi\, x B_2}{8\pi^3 (A_1 - A_2)\phi(A_2 + B_2)} & , \quad \frac{\mu}{2k\pi B_1} < x < \frac{\mu}{2k\pi B_2} \end{cases} , \tag{10.15}$$

where, without loss of generality, we assumed that $a_1 > a_2$ and $a_4 > a_5$. Furthermore, we used (10.13) and for the ease of notation we denoted $\psi \equiv \Delta_\psi$ for a field ψ. In the outer regions of (10.15), u is frozen to $u_{min} = -2\pi(a_2+a_4)$ and $u_{\max} = 2\pi(a_2+a_5)$, respectively. In the middle region we find

$$u(x) = \frac{kP_{a_i}x}{a_3\left(\mu + k\pi x\left(a_4 - a_5\right)\right)} - \pi\left(a_4 - a_5\right) \,. \tag{10.16}$$

The Lagrange multiplier μ is fixed by $\int \rho = 1$ and the free energy finally reads

$$\left(\frac{F}{N^{3/2}}\right)^2 = \frac{32\pi^2 k\, a_3\left(a_1+a_4\right)\left(a_2+a_4\right)\left(a_1+a_5\right)\left(a_2+a_5\right)}{9\left(2-a_3\right)} \tag{10.17}$$

We want to compare this to the Hilbert series. From the toric data in figure 10.1 and (C.1), we read off the monomial weights

$$(Z_i) = (z, z, 1, z^{-1}, z^{-1})\,. \tag{10.18}$$

We can then compute the Hilbert series (9.13),

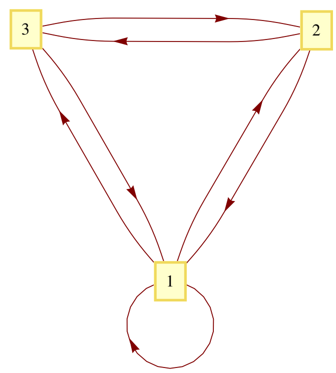
$$\mathrm{Hil}(Y_{\mathbb{C}\times\mathcal{C}};\, t_i) = \oint \frac{\mathrm{d}z}{2\pi i z}\frac{1}{(1-t_1 z)(1-t_2 z)(1-t_3)(1-t_4/z)(1-t_5/z)}\,, \tag{10.19}$$

whose pole for $t_i = e^{-2\varepsilon a_i} \to 1$ indeed reveals $1/F^2$ from (10.17).

That this is a good description of the mesonic moduli space might look puzzling, when only counting parameters. As mentioned above, there are generically more a_i's, namely (g + 2 + (# of relations on the external p.m.'s)) [2] than there are mesonic symmetries, namely 4. The key observation is though, that the a_i's appear only in combinations of meson charges, hence modulo baryonic and spurious symmetries. In our example this is particularly easy since, given $g = 2$, there are no baryonic symmetries in the game. We do, nevertheless, identify the non-physical, spurious symmetry

$$a_{1/2} \to a_{1/2} + s\,, \qquad a_{4/5} \to a_{4/5} - s\,,$$

[2] This redundancy amongst the external perfect matchings originates from the splitting of points in the parent 2d diagram, which have a multiplicity [152]. These points may sit on the perimeter or in the internal of the diagram. Depending on this, the new external points of the 3d diagram may not be in 1 : 1 correspondence with the ($g+2$ many) global symmetries of the CFT$_3$. This is opposed to four dimensional theories, where the number of the external points is always identical with the number of non-anomalous global symmetries. When going to 3d, the anomalies disappear and all $g+2$ global symmetries are physical. Pick's theorem relates the number of these symmetries to the properties of the 2d toric diagram, $g+2 = \text{Perimeter} + 2(\text{Internal Points})$. We see that precisely in the cases in which all internal points split, the number of external points of the split3d diagram is $g+2$. Else, there are extra points coming from split points on the perimeter. This is the case for the vector-like theories discussed in this section. We marked the split points by green dots.



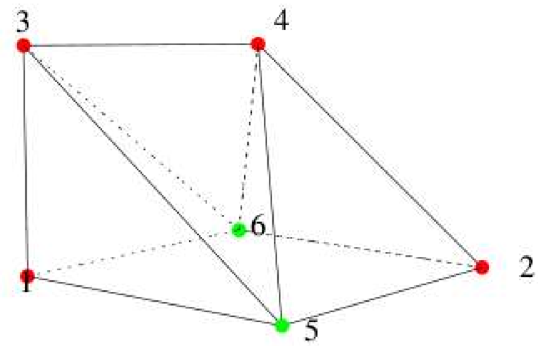


Figure 10.2: Quiver and toric diagram for $\widetilde{SPP}$. The diagram is plotted for CS levels $(2,-1,-1)$.

which reflects the relation $p_1 + p_2 = p_4 + p_5$ of the perfect matchings and reduces the number of independent a_i's to 4. These can in principle be mapped to the charges under the Cartan-part of the global symmetry

$$SU(2)_1 \times SU(2)_2 \times U(1)_1 \times U(1)_2 \,.$$

In absence of baryonic symmetries the bifundamental fields by themselves are mesonic operators, and their dimensions appear in the final result for F.

We conclude this example by observing that in [18] the authors discussed a dual phase of this theory, which involves fundamental flavor fields. Upon the identifications of the PM the two expressions for the free energy coincide.

10.3.2 $\widetilde{SPP}$

Next, we want to study the quiver in figure 10.2 with gauge group $U(N)_1 \times U(N)_2 \times U(N)_3$, one adjoint ϕ of the $U(N)_1$, and three pairs A_i, B_i, C_i of (anti) bifundamentals in the $(\mathbf{N}, \bar{\mathbf{N}}, 1), (1, \mathbf{N}, \bar{\mathbf{N}}), (\bar{\mathbf{N}}, 1, \mathbf{N})$ representation of the gauge group, respectively, and Chern-Simons couplings $(-k_2 - k_3, k_2, k_3)$. The superpotential reads

$$W = \phi(A_1A_2 - C_2C_1) - A_2A_1B_1B_2 + C_1C_2B_2B_1 \,. \tag{10.20}$$

As special cases, the family includes $\widetilde{SPP}_{-211}$ and $D_3 = \widetilde{SPP}_{1-10}$. From the perfect matching matrix, we again infer the charge assignment [152]

$$\Delta_{A_i} = a_i + a_{i+4}, \;\; \Delta_{B_i} = a_{i+2}, \;\; \Delta_{C_i} = a_i + a_{7-i}, \;\; \Delta_\phi = a_3 + a_4 \,, \tag{10.21}$$

where the six a_i's include one redundancy and one baryonic direction. For the ease of notation, let us introduce the combinations

$$A_- = \Delta_{A_1} - \Delta_{A_2}, \quad B_- = \Delta_{B_1} - \Delta_{B_2}, \quad C_- = \Delta_{C_1} - \Delta_{C_2}, \quad B_+ = \Delta_{B_1} + \Delta_{B_2}\,, \tag{10.22}$$

a convenient parameterisation for solving the saddle point equations. The free energy functional is given by

$$\begin{aligned}\frac{F[\rho,u,v,\mu]}{N^{3/2}} = &-\frac{\mu(\int\rho-1)}{2\pi} + \int\Bigg[-\frac{x\rho\,(k_2u_1-k_3u_3)}{2\pi} \\ &+\frac{\rho^2}{2}\left(P - B_+(\pi A_-+u_1)^2-(2-B_+)(\pi B_- - u_1-u_3)^2-B_+(\pi C_-+u_3)^2\right)\Bigg]\,,\end{aligned}$$

with

$$P = \pi^2\,(-4+B_+)\,(-2+B_+)\,B_+\,. \tag{10.23}$$

Here $u = y^1 - y^2$ and $v = y^3 - y^1$. For arbitrary CS levels and R-charges, the eigenvalue distribution is generically divided in five regions. We refrain from writing down the explicit functions $u(x), v(x)$ and $\rho(x)$,[3] since their expressions are cumbersome and not illuminating. We have computed F with arbitrary levels k_2 and k_3, where we had to made a choice on the relative sign. We skip the general expression because it is too cumbersome, and we focus on two specific examples. Nevertheless, we checked the agreement with the geometry at arbitrary levels k_i. Consider $\widetilde{SPP}_{-211}$

$$\left(\frac{F}{N^{3/2}}\right)^2 = \frac{\pi^2(4+A_--C_--2B_+)(4+2A_-+B_--B_+)(4+B_-+2C_--B_+)B_+(4-2A_--B_--B_+)(4-B_--2C_--B_+)(4-A_-+C_--2B_+)}{9\left(128+2A_-^2(-4+B_+)+2C_-^2(-4+B_+)-112B_+-B_-^2B_+-2B_-C_-B_++32B_+^2-3B_+^3-2A_-(4C_-(-2+B_+)+B_-B_+)\right)}$$

and $D_3 = \widetilde{SPP}_{1-10}$

$$\left(\frac{F}{N^{3/2}}\right)^2 = \frac{1}{9}\pi^2\,(2-B_--C_-)\,(2+B_-+C_-)\,(2-A_--B_+)\,(2+A_--B_+)\,B_+ \tag{10.24}$$

Note that upon (10.21), these are expressions in terms of the a_i's. The monopole charge is included along

$$\begin{aligned}&\delta a_1 \sim k_3\Delta_m, &&\delta a_3 \sim -2\,(k_2+k_3)\,\Delta_m, &&\delta a_5 \sim (k_2-k_3)\,\Delta_m,\\ &\delta a_2 \sim k_2\Delta_m, &&\delta a_4 \sim (k_2+k_3)\,\Delta_m, &&\delta a_6 \sim (k_3-k_2)\,\Delta_m,\end{aligned}$$

[3] Beyond the central region where (10.23) is extremised, there's a middle region with constant u or v (depending on relations amongst the k_a's and the a_i's). Finally, in the outer regions, both u and v are constant and ρ is eventually becoming zero.

which corresponds to the direction in gauge space parallel to the k's. The over-counting is reflected by the spurious symmetry

$$\delta a_{1/2} \sim s\,, \qquad \delta a_{3/4} \sim -s$$

and also the contribution of the baryonic symmetry $k_3 U(1)_2 - k_2 U(1)_3$,

$$\begin{aligned}
&\delta a_1 \sim -b k_2, && \delta a_3 \sim b\,(k_2 - k_3)\,, && \delta a_5 \sim b\,(k_2 + k_3)\,, \\
&\delta a_2 \sim b k_3, && \delta a_4 \sim 0, && \delta a_6 \sim -b\,(k_2 + k_3)\ ,
\end{aligned}$$

is indeed a symmetry of F.

For the Hilbert series, we extract the weights of the quotient from the toric data in figure 10.2 and (C.3),

$$(Z_i) = (w, w z^{-k_2-k_3}, z^{-k_2-k_3}, z^{k_2+k_3}, w^{-1} z^{k_3}, w^{-1} z^{k_2})\,, \tag{10.25}$$

from which we compute

$$\mathrm{Hil}(Y_{\mathrm{SPP}};\, t_i) = \oint \frac{\mathrm{d}z\,\mathrm{d}w}{(2\pi i)^2 z w} \frac{1}{(1-t_1 w)(1-t_2 w/z^{k_2+k_3})(1-t_3/z^{k_2+k_3})(1-t_4 z^{k_2+k_3})(1-t_5 z^{k_3}/w)(1-t_6 z^{k_2}/w)}\,. \tag{10.26}$$

The $t_i = e^{-2\varepsilon a_i} \to 1$ pole of (10.26) reproduces the free energy, where one again has to make choices on the signs of the k_i's.

10.3.3 $\widetilde{\mathcal{C}/\mathbb{Z}_2}$ and an RG flow

The generalised conifold with $\mathcal{N} = 3$ supersymmetry has been studied in [148], here we do not want to assume $\mathcal{N} = 3$ supersymmetry and consider the quiver as a $\mathcal{N} = 2$ model, i.e. we assign arbitrary R charges to the fields. The field content is shown in figure 10.3, the CS couplings are $(k, k, -k, -k)$ and we parameterise the R-charges of the fields corresponding to the perfect matchings

$$\Delta_{A_i/C_i} = a_i\,,\ \ \Delta_{B_i} = a_{i+2} + a_{i+4}\,,\ \ \Delta_{D_i} = a_{i+2} + a_{7-i}\,. \tag{10.27}$$

There is no redundancy but two baryonic symmetries, which are no actual degrees of freedom in the free energy. The eigenvalue distribution is divided in five parts, again we refrain from giving all the formulae and just present the result

$$\left(\frac{F}{N^{3/2}}\right)^2 = \frac{2k\pi^2 A_+ \left(4-(A_-+B_-)^2\right)\left(4-(A_-+D_-)^2\right)\left((B_--D_-)^2 - 4\,(2-A_+)^2\right)}{9\left(A_+\left(4 + A_-^2 + B_- D_- + A_-\,(B_- + D_-)\right) + (B_- - D_-)^2 - 16\right)}\,, \tag{10.28}$$

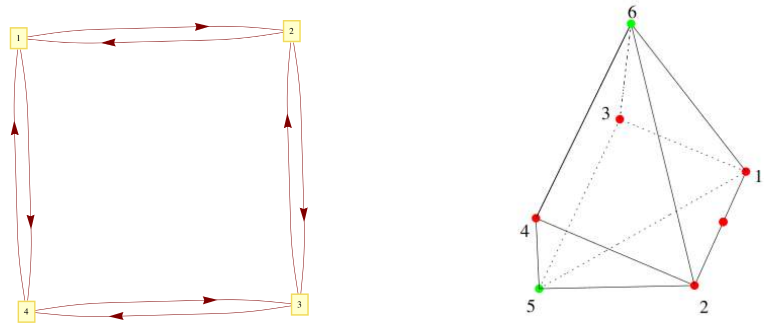


Figure 10.3: Quiver and toric diagram for $\widetilde{\mathcal{C}/\mathbb{Z}_2}$.

where we used a similar rewriting as in (10.22).

From the toric diagram in figure 10.3 and (C.5), we read off the charge matrix for the Hilbert series

$$\begin{aligned}&\mathrm{Hil}(Y_{\text{gen.Con.}};\,t_i) = \\ &\quad \oint \frac{\mathrm{d}z\,\mathrm{d}w}{(2\pi i)^2 zw}\frac{1}{(1-t_1 w)(1-t_2/w)(1-t_3/(wz))(1-t_4 w/z)(1-t_5 z)(1-t_6 z)}\,. \end{aligned} \tag{10.29}$$

Its pole for $t_i = e^{-2\varepsilon a_i} \to 1$ immediately reproduces (10.28).

Let us comment on the RG flow between the three theories discussed so far. We can follow the flow between the fixed points $\widetilde{\mathcal{C}/\mathbb{Z}_2} \to \widetilde{SPP} \to \mathbb{C}\times\mathcal{C}$ by partially resolving the singular spaces. This corresponds to removing points in the toric diagram [162]. More explicitly, upon removing point 1 and one of the internal points in figure 10.3, we obtain the diagram of figure 10.2, up to renaming. In the field theory, this corresponds to giving a vev and integrating out $A_1 = p_1 p_7$. Note that p_7 is an internal perfect matching, which is the reason we have omitted it in the discussion so far. The groups $U(N)_{k_1}$ and $U(N)_{k_2}$ are identified to $U(N)_{k_2+k_3}$ and A_2 becomes the adjoint field in $\widetilde{SPP}$. If we now in figure 10.2 remove also point 4, we end up with the diagram of $\mathbb{C}\times\mathcal{C}$ in figure 10.1, modulo relabeling the points. In the field theory this is achieved by higgsing $B_2 = p_4$.

10.3.4 ABJM/$\mathbb{Z}_2$

We consider the theory with product gauge group $\prod_{i=a}^{4} U(N)_a$, Chern-Simons levels $(k,-k,k,-k)$ and four pairs of bifundamental fields A_i, B_i, C_i, D_i as shown in figure 10.4.

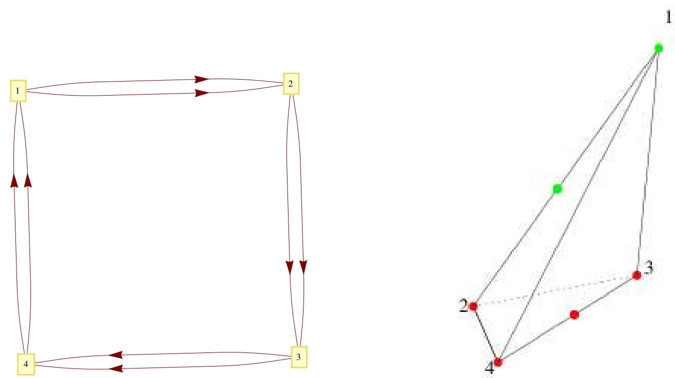


Figure 10.4: (Left) Quiver for $\widetilde{\mathbb{F}}_0$. According to the choice of the CS levels, it gives several theories studied in the thesis. (Right) Toric diagram for ABJM/$\mathbb{Z}_2$, which corresponds to CS levels $(k, -k, k, -k)$.

At a first look, the theory seems chiral and it is not clear how the long-range forces vanish without modifications. Taking into account the symmetry of the quiver, though, we find that the contribution to the long range forces coming from A/B cancels with that of C/D, respectively. In fact, the theory can be seen as a $\mathbb{Z}_2$ quotient of ABJM, effectively being vector-like and having a saddle point solution following the ansatz used so far. We assign to the fields charges under the perfect matchings,

$$\Delta_{A_i/C_i} = a_i \,, \quad \Delta_{B_i/D_i} = a_{i+2} \,, \tag{10.30}$$

where the affiliation to ABJM is manifest: Both baryonic directions are killed by the $\mathbb{Z}_2$ flip symmetry of the quiver and we are left with the 4 mesonic charges only. Imposing the symmetry $y^1 = y^3$ and $y^2 = y^4$, makes the orbifold of ABJM obvious even at the level of the free energy functional. As a solution we find consequently

$$\left(\frac{F}{N^{3/2}}\right)^2 = \frac{128}{9} k\pi^2 a_1 a_2 a_3 a_4 \,. \tag{10.31}$$

The toric diagram is given in figure 10.4 and (C.7). Modulo a discrete $\mathbb{Z}_2$, the Hilbert series is trivial

$$\mathrm{Hil}(S^7;\, t_i) = \frac{1}{(1-t_1)(1-t_2)(1-t_3)(1-t_4)} \,, \tag{10.32}$$

matching with (10.31) as $t_i = e^{-2\varepsilon a_i} \to 1$.

10.3.5 Chiral-Like models

As we have mentioned in the introduction, all the beauty that we have seen for the vector-like models does not apply anymore when the field theory has a 'chiral' field content, i.e. matter that does not come in charge conjugate pairs. Certain long range forces do not cancel in the large N saddle point approach of [18], the ansatz (10.5) is not sensible anymore and one cannot even reconstruct the predicted $N^{3/2}$ scaling. We have also put the problem on a computer, with a standard approach the numerical analysis indeed picks a saddle point where F scales as N^2, if it scales homogeneously at all.

It is not clear if there is a problem with the matrix model, with the chiral models per se[4] or with the large N ansatz that we applied for solving the matrix model.

Here we want to comment very briefly on some results we have obtained by modifying the large N ansatz, for few more details see appendix D and for a full discussion the paper [4]. By applying a proposal put forward in [171] we found indeed the correct scaling for various theories. However, while in a couple of theories we could even match the volumes with the on-shell value of the free energy, other theories did not give the expected result. The modified ansatz is technically very hard to evaluate and we need to make some assumptions for our calculations. It is not clear if the modified ansatz is wrong, or if the assumptions were too strong. Many open questions remain for future work. We put a quick summary for the interested reader in appendix D.

[4] A quantum analysis of chiral quiver theories indeed shows some particularities [170].

Chapter 11

Seiberg duality in vector-like theories

In this section we show that the large N free energy preserves the rules of Seiberg duality for vector like gauge theories worked out in [155]. First of all, we review the rules of Seiberg duality in three dimensional vector like CS matter theories with product groups, and their relation with toric duality. In three dimensions a vector multiplet can have either a YM or a CS term in the action. In the first case the theory is similar to the four dimensional parent but the rules of duality cannot be extended straightforwardly. Indeed, the vector multiplet has an additional scalar coming from the dimensional reduction which modifies the moduli space. As a consequence it was observed in [172] that the rules of Seiberg duality are modified by adding new gauge invariant degrees of freedom in the dual magnetic theory, which take into account the extra constraints on the moduli space. On the other hand, YM-CS (or even CS) theories do have a dual description with the same field content as their four dimensional parents. The only difference is on the gauge group. Indeed for CS SQCD with $U(N)_k$ gauge group and N_f pairs of Q and $\tilde{Q}$, the dual field theory has $U(N_f + |k| - N_c)_k$ gauge group, as shown in [155]. The partition function has already been used to check this extension of CS $\mathcal{N} = 2$ Seiberg duality in three dimensions in [29, 141, 144, 173–176].

One may then wonder if the same rules can be extended to more complicated gauge theories, like the ones related by AdS/CFT to the motion of M2 branes on CY_4. The first generalisation of Seiberg like dualities on CS quiver gauge theories appeared in [154] for the ABJM model. It was observed that the field content transforms as in $4d$ while the gauge group transforms as

$$U(N)_k \times U(N+M)_{-k} \to U(N)_{-k} \times U(N-M)_k \tag{11.1}$$

Differently from the four dimensional case, also the gauge group spectator feels the duality, since its CS level is modified. The above rule can be derived by looking at the system of branes engineering the gauge theory. This consists of a stack of N D3 on a circle and two pairs of $(1,p)$ branes orthogonal to them. Moreover M fractional D3 branes on a semicircle connecting the fivebranes are added. By moving the fivebranes on the circle and by applying the s-rule [177], when one stack of $(1,p)$ branes crosses the other, the rule above is derived.

It is then natural to extend these ideas to theories with a higher number of gauge groups. When these theories can be described as a set of $(1,p_i)$ and D3 branes on a circle, they are the extension of the four dimensional L^{aba} gauge theories [151, 178, 179]. They consist of a product of gauge groups $U(N_i)$ with bifundamentals and adjoints (the presence of the adjoint is related to the choice of the angles between the $(1,p_i)$ fivebranes). In absence of an adjoint field on the node N_i the interaction in the superpotential is $W_i = X_{i-1,i}X_{i,i+1}X_{i+1,i}X_{i,i-1}$ while if there is an adjoint on N_i we have $W_i = X_{i-1,i}X_{i,i}X_{i,i-1} - X_{i+1}X_{i,i}X_{i,i+1}$. The signs as in four dimensions alternate between $+$ and $-$.

Even in this case the duality rules are found by exchanging the $(1,p_i)$ and the $(1,p_{i+1})$ fivebranes. The final rule is

$$\begin{array}{ccl} U(N_{i-1})_{k_{i-1}} & \to & U(N_{i-1})_{k_{i-1}+k_i} \\ U(N_i)_{k_i} & \to & U(N_{i-1}+N_{i+1}+|k_i|-N_i)_{-k_i} \\ U(N_{i+1})_{k_{i+1}} & \to & U(N_{i+1})_{k_{i+1}+k_i} \end{array} \tag{11.2}$$

while the matter field content and the interactions transform as in four dimensions.

11.1 Matching the free energy

In this section we provide the rules for the action of Seiberg duality (11.2) on the eigenvalues of non chiral theories and we show that the free energy matches even before the large N integrals are performed. Consider a duality on the i-th node. The CS level of this group becomes $-k_i$ and the imaginary part of this eigenvalue y_i becomes $-y_i$. Moreover the CS levels $k_{i\pm 1}$ (here we just refer to necklace quivers) become $k_{i\pm 1}+k_i$. This rule and the constraint that the sum of the CS level is vanishing provide the duality action on the eigenvalues. We have

$$
\begin{aligned}
y_{i+1} &\to \tilde{y}_{i-1} = -y_{i-1} & \qquad y_{i-1} &\to \tilde{y}_{j-1} = -y_{i+1} \\
y_j &\to \tilde{y}_j = y_j - y_{i-1} - y_{i+1} & \qquad j &\neq i, i \pm 1
\end{aligned}
\tag{11.3}
$$

Note that the shift in the rank of the gauge group has a subleading effect at large N. This apparently trivial statement is subtle, since naively one finds new fundamental-like terms scaling like $N^{3/2}$, which descend from the $N^{5/2}$ contributions to F. To see this, let us consider a shift of the i-th rank by δN and collect the additional contributions to the free energy following (2.9) of [18]. At $N^{3/2}$, one has extra contributions $\rho\, x\, \delta N$ from the gauge sector, $\rho\, x\, \delta N/2(\Delta_{j,i} - 1 + y^j/4\pi)$ from each incoming and $\rho\, x\, \delta N/2(\Delta_{i,j} - 1 - y^j/4\pi)$ from each outgoing matter field, where $j = i \pm 1$. We see that the net contribution cancels for the non-chiral theories at hand. The dependency on the y's drops out due to the vectorial nature of the quiver and the y-independent part is the anomaly cancellation of the $4d$ parent, which has already been used in the treatment of the long-range forces.

In the case of a $\widetilde{L^{aba}}_{k_i}$ theory, we distinguish between the duality action on the classical term of the free energy (10.6) and the one on the loop contribution (10.7). By supposing that the duality acts on the i-th node, the CS levels transform as in (11.2) while the sum in the integral (10.6) becomes

$$
\sum_{a=1}^{G} k_a y_a \to -k_i y_i + k_{i-1} y_{i-1} + k_{i+1} y_{i+1} + \sum_{a \neq i, i \pm 1} k_a y_a \tag{11.4}
$$

By applying (11.2) this last formula becomes

$$
\sum_{a=1}^{G} k_a y_a \to k_i \tilde{y}_i \sum_{a \neq i, i \pm 1} k_a \tilde{y}_a - \tilde{y}_{i-1} \sum_{b \neq i-1} k_b - \tilde{y}_{i+1} \sum_{c \neq i+1} k_c = \sum_{a=1}^{G} k_a \tilde{y}_a \tag{11.5}
$$

The second term is the one loop contribution coming from the vector and the matter fields. In the non chiral case of $\widetilde{L^{aba}}_{k_i}$ theories this contribution is

$$
F_{1L} = \sum_{a,b} F_{1L}^{a,b} \tag{11.6}
$$

where $F_{1L}^{a,b}$ has been defined in (10.7) and the sum extends to the pairs of bifundamentals (a,b) and adjoint fields (a, a) (counted twice). We are going to show that the rules (11.2) leave F_{1L} invariant.

This result is proven by distinguishing two cases. In the first case, shown in figure 11.1, the theory does not posses adjoint fields in the first phase and after duality two adjoints arise. In the second case there is one adjoint field and the duality acts as in figure 11.2.

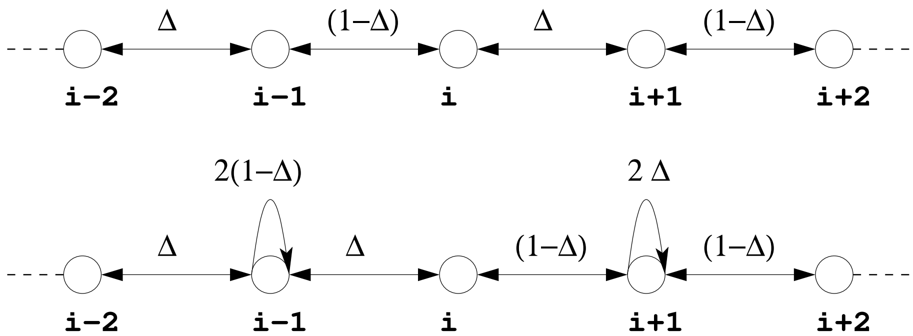


Figure 11.1: Quiver diagram for the first type of the Seiberg duality

Let us discuss the first case in more detail, where in the quiver before duality there are no adjoint fields (at least next to the group which undergoes the duality). The dual theory has instead two adjoint fields on the nodes $N_{i\pm1}$ if the $U(N_i)$ group is dualised. The superpotential

$$\begin{aligned} W &= X_{i-2,i-1}X_{i-1,i}X_{i,i-1}X_{i-1,i-2} - X_{i-1,i}X_{i,i+1}X_{i+1,i}X_{i,i-1} \\ &+ X_{i,i+1}X_{i,i+2}X_{i+2,i+1}X_{i+1,i} + \Delta W \end{aligned} \tag{11.7}$$

becomes

$$\begin{aligned} W &= X_{i-2,i-1}Y_{i-1,i-1}X_{i-1,i-2} - Y_{i,i-1}Y_{i-1,i-1}Y_{i-1,i} + Y_{i-1,i}Y_{i,i+1}Y_{i+1,i}Y_{i,i-1} \\ &- Y_{i,i+1}Y_{i+1,i+1}Y_{i+1,i} + X_{i+2,i+1}Y_{i+1,i+1}X_{i+1,i} + \Delta W \end{aligned} \tag{11.8}$$

where in ΔW we collected all the superpotential terms which are not involved in the duality. Moreover there is a relation between the R charges Δ in the two phases. Indeed the adjoint fields $Y_{i\pm1,i\pm1}$ are related to the original fields as

$$Y_{i\pm1,i\pm1} = X_{i\pm1,i}X_{i,1\pm1} \tag{11.9}$$

and the R charges become

$$\widetilde{\Delta}_{i\pm1,i\pm1} = \Delta_{i,i\pm1} + \Delta_{i\pm1,i} = 2\Delta_{i,i\pm1} \tag{11.10}$$

where the last equality follows from the symmetry of the $\widetilde{L^{aba}}$ quivers. From (11.10) and from the constraints imposed by the superpotential the other R charges are assigned as in the figure 11.2. The fields which are not directly involved in the duality (mesons and dual quarks) have the same R charge in both the theories. At this stage of the discussion one can apply the rules (11.3) and check that even the matter content of the dual theories

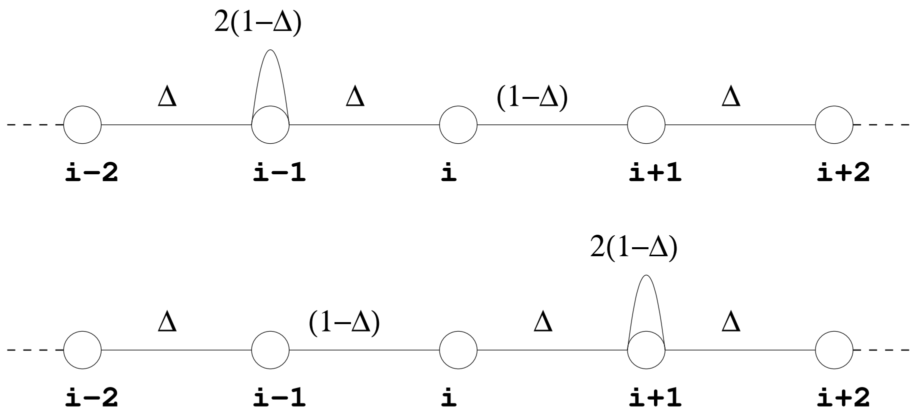


Figure 11.2: Quiver diagram for the second type of the Seiberg duality

gives the same contribution to the free energy. We distinguish three sectors: the fields charged under N_i, the adjoints and the bifundamentals uncharged under N_i, and we show that each sector separately contributes with the same terms.

The two pairs of bifundamental fields $X_{i,i\pm 1}$ and $X_{i\pm 1,i}$ contribute to the free energy as

$$\Delta F_i = -(1-\Delta)\int \rho^2(\delta y_{i-1,i}^2 - \frac{4}{3}\pi^2\Delta(2-\Delta)) - \Delta\int \rho^2(\delta y_{i,i+1}^2 - \frac{4}{3}\pi^2\Delta(1+\Delta)) \quad (11.11)$$

while in the dual phase this contribution is

$$\Delta F_i = -\Delta\int \widetilde{\rho}^2(\widetilde{\delta y}_{i-1,i}^2 - \frac{4}{3}\pi^2\Delta(1+\Delta)) - (1-\Delta)\int \widetilde{\rho}^2(\widetilde{\delta y}_{i,i+1}^2 - \frac{4}{3}\pi^2\Delta(2-\Delta)) \quad (11.12)$$

The rules (11.3) map the new $\tilde{y}$ variables in the former ones as

$$\widetilde{\delta y}_{i\pm 1,i} = -\delta y_{i\mp 1,i} \quad (11.13)$$

By substituting (11.13) in (11.12) formula (11.11) is recovered (with $\rho = \tilde{\rho}$ as in F_{cl}).

The second contribution to the one loop free energy comes from the adjoint fields. In the electric theory this contribution vanishes, because there are no adjoints for the nodes $N_{i\pm 1}$. In the dual theory there are two adjoint fields and their contribution is

$$\Delta F_{i\pm 1} = \frac{2}{3}\pi^2\sum_{\alpha=1}^{2}\Delta_\alpha(1-\Delta_\alpha)(2-\Delta_\alpha)\int \rho^2 dx \quad (11.14)$$

In this case $\Delta_1 = 2\Delta$ and $\Delta_2 = 2(1-\Delta)$ and the sum is vanishing, as in the other phase.

The last contribution comes from the other matter fields. The integrals are the same in both the phases and the relation (11.2) guarantees that $\delta y_{\alpha\beta} = \widetilde{\delta} y_{\alpha\beta}$ if $\alpha \neq i \neq \beta$. This proves that the dual theories have the same F even before the extremisation.

The second case is similar to the former one and we refer to it in the figure 11.2. By repeating the analysis on the superpotentials above one finds a distribution of R charges as in figure 11.2. Then the analysis in straightforward. Indeed the contributions of the CS term and the one loop contribution of the bifundamental fields are exactly as before, while the contribution from the adjoints is trivially the same, since there is no y dependence for the adjoints and they have the same R charge.

Let us conclude this chapter with some comments. An extension of our work is studying the role of the subleading contributions in the dualities that we checked here. Indeed, as we observed, the finite k contribution in the dual gauge group gives a leading contribution at large N which cancels because the theory is vector like. A deeper check should consist of matching the subleading contributions between the dual phases. Moreover one should study the existence of similar dualities among theories completely unrelated in four dimensions. Usually the dual phases are obtained by unhiggsing. In many cases the unhiggsing involves a bifundamental field and a chiral like theory is generated,. Anyway by unhiggsing an adjoint field the daughter theory is still vector like. For example this is the case for the third phase of D_3 discussed in [180]. This duality relates the classical mesonic moduli spaces, but a better check should be the matching of the free energy at large N.

Chapter 12

An alternative formula

In this section we construct a function quartic in the R charges, that reproduces the volume formula from the toric diagram / the free energy. It is an attempt for a three dimensional generalisation of [20], in which it was shown that the volume minimisation is equivalent to the a-maximisation in field theory. In [20] the authors computed the 'geometrical R charges' from the toric data and provided a formula for the a-function in terms of these geometrical R charges and the data of the toric diagram. Then in [158] it was observed that this formula could be simplified by imposing the constraints of the superpotential. The final result for the geometrical version of the a-function is

$$a_g = \frac{1}{4} \sum \langle v_i, v_j, v_k \rangle R_i R_j R_k \tag{12.1}$$

where v_i represent the external points of the toric diagram, $\langle \cdot, \cdot, \cdot \rangle$ is the area of the oriented surface generated by every set i, j, k of external points and R_i are the R-charges associated to each point of the diagram, which represent the set of fields in a given perfect matching. This cubic formula corresponds to the sum of the areas among the external points weighted by the R charges of their PM.

One may be tempted to extend (12.1) to the three dimensional case. Here the field theory candidate for the matching with the geometric data is the free energy. Then, our candidate geometrical version of the free energy, F_g, is

$$F_g^2 = \frac{1}{6} \sum \langle v_i, v_j, v_k, v_l \rangle R_i R_j R_k R_l \tag{12.2}$$

We observe that (12.2) reproduces the Z function in all cases without internal lines or surfaces in the toric diagram. If there are internal lines or surfaces, we did not find any example in which (12.2) reproduces the Z function. Surprisingly, by adding some contribution related to the internal lines and surfaces we have reproduced the geometric Z as a function of the Reeb vector. We have not found a derivation for a general formula but we will show the equivalence in several examples. The most interesting result is that in all the examples only a quartic correction in the R charges associated to the external points of the toric diagram is needed in order to identify $F_g^2 \simeq 1/Z$.

We therefore conjecture that this can always be done. If proven to be true, and if the corrected F_g equals the field theoretical free energy, our discussion would offer a simpler extremisation problem than the large N limit of the localised free energy. Some extensions of the ideas in this chapter have appeared in [149].

Before we show our ideas with some examples, let pause for two comments. It is important to stress that this relation between F_g^2 and Z does not involve any information about the dual field theory and applies directly to the toric diagram. This implies that it is not necessary to know the field theory dual, to state the correspondence between F_g^2 and Z it is enough to have control over the geometry of the mesonic moduli space. It follows that the F_g function we define cannot solve the problems discussed in [18] for the large N scaling of the free energy in chiral theories.

Another observation is that here we simply define the R charges of the perfect matchings associated to the external points of the toric diagram, and we do not relate them to any field theory description. With this procedure our candidate F_g is always polynomial in the R charges, contrary with the known examples computed in the literature [18, 138] and in chapter 10, where the free energy at large N is a rational function of the R charges of the fields and monopoles. We checked in every example that the large N free energy and our geometrical F_g coincide once the symmetries among the perfect matchings are imposed.

The existence of this quartic object on the geometric side makes it very tempting to speculate about a quartic function that one can define in the field theory. This would be computationally much simpler than the current F and might overcome the problems with the chiral theory. It is, however, highly speculative to think about such a function.

12.1 Examples

The first case that we discuss is D_3. In this case the toric diagram is

$$G_T = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{12.3}$$

The Z function in terms of the trial Reeb vector is

$$Z = \frac{16}{(b_1 b_2 b_3 (4 + b_1 - b_2)(b_3 - 4))} \tag{12.4}$$

The R charges associated to the six external points become $R_i = 2\mathrm{Vol}(\Sigma_i)/Z$. In this case we find that the conjectured geometrical free energy becomes

$$F_g^2 = \frac{1}{6} \sum \langle v_i, v_j, v_k, v_l \rangle R_i R_j R_k R_l = \frac{b_1 b_2 b_3 (4 + b_1 - b_2)(b_3 - 4)}{16} \tag{12.5}$$

and $F_g^2 Z = 1$. Even if the (12.5) is a polynomial function while (10.24) is a rational function they match once the symmetries among the PM are imposed.

Consider the general class of toric diagrams [1]

$$G_T = \begin{pmatrix} x_1 & 0 & 0 & x_4 & 0 \\ x_2 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{12.6}$$

with the constraint coming from the convexity $x_1 x_3 + x_2 x_4 - x_3 x_4 > 0$. The Z function is given by

$$Z = \frac{((4x_5(x_2((b_2 - 4x_5)x_1 x_3 + b_1 x_5(x_3 - x_2))x_4 + b_3 x_5 x_1 x_3 (x_4 - x_1)))}{(b_1 b_2 b_3 (b_3 x_5 x_1 + (b_2 - 4x_5)x_1 x_3 + b_1 x_5 (x_3 - x_2))(b_3 x_5 (x_4 - x_1) + x_2 (b_1 x_5 + (b_2 - 4x_5) x_4))))} \tag{12.7}$$

By choosing $x_i > 0$ and $x_1 x_3 + x_2 x_4 - x_3 x_4 > 0$ the geometric F_g^2 becomes

$$F_g^2 = \frac{1}{6} \frac{(3b_1 b_2 b_3 (b_3 x_5 x_1 + (b_2 - 4x_5)x_1 x_3 + b_1 x_5 (x_3 - x_2))(b_3 x_5 (x_4 - x_1) + x_2 (b_1 x_5 + (b_2 - 4x_5) x_4))))}{((2x_5(x_2((b_2 - 4x_5)x_1 x_3 + b_1 x_5(x_3 - x_2))x_4 + b_3 x_5 x_1 x_3 (x_4 - x_1)))} \tag{12.8}$$

and again $Z F_g^2 = 1$.

[1] Up to $SL(4,\mathbb{Z})$ transformations this class generalises to every example of the class $\mathbb{C}^2 \times \mathbb{C}$ where the $\mathbb{C}^2$ basis refers to a four-dimensional parent theory with four external points.

We now move to a vector-like example which requires a correction. It can be obtained by modifying the toric diagram of the D_3 theory. We consider a basis with four points $(0,0,0)$ $(1,0,0)$ $(0,1,0)$ and $(1,1,0)$ as in D_3 but we modify the two points in the z directions, such that they are not associate to the splitting of two points on the same line on the plane (x,y). This is not associated to an SL transformation and the toric diagram should describe a different model (for example it can by obtained by an appropriate un-higgsing of the ABJM model). The toric diagram is

$$G_T = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{12.9}$$

The Z function is

$$Z = \frac{(16((b_1-b_2)^2-16)+8(8+b_1b_2-2b_1-2b_2)b_3)}{(b_1b_2b_3(b_1-4)(b_2-4)(b_1-b_2-b_3+4)(b_1-b_2+b_3-4))} \tag{12.10}$$

while F_g^2, if computed from (12.2), does not reproduce the expected result and it is a complicate expression. However we observe that in this case there exist two internal lines, connecting the points $1,6$ and $4,5$, and an internal plane which passes through the points $1,4,5$ and 6. As we discussed above a correction proportional to $\Delta F_g^2 = -2(R_1R_6 - R_4R_5)^2$ can be added to F_g^2. With this correction it is straightforward to observe that F_g^2 and $1/Z$ match.

We can consider another class of vector-like models in which (12.2) does not coincide with Z. We refer to this class as $\widetilde{SPP}_{(-m-1,m,1)}$ (with $m>0$). The toric diagram is given by

$$G_T = \begin{pmatrix} 0 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & m-1 & 0 & 0 & -1 & m \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{12.11}$$

In this case the F_g^2 function reproduces the Z function only after the deformation

$$\Delta F_g^2 = \frac{4(mR_1^2R_4^2 - 2R_1R_2R_3R_4 + mR_2^2R_3^2)}{m+1} \tag{12.12}$$

It is interesting to observe that the formula is still quartic and the deformation of F_g involves all the sets of coplanar and collinear external points .

The last examples that we analyse are associated to the chiral-like cases investigated in the thesis, M^{111} and Q^{222}. If the intuition that we got from the other examples is correct one must add a contribution proportional to all the possible internal planes and lines, by a quartic combination of their charges.

Let us turn to the first of the two examples, where the toric diagram is

$$G_T = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & k_1 & -k_2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{12.13}$$

with $k_1, k_2 > 0$. This diagram reduces to $M^{111}/\mathbb{Z}_k$ for $k_1 = k_2$. We found that the geometrical F_g^2 and the Z function may be identified if a correction

$$\Delta F_g^2 = -\frac{4(k_1+k_2)^3}{9k_1k_2} R_4^2 R_5^2 \tag{12.14}$$

is added to F_g^2 (where R_4 and R_5 refer to the points with k_1 and $-k_2$ splitting.
In the second case, Q^{222}, the expression for F_g reduces to the Z function only after adding the correction

$$\Delta F_g^2 = 4(R_1R_2 + R_3R_4 + R_5R_6)^2 - 8(R_1^2R_2^2 + R_3^2R_4^2 + R_5^2R_6^2) \tag{12.15}$$

We see that even in this case it is possible to express the free energy as a set of quartic combinations of the R charges.

It would be interesting to find a derivation of this result like in [20] and to see if it provides, at least in the toric case, a different way for the computation of the free energy instead of the localisation of [14].

Part III

Appendix

Appendix A

Notations and Conventions

A.1 Spinor conventions $\mathcal{N} = 1$ Lorentzian

In this appendix we collect our spinor conventions. The Clifford$(1,4)$ gamma matrices γ^α satisfy

$$\{\gamma_\alpha, \gamma_\beta\} = 2\,\eta_{\alpha\beta}\,, \qquad \gamma_\alpha^\dagger = \gamma_0\gamma_\alpha\gamma_0\,, \qquad \gamma_\alpha^t = C\gamma_\alpha C^{-1}\,, \tag{A.1}$$

where the five-dimensional charge conjugation matrix C satisfies $C = -C^t = C^* = -C^{-1}$. We adopt a representation of the Clifford algebra in which the first four gamma matrices are real, while $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is purely imaginary. Then $\gamma^1, \gamma^2, \gamma^3$ are symmetric, while γ^0 and γ^5 are anti-symmetric. In this case, a consistent choice of the charge conjugation matrix is $C = i\gamma^0\gamma^5$. Our spinors are commuting. Furthermore, for five-dimensional spinors, the symplectic-Majorana condition is

$$\bar{\epsilon}^I = (\epsilon^I)^t C\,, \tag{A.2}$$

where ϵ^I, $I = 1, 2$, are Dirac spinors and we define $\bar{\epsilon}^I = \epsilon^{IJ}\epsilon^{J\,\dagger}\gamma^0$, with ϵ^{IJ} being the antisymmetric symbol, such that $\epsilon^{12} = +1$. So a symplectic-Majorana pair ϵ^I carries in total eight real degrees of freedom.

A.2 Spinors & Conventions $\mathcal{N} = 2$

Throughout the thesis Greek indices denote Lorentz signature. The indices $\mu, \nu, \ldots$ are curved and $\alpha, \beta, \dots = 0, 1, 2, 3$ are flat. Similarly, we use Roman indices for Euclidean spaces. More explicitly, $m, n, \ldots$ are curved while $a, b, \dots = 1, 2, 3, 4$ are flat. We denote with $x, y, \ldots$ flat "spatial" indices running from 1 to 3, both in Euclidean and in Lorentzian signature.

A.2.1 Lorentzian Signature

We work with a mostly plus signature, $(\eta_{\alpha\beta}) = (-,+,+,+)$. The conventions for our gamma matrices are

$$\gamma_\mu^* = \gamma_\mu \qquad \gamma_\mu^\dagger = \gamma^0\gamma_\mu\gamma^0 \qquad \gamma = i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad \gamma^* = \gamma^T = -\gamma \; . \tag{A.3}$$

Following the conventions in [123], we have supersymmetry parameters that are chiral spinor doublets of $SU(2)$

$$\epsilon_+^i \qquad \epsilon_{-i} = \left(\epsilon_+^i\right)^* \qquad \eta_{+i} \qquad \eta_-^i = \left(\eta_{+i}\right)^* \tag{A.4}$$

and we define

$$\bar{\epsilon}_{+i} = \left(\epsilon_+^i\right)^\dagger \gamma^0 \qquad \bar{\epsilon}_-^i = \left(\epsilon_{-i}\right)^\dagger \gamma^0 \; . \tag{A.5}$$

We can use the $SU(2)$ invariant tensor to raise and lower indices

$$\epsilon_{+i} = \epsilon_{ij}\epsilon_+^j \qquad \epsilon_-^i = \epsilon^{ij}\epsilon_{-j} \qquad \eta_+^i = \epsilon^{ij}\eta_{+j} \qquad \eta_{-i} = \epsilon_{ij}\eta_-^j \; , \tag{A.6}$$

where $\epsilon_{12} = \epsilon^{12} = 1$. Note that this is different to the conventions in [123]. As opposed to there, for us the $SU(2)$ index position does not denote chirality but the $SU(2)$ representation: we allow for raising and lowering with ϵ_{ij} and put explicit labels $+/-$ to indicate chirality. The reader who wants to compare with [123] should contract with ϵ_{ij} whenever the index structure of a spinor here is different from its analogous spinor there.

In our conventions, the (anti-)selfduality conditions read

$$\epsilon_{\alpha\beta}{}^{\gamma\delta}\Omega^\pm_{\gamma\delta} = \pm 2i\Omega^\pm_{\alpha\beta} \; . \tag{A.7}$$

Note that $(\Omega^+)^* = \Omega^-$.

Let us also summarise some details on the symbols η^x. They enter our equation through the identity

$$\sigma^\alpha\sigma^\beta = -\eta_{x\alpha}{}^\beta\sigma^x + \delta_\alpha^\beta\mathbb{1}$$

and they obey the following orthogonality and self-duality relations

$$\begin{aligned} \eta_{x\alpha\beta}\eta_x^{\gamma\delta} &= i\epsilon_{\alpha\beta}{}^{\gamma\delta} - 2\delta_{[\alpha}{}^\gamma\delta_{\beta]}{}^\delta \qquad & \eta_{x\alpha\beta}\eta_y^{\alpha\beta} &= -4\delta_{xy} \\ \epsilon_{xyz}\eta_{y\alpha\beta}\eta_z^{\gamma\delta} &= -4i\delta_{[\alpha}{}^{[\gamma}\eta_{x\beta]}{}^{\delta]} \qquad & \epsilon_{\alpha\beta}{}^{\gamma\delta}\eta_{x\gamma\delta} &= 2i\eta_{x\alpha\beta} \end{aligned} \tag{A.8}$$

where $\epsilon_{xyz} = \epsilon_{0xyz}$ and $\epsilon_{0123} = 1$.

A.2.2 Euclidean Signature

We work with gamma matrices

$$
\begin{aligned}
\gamma_m^* &= \gamma_m^T = B\gamma_m B^{-1} && \text{with} \quad B^* = B^T = -B^{-1} = -B \\
\gamma &= \gamma^1\gamma^2\gamma^3\gamma^4 && \gamma^* = \gamma^T = \gamma \ .
\end{aligned}
\tag{A.9}
$$

The supersymmetry parameters are symplectic Majorana-Weyl

$$
\left(\epsilon_+^i\right)^c = i\epsilon_{ij}\epsilon_+^j \qquad (\epsilon_{-i})^c = -i\epsilon^{ij}\epsilon_{-j} \qquad (\eta_{+i})^c = -i\epsilon^{ij}\eta_{+j} \qquad \left(\eta_-^i\right)^c = -\epsilon_{ij}\eta_-^j, \tag{A.10}
$$

where $(\epsilon)^c \equiv B^{-1}\epsilon^*$ and the position of the $SU(2)$ index distinguishes the two representations. We also define

$$
\epsilon_{+i}^\dagger = \left(\epsilon_+^i\right)^\dagger \qquad\qquad \epsilon_-^{\dagger i} = (\epsilon_{-i})^\dagger \tag{A.11}
$$

Again, ϵ_{ij} is used to raise and lower indices

$$
\epsilon_{+i} = \epsilon_{ij}\epsilon_+^j \qquad \epsilon_-^i = \epsilon^{ij}\epsilon_{-j} \qquad \eta_+^i = \epsilon^{ij}\eta_{+j} \qquad \eta_{-i} = \epsilon_{ij}\eta_-^j \ , \tag{A.12}
$$

where $\epsilon_{12} = \epsilon^{12} = 1$.

We use conventions with (anti-)selfduality conditions as

$$
\epsilon_{ab}{}^{cd}\Omega^\pm_{cd} = \mp 2\Omega^\pm_{ab} \ . \tag{A.13}
$$

The t'Hooft symbols η in Euclidean signature appear in the identities

$$
\begin{aligned}
\sigma^a\bar\sigma^b &= i\bar\eta_x^{ab}\sigma^x + \delta^{ab}\mathbb{1} \\
\bar\sigma^a\sigma^b &= i\eta_x^{ab}\sigma^x + \delta^{ab}\mathbb{1} \ .
\end{aligned}
\tag{A.14}
$$

They obey the following orthogonality and self-duality relations

$$
\begin{aligned}
\bar\eta_{xab}\bar\eta_x^{cd} &= -\epsilon_{ab}{}^{cd} + 2\delta_{[a}{}^c\delta_{b]}{}^d & \bar\eta_{xab}\bar\eta_y^{ab} &= 4\delta_{xy} \\
\eta_{xab}\eta_x^{cd} &= \epsilon_{ab}{}^{cd} + 2\delta_{[a}{}^c\delta_{b]}{}^d & \eta_{xab}\eta_y^{ab} &= 4\delta_{xy} \\
\epsilon_{xyz}\bar\eta_{yab}\bar\eta_z^{cd} &= 4\delta_{[a}{}^{[c}\bar\eta_{xb]}{}^{d]} & \epsilon_{ab}{}^{cd}\bar\eta_{xcd} &= -2\bar\eta_{xab} \\
\epsilon_{xyz}\eta_{yab}\eta_z^{cd} &= 4\delta_{[a}{}^{[c}\eta_{xb]}{}^{d]} & \epsilon_{ab}{}^{cd}\eta_{xcd} &= 2\eta_{xab}
\end{aligned}
\tag{A.15}
$$

where $\epsilon_{xyz} = \epsilon_{xyz4}$ and $\epsilon_{1234} = 1$.

Appendix B

Intrinsic torsions and differential forms in $\mathcal{N} = 1$ Lorentzian

In this appendix we will explain how to obtain the system (4.13), which allows to compute the intrinsic torsions p and q by using differential forms and exterior differentials only, and not spinors. We will also give explicit expressions for the differentials and covariant derivatives of the vielbein $\{z, e^-, w, \bar{w}\}$ and the two form ω corresponding to a conformal Killing spinor.

We start with the derivation of system (4.13), consisting of the derivatives of the elements of the vielbein. The easiest to compute is dz, (4.13a). z is a spinor bilinear, as can be seen in (4.1); so its derivative can be computed in the standard way. We actually even gave its covariant derivative in (4.17); indeed by anti-symmetrising its μ and ν indices one obtains (4.13a).

dw and de^- are trickier because w and e^- are not directly expressed as bilinears of ϵ_+; as explained in section 4.1.1, they are an additional piece of data, subject to the ambiguity (4.8). The two-form ω, on the other hand, is a bispinor, defined in (4.1), and one can compute $d\omega$ again in a standard way; one gets (4.14b). Now, since w is that $\omega = z \wedge w$, from $d(z \wedge w) = z \wedge dw - dz \wedge w$ one sees that

$$z \wedge dw = -2i\mathrm{Im}p \wedge z \wedge w - 2q \wedge e^- \wedge z \; ; \tag{B.1}$$

from this, it follows that one can write dw as in (4.13b), for some one-form ρ.

We can give an alternative characterisation of ρ by computing dw in a different way: namely, by writing

$$\bar{\epsilon}_- \gamma_\mu e^- \epsilon_+ = -4(e^-)^\nu \omega_{\mu\nu} = 8w_\mu \tag{B.2}$$

and deriving the left hand side. For this, we need to compute

$$
\begin{aligned}
D_\mu(e^- \epsilon_+) &= [D_\mu, e^-]\epsilon_+ + e^-(p_\mu \epsilon_+ + q_\mu e^- \epsilon_-) = (D_\mu e^-_\nu + p_\mu e^-_\nu)\gamma^\nu \epsilon_+ = \\
&= -w^\mu D_\mu e^0_\nu \epsilon_- + \left(p_\mu + \frac{1}{2} z^\nu D_\mu e^-_\nu\right) e^- \epsilon_+ \,. \qquad \text{(B.3)}
\end{aligned}
$$

Here we have used the definition (4.12) of the intrinsic torsions, and (4.11). Using this and (B.2), we get again (4.13b), where now we see that

$$
\rho_\mu = \frac{1}{4} w^\nu D_\mu e^-_\nu \ , \qquad \mathrm{Re} p_\mu = -\frac{1}{4} z^\nu D_\mu e^-_\nu \ . \qquad \text{(B.4)}
$$

This now suggests a way of computing e^-. Using

$$
e^-_\mu = \frac{1}{16} \bar{\epsilon}_+ e^- \gamma_\mu e^- \epsilon_+ \ , \qquad \text{(B.5)}
$$

the expression for $D_\mu(e^- \epsilon_+)$ computed in (B.3), and the formula for ρ in (B.4), we obtain (4.13c).

The system of equations (4.13) is general and applies to any vielbein constructed from a chiral fermion ϵ_+ as explained in section 4.1.1; ϵ_+ is not assumed to satisfy any particular differential equation. It is of some interest to go on-shell and write the derivatives of the elements of a vielbein corresponding to a solution of the CKS equation (4.15). Imposing the constraints (4.16) on the torsions we have

$$
\begin{aligned}
dz &= 2\mathrm{Re}(q \cdot \bar{w}) e^- \wedge z + 2\mathrm{Im}(q \cdot \bar{w}) i w \wedge \bar{w} + 4\mathrm{Re}((q \cdot e^-) z \wedge \bar{w}) \ , \\
dw &= \big(2iA + (\mathrm{Re}(q \cdot \bar{w}) + 3i\mathrm{Im}(q \cdot \bar{w}))\, e^- - (q \cdot e^-)\bar{w}\big) \wedge w - 2\sigma \wedge z \ , \qquad \text{(B.6)} \\
de^- &= 4\mathrm{Re}(\sigma \wedge \bar{w}) \ ,
\end{aligned}
$$

where $\sigma = \rho - \frac{1}{2}(q \cdot e^-) e^-$. Equation (4.21) easily follows from these equations. By construction, the set of equations (B.6) implies that z is conformal Killing. These equations are also interesting because they can be used to determine the gauge field A.

In the new minimal case, using (4.37) and the definition $a = A + \frac{3}{2} v$ we find

$$
\begin{aligned}
dz &= i\, \iota_v (z \wedge w \wedge \bar{w}), \\
dw &= 2i \left(a - \frac{3}{4}(v \cdot e^-) z - \frac{1}{2}(v \cdot w)\bar{w}\right) \wedge w - 2\sigma \wedge z \ , \qquad \text{(B.7)} \\
de^- &= 4\mathrm{Re}(\sigma \wedge \bar{w}) \ ,
\end{aligned}
$$

from which equations (4.42) follows. The set of equations (B.7) implies that z is Killing. They allow to determine uniquely the background fields a and v.

Finally, we also give some expressions for the covariant derivatives of the forms z and ω corresponding to a solution of the CKS equation, which have been used in the bulk to boundary comparison in section 4.3. The expressions are not particularly nice in terms of the torsions p and q but become simple if we replace q with the vector v using (4.37). This formal redefinition can be used both in the case of solutions of the CKS equation and in the case of solutions of the new minimal conditions. As discussed in section 4.2, the only difference between the two cases is that, for conformal Killing spinors, v has a complex part given by (4.38). We also use $a = A + \frac{3}{2}v$. By explicitly differentiating the bilinears z and ω and using (4.16), we find

$$\nabla_\nu z_\mu = 2\,\mathrm{Im} v_{[\mu} z_{\nu]} + \mathrm{Re} v^\tau (*z)_{\mu\nu\tau} - g_{\mu\nu} z_\tau \mathrm{Im} v^\tau \,, \tag{B.8}$$

$$\nabla_\tau \omega_{\mu\nu} = 2iA_\tau \omega_{\mu\nu} + i(v \wedge \omega)_{\mu\nu\tau} + i(g_{\nu\tau} v^\sigma \omega_{\mu\sigma} - g_{\mu\tau} v^\sigma \omega_{\nu\sigma})\,. \tag{B.9}$$

As expected, by symmetrising and anti-symmetrising and using (4.37) and (4.38) we recover known formulae: (4.18), the first expression in (B.6) and (4.42).

Appendix C

The Z-function for arbitrary trial Reeb vector

Here we compute the volumes $12/\pi^4 Z(b_i)$ of the Sasaki-Einstein manifolds dual to the field-theoretical models we will be interested in the rest of the thesis. In the case of toric manifolds, the computations only need the knowledge of the toric diagram and the volumes are rational functions of the trial Reeb vector $\mathbf{b} = (b_i)_{i=1\ldots4}$. By identifying $a_i(\mathbf{b}) = 2\mathrm{Vol}_{\Sigma_i}/Z$, these results are in agreement with the ones discussed in the thesis.

$\mathbb{C} \times \mathcal{C}$. The toric diagram is shown in figure 10.1, it is spanned by the vectors

$$G_t = \begin{pmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \tag{C.1}$$

and, applying the standard techniques discussed above, we find the volumes at trial Reeb vector

$$
\begin{aligned}
\mathrm{Vol}_{\Sigma_1} &= \frac{-1}{b_2\left(4+b_2-b_3\right)\left(4-b_1+b_2+b_3\right)}\,, \\
\mathrm{Vol}_{\Sigma_2} &= \frac{1}{b_2 b_3\left(b_3-b_1\right)}\,, \\
\mathrm{Vol}_{\Sigma_3} &= \frac{4+b_2}{\left(4+b_2-b_3\right) b_3\left(b_1-b_3\right)\left(4-b_1+b_2+b_3\right)}\,, \\
\mathrm{Vol}_{\Sigma_4} &= \frac{1}{b_2\left(4+b_2-b_3\right)\left(b_3-b_1\right)}\,, \\
\mathrm{Vol}_{\Sigma_5} &= \frac{-1}{b_2 b_3\left(4-b_1+b_2+b_3\right)}\,, \\
Z &= \frac{4\left(4+b_2\right)}{b_2\left(4+b_2-b_3\right) b_3\left(-b_1+b_3\right)\left(4-b_1+b_2+b_3\right)}\,,
\end{aligned}
\tag{C.2}
$$

with $Z \equiv \sum_{i=1}^{5} \mathrm{Vol}_{\Sigma_i} = 12\mathrm{Vol}(H)/\pi^4$.

$\widetilde{SPP}$**.** The toric diagram for the SPP is

$$
G_t = \begin{pmatrix} 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & 1 \\ 0 & k_2-k_3 & 0 & 0 & -k_3 & k_2 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}\,, \tag{C.3}
$$

shown in figure 10.2 for the case $k_2 = k_3 = -1$. Note that we have chosen a different $SL(4,\mathbb{Z})$ frame then in chapter 12. We can compute the volumes

$$
\begin{aligned}
\mathrm{Vol}_{\Sigma_1} &= \frac{k_2+k_3}{b_1\left(b_2k_2-b_3\right)\left(b_3+b_2k_3\right)}\,, \\
\mathrm{Vol}_{\Sigma_2} &= \frac{k_2+k_3}{b_1\left(b_3+\left(4-b_2\right)k_2+\left(4+b_1\right)k_3\right)\left(\left(4+b_1\right)k_2+\left(4-b_2\right)k_3-b_3\right)}\,, \\
\mathrm{Vol}_{\Sigma_3} &= \frac{\left(k_2+k_3\right)\left(b_3\left(k_2-k_3\right)+\left(4+b_1+b_2\right)k_2k_3\right)}{\left(b_3-\left(4+b_1\right)k_2\right)\left(b_2k_2-b_3\right)\left(b_3+\left(4+b_1\right)k_3\right)\left(b_3+b_2k_3\right)}\,, \\
\mathrm{Vol}_{\Sigma_4} &= \frac{-\left(k_2+k_3\right)\left(\left(4+b_1\right)k_2^2+\left(4-b_2\right)k_2k_3+\left(4+b_1\right)k_3^2+b_3\left(k_3-k_2\right)\right)}{\left(b_3-\left(4+b_1\right)k_2\right)\left(b_3+\left(4+b_1\right)k_3\right)\left(b_3+\left(4-b_2\right)k_2+\left(4+b_1\right)k_3\right)\left(b_3-\left(4+b_1\right)k_2-\left(4-b_2\right)k_3\right)}\,, \\
\mathrm{Vol}_{\Sigma_5} &= \frac{b_3\left(k_2+k_3\right)+k_3\left(4k_2+\left(4+b_1\right)k_3\right)}{b_1\left(b_3+\left(4+b_1\right)k_3\right)\left(b_3+\left(4-b_2\right)k_2+\left(4+b_1\right)k_3\right)\left(b_3+b_2k_3\right)}\,, \\
\mathrm{Vol}_{\Sigma_6} &= \frac{b_3\left(k_2+k_3\right)-k_2\left(\left(4+b_1\right)k_2+4k_3\right)}{b_1\left(\left(4+b_1\right)k_2-b_3\right)\left(b_3-b_2k_2\right)\left(\left(4+b_1\right)k_2+\left(4-b_2\right)k_3-b_3\right)}\,.
\end{aligned}
$$

The formula for Z is lengthy, we refrain from an explicit expression here. Note, nevertheless the two special cases D_3, corresponding to CS levels $(1,-1,0)$, and SPP_{2-1-1},

$$
\begin{aligned}
Z_{D_3} &= \frac{16}{b_1 b_3 \left(4+b_1+b_3\right)\left(4-b_2-b_3\right)\left(b_2+b_3\right)}\,,\\
Z_{SPP_{2-1-1}} &= \frac{8\left(b_1^3+b_1^2\left(20-b_2\right)+b_1\left(128-8b_2+b_2^2-3b_3^2\right)+16\left(16-b_3^2\right)\right)}{b_1\left(\left(4+b_1\right)^2-{b_3}^2\right)\left(\left(8+b_1-b_2\right)^2-b_3^2\right)\left(b_3^2-b_2^2\right)}\,.
\end{aligned}
\tag{C.4}
$$

$\widetilde{\mathcal{C}/\mathbb{Z}_2}$. The toric diagram is spanned by

$$
G_t = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -2 & 0 & -2 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}\,,
\tag{C.5}
$$

which gives the volumes

$$
\begin{aligned}
\mathrm{Vol}(\Sigma_1) &= \frac{2\left(4-b_1-b_2\right)}{\left(\left(4-b_1\right)^2-b_3^2\right)\left(b_2^2-b_3^2\right)}\,,\\
\mathrm{Vol}(\Sigma_2) &= \frac{2\left(12-b_1+b_2\right)}{\left(\left(4-b_1\right)^2-b_3^2\right)\left(\left(8+b_2\right)^2-b_3^2\right)}\,,\\
\mathrm{Vol}(\Sigma_3) &= \frac{2}{b_1\left(b_2^2-b_3^2\right)}\,,\\
\mathrm{Vol}(\Sigma_4) &= \frac{2}{b_1\left(\left(8+b_2\right)^2-b_3^2\right)}\,,\\
\mathrm{Vol}(\Sigma_5) &= \frac{2\left(4+b_3\right)}{b_1\left(4-b_1+b_3\right)\left(b_3-b_2\right)\left(8+b_2+b_3\right)}\,,\\
\mathrm{Vol}(\Sigma_6) &= \frac{-2\left(4-b_3\right)}{b_1\left(4-b_1-b_3\right)\left(8+b_2-b_3\right)\left(b_2+b_3\right)}\,,\\
Z &= \frac{18\left(b_1\left(32+8b_2+b_2^2-b_3^2\right)+8\left(b_3^2-16\right)\right)}{k\pi^2 b_1((4-b_1)^2-b_3^2)((8+b_2)^2-b_3^2)(b_3^2-b_2^2)}\,.
\end{aligned}
\tag{C.6}
$$

ABJM/$\mathbb{Z}_2$. The toric diagram in figure 10.4 is spanned by

$$
G_t = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}\,,
\tag{C.7}
$$

which gives the volumes

$$
\begin{aligned}
\mathrm{Vol}(\Sigma_1) &= \frac{2}{(b_3-2b_1)\,(4+b_1-b_2-b_3)\,(4+b_1+b_2-b_3)}\,,\\
\mathrm{Vol}(\Sigma_2) &= \frac{2}{(4+b_1-b_2-b_3)\,(4+b_1+b_2-b_3)\,b_3}\,,\\
\mathrm{Vol}(\Sigma_3) &= \frac{2}{(b_3-2b_1)\,(4+b_1-b_2-b_3)\,b_3}\,,\\
\mathrm{Vol}(\Sigma_4) &= \frac{2}{(4+b_1+b_2-b_3)\,b_3\,(b_3-2b_1)}\,,\\
Z &= \frac{16}{(4+b_1+b_2-b_3)\,b_3\,(b_3-2b_1)\,(4+b_1-b_2-b_3)}\,.
\end{aligned}
\tag{C.8}
$$

$Q^{222}/\mathbb{Z}_k$. The toric diagram for $k=1$ is given by

$$
G_t = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}
\tag{C.9}
$$

The volume of $Q^{222}/\mathbb{Z}_k$ is proportional to the minimum of

$$
Z = \frac{32(b_1^2+b_2^2+b_3^2)+b_1^4+b_2^4+b_3^4-2(b_1^2b_2^2+b_1^2b_3^2+b_2^2b_3^2)-768}{-\sqrt{\prod_{\alpha,\beta,\gamma,\delta=\pm1}(4\alpha+\beta b_1+\gamma b_1+\delta b_3)}}
\tag{C.10}
$$

In this case no computation is actually needed: by the symmetry of the Z function, the minimum is found for $b_1=b_2=b_3\equiv b$ and the variational problem further sets $b=0$. Then, the volume of the compact manifold is given by

$$
\mathrm{vol}(Q^{222}) = \frac{\pi^4}{12}Z = \frac{\pi^4}{16}
\tag{C.11}
$$

$M^{111}/\mathbb{Z}_k$. The toric diagram for this geometry is specified by

$$
G_t = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 1 & & 1 & 1 & 1 \end{pmatrix}
\tag{C.12}
$$

and the corresponding Z function is

$$
Z = -\frac{72^3\,(b_3^2-3\,(b_1^2-b_1b_2+b_2^2-16))}{(b_3^2-(4+b_1-2b_2)^2)\,(b_3^2-(4-2b_1+b_2)^2)\,(b_3^2-(4+b_1+b_2)^2)}
\tag{C.13}
$$

The b_1 and b_2 components of the trial Reeb vector $b = (4, b_1, b_2, b_3)$ can be fixed by the symmetries to be equal. By minimising this function the components of b are $b_1 = b_2 = b_3 = 0$ and all the R charges of the fields become $2/3$ while the monopole charge vanishes. The volume of the compact manifold M^{111} is

$$\mathrm{vol}(M^{111}) = \frac{\pi^4}{12} Z = \frac{9\pi^4}{128} \tag{C.14}$$

Appendix D

Summary of the treatment for chiral quivers at large N

The main idea of [171] is to make manifest in the integrand of the matrix model the reflection $\lambda \to -\lambda$, which naively is a symmetry of the problem but which is lost in the standard saddle point approach. Roughly, this is done by re-writing $Z = \frac{1}{2}\int d\lambda[\exp(-F(\lambda)) + \exp(-F(-\lambda))]$, where we use a very condensed notation for the set of $\{\lambda_i^{(a)}\}$.[1] Since the integration limits are symmetric, this really is a mere re-writing.

However, computing the saddle point equations leads to a potentially different result. The equations one obtains are very complicated, in order to solve them we had to make the simplifying assumption of a symmetric eigenvalue distribution. This is a strong assumption, in all the vector-like examples above it holds on-shell, but not off-shell. It turns out that the equations of motion that follow from this explicitly symmetrised ansatz will not suffer from contributions of the problematic long range forces.

Note that this 'vectorialisation' of the field theory closely resembles the one found in the weak coupling case [171], where it was observed that at two loop order the contribution coming from a field in a representation of the gauge group is the same of that coming from a field in the conjugate representation, even at finite N.

Let us briefly browse through some of the results in [4] obtained at strong coupling. Two models of interest are the field theories with moduli spaces $C(Q^{222})/\mathbb{Z}_k$ and $C(M^{111})/\mathbb{Z}_k$. The quiver diagram for these theories is given in 10.4 and D.1 with Chern-Simons levels $(k, k, -k, -k)$ and $(k, k, -2k)$, respectively. With the assumption of a symmetric eigenvalue distribution, we computed the on-shell value of the volume via the free

[1] In fact, we should symmetrise with respect to all $\lambda_i^{(a)}$'s.

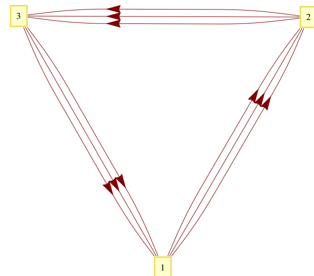


Figure D.1: Quiver diagram for M^{111}.

energy (10.10) for these two cases as

$$\mathrm{Vol}(Q^{222}/\mathbb{Z}_k) = \frac{\pi^4}{16k} \qquad\qquad \mathrm{Vol}(M^{111}/\mathbb{Z}_k) = \frac{9\pi^4}{128k}\ , \tag{D.1}$$

which agree with the values found in the literature. We did not observe a matching of F with the geometry *before* extremisation. Given that in the well understood vector-like theories the off-shell eigenvalue distribution is not symmetric anymore, we are not too surprised about this.

Despite this motivating result, we leave many open problems that deserve further studies. Since the symmetrised free energy functional is technically so hard to control, our computations relied on the simplifying assumption on the eigenvalue distribution. This is clearly a huge limitation. First, as we have seen does it make it impossible to check the proposal off-shell. Furthermore, we can trust our free energy computation only for theories with expected vanishing monopole charge, which is why we had to restrict our analysis to few models. It would be interesting to study the more general case, for instance on a large cluster of computers.

There are also other issues. The field theory dual to $AdS_4 \times Q^{111}/\mathbb{Z}_k$ has no expected monopole charge on-shell and in principle we should be able to understand it in our simplified approach. We found a solution with a well-behaved free energy scaling $F \propto N^{3/2}$, but the field theory computation for the volume does not match with the supergravity one.

Similarly, we would like to comment on the dual phases of our models proposed in [156, 180, 181]. Strictly speaking, these dualities have been derived for vector-like models, but they were shown to be also applicable to some chiral-like theories, namely $Q^{111}/\mathbb{Z}_k$ and $Q^{222}/\mathbb{Z}_k$. The latter has a toric phase which is the analog of the Phase II of F_0 in four dimensions. We applied our procedure to this phase as well and got a result which differs from the expected one.

These mismatches are a severe problem that might rule out the validity of our modified saddle point technique. However, notice that the full understanding of the quantum corrected moduli space of chiral theories is intricate. It certainly would be rewarding to see if an extension of the field theory models along the lines of [145] shed light on some of the open problems reported here.

The understanding of 'chiral' theories in three dimensions remains the big open question in this subject. We hope that our results can give a small contribution on the way to a more complete picture.

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