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SUPERSYMMETRY, SUPERGRAVITY THEORIES AND

THE DUAL SPINOR MODEL

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ABSTRACT

We study the connection between the dual spinor model and supersymmetry. We show that in the low-energy region, the dual spinor model yields a supersymmetric Yang-Mills theory with $O(4)$ internal symmetry and a supergravity theory with $O(4)$ internal symmetry.

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1. INTRODUCTION

The non-renormalizability of the V-A theory of weak interactions has led to a long search for renormalizable theories. The problem was not solved by finding a new method to deal with the divergences of the V-A theories. Rather, it was solved by considering the V-A theory as a special limit ($M_W = \infty$) of a unified gauge theory of weak and electromagnetic interactions¹⁾. The success of this approach was due to the theoretical proof that the gauge theories are renormalizable²⁾ and to the experimental fact that the most striking predictions of these models, namely the existence of neutral currents, and of charm³⁾, turned out to be true.

The same techniques used to quantize gauge theories were then applied to Einstein's theory of gravity⁴⁾. It was then found that one-loop diagrams in pure gravity were on-shell finite. However, as soon as interactions with matter fields were included, one obtained divergent, and non-renormalizable results⁴⁾.

It was then thought that the problem could be solved by making gravity more symmetric, thus including more fields in the gravitational multiplet. This led to the idea, and to the realization, of supergravity⁵⁾ (2, 3/2) and of complex supergravity⁶⁾ (2, 3/2, 3/2, 1). It is remarkable that in these models, one loop on-shell amplitudes do turn out to be finite⁷⁾. One can imagine more complex supersymmetric systems including the graviton. However, in this way one produces only a gravitational-like sector, as there is only one coupling constant, κ (the gravitational constant) present in such models. One is then led to introduce other multiplets of fields containing a Yang-Mills field⁸⁻¹⁰⁾, or scalar fields^{10,11)} in order to represent matter. In this case one recovers the faults of ordinary Einstein theory in interaction with matter. For instance, the locally supersymmetric system (2, 3/2) coupled to (1, 1/2)^{8,9)} turns out to give non-renormalizable one-loop divergences¹²⁾. A possibility is that by making the theory even more symmetric, one will get convergent results even in the matter-gravity interactions.

Another line of thinking is to consider supergravity and its possible refinements as only a first step. For instance nothing guarantees that 2-loop diagrams in such a scheme will be finite¹³⁾; interaction with matter may also not work. Then there would be an obvious parallelism between the V-A theory with coupling constant G_F and the gravitational theory with coupling constant $G_N \sim \kappa^2$ since both constants have the same dimensions. In the renormalizable gauge theories G_F is a phenomenological parameter, expressed as $G_F \sim g^2/M_W^2$ where g is the real dimensionless coupling constant of the theory and M_W^2 play the role of a

cut-off. So one may ask whether one could find a similar expression for $G_N \sim (g^2)^n / \Lambda^2$ and identify g and Λ .

Dual models¹⁴⁾ provide an answer to this kind of question. The problem of obtaining a cut-off Λ for gravity is far from obvious, as there already is a gauge boson (the graviton) associated with gravity. In the dual model there is a sector (open strings) which is essentially a short distance modified Yang-Mills theory¹⁵⁾, the distance $\Lambda^{-1/2}$ being $\alpha'^{1/2}$, α' being the slope of the Regge trajectories. Then a sector (closed strings) containing a spin-2 massless graviton, is obtained as a bound state of the open-string sector, already from the one-loop diagrams. Then $n = 2$, and $\Lambda^{-1/2}$ is essentially of the order of Planck's length¹⁶⁾.

As there are extra dimensions to be compactified¹⁷⁾ Λ is in fact expressed both in terms of α' and of the volume associated with the compactified dimensions so that the model contains more than one parameter of the dimension of a length, thus giving a certain amount of freedom.

Further, we would like to show that dual models are in accordance with the program of supergravity. As there is only one dual model containing both bosons and fermions (the dual spinor model)¹⁸⁾, we consider its supersymmetric properties. It is already known that the dual spinor model can be obtained from a classical action, which is locally supersymmetric invariant on the two-dimensional surface spanned by the string¹⁹⁾. Here we give arguments why the spinor dual model of open strings should be globally supersymmetric in the 10-dimensional space-time in which the model is consistent. We show that in the limit where both the slope and the size of the 6-dimensional compact go to zero, the dual model of open strings turns into a hypersymmetric Yang-Mills theory (1 spin 1, 3 scalars, 3 pseudoscalars, 4 Majorana spin $\frac{1}{2}$) with $SU(4)$ symmetry. A model with the same field content was studied by Fayet²⁰⁾. Similar arguments apply for the sector of closed strings. In the same limit, we show that the dual model must yield a hypergravity theory with internal $SU(4)$ symmetry, containing at least 1 spin 2, 3 spin 1 vectors, 3 spin 1 axial vectors, 2 scalars, 4 spin 3/2 (Majorana), 4 spin $\frac{1}{2}$. The dual model implies the existence of this field theory which has not yet been written down. To make contact with already known theories, we suppress some of the components present in the dual model spectrum and obtain two known theories:

- a) supergravity in interaction with a massless supermultiplet $[(2, 3/2), (1/2, 0)]^{10,11}$;
- b) complex supergravity⁶⁾ $(2, 3/2, 3/2, 1)$.

Most of these results were announced in Ref. 21). The correspondence between supergravity and the dual spinor model was also conjectured by Schwarz in Ref. 9). The most important result of this study is to reveal a hierarchy of theories.

The hypersymmetric theories can be labelled with $SU(N)$ in the case one takes all generators of the symmetry group, including those which contain γ^5 , or by $O(N)$ if those are excluded. By order of inclusion, we have the following list:

Theories of gravity:

Einstein's theory; supergravity $[O(1)]^{5)}$; complex supergravity $[O(2)]$; $O(4)$ supergravity (yet to be worked out); supergravity in 10 dimensions; the dual spinor model of closed strings (κ, α') .

Theories of matter:

Yang-Mills theory; supersymmetric Yang Mills theory²²⁾ $[O(1)]$; supersymmetric Yang-Mills with $O(2)^{22, 23)}$; supersymmetric Yang-Mills with $O(4)$, first presented here; supersymmetric Yang-Mills in 10 dimensions; the dual spinor model of open strings (g, α') .

The final unification of matter and gravity appears only at the last stage, as closed strings are bound states of open strings.

As far as renormalizability is concerned, theories of gravity always give finite one-loop results. This is true even for the dual spinor model of closed strings, as the same arguments used by Shapiro²⁴⁾ for the Virasoro-Shapiro model²⁵⁾ must apply. The same topological arguments should prove that the N-loop diagrams are also finite for closed strings.

For open strings, one-loop diagrams give rise to a divergence. There are arguments that the divergence can be removed by a slope renormalization²⁶⁾, but the subject is still under study.

In Section 2, we discuss the spectrum of the dual spinor model and give arguments for its global supersymmetry in 10 dimensions. In Section 3, we discuss its connection with the Yang-Mills theory with fermions in 10 dimensions, which is shown to be supersymmetric. Out of it we obtain the hypersymmetric Yang-Mills theory in 4 dimensions with internal $O(4)/SU(4)$ symmetry. It appears that even if one is interested only in that model, the 10-dimensional notation is more compact and elegant. In Section 4 we define and study the spectrum of the closed string sector of the dual spinor model. In Section 5 we establish the correspondence between supergravity theories and the dual model of closed strings.

2. THE SPECTRUM OF THE DUAL SPINOR MODEL

2.1 The spectrum of the Neveu-Schwarz model

The model is defined out of two sets of harmonic oscillators:

$$\text{Bose set} : [a_n^\mu, a_m^{\nu\dagger}]_- = \eta^{\mu\nu} \delta_{m,n} \quad n, m \text{ integers} \quad (2.1)$$

$$\text{Fermi set} : [b_r^\mu, b_s^{\nu\dagger}]_+ = \eta^{\mu\nu} \delta_{r,s} \quad r, s \text{ half integers} \quad (2.2)$$

$\eta^{\mu\nu} = (-+++)$, and - and + denote commutators and anticommutators, respectively.

The states $|\psi\rangle$ obtained by applying creation operators on the vacuum $|0\rangle$ do not form a Hilbert space because of the indefinite metric. However, one can define a sector of states, called the physical sector, which has semi-definite positive metric. They are defined by an infinite number of gauge conditions:

$$L_n |\phi\rangle = 0 \quad n > 0 \quad (2.3)$$

$$G_r |\phi\rangle = 0 \quad r > 0 \quad r \text{ half integer} \quad (2.4)$$

$$(L_0 - \frac{1}{2}) |\phi\rangle = 0 \quad (2.5)$$

The last equation being the mass shell condition. The gauge operators G_r , L_n form an infinite graded Lie algebra.

The mass shell condition gives a quantization condition for the masses:

$$\alpha' m^2 = n \quad n = -\frac{1}{2}, 0, +\frac{1}{2}, 1 \dots \text{etc} \quad (2.6)$$

The ground state is thus a tachyon. More generally one can distinguish two sectors of states labelled by the eigenvalues of the operator $G = (-1)^{(\sum_m b_m^\dagger b_m - 1)}$.

- The sector with $G = -1$ has $\alpha' m^2 = n$, $n = -1/2, 1/2, 3/2, \dots$ and contains the tachyon, represented by $|0\rangle$;
- The sector with $G = +1$ is tachyon free and has integrally quantized masses: $\alpha' m^2 = n$, $n = 0, 1, 2, \dots$.

The ground state of the $G = -1$ sector is a massless vector, represented by

$$b_{\frac{1}{2}}^{\dagger\mu} |0\rangle \epsilon_\mu(p); \quad p^2 = 0.$$

Because of the constraint (2.4) with $r = 1/2$, $\epsilon_\mu(p)$ must satisfy the Lorentz condition: $p \cdot \epsilon = 0$. Strictly speaking, this is not a gauge condition as it can be derived from the free Maxwell action for on-shell photons (see Ref. 10)). This equation decouples the time-component of the photon polarization vector ϵ_μ and guarantees the absence of the negative metric states associated with it.

The physical states of the dual model contain amongst themselves some zero norm states: they are both "physical" (i.e. they satisfy Eqs. (2.3)-(2.5) and spurious, i.e. can be written in the form $\sum_{n \geq 1} L_{-n} |\phi_n\rangle + \sum_{r \geq \frac{1}{2}} G_{-r} |\phi_r\rangle$. These states always decouple in the dual S-matrix because the external states are physical and the L_n, G_r operators have nice commutation relations with the vertices. In particular a very important property of the dual S-matrix is that an external photon polarization vector can be subjected to an on-shell gauge transformation: $\epsilon_\mu(p) \rightarrow \epsilon_\mu(p) + \lambda p_\mu$ without affecting the S matrix. This is because the variation of the photon state is

$$p_\mu b_{\frac{1}{2}}^{\dagger\mu} |0\rangle_\lambda = G_{-\frac{1}{2}} |0\rangle_\lambda \quad (2.7)$$

and such a zero norm state is decoupled by the above arguments. Thus on-shell gauge invariance, together with the Lorentz condition proves that only the transverse components of the photon are coupled in the dual model, at all orders in α' . The number of physical degrees of freedom of the photon are then $D-2$.

All these statements hold for $D \leq 10$. For $D = 10$, the zero norm states are more numerous and one can show that not only is the photon transverse, but also all the states in the dual model. One can then reformulate the model in such a way that oscillators with only $D-2 = 8$ Lorentz indices appear, so that the no-ghost theorem is trivial, but Lorentz invariance is less obvious (but true).

Finally if a $U(N)$ group is introduced via Chan-Paton factors, all states of the NS model belong to the regular representation of $U(N)$ and are N^2 degenerate.

2.2 The spectrum of the Ramond model

The model is defined out of two sets of oscillators:

$$\text{Bose set} : [a_n^\mu, a_m^{\dagger\nu}]_- = \eta^{\mu\nu} \delta_{mn} \quad m, n \text{ integers} \quad (2.8)$$

$$\text{Fermi set} : [d_n^\mu, d_m^{\dagger\nu}]_+ = \eta^{\mu\nu} \delta_{mn} \quad m, n \text{ integers} \quad (2.9)$$

The physical states satisfy an infinite number of gauge conditions:

$$F_n |\phi\rangle = L_n |\phi\rangle = 0 \quad \text{for all } n \geq 0 \quad (2.10)$$

The $n = 0$ equations give the mass-shell condition and a Dirac-like equation. The algebra of the F_n, L_n gauges differs only by its c-numbers from the Neveu-Schwarz case.

The masses are quantized through integers:

$$\alpha' m^2 = n \quad n = 0, 1, 2, \dots \quad (2.11)$$

In particular in such a model there is no tachyon. The ground state is $|0\rangle u$ where u is a (commuting) spinor satisfying the Dirac equation:

$$\not{p}u = 0 \quad (2.12)$$

One can also define two sectors in the Ramond model, according to the eigenvalues of the operator

$$\tilde{\gamma}^{D+1} = \gamma^{D+1} (-1)^{\sum d_n^\dagger \cdot d_n} \quad (2.13)$$

where $\gamma^{D+1} = \gamma^0 \gamma^1 \dots \gamma^{D-1}$. The Dirac γ matrices are conventionally defined to be in the representation of lowest dimension ($2^{D/2}$) of the Clifford algebra

$$[\gamma^\mu, \gamma^\nu] = 2 \eta^{\mu\nu} \quad (2.14)$$

in D dimensions (one time, $D-1$ space dimensions, D assumed even). Then one can show that:

$$(\tilde{\gamma}^{D+1})^2 = (-1)^{\left(\frac{D}{2} - 1\right)} \quad (2.15)$$

Left and right handed states can be defined for any even D .

Just as in the Neveu-Schwarz model, physical states are ghost free for $D \leq 10$, and for $D = 10$ only transverse (positive norm) states are coupled. The difference does not affect the ground-state spinor, which is not a gauge field.

Finally, under the $U(N)$ group the fermions can belong to a different representation to that of the bosons. In the following, we shall see that assigning them to the regular representation of $U(N)$ solves a problem one meets in computing fermion-fermion scattering.

2.3 Supersymmetry and the spectrum of the Neveu-Schwarz-Ramond model

a) Three problems of the NSR model

We want to review three problems which arise out of the calculations of the fermion-fermion scattering amplitudes, and how these three problems are solved by a supersymmetric assignment of the ground states.

1) The first problem has to do with fermion number conservation. When computing the graphs of Figs (1a,1b) depicting fermion-fermion scattering, one finds that the spectrum of mesonic states exchanged in the s-channel (namely the Neveu-Schwarz model) is almost identical to the spectrum of bound states one finds by duality in the t-channel. In fact the only difference is that the states with $G = -1$ exchange parity between s and t channels. If one wants to define a fermion number in the conventional way, one thus finds that for instance at $m = 0$ there are three photons, with fermion numbers 0 and ± 2 . It is likely that if one could construct amplitudes with an arbitrary number of fermions, each state would appear infinitely degenerate with respect to the fermion number. This is clearly bad for a naive quark-gluon interpretation of that model.

A possible solution, proposed in Ref. 17), was to quantize in the extra directions the momenta of the fermions in a way which would depend upon the fermion number: for example

$$p^4 = \frac{2\pi}{R} (n + \alpha Z) \quad (2.16)$$

n being an integer, Z the fermion number, R an arbitrary radius, α an arbitrary and small parameter. This works well for small Z, but when $n + \alpha Z$ approaches an integer, the contribution of p^4 to the mass falls again to zero. Another solution, somewhat unnatural (but maybe consistent) is to force the model to conserve fermion numbers by considering only graphs of the type of Fig. 1a rather than those of Fig. 1b.

2) A second problem, related to the first is the "representation explosion". As s and t channels are degenerate one may expect (as is also based on duality arguments) that the same mesons will turn up as bound states of $2n$ fermions. If one wants to put the fermions in the spinorial representation of the $U(N)$ group (for instance), an explosion of representations is bound to take place. The only known way to prevent this explosion is to put the spinor in the regular representation of the $U(N)$ group, just like the bosons, and to use Chan-Paton factors. Then only the regular representation of $U(N)$ appears as bound states, a well-known property of the Chan-Paton method.

3) The last problem is the strange doubling in parity of the $G = -1$ sector of the NS which distinguishes the two sides of a fermionic string. Because of this distinction, twisting fermionic strings is impossible and loops such as the one of Fig. 2 is forbidden. Last but not least, the $G = -1$ sector now contains two tachyons, both of which are undesirable²⁷⁾.

A nice way to solve the two first problems is to choose the fermions to be Majorana, and to belong to the regular representation of the $U(N)$ group. This does not mean that fermion number conservation is to be abandoned, but rather that it has to be obtained in a different way, just as in ordinary supersymmetry²⁸⁾.

The last problem is solved by making the ground state fermion left-handed. Since the sector of the NS model with $G = -1$ changes handedness, it is decoupled through such a choice and the tachyon disappears, and with it the problem of identifying the two sides of a fermionic string.

The correct model to be considered thus seems to be the $G = +1$ sector of the NS model together with the Weyl (left-handed) and Majorana (real) sector of the Ramond model. It seems indeed to be trouble-free (no tachyons, no ghosts).

In such a model fermion-fermion and boson-fermion scattering have exactly the same duality properties (see Fig. 3).

It may look surprising to see the Weyl condition arising. On the other hand, it is worth noticing that in 4 dimensions for massless spinors, the Majorana and Weyl conditions are equivalent. In the real representation of the γ matrices a Majorana fermion is real. Let us consider a complex Weyl spinor defined by

$$\gamma^5 \nu(x) = i\nu(x). \quad (2.17)$$

Decomposing $\nu(x)$ into Majorana (real) spinors one has:

$$\nu(x) = \frac{1}{\sqrt{2}} (\lambda(x) + i \chi(x)) \quad (2.18)$$

and

$$\chi = \gamma_5 \lambda \quad (2.19)$$

is equivalent to the Weyl condition. The Lagrangian of a "neutrino" field $\nu(x)$ is

$$\mathcal{L} = -\frac{1}{2} i \bar{\nu}(x) \gamma^r \overset{\leftrightarrow}{\partial}_r \nu(x) \quad (2.20)$$

After a brief calculation using (2.19) it can be recast into

$$\mathcal{L} = -i \bar{\lambda}(x) \gamma^\mu \partial_\mu \lambda(x) \quad (2.21)$$

which is the Lagrangian for a massless Majorana field $\lambda(x)$.

So for $D = 4$, the Majorana and Weyl conditions can be transformed into each other, and except for $m \neq 0$ one can use one formalism or the other. Note that for 4 dimensions imposing Majorana and Weyl on a spinor would imply that it vanishes.

b) Existence of a supersymmetric assignment of the NSR ground states

We now examine the problem of whether Majorana-Weyl spinors can exist at all. Assuming first that Majorana spinors exist for $D = 2d$, we see that since $(\gamma^{2d+1})^2 = (-1)^{d-1}$, in order for both conditions to be compatible, the eigenvalues of γ^{2d+1} must be ± 1 rather than $\pm i$, and therefore d must be odd.

We now look for which values of D Majorana spinors exist. We shall prove the following crucial theorem:

Theorem:

Majorana spinors, and a real representation of the γ matrices, exist if and only if D is 2 or 4 modulo 8.

Throughout this section $*$ denotes complex conjugation, \dagger denotes Hermitian conjugation and T the transposition.

Using Schur's lemma²⁹, since γ^μ and $\gamma^{*\mu}$ have the same algebra, and the representation of the γ^μ is of minimal dimension $2^{D/2}$, there must exist a matrix B such that

$$\gamma^{*\mu} = B \gamma^\mu B^{-1} \quad (2.22)$$

one can require further that

$$|\det B| = 1 \quad (2.23)$$

We note that B has involutive properties namely

$$\gamma^\mu = B^* \gamma^{*\mu} B^{*-1} \quad (2.24)$$

or

$$\gamma^{*\mu} = B^{*-1} \gamma^\mu B^* \quad (2.25)$$

Comparing equations we see that B^* and B^{-1} are proportional to each other. Using (2.23) one gets

$$B^* B = \epsilon I \quad |\epsilon| = 1 \quad (2.26)$$

Let us show that ϵ is real:

$$B B^* = \epsilon^* I \quad (2.27)$$

but multiplying Eq. (2.26) by B and B^{-1} one gets

$$B (B^* B) B^{-1} = \epsilon I = \epsilon^* I$$

So $\epsilon = \pm 1$. One can easily show that ϵ does not depend upon which representation of the γ matrices one uses. Therefore, it has an intrinsic significance, and depends only upon the space time metric $\eta^{\mu\nu}$.

ϵ is of crucial importance: if $\epsilon = -1$ no Majorana spinors exist, whilst for $\epsilon = +1$ they do exist.

To see this let us define Majorana spinors. It is easily seen that if ψ satisfies the Dirac equation, $B^{-1}\psi^*$ also satisfies it. Therefore, the charge conjugate of ψ is $B^{-1}\psi^* = \psi^c$. A Majorana spinor is defined such that

$$\psi = \psi^c = B^{-1} \psi^* \quad (2.28)$$

Now $\psi = B^{-1}\psi^*$ implies

$$\psi^* = B^{*-1} \psi = \epsilon B \psi \quad (2.29)$$

and therefore we get the consistency condition

$$(1-\epsilon)\psi = 0 \quad (2.30)$$

which implies that if ψ does not vanish, ϵ must be $+1$. A more general definition of a Majorana spinor, i.e. $\psi = \lambda\psi^c$ does not affect this conclusion.

Until now, the signature of the metric, and the Hermiticity properties of the γ matrices have not been used. We now bring them into the game by assuming one time dimension and $D-1$ space dimensions; then $\gamma^{0\dagger} = -\gamma^0$ and $\gamma^{i\dagger} = \gamma^i$.

Then one has

$$\gamma^{\mu\dagger} = -\gamma^0 \gamma^\mu (\gamma^0)^{-1} \quad (2.31)$$

Defining the charge conjugation matrix C such that

$$\gamma^{\mu T} = -C \gamma^\mu C^{-1} \quad (2.32)$$

C can be computed in terms of the B matrix and γ^0 in two different ways:

$$\gamma^{\mu T} = (\gamma^{\mu*})^\dagger = (\gamma^{\mu\dagger})^* \quad (2.33)$$

Using the first way of writing $\gamma^{\mu T}$ one obtains:

$$\gamma^{\mu T} = - (B^{\dagger-1} \gamma^0) \gamma^\mu (B^{\dagger-1} \gamma^0)^{-1} \quad (2.34)$$

And using the second way one obtains

$$\gamma^{\mu T} = - (B \gamma^0) \gamma^\mu (B \gamma^0)^{-1} \quad (2.35)$$

This shows that the matrices B and $(B^\dagger)^{-1}$ are proportional to each other, thus $BB^\dagger = \lambda$. Because of the normalization of B , $|\lambda| = 1$, and λ is positive. Thus $\lambda = 1$.

We recapitulate the results obtained so far: we have

$$BB^\dagger = B^\dagger B = I \quad (2.36)$$

and

$$BB^* = \epsilon I \quad (2.37)$$

This implies that

$$B = \epsilon B^T \quad (2.38)$$

so that B is either symmetric or antisymmetric.

Finally, we have

$$C = B \gamma^0 \quad (2.39)$$

and

$$C^T = -\epsilon C \quad (2.40)$$

We now go into the crucial phase of the proof which is the computation of ϵ^{29} . We achieve this by counting the number of symmetric and antisymmetric matrices which form a complete basis for the product of γ matrices.

Let us denote by Γ^n $n = 0, 1, \dots, D$ (assume here D even) the products of n γ matrices, completely antisymmetrized with respect to the indices of the γ matrices and normalized such that $(\Gamma^n)^2$ is either $+1$ or -1 . (We shall not use this normalization condition here, but it will be important later on for Fierz transformation.) For a given n there are $\binom{D}{n}$ Γ^n matrices, and the total number of these matrices is 2^D . Since the γ matrices are $2^{D/2} \times 2^{D/2}$, this shows that the Γ^n matrices form a complete basis for any combination of sum or products of the γ matrices, and the independence of these matrices is also obvious.

Let us now consider $C \Gamma^n C^{-1}$. It is easy to see that this is equal to Γ^{nT} , up to a sign which is given by $(-1)^n (-1)^{n(n-1)/2}$. The first factor comes from using (2.32) n times, and the second factors from reordering the n γ matrices inside Γ^n to get the correct ordering. In doing this reordering no contraction ever occurs as Γ^n is antisymmetrized with respect to the indices of the γ matrices inside. So we have

$$\Gamma^{nT} = (-1)^n (-1)^{\frac{n(n-1)}{2}} C \Gamma^n C^{-1} \quad (2.41)$$

or:

$$(C \Gamma^n)^T = \epsilon (-1)^{\frac{(n-1)(n-2)}{2}} (C \Gamma^n) \quad (2.42)$$

So the matrices $C \Gamma^n$ are either symmetric or antisymmetric. This equation allows one to count the number of independent antisymmetric matrices in two different ways. On one hand we know that as the γ matrices are $2^{D/2}$ dimensional, a complete basis of antisymmetric matrices must contain $2^{D/2}(2^{D/2}-1)/2$ matrices. On the other hand, using Eq. (2.42) the number of antisymmetric matrices can also be written as:

$$N = \sum_{n=0}^D \left(\frac{1 - \epsilon (-1)^{\frac{(n-1)(n-2)}{2}}}{2} \right) \binom{D}{n} = \frac{2^{D/2} (2^{D/2} - 1)}{2} \quad (2.43)$$

Thus we obtain an equation which determines ϵ :

$$\epsilon \sum_{n=0}^D (-1)^{\frac{(n-1)(n-2)}{2}} \binom{D}{n} = 2^{D/2} \quad (2.44)$$

The summation over n can be performed by noticing that:

$$(-1)^{\frac{(n-1)(n-2)}{2}} = -\frac{1}{2} \left[(1+i) i^n + (1-i) (-i)^n \right] \quad (2.45)$$

and one finally obtains ϵ :

$$\epsilon = -\sqrt{2} \cos \frac{\pi}{4} (D+1) \quad (2.46)$$

So

$$\epsilon = +1 \quad \text{for } D = 2, 4 \pmod{8} \quad (2.47)$$

and

$$\epsilon = -1 \quad \text{for } D = 6, 8 \pmod{8} \quad (2.48)$$

To complete our theorem, we now prove that when $\epsilon = +1$, a real representation of the γ matrices exists. Under a change of basis of the γ matrices $\gamma'^{\mu} = A\gamma^{\mu}A^{-1}$, B transforms as follows:

$$B' = A^* B A^{-1} \quad (2.49)$$

For $\epsilon = +1$, B is symmetric and also unitary. Therefore if we decompose B into real and imaginary parts $B = B_1 + iB_2$, B_1, B_2 are real, symmetric matrices, which satisfy because of the unitarity condition: $B_1^2 + B_2^2 = 1$ and $[B_1, B_2] = 0$. The last condition ensures that they can be simultaneously diagonalized. Thus B itself can be diagonalized, and since B is unitary it can be put in the form $B = \text{diag}(e^{i\alpha_1}, \dots, e^{i\alpha_D})$. Then choosing $A = \text{diag}(e^{\frac{1}{2}\alpha_1}, \dots, e^{\frac{1}{2}\alpha_D})$ brings B to the form $B = 1$, which proves the existence of a real representation of the γ matrices. In such a representation the charge conjugation matrix C is just given by γ^0 .

The requirement of the existence of Majorana-Weyl spinors gives thus the following condition on D : $D = 2[\text{mod } 8]$. The first non-trivial dimension (2 is somewhat trivial) is thus $D = 10$, which is a nice way to recover the critical dimension of the NSR model.

We end this Section by giving an explicit real representation of the γ matrices in 10 dimensions, which will be useful later on when making explicit the 4-dimensional content of a 10-dimensional theory. The 32-dimensional Dirac indices can be written as a direct product of a Dirac index running from 1 to 4 (ordinary space-time) and a Dirac index running from 1 to 8 ("internal" space). Thus the 32-dimensional γ matrices can be written as direct products of 4×4 ordinary Dirac matrices and of 8×8 ("internal symmetry" indices).

A real (Majorana) representation is then obtained, as given by

$$(a) \quad \gamma^{\mu} = \gamma^{\mu} \otimes \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix} ; \mu = 0, 1, 2, 3. \quad (2.50)$$

where γ^{μ} on the right are the 4×4 Dirac matrices in a real representation, I_4 the 4×4 identity matrix.

$$(b) \quad \gamma^{3+i} = \gamma^5 \otimes \begin{pmatrix} P_i & 0 \\ 0 & P_i' \end{pmatrix} ; i = 1, 2, 3. \quad (2.51)$$

$$\rho_1 = \rho_1' \equiv \gamma^0 \quad (2.52)$$

$$\rho_2 = \rho_2' \equiv \gamma^5 \quad (2.53)$$

$$\rho_3 = -\rho_3' \equiv \gamma^0 \gamma^5 \quad (2.54)$$

(c)

$$\gamma^{6+j} = I_4 \otimes \begin{pmatrix} 0 & \rho_j \\ \rho_j & 0 \end{pmatrix} \quad ; j=1,2,3. \quad (2.55)$$

$$\rho_1 \equiv \gamma^1 \quad \rho_2 \equiv \gamma^2 \quad \rho_3 \equiv \gamma^3 \quad (2.56)$$

When writing such equations such as (2.56), we only mean that the algebra of ζ^i and γ^i are identical. One can verify that this representation is (obviously) real and satisfies the correct transposition and anticommutation properties. The advantage of it is that the "extra" six γ matrices appear as three scalar and three pseudoscalar matrices in Dirac space. Computing $\gamma^{11} = \gamma^0 \gamma^1 \dots \gamma^9$ one obtains:

$$\gamma^{11} = I_4 \otimes \begin{pmatrix} 0 & \rho_3 \\ -\rho_3 & 0 \end{pmatrix} \quad (2.57)$$

A Dirac spinor in 10 dimensions can be written as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad (2.58)$$

where ψ_1, ψ_2 are four Dirac spinors in four dimensions. A Majorana spinor in 10 dimensions is such that $\psi = \psi^*$ and is a collection of four Majorana spinors in four dimensions. Finally, a Weyl-Majorana spinor in 10 dimensions in this representation is given by: $\psi = \gamma^{11} \psi$, i.e.:

$$\psi = \begin{pmatrix} \psi_1 \\ -\rho_3 \psi_1 \end{pmatrix} \quad (2.59)$$

$$\psi_1^* = \psi_1 \quad (2.60)$$

This shows explicitly the existence, in 10 dimensions, of Majorana-Weyl spinors and also reveals that the 10 dimensional Weyl condition does not imply parity breaking in four dimensions. A Majorana-Weyl spinor in 10 dimensions simply appears as a collection of four four-dimensional Majorana spinors.

2.4 Counting the states of the NSR model

Having decided that the ground state fermion is a Majorana-Weyl fermion, and that it must belong also to the regular representation of $U(N)$, we are left with the Majorana-Weyl-Ramond model and the $G = +1$ sector of the Neveu-Schwarz model. We shall show that this model has a good chance of being supersymmetric by showing that at each mass level, the number of physical states of the Bose and Fermi sectors are equal.

The quantization in masses for the model is now given by

$$\alpha' m^2 = n \quad ; \quad n = 0, 1, 2, \dots \quad (2.61)$$

and the spectrum has no tachyon. As both Fermi and Bose sectors belong to the same representation of $U(N)$, the factor N^2 will be dropped. We start with the ground state ($n = 0$). In the NS sector this is a "photon" with $D - 2 = 8$ physical degrees of freedom. In the R sector there is a spinor (32 components) subjected to two conditions, Majorana and Weyl, which decrease each by half the number of independent components. So the degeneracy $d(n)$ for $n = 0$ is 8 both for fermions and bosons.

We now compute the partition function of the Fermi sector for $D = 10$. Since we count only the coupled (i.e., transverse states), the partition function is obtained by using the counting operator made of transverse oscillators

$$\mathcal{P} = \prod_{n=1}^{\infty} n (\tilde{a}_n^+ \cdot \tilde{a}_n + \tilde{d}_n^+ \cdot \tilde{d}_n) \quad (2.62)$$

$$(\tilde{a}_n^+ \cdot \tilde{a}_n = \prod_{i=1}^{\infty} a_n^{+i} a_n^i , \text{ etc. })$$

Since the ground state fermion is eight-fold degenerate, one has:

$$f_R(q) = \sum_{n=0}^{\infty} d_R(n) q^{2n} = 8 \operatorname{Tr}(q^{2R}) = 8 \prod_{m=1}^{\infty} (1 - q^{2m})^{-8} (1 + q^{2m})^8 \quad (2.63)$$

A similar computation for the $G = +1$ sector of the NS model gives with

$$R = \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n + \sum_{r=1/2}^{\infty} r b_r^\dagger \cdot b_r$$

$$\begin{aligned} f_{NS}(q) &= \sum_{n=0}^{\infty} d_{NS}(q) q^{2n} \\ &= \frac{1}{q} \operatorname{Tr} \left[\left(\frac{1+G}{2} \right) q^{2R} \right] \end{aligned}$$

$$= \prod_{n=1}^{\infty} (1 - q^{2n})^{-8} \frac{1}{2q} \left\{ \prod_{n=1}^{\infty} (1 + q^{2n-1})^8 - \prod_{n=1}^{\infty} (1 - q^{2n-1})^8 \right\} \quad (2.64)$$

The equality of these two expressions was known to Jacobi³⁰⁾ who called it "aequatio identica satis abstrusa". For us its meaning is less abstruse as it means that $d_{NS}(n) = d_R(n)$ at each level, and therefore an important necessary condition for a linear realization of supersymmetry is satisfied.

A picture of the NSR spectrum is given in Fig. 4.

To prove supersymmetry completely, it remains to define supersymmetric transformations which exchange the NS and R states, and we plan to come back to this subject in a later publication.

3. SUPERSYMMETRIC YANG-MILLS THEORY AND THE DUAL SPINOR MODEL

We shall not review the arguments¹⁵⁾ which lead one to conclude that in the $\alpha' = 0$ limit the dual spinor model on-shell amplitudes in the tree approximation are identical with those of the (10 dimensional) Yang-Mills theory. These are similar to those used in the Veneziano model to prove the connection with the Yang-Mills theory without fermions. So we shall simply list the main ingredients of the proof:

- a) the fact that in the $\alpha' = 0$ limit dual diagrams turn into a sum of Feynman diagrams, each Feynman diagram being obtained with the correct weight³¹⁾;
- b) the dimensionality of the couplings obtained when $\alpha' = 0$ (only the dual coupling constant g remains and it is identified with the Yang-Mills coupling constant);
- c) the on-shell gauge invariance of the dual S matrix ($\delta S = 0$ when $\delta \epsilon_\mu = \alpha p_\mu$) which implies the non-Abelian Yang-Mills invariance;
- d) the invariance of the dual S matrix under the $U(N)$ group introduced by the Chan-Paton factors, the gauge group G of Yang-Mills being identified with $U(N)$. Because of duality requirement the gauge coupling constant g' of the $U(1)$ subgroup of $U(N)$ is identical with g the gauge coupling constant of the $SU(N)$.

The only non-trivial point which requires a bit of care is the first one. It actually requires that each external line of a Feynman graph can be obtained either from a twisted or untwisted line in a dual diagram. So it needs the definition of a fermion twist operator. For instance, the photon-fermion dual diagram of Fig. 5a should be accompanied with the analogous diagram of Fig. 5b. It is worth while to notice that it leads to problems when the $G = -1$ sector is included²⁷⁾, but that the problems disappear when this sector is removed and one can obtain a consistent dual definition of the graph of Fig. 5b.

So the theory one obtains in the zero slope limit is simply the Yang-Mills theory with fermions:

$$\mathcal{L} = \text{Tr} \left(-\frac{1}{4} G_{\mu\nu} G^{\mu\nu} - \frac{1}{2} i \bar{\lambda} \gamma^\mu \mathcal{D}_\mu \lambda \right) \quad (3.1)$$

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig [A_\mu, A_\nu] \quad (3.2)$$

$$A_\mu = A_\mu^0 \frac{1}{\sqrt{N}} + \sum_{i=1}^{N^2-1} A_\mu^i \chi^i \quad (3.3)$$

$$\mathcal{D}_\mu \lambda = \partial_\mu \lambda + ig [A_\mu, \lambda] \quad (3.4)$$

The X^i are the generators of $SU(N)$ in the $N \times N$ representation, normalized such that $\text{Tr}(X_i X_j) = \delta_{ij}$. Note that the $U(1)$ vector boson A_μ^0 is in fact a free field.

All indices are 10 dimensional and the γ matrices are defined as in the previous Section; the λ being 32 dimensional Majorana-Weyl spinors:

$$\lambda^* = \lambda \quad (3.5)$$

$$\gamma'' \lambda = \lambda. \quad (3.6)$$

We now show that the action $S = \int d^{10}x \mathcal{L}$ is invariant under global supersymmetric transformations in 10 dimensions. One indeed checks that if ϵ is a constant anticommuting Majorana-Weyl spinor, δS vanishes under the following infinitesimal transformation:

$$\delta A_\mu = -i \bar{\epsilon} \gamma_\mu \lambda \quad (3.7)$$

$$\delta \lambda = G_{\mu\nu} \sigma^{\mu\nu} \epsilon \quad (3.8)$$

where

$$\sigma^{\mu\nu} = \frac{1}{4} [\gamma^\mu, \gamma^\nu] \quad (3.9)$$

This transformation of the fields is the same as in four dimensions. The only difference in proving the vanishing of δS for four and 10 dimensions is that now one has to use the following identity:

$$\gamma^\mu \sigma^{\alpha\beta} = \frac{1}{2} [g^{\mu\alpha} \gamma^\beta - g^{\mu\beta} \gamma^\alpha - \frac{1}{7!} \epsilon^{\mu\alpha\beta\lambda_1 \dots \lambda_7} \gamma'' \gamma^{\lambda_1} \dots \gamma^{\lambda_7}] \quad (3.10)$$

where $\epsilon^{\mu_1 \dots \mu_{10}}$ is the completely antisymmetrized tensor in 10 dimensions.

One also verifies that the supersymmetric algebra closes on fields which satisfy the equations of motion:

$$[\delta_1, \delta_2] \lambda = 2i (\bar{\epsilon}_1 \gamma^\nu \epsilon_2) \mathcal{D}_\nu \lambda \quad (3.11)$$

$$[\delta_1, \delta_2] A_\mu = 2i (\bar{\epsilon}_1 \gamma^\nu \epsilon_2) G_{\nu\mu} \quad (3.12)$$

These commutation relations are the same as in four dimensions and therefore have the same interpretation. The non-trivial step to prove (3.12) is to use the 10 dimension Fierz transformation. It is actually rather simple because the spinors ϵ_1, ϵ_2 are both left handed. In that case the following formula holds:

$$\begin{aligned} (\bar{u}_1 \gamma_\nu u_2) \gamma^\nu u_3 &= \frac{1}{2} (\bar{u}_1 \gamma_\alpha u_3) \gamma^\alpha u_2 \\ &- \frac{1}{4} (\bar{u}_1 \sigma_{\alpha\beta\gamma} u_3) \sigma^{\alpha\beta\gamma} u_2 \end{aligned} \quad (3.13)$$

if $\gamma_{11} u_i = u_i$; $i = 1, 2, 3$. $\sigma_{\alpha\beta\gamma}$ is the completely antisymmetrized product of 3 γ matrices. The contribution of the second term vanishes in Eq. (3.11) because of the Majorana property.

So the 10 dimensional Yang-Mills theory with fermions in the regular representation of the group is automatically supersymmetric provided that the fermions are Majorana-Weyl.

We now show that a very interesting four dimensional action can be obtained from this 10 dimensional formalism. In the field theory we can assume that the extra six co-ordinates, x^{3+i} ($i = 1, \dots, 6$) are compact. A simple compactification is obtained by introducing six radii, R_i such that $0 \leq x^{3+i} \leq R_i$, all fields being periodic in x^{3+i} between 0 and R_i . A Fourier expansion of the 10 dimensional fields reveals that each of them represents an infinite number of four dimensional fields, of quantized masses given by $m^2(n_1, \dots, n_6) = 4\pi^2 \sum_1^6 n_i^2 / R_i^2$. Assuming the R_i to be small, we take in the Fourier expansion only the mode corresponding to $n_1 = n_2 = \dots = n_6 = 0$, i.e., we drop all the dependence of the fields upon the six extra co-ordinates. In this way, up to a rescaling of the fields and of the coupling constant g we obtain a four dimensional local action with a finite number of massless fields.

This action can also be obtained by a direct zero slope limit of the dual spinor model in the following way¹⁷⁾: quantize all momenta of the particles in the dual model by $p^{3+i} = 2\pi(n^i/R_i)$. The six n^i appear as six additive conserved quantum numbers. As α' goes to zero, let us also take R_i to zero with fixed ratios $\gamma_i = \alpha'/R_i^2$. Then one obtains the four dimensional action we are going to write down.

The non-trivial character of the four-dimensional action we obtain in this way is due to the fact that the components A_{3+i} are independent of the x^{3+i} , but are non-vanishing, so that three pseudoscalar fields $B_i(x) = A_{3+i}(x)$ ($i = 1,2,3$) and three scalar fields $A_i(x) = A_{6+i}(x)$ ($i = 1,2,3$) appear associated with the vector field $A_\mu(x)$ ($\mu = 0,1,2,3$). Further, the Majorana-Weyl spinor $\lambda(x)$ describes four Majorana spinors $\lambda_K(x)$ in four dimensions, the internal symmetry index K running from 1 to 4. The supersymmetry transformation parameter ϵ describes also four Majorana spinor parameters ϵ_K . So we obtain an action which contains one vector multiplet and three scalar multiplets, with only one coupling constant g ; invariant under a supersymmetry transformation with an internal symmetry index. In fact, the action is not only supersymmetric, but hypersymmetric in the sense defined by Fayet²³). It is straightforward to derive the action from Eq. (3.1) together with the representation of the γ matrices written in Section 2, and we shall only give the result where all symbols have a conventional, four-dimensional meaning:

$$\begin{aligned} \mathcal{L} = & \text{Tr} \left\{ -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} - \frac{1}{2} (\mathcal{D}_\mu A_i)^2 - \frac{1}{2} (\mathcal{D}_\mu B_i)^2 \right. \\ & + \frac{g^2}{4} \left([A_i, A_j]^2 + [B_i, B_j]^2 + 2[A_i, B_j]^2 \right) \\ & - \frac{i}{2} \bar{\lambda}_K \gamma^\mu \mathcal{D}_\mu \lambda_K \\ & \left. + \frac{g}{2} \bar{\lambda}_{K\ell} \left[(\alpha_{K\ell}^j A_j + \beta_{K\ell}^j \gamma^5 B_j), \lambda_\ell \right] \right\} \end{aligned} \quad (3.14)$$

In this action, K, ℓ indices run from 1 to 4, i and j indices from 1 to 3. The six matrices α^i and β^i satisfy the following algebra:

$$\{ \alpha^i, \alpha^j \} = \{ \beta^i, \beta^j \} = -2 \delta^{ij} \quad (3.15)$$

$$[\alpha^i, \beta^j] = 0 \quad (3.16)$$

they are antisymmetric and real, and are related to the previously defined matrices ρ_i and ζ_i :

$$\beta_i = \rho_i \quad \alpha_j = -\rho_3 \zeta^j \quad (3.17)$$

An explicit representation of these matrices is given in terms of Pauli matrices:

$$\beta^1 = \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \quad \beta^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \beta^3 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}$$

$$\alpha^1 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} \quad \alpha^2 = \begin{pmatrix} 0 & -\sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \quad \alpha^3 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}$$

(3.18)

Before we discuss the full invariance of the Lagrangian, let us consider a few simple examples.

- Let $A_i = B_i = 0$, and only one of the form λ_K non-vanishing, then we get

$$\mathcal{L} = \text{Tr} \left(-\frac{1}{4} G_{\mu\nu} G^{\mu\nu} - \frac{1}{2} \bar{\lambda} \gamma^\mu \mathcal{D}_\mu \lambda \right) \quad (3.19)$$

which is the supersymmetric (four-dimensional) non-Abelian vector multiplet action ²²⁾.

- Let us choose $\lambda_3 = \lambda_4 = 0$; λ_1, λ_2 non-vanishing. Choose $A_1 = A_2 = B_1 = B_2 = 0$, $A_3 = A, B_3 = B$ non-vanishing. Using the explicit representation of the α, β matrices, we see that we obtain

$$\mathcal{L} = \text{Tr} \left(-\frac{1}{4} G_{\mu\nu} G^{\mu\nu} - \frac{1}{2} (\mathcal{D}_\mu A)^2 - \frac{1}{2} (\mathcal{D}_\mu B)^2 \right. \\ \left. + \frac{g^2}{2} [A, B]^2 - \frac{i}{2} \bar{\lambda}_1 \gamma^\mu \mathcal{D}_\mu \lambda_1 - \frac{i}{2} \bar{\lambda}_2 \gamma^\mu \mathcal{D}_\mu \lambda_2 \right. \\ \left. + g \bar{\lambda}_1 [(A + \gamma^5 B), \lambda_1] \right) \quad (3.20)$$

This is precisely the action for a vector multiplet in interaction with a massless scalar multiplet ²²⁾, both belonging to the regular representation of the gauge group. This action was shown to have a hidden $SU(2) \otimes U(1)$ symmetry ²³⁾ by Fayet, and the supersymmetric invariance with internal indices one derives from this was called by him "hypersymmetry".

We now return to our action of Eq. (3.14) and list its invariances.

- a) Non-Abelian gauge invariance with respect to $U(N)$.

$$\delta A_\mu = \partial_\mu \Lambda + ig [A_\mu, \Lambda]$$

$$\delta A_j = ig [A_j, \Lambda] \quad j=1,2,3$$

Same for B_j, λ_κ .

- b) Invariance under supersymmetric $O(6) \sim SU(4)$ transformations, the supersymmetric transformation parameters being four Majorana spinors ϵ

$$\delta A_\mu = -i \bar{\epsilon}_\kappa \gamma_\mu \lambda_\kappa \quad \kappa=1,2,3,4.$$

$$\delta A_i = -i \bar{\epsilon}_\kappa (\alpha^i)_{\kappa\epsilon} \lambda_\epsilon$$

$$\delta B_i = -i \bar{\epsilon}_\kappa (\beta^i)_{\kappa\epsilon} \gamma^5 \lambda_\epsilon$$

$$\begin{aligned} \delta \lambda_\kappa = & \left\{ G_{\mu\nu} \sigma^{\mu\nu} + \gamma^\mu \partial_\mu (\alpha^i A_i + \gamma^5 \beta^i B_i) \right. \\ & + ig [B_i, B_j] \sigma^{ij} + ig [A_i, A_j] \sigma'^{ij} \\ & \left. + 2ig [B_i, A_j] k^{ij} \gamma^5 \right\} \epsilon_\kappa \end{aligned}$$

where

$$\sigma^{ij} = \frac{1}{4} [\beta^i, \beta^j]$$

$$\sigma'^{ij} = \frac{1}{4} [\alpha^i, \alpha^j] \quad k^{ij} = \frac{1}{4} \{\alpha^i, \beta^j\}$$

- c) Invariance under $SO(6) \sim SU(4)$ transformations [or $SU(2) \otimes SU(2)$ if the γ^5 transformations are not considered]

$$\delta A_\mu = 0$$

$$\delta \lambda_\kappa = \frac{1}{2} [\sigma^{ij} \Lambda_{ij} + \sigma'^{ij} \Lambda'_{ij} + \gamma^5 \kappa^{ij} \tilde{\Lambda}_{ij}]_{\kappa e} \lambda_e$$

$$\delta A_i = \Lambda'_{ij} A_j + \tilde{\Lambda}_{ie} B_e$$

$$\delta B_i = \Lambda_{ij} B_j + \tilde{\Lambda}_{ie} A_e$$

It is obvious that to derive the results, the notation of (3.1) is both elegant and time saving. The $O(6)$ invariance is trivially a consequence of the 10-dimensional Lorentz invariance of (3.1) which is broken down to four-dimensional Lorentz invariance times $O(6)$ by the ansatz that $\partial_{3+i} = 0$.

4. THE SPECTRUM OF THE CLOSED STRINGS OF THE DUAL SPINOR MODEL

The equations of motion of a classical spinning string can be derived from a well defined action which exhibits local supersymmetry in the two-dimensional surface spanned by the string¹⁹⁾. In a specific orthonormal gauge, these equations read:

$$\square \phi^\mu = 0 \tag{4.1}$$

$$\gamma^i \partial_i \psi^\mu = 0 \tag{4.2}$$

where μ is the space-time index ($\mu = 0, 1, \dots, 9$), $i = 1, 2$ referring to the two-dimensional surface. The coordinates on the surface being $y^0 = \tau$, $y^1 = \sigma$. ψ^μ is a two-dimensional spinor

$$\psi^\mu = \begin{pmatrix} \psi_1^\mu \\ \psi_2^\mu \end{pmatrix}$$

Using the representation of the γ matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

these equations are easily solved as:

$$\phi^{\mu} = \phi^{\mu}(\sigma + \tau) + \bar{\phi}^{\mu}(\sigma - \tau) \quad (4.3)$$

$$\psi_1^{\mu} = \psi_1^{\mu}(\sigma + \tau) \quad (4.4)$$

$$\psi_2^{\mu} = \psi_2^{\mu}(\sigma - \tau) \quad (4.5)$$

We wish now to discuss the boundary conditions for a closed spinning string. For the closed string one must have

$$\phi^{\mu}(\sigma, \tau) = \phi^{\mu}(\sigma + 2\pi, \tau) \quad (4.6)$$

as ϕ^{μ} represents the position of the string in space-time. Thus in the expansion of ϕ^{μ} in Fourier series, one obtains two sets of modes, α_n^{μ} [expansion in $e^{-in(\tau+\sigma)}$], $\bar{\alpha}_n^{\mu}$ [expansion in $e^{-in(\tau-\sigma)}$] with α_0^{μ} being related to the momentum of the whole string.

For the fermionic fields ψ_1, ψ_2 , there is however an arbitrariness. Looking at the gauge conditions, one can see that it would be inconsistent to identify $\psi_1(\sigma = 0)$ with $\psi_2(\sigma = 2\pi)$. However, there is a sign ambiguity in identifying $\psi_1(\sigma = 0)$ with $\psi_1(2\pi)$, and a similar one arises for ψ_2 . Thus one obtains three possible cases:

$$\text{a) } \left. \begin{aligned} \psi_1(\sigma = 0) &= -\psi_1(\sigma = 2\pi) \\ \psi_2(\sigma = 0) &= -\psi_2(\sigma = 2\pi) \end{aligned} \right\} \quad (4.7)$$

$$\text{b) } \left. \begin{aligned} \psi_1(\sigma = 0) &= -\psi_1(\sigma = 2\pi) \\ \psi_2(\sigma = 0) &= +\psi_2(\sigma = 2\pi) \end{aligned} \right\} \quad (4.8)$$

$$c) \left. \begin{aligned} \psi_1(\sigma=0) &= + \psi_1(\sigma=2\pi) \\ \psi_2(\sigma=0) &= + \psi_2(\sigma=2\pi) \end{aligned} \right\} (4.9)$$

Expanding ψ_1, ψ_2 into a Fourier series:

$$\psi_1 = \sum_n c_n \exp[-in(\sigma + \tau)] \quad (4.10)$$

$$\psi_2 = \sum_n \bar{c}_n \exp[-in(\tau - \sigma)] \quad (4.11)$$

we see that in the first case both oscillators c_n, \bar{c}_n are half-integrally moded, so that the model which results upon quantization is a doubling of the Neveu-Schwarz model. It has already been studied in detail in Refs. 32) and 33), and it appears in loop diagrams as bound states of the $G = +1$ sector of the Neveu-Schwarz model. The particles it describes are bosonic and it contains a massless graviton.

In the second case, the c_n oscillators are of the Neveu-Schwarz type (half-integrally moded), but the \bar{c}_n are of the Ramond type (integrally moded). This model describes fermions as the c_0^μ modes must be identified with the (32-dimensional) γ^μ matrices. It has not so far been studied and should, in principle, arise as a bound state of the interacting Neveu-Schwarz and Ramond models, through the graph of Fig. 2 which has not yet been computed due to the difficulties in defining a fermion twist operator. Now that these conceptual difficulties are removed, it should become possible to compute this graph and to verify this conjecture. We are going to see that the ground state of this model contains a massless spin 3/2 particle.

Finally, in the third case we get a model obtained by doubling the Ramond model which should describe bosons in a Kemmer-Duffin-like formalism. However, this sector is known to be obtained in loop diagrams as bound states of the $G = -1$ sector of the Neveu-Schwarz model³²⁾. Since this sector is eliminated by the Weyl condition, the model with two sets of Ramond oscillators should not appear as a bound state in our supersymmetric dual model, and we shall disregard it.

We would like now to discuss model (b) and its spectrum. To make the discussion more transparent we will first recall how model (a) is defined and what its spectrum is.

The Neveu-Schwarz model was defined through the equations:

$$L_n |\phi\rangle = 0; n > 0 \quad (L_0 - \frac{1}{2}) |\phi\rangle = 0 \quad (4.12)$$

$$G_r |\phi\rangle = 0 \quad r = \frac{1}{2}, \frac{3}{2}, \dots$$

These equations define a ghost-free sector of $D \leq 10$. Suppose now we want to "double" this model. We will get a ghost-free spectrum if we double all these conditions, i.e., assume that:

$$L_n |\phi\rangle = \bar{L}_n |\phi\rangle = 0 \quad n = 1, 2, \dots$$

$$G_r |\phi\rangle = \bar{G}_r |\phi\rangle = 0 \quad r = \frac{1}{2}, \frac{3}{2}, \dots \quad (4.13)$$

$$(L_0 - \frac{1}{2}) |\phi\rangle = (\bar{L}_0 - \frac{1}{2}) |\phi\rangle = 0$$

The last equation implies, in particular, that $(L_0 - \bar{L}_0) |\phi\rangle = 0$. As in the Neveu-Schwarz model, the ground state $(|0\rangle)$ is a tachyon, with $\alpha' m^2 = -1$. The intercept of the tachyon trajectory is $\alpha(0) = 1$. One has:

$$L_0 = \frac{1}{2} \alpha' p^2 + \sum_n n a_n^+ \cdot a_n + \sum_r r b_r^+ \cdot b_r \quad (4.14)$$

$$\bar{L}_0 = \frac{1}{2} \alpha' p^2 + \sum_n n \bar{a}_n^+ \cdot \bar{a}_n + \sum_r r \bar{b}_r^+ \cdot \bar{b}_r \quad (4.15)$$

As in the NS model one can define a conserved multiplicative quantum number G_p which distinguishes two sectors in this model:

$$G_p = (-1)^{\sum_m b_m^+ b_m - 1} = (-1)^{\sum_m \bar{b}_m^+ \bar{b}_m - 1} \quad (4.16)$$

The last equality follows from the fact that for on-shell states $L_0 - \bar{L}_0 = 0$. G_p should not be confused with the G operator of the Neveu-Schwarz model for open strings.

So we see that we have two sectors in that model, one with $G_p = -1$ which has intercept 1 and contains a tachyon, and one with $G_p = +1$. The ground state of the $G_p = +1$ sector is represented by:

$$b_{\frac{1}{2}}^{+\mu} \bar{b}_{\frac{1}{2}}^{+\nu} |0\rangle \epsilon_{\mu\nu} \quad (4.17)$$

This state has zero mass. We see that because of the Eqs. (4.13), $\epsilon_{\mu\nu}$ is subjected to two conditions:

$$p^\mu \epsilon_{\mu\nu} = p^\nu \epsilon_{\mu\nu} = 0 \quad (4.18)$$

$\epsilon_{\mu\nu}$ is neither symmetric, nor antisymmetric, nor traceless. So it describes:

- a massless antisymmetric tensor $A_{\mu\nu} = \frac{1}{2}(\epsilon_{\mu\nu} - \epsilon_{\nu\mu})$;
- a massless scalar $\epsilon_{\mu\mu}$ (spin 0);
- a massless symmetric tensor (spin 2): $e_{\mu\nu} = \frac{1}{2}(\epsilon_{\mu\nu} + \epsilon_{\nu\mu})$ ($\epsilon_{\mu\nu}$ traceless).

We have refrained from giving the spin of $A_{\mu\nu}$. In 10 dimensions it represents $8.7/2 = 28$ physical degrees of freedom, while in 4 dimensions it represents $2.1/2 = 1$ physical degree of freedom and can in fact be identified with a pseudo-scalar, as we are going to see in a moment.

The transversality of $A_{\mu\nu}$ and $e_{\mu\nu}$ follows from the fact that if we replace $\epsilon_{\mu\nu}$ by $\epsilon_{\mu\nu} + p_\mu \zeta_\nu + p_\nu \bar{\zeta}_\mu$ where ζ_ν and $\bar{\zeta}_\nu$ are two vectors subject to the condition $p \cdot \zeta = p \cdot \bar{\zeta} = 0$, the variation of the dual S matrix will be zero as it amounts to replacing the massless tensor state by

$$G_{-\frac{1}{2}} \bar{b}_{\frac{1}{2}}^{+\nu} |0\rangle \zeta^\nu + \bar{G}_{-\frac{1}{2}} b_{\frac{1}{2}}^{+\nu} |0\rangle \bar{\zeta}^\nu \quad (4.19)$$

and as this state is physical and spurious, it will be decoupled.

It would seem that for the symmetric part of $\epsilon_{\mu\nu}$, for instance, we get a gauge invariance which is less than general co-ordinate invariance since ζ^μ is constrained by the condition $p \cdot \zeta = 0$. But in fact the general equation derived from an on-shell spin 2 massless particle from the linearized Einstein action is $p^\mu e_{\mu\nu} = \frac{1}{2} p_\nu e_{\mu\mu}$. From Eq. (4.18) we see that the dual model picks up a particular gauge, namely $e_{\mu\mu} = 0$ to describe the graviton, and this is the reason why our on-shell gauge invariance looks restricted. This point was

discussed in detail in Refs. 16), and it was concluded that in fact general coordinate invariance is implied by the invariance of the dual S matrix under the restricted gauge transformation. Similarly, for the antisymmetric part, it was shown ¹⁶⁾ that it implies that the $A_{\mu\nu}$ field always appears through the gauge invariant, completely antisymmetric tensor:

$$F_{\mu\nu\rho} = \partial_\mu A_{\nu\rho} + \partial_\nu A_{\rho\mu} + \partial_\rho A_{\mu\nu} \quad (4.19)$$

Now we look at model (b). Again we shall obtain a ghost-free spectrum for $D \leq 10$, if we define as physical states those which satisfy both the Ramond and the Neveu-Schwarz conditions for physical states, i.e.,

$$F_n |\phi\rangle = 0 \quad n \geq 0 \quad (4.20)$$

$$L_n |\phi\rangle = 0 \quad n \geq 0 \quad (4.21)$$

$$\bar{G}_r |\phi\rangle = 0 \quad r = \frac{1}{2}, \frac{3}{2}, \dots \quad (4.22)$$

$$\bar{L}_n |\phi\rangle = 0 \quad n \geq 1 \quad (4.23)$$

$$(\bar{L}_0 - \frac{1}{2}) |\phi\rangle = 0 \quad (4.24)$$

A surprising feature obtained by comparing Eqs. (4.21) and (4.24) is that now

$$(L_0 - \bar{L}_0) |\phi\rangle = \frac{1}{2} |\phi\rangle \quad (4.25)$$

rather than zero. Thus the ground state in such a formalism is not the vacuum. Rather, it is given by

$$|\phi_0\rangle = \bar{b}_{\frac{1}{2}}^{\dagger\mu} |0\rangle \psi_\mu \quad (4.26)$$

which satisfies

$$L_0 |\phi_0\rangle = 0, \quad (\bar{L}_0 - \frac{1}{2}) |\phi_0\rangle = 0$$

for $p^2 = 0$. Hence it describes a massless state. It is a spin-vector state, as

the F operators contain γ matrices. The condition

$$F_0 |\phi_0\rangle = 0 \quad (4.27)$$

implies

$$\not{p} \psi_\mu = 0 \quad (4.28)$$

and the condition

$$\bar{G}_{\frac{1}{2}} |\phi_0\rangle = 0 \quad (4.29)$$

implies

$$p^\mu \psi_\mu = 0 \quad (4.30)$$

So the ground state is a massless spin-vector state ψ_μ subject to the conditions (4.28) and (4.30).

Now we relate this result with what is known about the Rarita-Schwinger equation which describes spin 3/2 particles.

For $D = 4$, the equation reads

$$\epsilon^{\mu\nu\rho\sigma} \gamma_5 \gamma_\rho \partial_\nu \psi_\sigma = (g^{\mu\nu} \gamma^\sigma - g^{\mu\sigma} \gamma^\nu + g^{\nu\sigma} \gamma^\mu - \gamma^\mu \gamma^\nu \gamma^\sigma) \partial_\nu \psi_\sigma = 0 \quad (4.31)$$

For $D \neq 4$, we cannot use the first form of the R.S. equation as it contains explicitly the $\epsilon^{\mu\nu\rho\sigma}$ symbol. However, we can use the second form which is independent of the space-time dimension. In the second form, it is still true that the equation is invariant under $\psi_\sigma \rightarrow \psi_\sigma + \partial_\sigma \epsilon$ whatever D . In this way one can obtain the properties of a spin 3/2 particle for any space-time dimension. Going to the momentum representation we obtain, by contraction with γ^μ , a system of two equivalent equations:

$$p \cdot \psi = \not{p} \gamma \cdot \psi \quad (4.32)$$

$$\not{p} \psi_\mu = p_\mu \gamma \cdot \psi \quad (4.33)$$

Assume now that ψ_μ is on-shell ($p^2 = 0$). Multiplying (4.33) by \not{p} we get

$$0 = p_\mu \not{p} \gamma \cdot \psi = p_\mu p \cdot \psi \quad (4.34)$$

Hence if $p_\mu \neq 0$, the generalized on-shell Rarita-Schwinger equation reduces to

$$p \cdot \psi = 0 \quad (4.35)$$

$$\not{p} \psi_\mu = p_\mu \gamma \cdot \psi \quad (4.36)$$

This system is still invariant under $\psi_\mu \rightarrow \psi_\mu + \epsilon p_\mu$: therefore we can further specify a gauge, for instance, $\gamma \cdot \psi = 0$. Then we get three simple equations:

$$p \cdot \psi = \gamma \cdot \psi = \not{p} \psi_\mu = 0 \quad (4.37)$$

Now we can compare these equations with those obtained for the ground state of the dual model. We see that in the dual model the condition $\gamma \cdot \psi = 0$ is missing. Therefore the ground state of this sector contains not only a spin 3/2 particle but also a spin 1/2 particle.

To separate more obviously the spin 3/2 and 1/2 content present in the dual spinor ψ_μ one can use a method which was used in Refs. 16) to separate the spin 2 and spin 0 content in $\epsilon_{\mu\nu}$. One introduces a momentum \bar{p}^μ conjugate to p^μ satisfying $p^2 = \bar{p}^2 = 0$; $p\bar{p} = 1$.

Then one defines the spin 1/2 part of ψ_μ as:

$$\psi_\mu^{\frac{1}{2}} = \frac{1}{\sqrt{D-2}} [\gamma_\mu - p_\mu (\bar{p} \cdot \gamma)] \chi \quad (4.38)$$

where χ is a Dirac spinor satisfying $\not{p}\chi = 0$, and notice that it satisfies the dual conditions $p^\mu \psi_\mu = 0$; $\not{p}\psi_\mu = 0$. Note that $\gamma \cdot \psi^{\frac{1}{2}} = (\sqrt{D-2})\chi$.

The spin 3/2 part of ψ_μ is defined as follows:

$$\psi_\mu^{\frac{3}{2}} = \psi_\mu - \frac{1}{2} p_\mu (\bar{p} \cdot \gamma) \gamma \cdot \psi \quad (4.39)$$

where ψ_μ satisfies the on-shell Rarita-Schwinger equation:

$$p \cdot \psi = 0 ; \quad \not{p} \psi_\mu = p_\mu \gamma \cdot \psi \quad (4.40)$$

one verifies easily that $\psi_\mu^{3/2}$ satisfies the dual equations, together with the additional condition

$$\gamma \cdot \psi^{3/2} = 0 \quad (4.41)$$

In this way one can explicitly separate the content in spins 3/2, 1/2 of an amplitude with an arbitrary number of external states with spin 3/2 or 1/2. Further, it can be shown that the dependence of these amplitudes upon the momenta \bar{p} will drop out so that the separation is in fact covariant.

Under a gauge transformation of $\psi \rightarrow \psi + p_\mu \epsilon$ where ϵ is an unrestricted spinor, $\psi_\mu^{3/2}$ transforms as

$$\delta \psi_\mu^{3/2} = p_\mu \epsilon' \quad (4.42)$$

with $\epsilon' = \epsilon - \frac{1}{2}(\bar{p} \cdot \gamma) \not{p} \epsilon$. Since $p^2 = 0$, ϵ' satisfy $\not{p} \epsilon' = 0$. Therefore

$$\delta |\phi_0\rangle = p \cdot \bar{b}_{\frac{1}{2}}^+ |\phi_0\rangle \epsilon' = G_{-\frac{1}{2}} |\phi_0\rangle \epsilon' \quad (4.43)$$

is a state which is both spurious and physical, and it will be decoupled in the S matrix. This allows us to assert that in the dual model the spin 3/2 particle will also be transverse.

It is now easy to introduce an interaction between bosonic and fermionic closed strings. As far as defining a self-interaction of bosonic closed strings, this was already done in Refs. 32) and 33). So we only need to define (in an analogous way to what one does in the NSR model) the vertices where a fermionic closed string line emits an arbitrary number of states of the bosonic closed strings. More specifically, we can require that the fermionic closed string line should emit an arbitrary number of massless states of the $G_p = +1$ sector of the bosonic closed string (graviton, scalar, antisymmetric tensor). Since the fermionic model is a product of the Neveu-Schwarz model in the \mathcal{F}_2 formalism by a Ramond model, the vertex for the emission of the tensor is constructed as a

product of the Neveu-Schwarz vertex for a photon emission (in the \mathcal{F}_2 formalism), by the vertex of the Ramond model for a photon emission. The selection of which formalism R_1 , or R_2 one makes for the Ramond part, depends upon the choice of the propagator. It seems most natural to use for propagator

$$P = \int_0^1 dx x^{\bar{L}_0 - \frac{1}{2} + L_0 - 1} \int_0^{2\pi} d\theta e^{i\theta(L_0 - \bar{L}_0 - \frac{1}{2})}$$

and this selects the R_1 formalism for the fermions.

So the vertex for a tensor emission will be given by:

$$V = \epsilon^{\mu\nu} V_\mu^{NS} V_\nu^R \quad (4.44)$$

where $\epsilon^{\mu\nu}$ is subjected to the gauge conditions $p^\mu \epsilon_{\mu\nu} = p^\nu \epsilon_{\mu\nu} = 0$ and

$$V_\mu^{NS} = :(\bar{P}_\mu - \bar{H}_\mu \frac{P \cdot \bar{H}}{P \cdot \bar{H}}) \exp i \frac{P}{2} \cdot \bar{Q} : \quad (4.45)$$

and

$$\begin{aligned} V_\mu^{R(1)} &= : (P_\mu + \frac{1}{2} \Gamma_\mu \frac{P \cdot \Gamma}{P \cdot \Gamma}) \exp i \frac{P}{2} \cdot Q : \\ &= \{ F_\mu, V_\mu^{R(2)} \} \end{aligned} \quad (4.46)$$

$$V_\mu^{R(2)} = : \Gamma_\mu \exp i \frac{P}{2} \cdot Q : \quad (4.47)$$

The peculiarity of the R_1 formalism is that all vertices, except the last one in the chain, are $V_\mu^{(1)}$ vertices; the last one being $V_\mu^{(2)}$.

So we get the expression for the process $3/2$ or $1/2 \rightarrow 3/2$ or $1/2 + N$ tensor states

$$\begin{aligned} A_N &= K^N \sqrt{\mathcal{V}_\alpha(z_i)} \langle 0; z_i | b_{\frac{1}{2}}^\alpha \epsilon^{\mu_1 \nu_1} V_{\mu_1}^{NS} V_{\nu_1}^{R(2)} \\ &\epsilon^{\mu_2 \nu_2} V_{\mu_2}^{NS} V_{\nu_2}^{R(1)} \dots \epsilon^{\mu_N \nu_N} V_{\mu_N}^{NS} V_{\nu_N}^{R(1)} b_{\frac{1}{2}}^{\alpha + \beta} | 0, z_i \rangle \mathcal{V}_\beta(z_i) \end{aligned} \quad (4.48)$$

where each x, θ integral is in fact extended to the whole complex $z = xe^{i\theta}$ plane to have non-planar duality.

We see that if the model is defined in this way, the $G_p = -1$ sector, which contains a tachyon, will never appear. Also we see that in such a process, the handedness of $\psi_\alpha(q_1), \psi_\beta(q_2)$ is the same, and to define both to be left- or right-handed is consistent with our elimination of the $G_p = -1$ sector. Similarly as in the Ramond model, scattering of two spin vector states can also be defined and the same troubles would arise if the $G_p = -1$ sector was included. It is also easy to predict the relative handedness between the open and closed string fermions: since the Noether current of the 10 dimensional Yang-Mills is

$$J^\mu = \gamma^\alpha \gamma^\beta \gamma^\mu \text{Tr}(\lambda F_{\alpha\beta}) \quad (4.49)$$

As we need a non-vanishing coupling $\bar{\psi}_\mu J^\mu$, it follows that the closed string fermions ψ_α must have the same handedness as the open string fermions λ . A word of caution, however, is needed: when we decompose ψ_μ into its spin 3/2 and spin 1/2 content, we easily see in Eq. (4.49) that ψ_μ and χ will have opposite handednesses. So what appears as same handedness in the dual formalism for closed string will appear as opposite in the field theory: again this result is consistent with the generalization to 10 dimensions of a coupling of the type $\bar{\psi}_\mu \gamma^\mu \chi$ present in supergravity.

As far as quantum numbers are concerned, as we are dealing with closed string states, all are $U(N)$ singlets, and the graviton and hemitron do not carry any internal quantum numbers. Further as the fermion-fermion scattering will automatically have s, t, u singularities, it is impossible to assign (at least in a trivial way) a fermionic number to the fermionic closed string states and one is led to make them Majorana fermions.

To summarize our model of closed string states consists of:

- a) a doubled Neveu-Schwarz model, from which the $G_p = -1$ sector has been eliminated, which contains no tachyon nor ghosts, is bosonic, and where the ground state is a massless graviton, a massless scalar, and a massless anti-symmetric tensor;
- b) a Neveu-Schwarz \otimes Ramond model describing Majorana fermions, which are left handed (in the dual notation), contains no tachyon nor ghosts and whose ground state is a massless hemitron (spin 3/2) and a massless scalar.

A pictorial description of this model is given in Fig. 6.

Let us start counting the physical degrees of freedom in such a model for bosons and fermions. Let us first do it for $D = 4$, which can be obtained either by defining the model for $D = 4$ or doing it for $D = 10$, but putting to zero all unwanted components (in the first case one drops the left-handedness condition).

Graviton	:	$\frac{(D-1)(D-2)}{2}$	$-1 = 2$	degrees of freedom:	e_{μ}^a	fields
Dilaton	:		1		A	
Antisym. tensor	:	$\frac{(D-2)(D-3)}{2}$	$= 1$		$A_{\mu\nu}$	B
Spin 3/2		$(D-3)2^{D/2} \frac{1}{2}$	$= 2$		ψ_{μ}	
Spin 1/2		$2^{D/2} \frac{1}{2}$	$= 2$		χ	

We see that again the numbers of fermions and bosons are equal. Further, one can group together the (e_{μ}^a, ψ_{μ}) and the (A, B, χ) into a gravitational and a "matter" scalar multiplet.

Let us now do the counting for $D = 10$, where the left-handed condition is imposed in addition to the Majorana condition.

$$e_{\mu}^a : 35 \quad A : 1 \quad A_{\mu\nu} : 28$$

$$\psi_{\mu} : 56 \quad \chi : 8$$

Again we get the same number of the degrees of freedom for bosons and fermions³⁴⁾ but we can no longer split these fields into two multiplets, so that all of them appear necessary to define "pure" supergravity in 10 dimensions. If one can define a supergravity theory in 10 dimensions (which looks now very plausible) by using the same process as in the 10 dimensional Yang-Mills theory, one will get "ordinary" supergravity in interaction with a very large number of matter multiplets. For instance, each 10 dimensional ψ_{μ} will appear as four dimensional $\psi_{\mu k}$ and a large number of spinors. Vector fields will appear from many sources, and one should recover as a particular case the recently discovered hypergravity with multiplets $(2, 3/2)$ and $(3/2, 1)$.

The general counting of fermions and boson states at each level will also work out if we notice that the fermionic closed strings also have their own G_p operator. This G_p operator is, in fact, the generalized γ^{11} operator: $\Gamma^{11} = \gamma^{11}(-1)^{\sum d_n}$. All states have $\Gamma^{11} = +1$, but massive states have both $\gamma^{11} = +1$ or -1 . The identity one has to prove is then easily deduced from the Jacobi identity.

Finally, we note that we have made many assumptions which could, in principle, all be derived from the loop computations for open strings, and which should be, in principle, checked explicitly in a unified model of open and closed strings. In such a model the constant κ which appears in Eq.(4.48) is in fact related to g through a relation of the type:

$$\kappa \sim g^2 \alpha'^{-1} \quad (4.50)$$

In order to have a four-dimensional interpretation of this equality, one compactifies six dimensions. Let V_0 be the volume of the six-dimensional compact space. Then κ_4 and g_4 (the four-dimensional observable coupling constants) are given by

$$\kappa_4 = \kappa (V_0)^{-\frac{1}{2}} ; \quad g_4 = g (V_0)^{-\frac{1}{2}} \quad (4.51)$$

So we obtain the relation

$$\kappa_4 \sim g_4^2 \frac{(V_0)^{\frac{1}{2}}}{\alpha'} \quad (4.52)$$

If $V_0 \sim \alpha'^3$ we get $\kappa_4 \sim g_4^2 (\alpha')^{\frac{1}{2}}$, i.e., α' is of the order of Planck's length. On the other hand, if there is an arbitrariness in the scale of V_0 , α' is more or less arbitrary. Note that Eq. (4.52) is very similar to the expansion for the Fermi constant $\sqrt{G_F}$ in a unified theory of weak and electromagnetic interactions. Only the power of g differs, and this is because in our case the graviton appears as a bound state of photons.

5. SUPERGRAVITY AND THE INTERACTING CLOSED STRINGS OF THE DUAL SPINOR MODEL

Now we consider the dual amplitudes for the scattering of closed string states. The external states are any of the ground states of the bosonic and fermionic closed string sectors, and depend upon the coupling constant α' of three closed strings and on α' . We ask whether in the limit $\alpha' \rightarrow 0$ these amplitudes can be recovered from a local action, and if this action has anything to do with supergravity. As we do not know (yet) the supergravity theory in 10 dimensions, we can restrict ourselves to establishing the connection with supergravity in four dimensions. To do this, we consider only the sector of the S matrix where the extra momenta are zero (after quantization), and further assume that all polarization tensors with extra indices vanish. As we have seen in Yang-Mills in 10 dimensions, in this case we must assume also that out of each of the four Majorana spinors which describe a 10-dimensional left-handed spinor, only one of them is non-vanishing:

$$\psi_{\mu\kappa} = \psi_{\mu} \delta_{\kappa 1} \quad (5.1)$$

$$\chi_{\kappa} = \chi \delta_{\kappa 1} \quad (5.2)$$

In this way we obtain a system of fields which are: e_{μ}^a (graviton), $A(x)$ (scalar), $A_{\mu\nu}(x)$ (pseudoscalar), $\psi_{\mu}(x)$ (spin 3/2), $\chi(x)$ (spin 1/2) where all symbols are four dimensional. We want to show that in the limit $\alpha' = 0$, this subsector of the dual S matrix interacts just like supergravity with a scalar multiplet.

The subsector of that system which consists of Bose fields and its zero slope limit was studied in Refs. 16). It was shown that the zero slope limit was correctly given by the action:

$$S = \int d^4x e \left[-\frac{1}{2\kappa^2} R(e) - \frac{1}{2} g^{\mu\nu} \partial_{\mu} A \partial_{\nu} A - \frac{1}{12} F_{\mu\nu\rho} F^{\mu\nu\rho} f(A) \right] \quad (5.3)$$

A few differences with respect to Refs. 16) are that we now use a vierbein formalism to describe the spin 2 (which does not affect the purely Bose part of the theory), rather than the metric formalism. Also we have rescaled the action so as to agree with the conventions of Refs. 10) and 11).

Now we explain why for $D = 4$ we can replace the $A_{\mu\nu}$ antisymmetric tensor by a pseudoscalar field $B(x)$. Let us define $B_{\mu} = 1/6 \epsilon_{\mu\alpha\beta\gamma} F^{\alpha\beta\gamma}$. As $F^{\alpha\beta\gamma}$ is completely antisymmetric, this formula can be inverted as $F_{\alpha\beta\gamma} = \epsilon_{\sigma\alpha\beta\gamma} B^{\sigma}$. So in the action we can replace $F_{\alpha\beta\gamma}$ by B^{σ} , provided that we take care of the fact that B_{μ} is constrained by the condition $D^{\mu} B_{\mu} = 0$. This we can do by adding to the action a Lagrange multiplier field $B(x)$. The action then becomes

$$S = \int d^4x e \left\{ -\frac{1}{2\kappa^2} R(e) - \frac{1}{2} g^{\mu\nu} \partial_{\mu} A \partial_{\nu} A + \right. \\ \left. + \frac{1}{2} B_{\sigma} B^{\sigma} f(A) - B^{\mu} \partial_{\mu} B \right\} \quad (5.4)$$

Using the equations of motion, we see that this is just the first-order formalism for a pseudoscalar field $B(x)$, the second-order formalism being obtained by replacing $B_{\sigma} f(A) = \partial_{\sigma} B$ in the action

$$S = \int d^4x e \left\{ -\frac{1}{2\kappa^2} R(e) - \frac{1}{2} g^{\mu\nu} \partial_{\mu} A \partial_{\nu} A \right. \\ \left. - \frac{1}{2} g^{\mu\nu} \partial_{\mu} B \partial_{\nu} B f^{-1}(A) \right\} \quad (5.5)$$

So indeed we can replace the formalism with an antisymmetric tensor by a formalism with a pseudoscalar, as was already noticed in Ref. 35). More generally suppose that in the action, additional terms of the form

$$S_{\pm} = - \int d^4x e F_{\mu\nu\rho} J^{\mu\nu\rho} \quad (5.6)$$

appeared with $J^{\mu\nu\rho}$ being a completely antisymmetric tensor depending on other fields. Then the second-order action we would get would be (dropping out the Einstein and kinetic term for the A field which are irrelevant for this problem:

$$S = \int d^4x e \left\{ -\frac{1}{2} g^{\mu\nu} \partial_{\mu} B \partial_{\nu} B f^{-1}(A) \right. \\ \left. + \partial_{\mu} B J^{\mu} f^{-1}(A) - J_{\mu} J^{\mu} f^{-1}(A) \right\} \quad (5.7)$$

where

$$J_{\mu} = \epsilon_{\mu\alpha\beta\gamma} J^{\alpha\beta\gamma} \quad (5.8)$$

So we see that if we have action, depending on $A_{\mu\nu}$, invariant under $\delta A_{\mu} = \partial_{\mu} \zeta_{\nu} - \partial_{\nu} \zeta_{\mu}$ such that only $F_{\alpha\beta\gamma}$ appears in the action and containing at most two derivatives, we can replace $A_{\mu\nu}$ by a pseudoscalar field $B(x)$. It is a pseudoscalar if $J^{\mu\nu\rho}$ is a true tensor, which will be the case. Further, we see that in the resulting action B appears only through its derivatives, and that new contact terms $(J_{\mu} J^{\mu})$ can arise from the transition from one formalism to the other.

Similarly, in the dual S matrix formulation, one can also replace $A_{\mu\nu}$ by a pseudoscalar. The gauge invariance $\delta A_{\mu\nu} = p_{\mu} \zeta_{\nu} - p_{\nu} \zeta_{\mu}$ is quite different from the graviton gauge invariance as there are no self couplings of the antisymmetric tensor $A_{\mu\nu}$. In fact there is no non-trivial antisymmetric conserved current to which $A_{\mu\nu}$ can couple, so that an S matrix amplitude can always be rewritten in such a way that only $F_{\mu\nu\rho}(p)$ appears in the external states. As before we can replace $F_{\mu\nu\rho}$ by $B_{\sigma}(p) = 1/6 \epsilon_{\sigma\alpha\beta\gamma} F^{\alpha\beta\gamma}$. Now B_{σ} can be expressed as p_{σ} multiplied by a pseudoscalar quantity. Indeed in four dimensions we can introduce a system of four vectors:

$$p_{\mu}; \bar{p}_{\mu}; \epsilon_{\mu}^i \quad i=1,2 \quad \text{such that} \quad p \cdot \bar{p} = 1$$

$$p \cdot \epsilon^i = \bar{p} \cdot \epsilon^i = \epsilon^1 \cdot \epsilon^2 = 0 \quad p^2 = \bar{p}^2 = 0 \quad \epsilon_1^2 = \epsilon_2^1 = 1$$

$A_{\mu\nu}$ can be represented in this basis of vectors. Using the condition that $p^{\mu} A_{\mu\nu} = 0$ one gets

$$A_{\mu\nu} = A(\epsilon_{\mu}^1 \epsilon_{\nu}^2 - \epsilon_{\nu}^1 \epsilon_{\mu}^2) + B \sum_{i=1}^2 (\epsilon_{\mu}^i p_{\nu} - \epsilon_{\nu}^i p_{\mu}) \quad (5.9)$$

and

$$B_{\sigma} = 2 \epsilon_{\sigma\alpha\beta\gamma} A \epsilon_1^{\alpha} \epsilon_2^{\beta} p^{\gamma} \quad (5.10)$$

expanding

$$B_{\sigma} = a p_{\sigma} + b \bar{p}_{\sigma} + \sum_{i=1,2} c_i \epsilon_{\sigma}^i \quad (5.11)$$

one sees that

$$b = c_1 = c_2 = 0$$

$$a = 2 \epsilon_{\sigma\alpha\beta\gamma} \epsilon_1^\alpha \epsilon_2^\beta p^\gamma \bar{p}^\sigma \quad (5.12)$$

and therefore

$$B_\sigma = p_\sigma (\epsilon^{\mu\alpha\beta\gamma} A^{\alpha\beta} p^\gamma \bar{p}^\mu) \quad (5.13)$$

This result is rather important as it shows that in four dimensions a lot of couplings of the $A_{\mu\nu}$ field vanish on the mass shell. To give an example, in Eq. (5.3) a function $f(A)$ was included because in general a coupling of order κ in $F_{\mu\nu\rho} F^{\mu\nu\rho} A$ is found, even on shell. However, in four dimensions this coupling vanishes on mass shell as it gives through the previous arguments a coupling proportional to $p^{(1)} \cdot p^{(2)} = 0$. So the function $f(A)$ can be got rid of (at least at first order in κ) by a field redefinition.

Now that the pseudoscalar character of $A_{\mu\nu}(x)$ is well established, we consider the zero slope limit of the dual model restricted to its four-dimensional sector. It is now obvious that one gets (up to canonical field redefinitions which do not affect the S matrix) the supergravity theory in interaction with the scalar multiplet written down in Refs. 10) and 11). This is because for $\alpha' = 0$ all the couplings have dimensions fixed by κ , and because we have both on-shell general covariance ($\epsilon_{\mu\nu} \rightarrow \epsilon_{\mu\nu} + p_\mu \zeta_\nu + p_\nu \zeta_\mu$) and local supersymmetry invariance ($\psi_\mu \rightarrow \psi_\mu + \epsilon p_\mu$), and the same system of local fields ($e_\mu^a, \psi_\mu, A, B, X$) as starting point. As explained in Section 5 of Ref. 10), on-shell gauge invariance of the S matrix is sufficient alone to generate all trilinear and higher couplings needed in the theory.

It is, however, both pedagogical and amusing to verify that the three-point couplings do agree. Also we shall learn about supergravity in 10 dimensions in doing so. As the three-point couplings for Bose fields have already been studied, we shall consider only the trilinear couplings where a fermionic line emits a boson. All these couplings are expressed by a single expression in the dual model:

$$V_3 = \kappa \bar{\psi}_\alpha(z_1) \langle 0, z_1 | b_{\frac{1}{2}}^\alpha \in^{M\mu} (p) V_\mu^{NS} V_\nu^{R_2} b_{\frac{1}{2}}^{\dagger\beta} | 0, z_2 \rangle \psi_\beta(z_2) \quad (5.14)$$

$$V_3 = \kappa \bar{\psi}_\alpha(z_1) [\delta_{\alpha\mu} p^\beta + \delta_{\beta\mu} z_2^\alpha + \delta_{\alpha\beta} z_1^\mu] \gamma^\nu \psi_\beta(z_2) \epsilon_{\mu\nu}(p) \quad (5.15)$$

The on-shell gauge invariance of this expression, both with respect to $\epsilon_{\mu\nu}$ and ψ_μ , is easy to verify. Note that the dependence on α' in V_3 disappears completely, so that the $\alpha' = 0$ limit is trivial on the trilinear couplings.

As ψ_μ represents both spin 3/2, 1/2 and $e_{\mu\nu}$ both spin 2 and two spin 0, the expression contains a lot of information. If we look at the vertex emission of a graviton ($\epsilon_{\mu\nu}$ symmetric and traceless) it is easy to verify that it couples to the energy momentum tensors of the spin 3/2 and spin 1/2 as it should. The graviton makes no transition between spin 3/2 and spin 1/2 for such a transition using the expression (4.38) for $\psi_\beta^{1/2}$, and the fact that $\psi^{3/2} \cdot \gamma = 0$, one has

$$V_3 \left(\frac{3}{2}; \frac{1}{2} \right) = \frac{\kappa}{\sqrt{D-2}} \bar{\psi}_\alpha^{\frac{3}{2}}(z_1) \gamma^\nu [\delta_{\alpha\mu} p \cdot \gamma + \gamma_\mu z_{2\alpha} \quad (5.16)$$

$$+ z_{1\mu} \gamma_\alpha] \chi \epsilon_{\mu\nu}(p) .$$

$$= \frac{\kappa}{\sqrt{D-2}} \bar{\psi}_\alpha^{\frac{3}{2}}(z_1) [2(z_1^\mu z_{2\nu} - z_2^\mu z_{1\nu}) - \eta^{\mu\nu} p^\alpha] \chi \epsilon_{\mu\nu} \quad (5.17)$$

In this expression the traceless symmetric part does not contribute. So only the antisymmetric tensor and the scalar contribute to the 3/2, 1/2 transition.

Before we look in greater detail into this transition we can look at the diagonal (3/2, 3/2) or (1/2, 1/2) processes. It is easy to check that the scalar A (i.e., the trace part of $\epsilon_{\mu\nu}$) gives vanishing contribution to these processes: therefore no term of the form $\bar{\psi}_\mu \dots \psi_\nu A$ or $\bar{\chi} \dots \chi A$ will appear in the action. The case of the antisymmetric part is much less obvious. If one looks, for instance, at a vertex like $A_{\mu\nu} \bar{\chi} \dots \chi$, one finds it a priori non-vanishing and proportional to the expression

$$A_{\mu\nu} \bar{\chi}(z_1) (\gamma^\mu z_2^\nu - \gamma^\nu z_2^\mu) \chi(z_2) \quad (5.18)$$

So a trilinear coupling of the form

$$A_{\mu\nu} \bar{\chi} (\gamma^\mu \partial_\nu - \gamma^\nu \partial_\mu - 2 \sigma^{\mu\nu} \not{\partial}) \chi \quad (5.19)$$

should appear in the action. However, for a Majorana spinor χ , this expression is equivalent to

$$-\frac{1}{2} A_{\mu\nu} \partial_\rho [\bar{\chi} \gamma^\mu \gamma^\nu \gamma^\rho \chi] \quad (5.20)$$

$$= \frac{1}{6} F_{\mu\nu\rho} (\bar{\chi} \gamma^\mu \gamma^\nu \gamma^\rho \chi) \quad (5.21)$$

by integrating by parts. The same manipulation can be made on the S matrix element which takes this form. In general ($D \neq 4$) this is non-vanishing even on-shell. For $D = 4$, however, it does, because it is proportional to $B_\sigma \bar{\chi} \gamma_5 \gamma^\sigma \chi$ and on-shell, as we have seen B_σ is proportional to p_σ . Using then the Dirac equation for $\chi, \bar{\chi}$, one obtains zero on-mass shell. The same result holds after a more complicated computation for the $3/2, 3/2, A_{\mu\nu}$ vertex: it does vanish on-shell, but only for $D = 4$.

Therefore the only non-vanishing couplings, apart from the gravitational couplings, are the $3/2 - 1/2 - 0^+$ and $3/2 - 1/2 - 0^-$ couplings. The first one is very easily seen to be proportional to $\bar{\psi}_\mu \beta A \gamma^\mu \chi$. For the second one, one has to use manipulations similar to the previous one, but one obtains a non-zero result, namely a $\bar{\psi}_\mu \gamma_5 \phi B \gamma^\mu \chi$ coupling. A careful study reveals that also the magnitudes of the coupling agree with those of Refs. 10) and 11).

A non-trivial check of our results is that the supergravity action in four dimensions, in interaction with a massless scalar multiplet, can be written in such a way that B appears only through its derivative. From the dual model we know that a consistent formalism in four dimensions would be to replace the scalar multiplet (A, B, χ) by an $(A, A_{\mu\nu}, \chi)$ multiplet. The gauge invariance $\delta A_{\mu\nu} = \partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu$ would imply, through the reasoning of the beginning of this Section, that in the (A, B, χ) formalism only $\partial_\sigma B$ appears.

Looking at the action first written in Ref. 10) and proved to be exact to all orders in Ref. 11), we see that there is only one offending term in the action which does not contain a derivative on B . It is [we make a temporary return to the notations and conventions of Ref. 10) to avoid notational confusion]:

$$-i \frac{k^2}{8} B \partial_\rho A [e \bar{\chi} \gamma_5 \gamma^\rho \chi - \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_\nu \psi_\sigma] \quad (5.22)$$

Integrating by parts, it gives a ∂_ρ^B term, and the still offending term:

$$\frac{i k^2}{4} AB \left[e \bar{\chi} \gamma_5 \gamma^\mu D_\mu \chi - \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_\nu D_\rho \psi_\sigma \right] \quad (5.23)$$

This term can be grouped with the kinetic terms of χ and ψ_μ and now reads:

$$\begin{aligned} & -\frac{1}{2} \bar{\chi} \left(1 - \frac{i k^2}{2} \gamma_5 AB \right) \gamma^\mu D_\mu \chi \\ & -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \left(1 + \frac{i k^2}{2} \gamma_5 AB \right) \gamma_\nu D_\rho \psi_\sigma \end{aligned} \quad (5.24)$$

We can now get rid of it through a field redefinition. Setting:

$$\chi' = \exp\left(-\frac{i k^2}{4} \gamma_5 AB\right) \chi \quad (5.25)$$

$$\psi'_\mu = \exp\left(\frac{i k^2}{4} \gamma_5 AB\right) \psi_\mu \quad (5.26)$$

One sees that one obtains for these four terms the expression

$$-\frac{1}{2} \bar{\chi}' \gamma^\mu D_\mu \chi' - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}'_\mu \gamma_\nu D_\rho \psi'_\sigma \quad (5.27)$$

Further, all other terms in the action turn into their counterparts. Apart from the Noether current, all bilinears appearing in the action are of the form $\bar{\psi}_\mu \gamma^\alpha \psi_\nu$ or $\bar{\psi}_\mu \gamma^\alpha \gamma^5 \psi_\nu$, or $\bar{\chi} \gamma^\alpha \chi$. For instance

$$\begin{aligned} \bar{\psi}_\mu \gamma^\alpha \psi_\nu &= \bar{\psi}'_\mu \exp\left(\frac{i k^2}{4} \gamma_5 AB\right) \gamma^\alpha \\ \exp\left(\frac{i k^2}{4} \gamma_5 AB\right) \psi'_\nu &= \bar{\psi}'_\mu \gamma^\alpha \psi'_\nu \end{aligned}$$

The only exception is the Noether current where an even number of γ 's appear. This is, however, just what is needed, as χ and ψ transform with the opposite sign in the exponential. So the supergravity in interaction with matter can indeed be written in the following way:

$$\begin{aligned}
 \mathcal{L} &= \mathcal{L}^G(e_r^a, \psi_r') + \mathcal{L}^M \\
 \mathcal{L}^M &= -\frac{1}{2} e g^{\mu\nu} (\partial_\mu A \partial_\nu A + \partial_\mu B \partial_\nu B) \\
 &+ \frac{1}{2} e \kappa \bar{\psi}' (\not{\partial} A + i \gamma_5 \not{\partial} B) \gamma^r \chi' - \frac{1}{2} e \bar{\chi}' \gamma^r D_r \chi' \\
 &+ \frac{\kappa^2}{32} (\bar{\chi}' \gamma_5 \gamma_T \chi') \left[\epsilon^{\mu\alpha\beta\tau} \bar{\psi}'_r \gamma_\alpha \psi'_b - 2e (\bar{\psi}'_\alpha \gamma_5 \gamma^\tau \psi'^\alpha) \right] \\
 &+ \frac{i\kappa^2}{4} A \partial_\rho B \left[e \bar{\chi}' \gamma_5 \gamma^\rho \chi' - \epsilon^{\alpha\beta\mu\tau} \bar{\psi}'_\alpha \gamma_\tau \psi'_\mu \right] \\
 &- \frac{e\kappa^2}{16} (\bar{\chi}' \chi')^2
 \end{aligned}$$

(5.28)

Using the Eqs. (5.3)-(5.5), it is now trivial to go from the (A,B,χ) formalism to the (A,A_{μν},χ) formalism. Note that the existence of that formalism, deduced from the dual model, gives an interesting insight as to why in the m = 0 case no infinite series occurred in the scalar multiplet action.

Now we notice that to obtain the (3/2, 2); (1/2, 0) supergravity theory we have enormously restricted the dual spectrum at zero mass. The O(2) supergravity theory⁶⁾ can also be obtained out of the dual S matrix. It contains one spin 2, two spin 3/2 and one spin 1. The spin 2 is simply obtained out of the vierbein components e_μ^a of the 10 dimensional theory with 0 ≤ μ, a ≤ 3. The 2 spin 3/2 are obtained by setting to zero only 2 of the 4 Majorana spin 3/2 which compose the 10 dimensional spin 3/2: ψ = (ψ_μ⁽¹⁾, ψ_μ⁽²⁾, 0, 0). The vector field A_μ is harder to obtain as it arises from various sources. It can come out of extra components of the vierbein, namely e_μ⁶, e_μ⁹ which are seen to make transitions between ψ_μ⁽¹⁾, ψ_μ⁽²⁾, from the vertex of Eq. (5.15). Actually, the computation of this vertex reveals that these components of the vierbein have the same Noether coupling as in Ref. 6). Similarly, the A_{μ6} or A_{μ9} fields are also candidates to contribute to A_μ. F_{μν9} = ∂_μA_{ν9} - ∂_νA_{μ9} is indeed a Maxwell tensor and one can verify that A_{μ6}, A_{μ9} couple again to ψ_ρ⁽¹⁾, ψ_ρ⁽²⁾ with the same Noether coupling as in Ref. 6). So the A_μ field is in fact a combination of those four fields. Once the Noether coupling is correctly obtained, the previous arguments imply that the quartic couplings are the same as in field theory.

By restricting less and less the dual spectrum one will obtain a model with 3 spin 3/2, 3 spin 1 and one spin 1/2 (Ref. 34)), and internal symmetry $O(4)$. Next a model with 4 spin 3/2, 6 vectors, 4 spin 1/2, and two spin zero, and internal symmetry $O(4)$. In this model which forms an irreducible representation of $O(4)$, the degrees of freedom inherent in the 10 dimensional model are not yet exhausted, as in principle we can have 6 vectors $e_{\mu i}$ and 6 vectors $A_{\mu i}$. So the largest model one can in principle obtain out of 10 dimensions will be $O(4)$ invariant but will contain several representations of $O(4)$ supergravity. Counting reveals that it will contain 6 times the representation of $O(4)$ supersymmetry with $J_{\max} = 1$, which itself contains one spin 1, four spin 1/2 and six spin 0. Even though these models have not yet been constructed, the dual theory is seen to imply their existence.

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FIGURE CAPTIONS

Fig. 1a : The duality equation for fermion-antifermion scattering.

1b : The duality equation for fermion-fermion scattering.

The \sim sign indicates that the equality holds up to parity reversal of the $G = -1$ states.

Fig. 2 : The non-planar fermionic loop, which should give fermionic closed string bound states.

Fig. 3a : Duality property for boson-boson scattering.

3b : Duality property for Majorana-Weyl fermion-fermion scattering.

Fig. 4 : The spectrum of the supersymmetric NSR model (open strings).

Fig. 5a : Two of the graphs which must contribute to boson-fermion
5b : scattering.

Fig. 6 : The spectrum of the supersymmetric NSR model of closed strings.



FIG. 1a

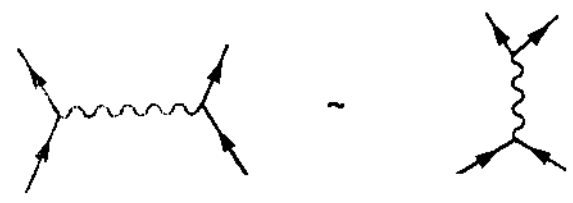


FIG. 1b

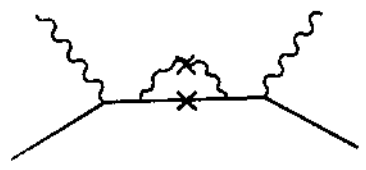


FIG. 2

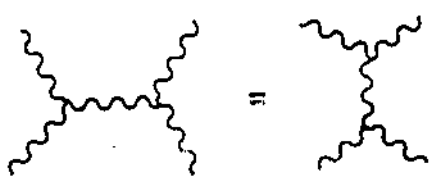


FIG. 3a

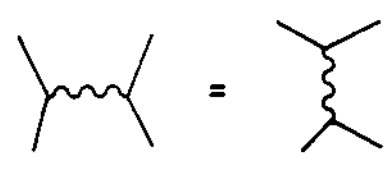


FIG. 3b

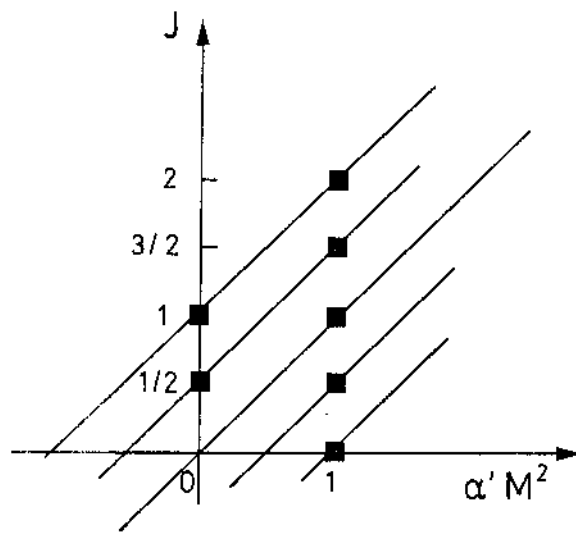


FIG. 4

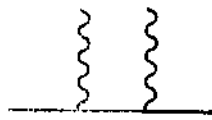


FIG. 5 a

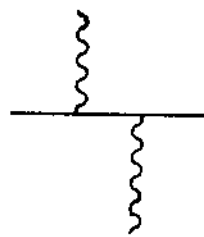


FIG. 5 b

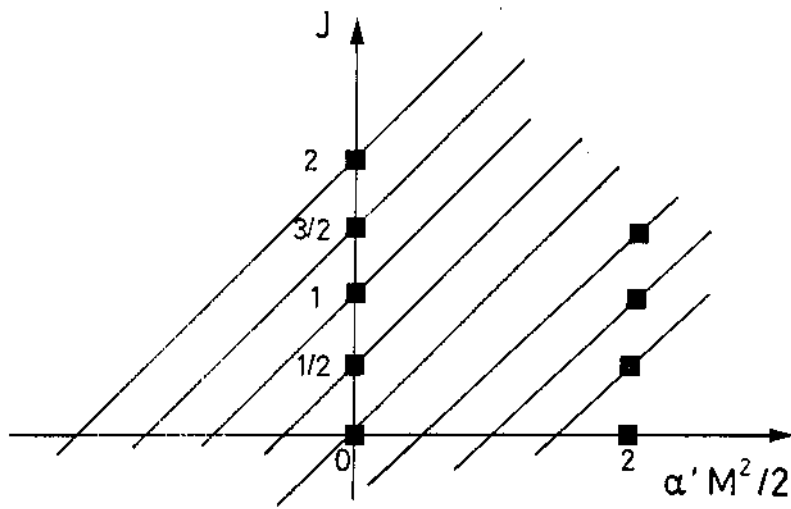


FIG. 6