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SUPPORTABILITY, SUSTAINABILITY AND SUBSIDY FREE PRICES

by

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In the case of a multiproduct firm the notions of subsidy free prices and anonymously equitable prices (see Faulhaber (1975), Faulhaber and Levinson (1981) and ten Raa (1982)) as well as the notions of supportability (see Sharkey and Telser (1978)) and sustainability (see Baumol, Bailey and Willig (1977), Panzar and Willig (1977) and Baumol, Panzar and Willig (1982)) are known to be closely related. Faulhaber and Levinson (1981) pointed out that supportability is necessary for the existence of an anonymously equitable price vector and ten Raa (1982) proved under general conditions that supportability is also sufficient. Sharkey and Telser (1978) argued that supportability is necessary for sustainability. They also suggested that supportability together with inelastic demands are sufficient for the existence of a sustainable price vector. It is shown in this paper that this is indeed the case when goods are weak gross substitutes but that a counter example can be constructed in the case in which goods are gross complements.

It is also shown that, in the multiproduct case where cost complementarities exist between outputs, any subsidy free price vector is anonymously equitable (and obviously vice versa). The second part of this paper deals with the relationship between anonymous equity and sustainability (see ten Raa (1982)). Indeed sufficient conditions are given for an anonymously equitable price vector to be sustainable. These sufficient conditions are used by Spulber (1983), in a sequel paper, to show that under local cost complementarities Aumann-Shapley sustainable prices exist.

Preliminaries Let E^n_+ denote the nonnegative orthant of n-dimensional Euclidian space. Consider a firm producing n infinitely divisible goods with a technology, expressed by a joint cost function $C: E_+^n \to E_+^1$ where $C(y)$ is the minimum cost of producing the output vector $y \in E_+^n$. Let $Q(p)$ be the inverse demand function, i.e., for each $p \in E_+^n$

$$Q(p) = (Q_1(p), \ldots, Q_n(p)) .$$

Denote by $N = \{1, \ldots, n\}$ the set of all goods and let $S \subseteq N$ be a subset of N. Let s denote the number of goods in S (or the cardinality of S) then, for a given $S \subseteq N$, y^S (or similarly $Q^S(p)$) and p^S are vectors in E_+^s denoting quantities and prices, respectively, of goods in S. For $S = N$ the subscript S is omitted. Thus y^S and p^S are the projections of y and p, respectively, on E_+^s. For convenience the notation $z|y^S$ with $y,z \in E^n$, will sometimes be used to denote the vector $(y^S, z^{N \setminus S})$ where $N \setminus S$ denotes the complement of S with respect to N, i.e., both $z|y^S$ and $(y^S, z^{N\setminus S})$ are the vector z except that the coordinates in S are replaced by y^S. Finally pz denote the inner product of p and z.

Definition:

A price vector p is subsidy free if

(i) $py = C(y)$

(ii) $p^S y^S \leq C(y^S)$ for each $S \subseteq N$ and

(iii) $y = Q(p)$

Definition:

The price vector p is anonymously equitable if

(i) $py = C(y)$

(ii) $pz \leq C(z)$ for any $z \in E_+^n$ such that $z \leq y$ and

(iii) $y = Q(p)$

Notice that any anonymously equitable price vector is subsidy free.

Definition:

The cost function C is supportable at y_0 if there exists a $p \in E_+^n$ such that

(i) $py_0 = C(y_0)$ and

(ii) $pz \leq C(z)$ for any $z \in E_+^n$ such that $z \leq y_0$

The cost function C is supportable if for any $y \in E_+^n$, $y \neq 0$, C is supportable at y. Thus if for some $p \in E_+^n$, C is supportable at Q(p) then p is an anonymously equitable price vector.

To define the notion of sustainability, consider a potential entrant having access to the same technology, expressed by the cost function C(y), as possessed by the firm and incurring zero entry and exit costs regardless of the goods and quantities produced. The entrant may produce any vector of quantities $\hat{y}^S$ of any subset $S \subseteq N$ of the goods at price $\hat{p}^S$. Panzar and Willig (1977) considered two types of entry behavior and their corresponding sustainability concepts. The first one is sustainability against partial entry.

Definition:

Sustainability against partial (quantity) entry. The price vector $\bar{p}$ is PE sustainable if every triple $(S,\hat{y}^S,\hat{p}^S)$ satisfying

(I) $$p^S \leq \bar{p}^S$$

and

(II) $$\hat{y}^S \leq Q^S(\hat{p}^S,\bar{p}^{N\setminus S})$$

also satisfies

$$\hat{p}^S\hat{y}^S - C(\hat{y}^S) \leq 0 .$$

Conditions (I) and (II) describe the behavior of a partial (quantity) entrant: for the goods in S, prices are offered which are not greater than those already prevailing in the market (condition I). At these prices any quantities up to those determined by the market demand functions evaluated at the new (lower) prices $\hat{p}^S$, for goods in S and the prevailing prices $\bar{p}^{N\setminus S}$, for he test of the goods (condition II) may be sold. Thus, $\bar{p}$ is PE sustainable if a potential entrant cannot anticipate positive profits by lowering some or all of the prices. The second sustainability concept is weaker and specifies that entrants must supply the entire market demand generated by the lower prices they offer:

Definition:

Sustainability against full (quantity) entry. The price vector $\bar{p}$ is FE sustainable if every triple $(S,\hat{y}^S,\hat{p}^S)$ satisfying (I) and

$$\text{(III)} \qquad \hat{y}^S = Q^S(\bar{p}|\hat{p}^S) \ ,$$

also satisfies,

$$\hat{p}^S\hat{y}^S - C(\hat{y}^S) \leq 0 \ .$$

Clearly, PE sustainability implies FE sustainability. Mirman, Tauman and Zang (1983) showed that under cost complementarity and weak gross substitutability the two notions are equivalent. To state this result consider the following two assumptions.

Assumptions

(a) Cost Complementarity. The cost function C is twice differentiable on $E^n_+\setminus\{0\}$ and $C_{\ell j} \leq 0$ where $C_{\ell j} = \frac{\partial^2 C}{\partial y_\ell \, \partial y_j}$

(b) Weak Gross Substitutability. For $j \in N$, $Q_j(\cdot)$ is differentiable on $E^n_+ \setminus \{0\}$ and $\frac{\partial Q_j}{\partial p_\ell} \geq 0$ for each $\ell \neq j$.

Proposition 1. (Mirman, Tauman and Zang (1983)). Let $p \in E^n_+$. Then under assumptions (a) and (b), p is FE sustainable if and only if it is PE sustainable.

II. Subsidy Free and Anonymous Equity

As Panzar and Willig (1977) observed, the core of a certain cost game plays an important role in the analysis of sustainability. However, there are two cost games which are of interest. To describe these games, let $y \in E^n_+$ be a vector of quantities of outputs. Output j, j = 1, ..., n is represented by an interval of size y_j--its quantity. Namely the first output is represented by the interval $[0,y_1]$, the second output by the interval $[y_1,y_1+y_2]$, etc. A player can be either one of the n intervals or an infinitesimal portion of one of the n intervals. This formulation leads to two games: the cost game between a finite number n of players and the cost game between a continuum of players.

In the first game each of the n intervals represents one player. The j-th player in this game is represented by the interval $[\sum_{k=1}^{j-1} y_k, \sum_{k=1}^{j} y_k]$, and a coalition in this game is a collection of these intervals. The cost game v_D (D stands for discrete) is defined for each $S \subseteq N$ by

$$v_D(S) = C(y^S) \ .$$

Namely the worth of a coalition S is the cost of producing the entire quantities vector y^S of the outputs in S.

The other game is the continuous game which is played among the infinitesimal players. In this game the coalitions are subsets of the interval $[0, \sum_{j=1}^{n} y_j]$. Each coalition S is associated with a vector $z = (z_1, \ldots, z_n) \in E_+^n$ where z is the portion of the j-th interval which belongs to S (i.e., z_j is the length of $S \cap [\sum_{k=1}^{j-1} y_k, \sum_{k=1}^{j} y_k]$). The continuous game v_C is defined by

$$v_C(S) = C(z) ,$$

i.e., $v_C(S)$ is the cost of producing the vector $z = (z_1, \ldots, z_n)$ of quantities associated with the coalition S. Thus, while the game v_D deals only with the production cost of the entire quantities of subsets of outputs, the game v_C deals with the production cost of any partial vector z of y ($z \leq y$).

An <u>imputation</u> of C(y) is a vector $p = (p_1, \ldots, p_n) \in E_+^n$ such that p_j is the per unit cost attributed to the j-th output and

$$\sum_{j=1}^{n} p_j y_j = C(y) .$$

Thus, all units of the same output are treated equally. The set of all imputations of C(y) is denoted by P(y).

The core $H_D(y)$ of the discrete game v_D consists of those imputations $p \in P(y)$ of C(y) for which no subset of outputs can be produced cheaper than the amount they are charged under p, i.e.,

$$H_D(y) = \{p \in P(y) | p^S y^S \leq C(y^S), \text{ for each } S \subseteq N\} .$$

Similarly the core $H_C(y)$ of the game v_C consists of those imputations $p \in P(y)$ of C(y) for which no partial quantities can be produced cheaper than the amount they are charged under p, i.e.,

$$H_C(y) = \{p \in P(y) | pz \leq C(z), \text{ for each } z \leq y\} .$$

Clearly $H_C(y) \subseteq H_D(y)$ for each $y \in E^n_+$. It is also easy to verify that $\bar{p} \in H_C(Q(\bar{p}))$ is a necessary condition for $\bar{p}$ to be PE sustainable.[1/] Similarly it can be shown that $\bar{p} \in H_D(Q(\bar{p}))$ is a necessary condition for $\bar{p}$ to be FE sustainable. Notice now that the core concepts considered here are closely related to the core concepts introduced by Faulhaber (1975), Faulhaber and Levinson (1981) and Sharkey and Telser (1978). In particular, the set $H_D(y)$ is related to the concept of subsidy free prices (Faulhaber (1975)). Accordingly, a price vector p is subsidy free iff $p \in H_D(Q(p))$. In addition, the set $H_C(y)$ is related to the concept of anonymous equity (Faulhaber and Levinson (1981)). Accordingly, a price vector p is anonymously equitable iff $p \in H_C(Q(p))$, while the notion of supportability (Sharkey and Telser, 1978) is related to the nonemptiness of the set $H_C(y)$.[2/]

The following result, inspired by Proposition 6 of Faulhaber and Levinson (1981), shows that under cost complementarities the requirement $p \in H_C(Q(p))$ is actually not stronger than the requirement $p \in H_D(Q(p))$ since in this case, for each $y \in E^n_+$, $H_C(y)$ and $H_D(y)$ coincide. This means that under cost complementarities the notions of subsidy free and anonymous equity are equivalent.

[1/] Panzer and Willig (condition 7, 1977) showed that a weaker condition is necessary for $\bar{p}$ to be a sustainable price vector. Their condition is $\bar{p} \in H_D(Q(\bar{p}))$.

[2/] Note, however, that the supportability concept, as globally defined by Sharkey and Telser (1978), is not necessary for sustainability as is claimed there. Actually, supportability as defined in Sharkey and Telser (1978) is equivalent to nonemptiness of $H_C(y)$ for all $y > 0$, while a necessary condition for sustainability is $H_C(\bar{y}) \neq \phi$ only for $\bar{y} = Q(\bar{p})$ (and that in this case $\bar{p} \in H_C(\bar{y})$).

Proposition 2. *If $C_{j\ell} \leq 0$, for all j and ℓ in N and $y \in E^n_+ \setminus \{0\}$ then,*

$$H_D(y) = H_C(y) ,$$

for all $y \in E^n_+$.

Notice that while Proposition 2 asserts that when $C_{j\ell} \leq 0$ the set of subsidy free price vectors coincides with the set of anonymously equitable price vectors (namely, $p \in H_D(Q(p))$ iff $p \in H_C(Q(p))$), it does not assert that these two sets are not empty. In this respect ten Raa (1982) used a fixed point argument to prove that when the cost function is supportable then, under some assumptions on the demands, there exists an anonymously equitable price vector (i.e., there exists p with $p \in H_C(Q(p))$). In the next section we prove (Proposition 7, below) that under cost complementarities there exists an Aumann-Shapley average cost price vector which is anonymously equitable.

Proof. The nontrivial part of the proposition is that $H_D(y) \subseteq H_C(y)$, for each $y \in E^n_+ \setminus \{0\}$. Let $p \in H_D(y)$ and consider the maximization problem

$$\begin{aligned} w(y) = \max_{z}\ & pz - C(z) \\ \text{subject to } & z \leq y , \\ & z \geq 0 . \end{aligned}$$

To establish that $p \in H_C(y)$ it must be shown that $w(y) \leq 0$. The Karush-Kuhn-Tucker (KKT) necessary conditions are

$$(1) \qquad p_j - \frac{\partial C}{\partial y_j}(z^*) - \lambda_j^* + \mu_j^* = 0 \ , \quad j \in N \ ,$$

$$(2) \qquad \mu_j^* z_j^* = 0 \ , \ \lambda_j^*(y_j - z_j^*) = 0 \ , \quad j \in N$$

$$\mu_j^* \geq 0 \ , \quad \lambda_j^* \geq 0 \ , \quad j \in N \ ,$$

where (z^*, λ^*, μ^*) is an optimal KKT pair. Let

$$S = \{j \mid 0 < z_j^* < y_j \ , \ j \in N\} \ ,$$

$$T = \{j \mid z_j^* = 0\}$$

$$R = N \setminus (S \cup T)$$

Then

$$(3) \qquad z^{*T} = 0 \ , \ z^{*S} < y^S \ \text{ and } \ z^{*R} = y^R \ .$$

By (3), $\mu^{*S} = \lambda^{*S} = 0$ and by (1) and (2)

$$(4) \qquad p_j = \frac{\partial C}{\partial y_j}(z^*) = \frac{\partial C}{\partial y_j}(z^{*S}, y^R) \ ,$$

for each $j \in S$. Now by (3) and (4),

$$w(y) = pz^* - C(z^*) = \sum_{j \in S} z_j^* \frac{\partial C}{\partial y_j}(z^{*S}, y^R) + \sum_{j \in R} p_j y_j - C(z^{*S}, y^R) + [C(y^R) - C(y^R)] \ .$$

After reordering we obtain by Euler's formula,

$$(5) \qquad w(y) = [\sum_{j \in R} p_j y_j - C(y^R)] + \sum_{j \in S} z_j^* \frac{\partial C}{\partial y_j}(z^{*S}, y^R) + \sum_{j \in S} - z_j^* \int_0^1 \frac{\partial C}{\partial y_j}(z^{*S} + t(0 - z^{*S}), y^R)dt$$

The first term of (5) is ≤ 0 since $p \in H_D(y)$. Hence

$$w(y) \leq \sum_{j\in S} z_j^*[\frac{\partial C}{\partial y_j}(z^{*S},y^R) - \int_0^1 \frac{\partial C}{\partial y_j}((1-t)z^{*S},y^R)dt] \leq 0 ,$$

where the last inequality follows by $C_{j\ell} \leq 0$. The proof of Proposition 2 is thus complete.

III. Subsidy Free and Sustainability

In this section sufficient conditions for a subsidy free price vector $\bar{p}$ (i.e., $\bar{p} \in H_D(Q(\bar{p}))$) to be FE or PE sustainable are established.

For each $S \subseteq N$ define,

$$\pi^S(p) = p^S Q^S(p) - C(Q^S(p)) .$$

Proposition 3. Let $\bar{p} \in E^n_+$. Then under assumptions (a) (cost complementarities), (b) (weak gross substitutes) and

(c) Subsidy free, i.e., $\bar{p} \in H_D(Q(\bar{p}))$,

(d) $\frac{\partial \pi^S}{\partial p_j}(p) \geq 0$, for each $j\in S$, $S \subseteq N$ and $p \leq \bar{p}$,

the price vector $\bar{p}$ is PE (and hence FE) sustainable.

Proof. By Proposition 1, it is sufficient to consider a triple $(S,Q^S(\hat{p}^S,\bar{p}^{N\setminus S}),\hat{p}^S)$ such that $\hat{p}^S \leq \bar{p}^S$. By (d) and (c)

$$\hat{p}^S Q^S(\hat{p}^S,\bar{p}^{N\setminus S}) - C(Q^S(\hat{p}^S,\bar{p}^{N\setminus S})) \leq \bar{p}^S Q^S(\bar{p}) - C(Q^S(\bar{p})) \leq 0$$

and the proof is complete.

Proposition 4 Under assumptions (a), (b), (c) and

(d') $\frac{\partial \pi^S}{\partial p_j}(\bar{p}) \geq 0$, for each $j \in S$ and $S \subseteq N$,

(e) $\pi^S(\bar{p}|p^S)$ is a pseudo concave function of p^S over $p^S \leq \bar{p}^S$,[3]

for each $S \subseteq N$,

the price vector $\bar{p}$ is PE (hence FE) sustainable.

Proof. By Proposition 1 we can, without loss of generality, consider the triple $(S, Q^S(\hat{p}^S, \bar{p}^{N\setminus S}), \hat{p}^S)$ for $\hat{p}^S \leq \bar{p}^S$. Then

$$(\hat{p}^S - \bar{p}^S)\nabla^S \pi^S(\bar{p}) \leq 0$$

holds by (d'), and consequently

$$\pi^S(\hat{p}^S, \bar{p}^{N\setminus S}) \underset{(e)}{\leq} \pi^S(\bar{p}) \underset{(c)}{\leq} 0 \ .$$

Thus $\bar{p}$ is PE sustainable.

Proposition 5. Let $C(\cdot)$ be a nondecreasing function on E^n_+ and let $\bar{p} \in E^n_+$. Then under assumptions (a), (b), (c) and

(f) The demands are inelastic with respect to prices below $\bar{p}$, i.e.,

$$\frac{\partial Q_j(p)/Q_j(p)}{\partial p_j/p_j} \geq -1, \text{ for each } j \in N \text{ and } p \leq \bar{p} \ ,$$

the price vector $\bar{p}$ is PE (and hence FE) sustainable.

Proof. Again, by Proposition 1 it is sufficient to establish FE sustainability. Consider the triple $(S, Q^S(\hat{p}^S, \bar{p}^{N\setminus S}), \hat{p}^S)$ for $\hat{p}^S \leq \bar{p}^S$. Let

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[3] A differentiable function f is pseudo concave over a convex set B if $y, z \in B$ and $(z-y)\nabla f(y) \leq 0$ imply $f(z) \leq f(y)$. A differentiable concave function is pseudo concave.

$$(6) \qquad S_1 = \{j\epsilon S | Q_j(\hat{p}^S, \bar{p}^{N\setminus S}) \leq Q_j(\bar{p})\}, \text{ and}$$

$$S_2 = S\setminus S_1$$

Then

$$(7) \qquad \pi^S(\hat{p}^S, \bar{p}^{N\setminus S}) = \hat{p}^S Q^S(\hat{p}^S, \bar{p}^{N\setminus S}) - C(Q^S(\hat{p}^S, \bar{p}^{N\setminus S})) .$$

Now since C is nondecreasing, by (6),

$$C(Q^S(\hat{p}^S, \bar{p}^{N\setminus S})) \geq C(Q^{S_1}(\hat{p}^S, \bar{p}^{N\setminus S}), Q^{S_2}(\bar{p})) .$$

By (c) and Proposition 2, since $Q^{S_1}(\hat{p}^S, \bar{p}^{N\setminus S}) \leq Q^{S_1}(\bar{p})$ and $\hat{p}^S \leq \bar{p}^S$,

$$C(Q^{S_1}(\hat{p}^S, \bar{p}^{N\setminus S}), Q^{S_2}(\bar{p})) \geq \hat{p}^{S_1} Q^{S_1}(\hat{p}^S, \bar{p}^{N\setminus S}) + \bar{p}^{S_2} Q^{S_2}(\bar{p}) .$$

Hence by (7) we obtain

$$\pi^S(\hat{p}^S, \bar{p}^{N\setminus S}) \leq \hat{p}^{S_2} Q^{S_2}(\hat{p}^S, \bar{p}^{N\setminus S}) - \bar{p}^{S_2} Q^{S_2}(\bar{p}) .$$

Using (b) we have

$$\pi^S(\hat{p}^S, \bar{p}^{N\setminus S}) \leq \sum_{j\epsilon S_2} [\hat{p}_j Q_j(\hat{p}_j, \bar{p}^{N\setminus \{j\}}) - \bar{p}_j Q_j(\bar{p})] ,$$

and thus (f) implies $\pi^S(\hat{p}^S, \bar{p}^{N\setminus S}) \leq 0$, as claimed.

Remark In their paper Sharkey and Telser (1978) suggested that supportability together with inelastic demands are sufficient for sustainability. In view of Proposition 3 condition (c) replaces supportability in their conjecture. However condition (c) together with inelastic demands are not sufficient to guarantee sustainability. In fact it can be shown that whenever costs are separable, average costs are declining and goods j and ℓ in N, are strict gross complements $(\frac{\partial Q_\ell}{\partial p_j} < 0, \frac{\partial Q_j}{\partial p_\ell} < 0)$, no price vector can be sustainable. This explains the requirement that outputs are weak gross substitutes (assumption (b)).

The next proposition asserts that any cost function satisfying $C_{j\ell} \leq 0$ is supportable.

Proposition 6. Under assumption (a) the cost function C is supportable by Aumann-Shapley average cost prices.

Proof We shall show that Aumann-Shapley average cost prices support C at any point $y \in E_+^n \setminus \{0\}$. (In fact this property was pointed out by Billera and Heath (1982)). Let $y \in E_+^n \setminus \{0\}$. Define the price vector $p = (p_1, \ldots, p_n)$ by

$$p_j = \int_0^1 \frac{\partial C}{\partial y_j} (ty)dt .$$

Let $z \in E_+^n$ such that $z \leq y$. Then since $C_{j\ell} \leq 0$

$$C(z) = \sum_{j=1}^{n} z_j \int_0^1 \frac{\partial C}{\partial y_j} (tz)dt$$

$$\geq \sum_{j=1}^{n} z_j \int \frac{\partial C}{\partial y_j} (ty)dt = pz .$$

Also

$$C(y) = \sum_{j=1}^{n} y_j \int \frac{\partial C}{\partial y_j} (ty)dt .$$

Thus p supports C at y.

Remark It could be shown that the prices calculated from the Shapley value (Shapley (1953)) corresponding to the game v_D also support the cost function C. Indeed let $y \in E^n_+$ and let $x(y) = (x_1(y), \ldots, x_n(y))$ be the Shapley value corresponding to v_D. Let $p_j = x_j(y)/y_j$ be the Shapley price of the j-th commodity. Then under (a), $p \in H_C(y) = H_D(y)$. This follows by the facts that the discrete cost game v_D is a concave game and that the Shapley value of a concave game is in the core of this game (Shapley (1971)). The same remark is applicable to the prices suggested by Sharkey and Telser (1978).

The next proposition asserts that under cost complementarity there exists an Aumann-Shapley average cost price vector which is anonymously equitable.

Proposition 7. Under assumption (a) and under semicontinuity of demand there exists an Aumann-Shapley price vector $\bar{p}$ which is anonymously equitable, i.e., there exists $\bar{p} \in E^n_+$, such that,

(i) $\bar{p} \in H_C(Q(\bar{p}))$

(ii) $\bar{p}_j = \int_0^1 \frac{\partial C}{\partial y_j} (tQ(\bar{p}))dt \ , \ j = 1, \ldots, n$.

Proof The existence of an equilibrium Aumann-Shapley price vector $\bar{p}$ (i.e., the existence of $\bar{p}$ satisfying (ii)) is proved in Mirman-Tauman (1982) under weak differentiable assumptions on C. Condition (i) is thus an immediate consequence of Proposition 6.

Proposition 8. Let $C(\cdot)$ be a nondecreasing function on E^n_+ and let $\bar{p}$ be an Aumann-Shapley equilibrium price vector. Then under assumptions (a), (b) and (f), $\bar{p}$ is a PE (or FE) sustainable price vector

Proof Follows immediately from Proposition 5.

Corollary: Under the conditions of Proposition 8 and the requirement that condition (f) holds globally, namely the inequality in (f) holds for all $p \in E^n_+$, there exists an Aumann-Shapley equilibrium sustainable price vector.

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