Open access • Journal Article • DOI:10.3150/BJ/1193667816

Supports of doubly stochastic measures — Source link

Kevin Hestir, Stanley C. Williams

Published on: 01 Sep 1995 - Bernoulli (Bernoulli Society for Mathematical Statistics and Probability)

Topics: Disjoint sets, Tree (graph theory) and Measure (mathematics)

Bernoulli 1(3), 1995, 217–243

Supports of doubly stochastic measures

KEVIN HESTIR and STANLEY C. WILLIAMS

Department of Mathematics and Statistics, Utah State University, Logan UT 84322, USA

Recent work has shown that extreme doubly stochastic measures are supported on sets that have no axial cycles. We give a new proof of this result and examine the supporting set structure more closely. It is shown that the property of no axial cycles leads to a tree like structure which naturally partitions the support into a collection of disjoint graphs of functions from the x-axis to the y-axis and from the y-axis to the x-axis. These functions are called a *limb numbering system*. It is shown that if the disjoint graphs in the limb numbering system are measurable, then the supporting set supports a unique doubly stochastic measure. Further, the limb structure can be used to develop a general method for constructing sets which support a unique doubly stochastic measure.

Keywords: extreme point, sets of uniqueness

1. Introduction

Doubly stochastic measures are measures on the unit square $[0, 1]^2$ which have uniform marginals. These measures are important because they embody all joint distributions. To see why, suppose that X and Y are jointly distributed random variables. Then one can find functions f: $[0, 1] \to \mathbb{R}$ and g: $[0, 1] \to \mathbb{R}$ and jointly distributed uniform random variables U_1 and U_2 so that $(f(U_1), g(U_2))$ has the same distribution as (X, Y). Hence a joint distribution can be represented with a pair of functions and a doubly stochastic measure. In this sense the set of doubly stochastic measures, which we denote by DSM, generates all possible joint distributions. It is therefore an important problem to study the probabilistic and analytic structure of DSM. Because DSM is convex one is led, via Choquet's theorem, to examine extreme points of DSM. This set we denote by $EDSM$.

Two natural questions to ask about an extreme doubly stochastic measure are what its support looks like and, given this support, whether the support determines a unique element of DSM? In this paper we examine these questions.

One obvious analogy to doubly stochastic measures is doubly stochastic matrices with uniform marginals, or probability measures on $X \times X$ with uniform marginals and X finite. The Birkhoff–von Neumann theorem says that a doubly stochastic matrix with uniform marginals is extreme if and only if it is a scalar times a permutation matrix; that is, a doubly stochastic matrix whose entries are only 0's and $1/n$'s. One way to prove this theorem is to employ the concept of an *axial path* and an *axial cycle*. To construct an axial path in a doubly stochastic matrix we start with a non-zero entry as step 1 in the path. Next we move horizontally to another non-zero entry in the same row; this is step 2. From here we move vertically to step 3, which is a non-zero entry in the same column as step 2. The axial path continues in this way, with alternating horizontal and vertical steps. An axial cycle is an axial path of length $n > 1$ with step n equal to step 1. Using the concept of axial paths and axial cycles, an equivalent statement of the Birkhoff–von Neumann theorem says that: a doubly stochastic matrix is extreme if and only if it contains no axial cycles. Denny (1980) showed that this concept generalizes to extreme points of joint distributions on $X \times Y$ with fixed marginals and X and Y countable.

Axial paths and cycles can also be defined on subsets of $[0,1]^2$ in an obvious way. Beneš and Štěpán (1987) found a result for doubly stochastic measures corresponding to necessity in the Birkhoff–von Neumann theorem. They showed that an extreme doubly stochastic measure is supported on a subset of $[0,1]$ with no axial cycles. No axial cycles, however, cannot be a sufficient condition for a doubly stochastic measure to be extreme, as Losert (1982) found an example of a subset of $[0,1]^2$ with no axial cycles which supports two distinct doubly stochastic measures.

Several authors have explored supports of doubly stochastic measures which are the union of the graphs of two functions, one from the x-axis to the y-axis and one from the y-axis to the x-axis (Seethoff and Shiflett 1978; Kamiński *et al.* 1990). In fact we show here that the property of no axial cycles leads to a tree-like structure which can be written as the disjoint union of the graphs of two functions. Denny (1980) showed that an extreme measure on $X \times Y$ with X and Y countable is supported on the graphs of two such functions, and Beneš and Štěpán (1987) showed that the same is true for extreme doubly stochastic measures. Also, Losert (1982) has found an example which shows that the functions need not be measurable.

One final concept of importance to us is that of a *set of uniqueness*. Call $F \subseteq [0,1]^2$ a set of uniqueness if there is a unique doubly stochastic measure supported on F. Štěpán (1979) uses a generalization of this concept for distributions on $X \times Y$ with fixed marginals, calling it an A-set. Beneš and Štěpán (1987) give a characterization of A-sets using axial cycles and the union of graphs of two functions for the case of X and Y Polish spaces with σ-fields equal to Borel σ-fields.

In this paper we give some new results about the support of measures in *EDSM*. We give a new proof of the Beneš–Štěpán result that each element of *EDSM* has a support which has no axial cycles. From this it follows that every element of *EDSM* is supported on a union of tree-like sets which we call an *axial forest*. This tree structure naturally partitions the support into a collection of disjoint graphs of functions from the x-axis to the y-axis and from the y-axis to the x-axis. We call these functions a *limb numbering system*. The limb numbering system structure of the *EDSM* support leads easily to the Beneš–Štěpán result that every element of *EDSM* is supported on the graphs of two (possibly non-measurable) functions. The limb numbering system structure also leads to two further results about sets of uniqueness. First, if the disjoint graphs in the limb numbering system are measurable, then the supporting set is a set of uniqueness for *DSM* or in fact a set of uniqueness for *GDSM*, the set of signed measures with uniform marginals. Second, the limb structure is used to develop a general method for constructing a set of uniqueness for a doubly stochastic measure. We also used this method to construct a set of uniqueness for *GDSM* which is not the support of a doubly stochastic measure. Finally, we give an example of a limb structure which is a set of uniqueness but does not have measurable limbs.

2. Existence of support with no cycles

In what follows we let λ denote Lebesgue measure on $[0,1]$. The basic tool for analysis of *EDSM* is a characterization of extreme doubly stochastic measures discovered by Lindenstrauss (1965) and Douglas (1964):

Theorem 1 (Lindenstrauss–Douglas) Let $\mu \in DSM$. Let $CL_1(\mu)$ be the subspace of $L_1(\mu)$ consisting of all functions of the form $f(x) + g(y)$ with $f, g \in L_1(\lambda)$. Then $\mu \in EDSM$ if and only if the subspace $CL_1(\mu)$ is dense in $L_1(\mu)$.

We use the Lindenstrauss–Douglas theorem to prove that a given element of *EDSM* has a support with no cycles. First some definitions are given.

Definition The sequence of points $\{(x_i, y_i)\}_{i=1}^{2n} \subseteq [0,1]^2$ is called an *axial cycle* if and only if the (x_i, y_i) are distinct, $y_{2i} = y_{2i-1}$, $x_{2i+1} = x_{2i}$, and $x_1 = x_{2n}$, for $1 \le i \le n$. Let $P_n = \{[0, 2^{-n}]\} \cup \{(i2^{-n}, (i+1)2^{-n}]\}_{i=1}^{2^n-1}$ be the diadic partition of $[0,1]$. Also, let $P_n \times P_n$ be the induced diadic partition of $[0,1]^2$ into diadic squares. A *basic $P_n \times P_n$-axial cycle* is a subset Γ of $P_n \times P_n$ such that the centres of the diadic squares of Γ form an axial cycle. Finally, for Γ a basic $P_n \times P_n$-axial cycle, with $\#(\Gamma)$ denoting the cardinality of Γ, the axial cycle $\{(x_i, y_i)\}_{i=1}^{\#(\Gamma)}$ is called a *Γ-axial cycle* if each (x_i, y_i) is in a distinct element of Γ. In this case, we also say that $\{(x_i, y_i)\}_{i=1}^{\#(\Gamma)}$ is a $P_n \times P_n$-axial cycle.

From the above definition it should be clear that if an axial cycle is a $P_n \times P_n$-axial cycle for some n, then it is a $P_m \times P_m$-axial cycle for all $m \ge n$.

One can similarly define an axial cycle in a doubly stochastic matrix as a sequence of non-zero matrix entries with indices $\{(p_i, q_i)\}_{i=1}^{2n}$ such that (p_i, q_i) are distinct, $q_{2i} = q_{2i-1}$, $p_{2i+1} = p_{2i}$, and $p_1 = p_{2n}$ for $1 \le i \le n$. It is a fact that a doubly stochastic matrix M is an extreme point in the set of all doubly stochastic matrices if and only if M has no axial cycles. The corresponding result for doubly stochastic measures is not quite as strong. Indeed, we can prove below that every element of *EDSM* has a support with no axial cycles. However, the converse fails as shown in the example in Losert (1982, p. 391).

To prove that every element of *EDSM* has support with no axial cycles first requires the next lemma, which is based on an idea from Lindenstrauss (1965).

Lemma 2 If $\mu \in EDSM$ then given a positive integer m and $\epsilon > 0$ there exists an $n \ge m$ and $\Pi \subseteq P_n \times P_n$ such that $\bigcup_{O \in \Pi} O$ contains no $P_m \times P_m$-axial cycle and $\mu(\bigcup_{O \in \Pi} O) > 1 - \epsilon$.

Proof

Fix m and ϵ. By Theorem 1, for a given $\delta > 0$, there exist for each $L \in P_m \times P_m$ functions f_L and $g_L \in L_1([0,1], \lambda)$ such that

$$\int |I_L(x,y) - f_L(x) - g_L(y)| \, d\mu < \delta.$$

Because simple functions on $\bigcup_k P_k$ are dense in $L_1([0,1], \lambda)$, f_L and g_L can be chosen to be P_n simple functions for some $n \ge m$. Let $a(m)$ be a finite integer such that $\#(\Gamma) < a(m)$ for all $P_m \times P_m$-axial cycles Γ. Temporarily fix a basic $P_m \times P_m$-axial cycle Γ, let $\{(x_i, y_i)\}_{i=1}^{2n}$ be the axial cycle of centres and choose $L(\Gamma) \in \Gamma$ so that $(x_1, y_1) \in L(\Gamma)$. Define

$$D(\Gamma) = \left\{ B \in P_n \times P_n \colon I_B \, |I_{L(\Gamma)} - f_{L(\Gamma)}(x) - g_{L(\Gamma)}(y)| \le \frac{1}{a(m)} \right\}.$$

Let $U(\Gamma) = \bigcup_{B \in D(\Gamma)} B$. We have

$$\frac{1}{a(m)}\mu\{(U(\Gamma))^c\} < \int_{(U(\Gamma))^c} |I_{L(\Gamma)} - f_{L(\Gamma)}(x) - g_{L(\Gamma)}(y)| \, d\mu \leq \delta,$$

so

$$1 - \delta a(m) < \mu\{U(\Gamma)\}.$$

Now it turns out that $U(\Gamma)$ will have no Γ-axial cycles. To see this, suppose $\{(x_i, y_i)\}_{i=1}^{\#(\Gamma)}$ is a Γ-axial cycle contained in $U(\Gamma)$. Because of the way $L(\Gamma)$ is defined, indexing can be chosen such that $(x_i, y_1) \in L(\Gamma)$. Existence of a Γ-axial cycle then leads to the following contradiction:

$$\begin{aligned} 1 = I_{L(\Gamma)}(x_1, y_1) &= \sum_{i=1}^{\#(\Gamma)} \{I_{L(\Gamma)}(x_i, y_i) - f_{L(\Gamma)}(x_i) - g_{L(\Gamma)}(y_i)\}(-1)^{i+1} \\ &\leq \sum_{i=1}^{\#(\Gamma)} |I_{L(\Gamma)}(x_i, y_i) - f_{L(\Gamma)}(x_i) - g_{L(\Gamma)}(y_i)| \leq \#(\Gamma)\frac{1}{a(m)} < 1. \end{aligned}$$

To finish the proof, let $\{\Gamma_i\}_{i=1}^{M}$ be the collection of all $P_m \times P_m$-axial cycles. For each Γ_i pick $\delta = \delta_i$ in the above argument with $0 < \delta_i < \epsilon/\{2^i a(m)\}$. Form $U(\Gamma_i) = \bigcup_{B \in D(\Gamma_i)} B$. Then $U(\Gamma_i) \subseteq P_{n_i} \times P_{n_i}$ for some n_i, $U(\Gamma_i)$ has no Γ_i-axial cycles, and

$$1 - \frac{\epsilon}{2^i} < 1 - \delta_i a(m) < \mu\{U(\Gamma_i)\}.$$

Finally, set $D = \bigcap_{i=1}^{M} U(\Gamma_i)$, check that $\mu(D) > 1 - \epsilon$, D has no $P_m \times P_m$-axial cycles, and $D = \bigcup_{O \in \Pi} O$ for $\Pi \subseteq P_n \times P_n$ with $n = \max_i\{n_i\}$. □

Theorem 3 Let $\mu \in EDSM$. There is a Borel set B with no axial cycles such that $\mu(B) = 1$.

Proof
By Lemma 2, for each $m \geq 1$ find K_m such that K_m has no $P_m \times P_m$-axial cycles and $\mu(K_m) > 1 - 2^{-m}$. Then $B = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} K_m$ has no $P_n \times P_n$-axial cycles for any $n \geq 1$ and $\mu(B) = 1$. Lastly, note that any axial cycle must be a $P_n \times P_n$-axial cycle for some n, so B has no axial cycles. □

One 'well-known' fact about Lebesgue measure is the following. Let $A \subseteq [0, 1]^2$ be a Borel set which does not contain the corners of any rectangle (that is, A contains no axial cycle of length 4); then the Lebesgue measure of A is zero. Together with Theorem 3 this fact yields the following result, proved in Lindenstrauss (1965).

Corollary 4 Let $\mu \in EDSM$. Then μ is singular with respect to Lebesgue measure on $[0, 1]^2$.

Proof
Except for establishing the 'well-known' fact, the proof is given in comments above. To see that a subset of $[0, 1]^2$ which does not contain the corners of any rectangle has Lebesgue measure zero one can prove the contrapositive. Let $\lambda \times \lambda$ denote Lebesgue measure on $[0, 1]^2$ and let $A \subseteq [0, 1]^2$ with

$\lambda \times \lambda(A) > 0$. We show that A contains the corners of a rectangle. Let $(x, y) \in A$ be a point of Lebesgue density of A (Halmos 1950, p. 268). Then for $\delta > 0$ there exists an $\epsilon > 0$ such that

$$\lambda \times \lambda(A \cap [x-\epsilon, x+\epsilon] \times [y-\epsilon, y+\epsilon]) > 4\epsilon^2(1-\delta).$$

Let $B = A \cap [x-\epsilon, x+\epsilon] \times [y-\epsilon, y+\epsilon]$. Divide B into four squares of width ϵ, $B_1 = B \cap [x-\epsilon, x] \times [y-\epsilon, y]$, $B_2 = B \cap [x-\epsilon, x] \times [y, y+\epsilon]$, $B_3 = B \cap [x, x+\epsilon] \times [y-\epsilon, y]$, and $B_4 = B \cap [x, x+\epsilon] \times [y, y+\epsilon]$. Then $\lambda \times \lambda(B_i) \geq \epsilon^2(1-4\delta)$. Now shift each B_i by $(\pm\epsilon/2, \pm\epsilon/2)$ so that it is centred on (x, y). Let C equal the intersection of the four shifted B_i's. By the shift invariance of $\lambda \times \lambda$,

$$\lambda \times \lambda(C) \geq \epsilon^2(1-16\delta).$$

Hence $C \neq \emptyset$ for small enough δ. Choose $(x_0, y_0) \in C$ and trace it back to the point $(x_i, y_i) \in B_i$ for each $i = 1, 2, 3, 4$. The four points, (x_i, y_i), $i = 1, 2, 3, 4$, all lie in A and are corners of a square. □

3. Doubly stochastic measures with rooted support

In this section we examine sets with no axial cycles which by Theorem 3 are the supports of measures in *EDSM*. The main result is that such sets have a structure which is the union of trees. We call this an *axial forest*. Further, the axial forest structure can be written as the union of the graphs of two functions; one from the x-axis to the y-axis and the other from the y-axis to the x-axis. Our analysis begins with the concept of a rooting set given in the next definition.

Definition Let $F \subseteq [0,1]^2$. Let $\mathbf{P}(\cdot)$ denote power set and define $T_F\colon \mathbf{P}(F) \to \mathbf{P}(F)$ by

$$T_F(A) = \{(x, y) \in F\colon \exists z \in [0,1] \text{ with } (x, z) \in A \text{ or with } (z, y) \in A\}.$$

For $(x, y) \in F$, $[(x, y)]_F = \bigcup_{k=1}^{\infty} T_F^k(x, y)$ is called the *orbit of* (x, y) *under* T_F. Let $A \subseteq F$; A is called a *rooting set* for F if and only if

(i) $F = \bigcup_{k=1}^{\infty} T_F^k(A)$.
(ii) For each (x, y), $(z, w) \in A$, $(x, y) \neq (z, w)$ implies $[(x, y)]_F \cap [(z, w)]_F = \emptyset$.
(iii) For all $(x, y) \in A$, $[(x, y)]_F$ has no axial cycles.

If A is Borel then A is called a *Borel rooting set*. If (i) holds μ-a.s. then A is called a *μ-essential rooting set*.

Condition (iii) says that each $[(x, y)]_F$ has a tree-like structure, and condition (ii) says that orbits of elements from A generate disjoint trees. By (i) F is the union of these disjoint trees, which motivates the following definition.

Definition Let $F \subseteq [0,1]^2$. F is called an *axial forest* if and only if F has a rooting set.

Note that axial forests and rooting sets are set-theoretic structures in a product space and are not necessarily related to any measures.

Theorem 5 $F \subseteq [0,1]^2$ is an axial forest if and only if F contains no axial cycles. In particular, every $\mu \in EDSM$ has axial forest support.

Proof
Clearly if F has a rooting set then it has no cycles. To go the other way, use Zorn's lemma to pick a maximal subset from F with points having disjoint orbits. Finally, Theorem 3 shows that every $\mu \in EDSM$ has support with no axial cycles. □

Definition Let $F \subseteq [0,1]^2$. The sequence of points $\{(x_i, y_i)\}_{i=1}^n \subseteq F$ is called an *F-axial path of length n* if and only if (x_i, y_i) are distinct and one of the following holds:

(i) $y_{2i} = y_{2i-1}$ and $x_{2i+1} = x_{2i}$ for $i \geq 1$.
(ii) $x_{2i} = x_{2i-1}$ and $y_{2i+1} = y_{2i}$ for $i \geq 1$.

In case (i), $\{(x_i, y_i)\}_{i=1}^n$ is called an *axial path starting horizontally from* (x_1, y_1) *to* (x_n, y_n). In case (ii), $\{(x_i, y_i)\}_{i=1}^n$ is called an *axial path starting vertically from* (x_1, y_1) *to* (x_n, y_n).

Let A be a rooting set for F and consider $(x, y) \in [A]_F$. Then there is a unique $(z, w) \in A$ such that $(x, y) \in [(z, w)]_F$. Hence there is a unique minimal k such that $(x, y) \in T_F^k(z, w)$. Because $[(z, w)]_F$ has no cycles there is a unique axial path of minimal length k from (z, w) to (x, y). One can then define the following subsets of orbits of elements of A:

$V_k = \{(x, y) \in [A]_F$: there is an axial path of minimal length k starting vertically from $(z, w) \in A$ to $(x, y)\}$
$H_k = \{(x, y) \in [A]_F$: there is an axial path of minimal length k starting horizontally from $(z, w) \in A$ to $(x, y)\}$.

Let π_1 and π_2 denote the canonical projections from $[0,1]^2$ onto $[0,1]$ defined by $\pi_1((x, y)) = x$ and $\pi_2((x, y)) = y$. We have the following lemma.

Lemma 6 Let F, A, H_k and V_k be defined as in the discussion above. Then by definition $H_1 = V_1 = A$. We have:

(i) $[A]_F = \bigcup_{k=2}^{\infty} H_k \cup \bigcup_{k=1}^{\infty} V_k$.
(ii) $H_k \cap V_j = \emptyset$ for all k, j (unless $j = k = 1$).
(iii) $j \neq k$ implies $H_j \cap H_k = \emptyset$ and $V_j \cap V_k = \emptyset$.
(iv) $\pi_i(H_k) \cap \pi_i(V_j) = \emptyset$ for all j, k and $i = 1, 2$, (unless $j = k = 1$).
(v) The map π_1 is an injection on H_{2k} and V_{2k-1}. Similarly, the map π_2 is an injection on H_{2k-1} and V_{2k}.
(vi) $H_{2k} \subseteq \pi_2^{-1}\pi_2 H_{2k-1}$, $H_{2k+1} \subseteq \pi_1^{-1}\pi_1 H_{2k}$, $V_{2k} \subseteq \pi_1^{-1}\pi_1 V_{2k-1}$, $V_{2k+1} \subseteq \pi_2^{-1}\pi_2 V_{2k}$.
(vii) $\pi_1(H_{2k+1}) \subseteq \pi_1(H_{2k})$, $\pi_1(V_{2k}) \subseteq \pi_1(V_{2k-1})$, $\pi_2(H_{2k+2}) \subseteq \pi_2(H_{2k+1})$, $\pi_2(V_{2k+1}) \subseteq \pi_2(V_{2k})$.
(viii) $j \neq k$ implies $\pi_1(H_{2k}) \cap \pi_1(H_{2j}) = \emptyset$, $\pi_1(V_{2k-1}) \cap \pi_1(V_{2j-1}) = \emptyset$, $\pi_2(H_{2k-1}) \cap \pi_2(H_{2j-1}) = \emptyset$, $\pi_2(V_{2k}) \cap \pi_2(V_{2j}) = \emptyset$.

Proof
Parts (i)–(iii) follow from uniqueness of a minimal path for each $(x, y) \in [A]_F$. Parts (iv) and (v)

follow from disjointness of orbits of points in A and the fact that these orbits have no cycles. Part (vi) follows from the fact that a minimal path to $(x_{k+1}, y_{k+1}) \in H_{k+1}$ contains a minimal path to some $(x_k, y_k) \in H_k$ and likewise for the V_k's. Statement (vii) follows from (vi). Finally, (viii) is a result of minimality of paths. □

Lemma 7 below shows that the H_k's and V_k's can be used to construct a collection of functions called a limb numbering system which is defined next. From an illustration of a limb numbering system see Figs 1 and 2. In what follows we will use *italic* type to denote a function and **bold** type to denote the graph of a function. So if $f: A \to B$ then $\mathbf{f} \subseteq A \times B$.

Definition Let $\{C_k\}_{k=1}^{\infty}$ be a collection of subsets of $[0, 1]$ such that $\{C_{2k+1}\}_{k=0}^{\infty}$ are disjoint and $\{C_{2k}\}_{k=0}^{\infty}$ are disjoint. For each $k \geq 1$, let $D_k \subseteq C_k$ and h_k be a function, $h_k: D_k \to C_{k-1}$. Set

$$\mathbf{f}_{2k-1} = \{(x, h_{2k-1}(x)): x \in D_{2k-1}\}$$

and

$$\mathbf{g}_{2k} = \{(h_{2k}(x), x): x \in D_{2k}\}.$$

Then $\mathbf{L} = \{\mathbf{f}_{2k-1}, \mathbf{g}_{2k}\}_{k=1}^{\infty}$ is called a *limb numbering system*. Also the functions f_{2k-1} and g_{2k} are called the *limbs* of $\mathbf{L}$. If $\mathbf{f}_{2k-1}$ and $\mathbf{g}_{2k}$ are measurable subsets of $[0, 1]^2$ for all k, then $\mathbf{L}$ is said to have *measurable limbs*. For ease of notation we will write $\cup\, \mathbf{L}$ for $\bigcup_{k=1}^{\infty}(\mathbf{f}_{2k-1} \cup \mathbf{g}_{2k})$.

Lemma 7 For $k \geq 1$, $V_{2k-1} \cup H_{2k}$ is the graph of a single-valued function f_{2k-1}, from $\pi_1(V_{2k-1} \cup H_{2k})$ to $\pi_2(V_{2k-1} \cup H_{2k})$. Also, $V_{2k} \cup H_{2k+1}$ is the graph of a single-valued function g_{2k}, from $\pi_2(V_{2k} \cup H_{2k+1})$ to $\pi_1(V_{2k} \cup H_{2k+1})$. Further,

$$\begin{aligned} &\text{range } (f_{2k+1}) \subseteq \text{domain } (g_{2k}) \\ &\text{range } (g_{2k}) \subseteq \text{domain } (f_{2k-1}) \end{aligned} \tag{1}$$

and $k \neq j$ implies

$$\begin{aligned} &\text{domain } (f_{2k-1}) \cap \text{domain } (f_{2j-1}) = \emptyset \\ &\text{domain } (g_{2k}) \cap \text{domain } (g_{2j}) = \emptyset. \end{aligned} \tag{2}$$

So $\{\mathbf{f}_{2k-1}, \mathbf{g}_{2k}\}_{k=1}^{\infty}$ is a limb numbering system.

Proof

Notice that f_{2k-1} being single-valued is equivalent to π_1 being an injection on $V_{2k-1} \cup H_{2k}$ which follows from (iv) and (v) of Lemma 6. Similarly, g_{2k} is single-valued.

To show that the relationships in (1) hold use (vii) of Lemma 6. To prove disjointness in (2) use parts (iv) and (viii) of Lemma 6. □

Theorem 8 Let $F \subseteq [0, 1]^2$. Then F is an axial forest if and only if there exists a limb numbering system $\mathbf{L} = \{\mathbf{f}_{2k-1}, \mathbf{g}_{2k}\}_{k=1}^{\infty}$ with $F = \cup\, \mathbf{L}$.

Proof

By Lemma 7 an axial forest can be written as the union of functions from a limb numbering system. To go the other way, suppose $\mathbf{L} = \{\mathbf{f}_{2k-1}, \mathbf{g}_{2k}\}_{k=1}^{\infty}$ is a limb numbering system and let $F = \cup\, \mathbf{L}$. By

Theorem 5 we need to show that F has no cycles. To do this, suppose there exists $\{\mathbf{x}_i\}_{i=1}^{2n}$ which is an F-axial cycle.

As in the definition of a limb numbering system, let $\mathbf{h}_k$ denote a generic element of $\mathbf{L}$, $\mathbf{h}_k = \mathbf{g}_k$ if k is even and $\mathbf{h}_k = \mathbf{f}_k$ if k is odd. Let k_i be chosen so that $\mathbf{x}_i \in \mathbf{h}_{k_i}$. We want to show first that $\{\mathbf{x}_i\}_{i=1}^{2n}$ can be indexed so that $k_1 \leq k_i$ for all $1 \leq i \leq 2n$. To do this, note that $\mathbf{y}_i = \mathbf{x}_{(i+2) \bmod (2n+1)+1}$ is an axial cycle. Hence we can index $\{\mathbf{x}_i\}_{i=1}^{2n}$ so that either k_1 or k_2 is minimal. Now $\mathbf{y}_i = \mathbf{x}_{2n+1-i}$ is an axial cycle so we can indeed index $\{\mathbf{x}_i\}_{i=1}^{2n}$ so that k_1 is minimal.

Now we argue that in either of the two cases, k_1 even or k_1 odd, $k_2, \ldots, k_{2n}, k_1$ is an increasing sequence, which leads to a contradiction.

In the first case, suppose $k_1 = 2k - 1$ for some k. Then by definition the step from $\mathbf{x}_1$ to $\mathbf{x}_2$ is horizontal so $\mathbf{x}_1 = (x_1, y_1) \in \mathbf{f}_{2k-1}$ and $\mathbf{x}_2 = (x_2, y_2)$ with $y_1 = y_2$. So $\mathbf{x}_2 \in \mathbf{f}_{2k-1}$ or $\mathbf{x}_2 \in \mathbf{g}_{2k-2}$. But $2k - 1 = k_1$ is minimal so $\mathbf{x}_2 \in \mathbf{f}_{2k-1}$.

In the second case, suppose $k_1 = 2k - 2$ for some k. Then again the step from $\mathbf{x}_1$ to $\mathbf{x}_2$ is horizontal, so $\mathbf{x}_1 = (x_1, y_1) \in \mathbf{g}_{2k-2}$ and $\mathbf{x}_2 = (x_2, y_2)$ with $y_1 = y_2$. Because g_{2k} is single-valued, $\mathbf{x}_2 \in \mathbf{f}_{2k-1}$.

Now in both cases the following inductive argument shows that $k_2, \ldots, k_{2n}, k_1$ is an increasing sequence, which yields a contradiction. First, suppose $\mathbf{x}_m = (x_m, y_m) \in \mathbf{f}_{2k+m-3}$ for some fixed even m with $2 \leq m \leq 2n$. Then $\mathbf{x}_{m+1} = (x_{m+1}, y_{m+1})$ with $x_{m+1} = x_m$. In other words, the step from $\mathbf{x}_m$ to $\mathbf{x}_{m+1}$ is vertical. Hence $\mathbf{x}_{m+1} \in \mathbf{g}_{2k+m-2}$ because f_{2k+m-3} is single-valued. Next, suppose $\mathbf{x}_m = (x_m, y_m) \in \mathbf{g}_{2k+m-3}$ for some fixed odd m with $3 \leq m \leq 2n - 1$. Then $\mathbf{x}_{m+1} = (x_{m+1}, y_{m+1})$ with $y_{m+1} = y_m$ and the step from $\mathbf{x}_m$ to $\mathbf{x}_{m+1}$ is horizontal. Hence $\mathbf{x}_{m+1} \in \mathbf{f}_{2k+m-2}$ because g_{2k+m-2} is single-valued. This proves that $k_2, \ldots, k_{2n}$ is an increasing sequence with $\mathbf{x}_{2n} \in \mathbf{f}_{2k+2n-3}$. Because $\{\mathbf{x}_i\}_{i=1}^{2n}$ is an axial cycle, there is a vertical step from $\mathbf{x}_{2n}$ to $\mathbf{x}_1$. This implies $\mathbf{x}_1 \in \mathbf{g}_{2k+2n-2}$, which contradicts the minimality of k_1. □

Theorem 9 Every $\mu \in EDSM$ is supported on the graphs of functions from a limb numbering system.

Proof

Let $\mu \in EDSM$. Use Theorem 5 to find a rooting set A for the support of μ. Define limb numbering system $\mathbf{L} = \{\mathbf{f}_{2k-1}, \mathbf{g}_{2k}\}_{k=1}^{\infty}$ with $\mathbf{f}_{2k-1}$ and $\mathbf{g}_{2k}$ as in Lemma 7. Then μ is supported on $[A]_F$ which is equal to the union of graphs of functions from $\mathbf{L}$ by Lemma 6 (i). □

Corollary 10 (Beneš–Štěpán) Every $\mu \in EDSM$ is supported on the graphs of two functions. In particular, there exist functions f, g with domain $(f) \subseteq \pi_1[0,1]^2$, range $(f) \subseteq \pi_2[0,1]^2$, domain$(g) \subseteq \pi_2[0,1]^2$, range$(g) \subseteq \pi_1[0,1]^2$, such that μ is supported on $\mathbf{f} \cup \mathbf{g}$. Further, $\mathbf{f}$ and $\mathbf{g}$ are disjoint.

Proof

Use Theorem 9 to find the support of $\mu \in EDSM$ which is a limb numbering system, $\mathbf{L} = \{\mathbf{f}_{2k-1}, \mathbf{g}_{2k}\}_{k=1}^{\infty}$. Because the domains of the f_{2k-1}'s are disjoint we can define single-valued function f on $\cup\text{domain}(f_{2k-1})$ in the obvious way: for $x \in \cup\text{domain}(f_{2k-1})$, set $f(x) = f_{2k-1}(x)$ where $x \in \text{domain}(f_{2k-1})$. Clearly g can be defined in a similar manner. Lastly, disjointness of $\mathbf{f}$ and $\mathbf{g}$ follows from parts (ii) and (iii) of Lemma 6. □

Corollary 11 Let $\mu \in DSM$ be supported on a limb numbering system $\mathbf{L} = \{\mathbf{f}_{2k-1}, \mathbf{g}_{2k}\}_{k=1}^{\infty}$. Then for every G: $[0,1]^2 \to \mathbb{R}$ there exist f: $[0,1] \to \mathbb{R}$ and g: $[0,1] \to \mathbb{R}$ such that $G(x,y) = f(x) + g(y)$ on $\cup\mathbf{L}$ (or μ-a.s.) with $g|_{C_0} = 0$. Further, if G is Borel measurable and if $\mathbf{f}_{2k-1}$ and $\mathbf{g}_{2k}$ are Borel subsets of $[0,1]^2$ for all k, then f and g can be taken to be Borel measurable.

Proof

Notice that we need $G(x,y) = f(x) + g(y)$ only on the limbs $\mathbf{f}_{2k-1}$ and $\mathbf{g}_{2k}$. This can easily be done first for $\mathbf{f}_1$ by setting $g(y) = 0$ for $y \in C_0$ and $f(x) = G(x, f_1(x))$ for $x \in C_1$. Then $G(x,y) = f(x) + g(y)$ on $C_1 \times C_0$. Now consider the limb $\mathbf{g}_2$. For $(x,y) \in \mathbf{g}_2$, $y \in C_2$ and $x = g_2(y) \in C_1$. So for $y \in C_2$ set $g(y) = G(g_2(y), y) - f(g_2(y))$. Then $G(x,y) = f(x) + g(y)$ on $C_1 \times C_0 \cup C_1 \times C_2$.

The above can be extended to simple inductive construction. To do this, assume that for all $k \le j$, f has been defined on C_{2k-1} and g has been defined on C_{2k} with $G(x,y) = f(x) + g(y)$ on $C_{2k-1} \times C_{2k-2} \cup C_{2k-1} \times C_{2k}$. Now for $x \in C_{2j+1}$ set

$$f(x) = G(x, f_{2j+1}(x)) - g(f_{2j+1}(x))$$

and for $y \in C_{2j+2}$ set

$$g(y) = G(g_{2j+2}(y), y)) - f(g_{2j+2}(y)).$$

Then for all $k \le j+1$, f has been defined on C_{2k-1} and g has been defined on C_{2k} with $G(x,y) = f(x) + g(y)$ on $C_{2k-1} \times C_{2k-2} \cup C_{2k-1} \times C_{2k}$ and the inductive construction is complete. This proves the first statement of the corollary.

To prove the second assertion, note that if $\mathbf{f}_{2k-1}$ and $\mathbf{g}_{2k}$ are Borel subsets of $[0,1]^2$, then f_{2k-1} and g_{2k} are Borel measurable functions: an easy proof can be formulated using Theorem 3.9 of Parthasarathy (1967, p. 10). Finally, it is clear from the above construction that if G, f_{2k-1}, and g_{2k} are all Borel measurable, then the resulting f and g are Borel measurable as well. □

Note that in Theorem 9 and Corollary 10 the rooting set was chosen using Zorn's lemma so in general nothing can be said about the measurability of the rooting set or of the limbs in the limb numbering system. In what follows we explore the case of Borel rooting sets and Borel limbs.

4. Doubly stochastic measures with Borel limb numbering system support

So far we have explored the general structure of *EDSM* supporting sets using the concept of a rooting set. More can be said if the rooting set is Borel. The main result of this section is Theorem 17, which gives several equivalent characterizations of what we call a Borel rooted Borel limbing system. Lemmas 12 and 13 are some technical results about measurability and orbits required for the proof of Theorem 17. Lemmas 14–16 establish the non-trivial assertions in Theorem 17.

We begin by introducing the concept of uniformization (Parthasarathy 1972, Chapter 8), which is useful for finding rooting sets.

Lemma 12 (uniformization lemma) Let C_1 and C_2 be subsets of $[0,1]$ and π_1, π_2 the canonical projections from $[0,1]^2$ onto $[0,1]$. Let $\mathbf{f} \subseteq [0,1]^2$ be the graph of a function $f: C_1 \to C_2$ and $\mathbf{f}'$ an F_σ subset of $\mathbf{f}$. Then there exists a Borel $\mathbf{r} \subseteq \mathbf{f}'$ such that $\pi_2\mathbf{r} = \pi_2\mathbf{f}'$ (that is, the range of r equals the range of f') and $\mathbf{r}$ is the graph of an injective function.

Proof
First suppose $\mathbf{f}'$ is a closed subset of $[0,1]^2$. For each $y \in \pi_2\mathbf{f}'$ define $r^{-1}(y) = \inf\{x: (x,y) \in \mathbf{f}'\}$. Now $(r^{-1}(y), y) \in \mathbf{f}'$. To see this, pick a sequence converging to $r^{-1}(y)$, $x_n \to r^{-1}(y)$, with $(x_n, y) \in \mathbf{f}'$ for all n. Then $(x_n, y) \to (r^{-1}(y), y)$ and $\mathbf{f}'$ is closed, so $(r^{-1}(y), y) \in \mathbf{f}'$. Set $\mathbf{r} = \{(r^{-1}(y), y): y \in \pi_2\mathbf{f}'\}$. Then $\mathbf{r} \subseteq \mathbf{f}'$ is the graph of an injective function with $\pi_2\mathbf{r} = \pi_2\mathbf{f}'$. To see that the graph of $\mathbf{r}$ is Borel, consider $r(0,\alpha)$ for $\alpha \in [0,1]$. Check that

$$r(0,\alpha) = \pi_2(\mathbf{f}' \cap \pi_1^{-1}(0,\alpha))$$

and $\mathbf{f}' \cap \pi_1^{-1}(0,\alpha)$ is an F_σ set, so $r(0,\alpha)$ is a Borel set and r^{-1} is a Borel map from $\pi_2\mathbf{f}'$ to $[0,1]$. Hence $\mathbf{r}$ is a Borel subset of $[0,1]^2$.

Now suppose $\mathbf{f}' = \bigcup_{i=1}^\infty \mathbf{f}'_i$ with each $\mathbf{f}'_i$ a closed subset of $[0,1]$. For each $\mathbf{f}'_i$ construct Borel $\mathbf{r}_i$ as done above. Partition $\pi_2\mathbf{f}'$ via $\pi_2\mathbf{f}' = \bigcup_{i=1}^\infty A_i$, with

$$A_i = \{y \in \pi_2\mathbf{f}': y \in \pi_2\mathbf{f}'_i \text{ and } y \notin \pi_2\mathbf{f}'_k \text{ for } k < i\}.$$

Set $r^{-1}(y) = \sum_{i=1}^\infty I_{A_i}(y) r_i^{-1}(y)$ and $\mathbf{r} = \{(r^{-1}(y), y): y \in \pi_2\mathbf{f}'\}$. Then r^{-1} is a Borel map from $\pi_2\mathbf{f}'$ to $[0,1]$, so $\mathbf{r}$ is a Borel subset of $[0,1]^2$. It is easy to check that $\mathbf{r} \subseteq \mathbf{f}'$ is the graph of an injective function with $\pi_2\mathbf{r} = \pi_2\mathbf{f}'$. □

The next lemma shows that minimal length axial paths are monotone.

Lemma 13 Let $\mathbf{L} = \{\mathbf{f}'_{2k-1}, \mathbf{g}_{2k}\}_{k=1}^\infty$ be a limb numbering system and let $\mathbf{h}_j$ denote $\mathbf{f}_j$ when $j = 2k-1$ and $\mathbf{g}_j$ when $j = 2k$. Suppose $\mathbf{x}_1, \ldots, \mathbf{x}_m$ is a minimal length axial path from $\mathbf{x}_1$ to $\mathbf{x}_m$. If $\mathbf{x}_1 \in \mathbf{h}_n$ and $\mathbf{x}_2 \in \mathbf{h}_{n+1}$, then $\mathbf{x}_j \in \mathbf{h}_{n+j-1}$ for $1 \le j \le m$.

Proof
Fix $k \ge 2$ and suppose $\mathbf{x}_j \in \mathbf{h}_{n+j-1}$ for $1 \le j \le k$. Assume $n+k-1$ is odd. There are three cases: $\mathbf{x}_{k+1} \in \mathbf{h}_{n+k-1}$, $\mathbf{x}_{k+1} \in \mathbf{h}_{n+k-2}$, and $\mathbf{x}_{k+1} \in \mathbf{h}_{n+k}$.

If $\mathbf{x}_{k+1} \in \mathbf{h}_{n+k-1}$, then because h_{n+k-1} is single-valued $\pi_1(\mathbf{x}_{k+1}) \ne \pi_1(\mathbf{x}_k)$. Hence, $\pi_2(\mathbf{x}_{k+1}) = \pi_2(\mathbf{x}_k)$ and $\mathbf{x}_{k+1}, \mathbf{x}_k \in \pi_2^{-1}\pi_2(\mathbf{x}_{k-1}) \cap \mathbf{h}_{n+k-1}$. This implies that $\mathbf{x}_1, \ldots, \mathbf{x}_m$ with $\mathbf{x}_k$ deleted is an axial path, which contradicts minimality. This case is therefore impossible.

If $\mathbf{x}_{k+1} \in \mathbf{h}_{n-k-2}$, then $\mathbf{x}_k \in \pi_2^{-1}\pi_2(\mathbf{x}_{k-1}) \cap \mathbf{h}_{n+k-1}$ and $\mathbf{x}_{k-1} \in \pi_2^{-1}\pi_2(\mathbf{x}_k)$, which yields $\pi_2(\mathbf{x}_{k-1}) = \pi_2(\mathbf{x}_k) = \pi_2(\mathbf{x}_{k-1})$. Because h_{n+k-2} is single-valued $\mathbf{x}_{k+1} = \mathbf{x}_{k-1}$, which again contradicts minimality. This case is also impossible.

We therefore have $\mathbf{x}_{k+1} \in \mathbf{h}_{n+k}$, which completes the induction step for $n+k-1$ odd. The argument for $n+k-1$ even is similar. □

Lemma 14 Let $\mu \in DSM$ be supported on a limb numbering system $\mathbf{L} = \{\mathbf{f}_{2k-1}, \mathbf{g}_{2k}\}_{k=1}^\infty$ with measurable limbs. Then μ has an F_σ support contained in $\cup\mathbf{L}$ with essential F_σ rooting set.

Proof
For each k one can find F_σ sets $\mathbf{f}'_{2k-1} \subseteq \mathbf{f}_{2k-1}$ and $\mathbf{g}'_{2k} \subseteq \mathbf{g}_{2k}$ such that $\mu(\mathbf{f}'_{2k-1}) = \mu(\mathbf{f}_{2k-1})$ and

$\mu(\mathbf{g}'_{2k}) = \mu(\mathbf{g}_{2k})$. Now form

$$\mathbf{L}' = \{\mathbf{f}_{2k-1}, \mathbf{g}_{2k}\}_{k=1}^{\infty}.$$

Then $F' = \cup \mathbf{L}'$ is an F_σ set, $\mu(F') = 1$, and $F' \subseteq F$. Because $F' \subseteq F$, F' has no axial cycles and by Theorem 5 is therefore an axial forest. Hence F' is an F_σ axial forest support for μ.

Next we find a Borel rooting set for F'. For each k and h'_k, ($h'_k = f'_k$ when k is odd and $h'_k = g'_k$ when k is even), construct an injection r_k with range of r_k equal to range of h_k as in Lemma 12. Inductively define subsets R_k of $[0,1]^2$ by $R_1 = \mathbf{r}_1$, and $R_k = (\mathbf{r}_k \setminus [R_{k-1}]_{F'}) \cup R_{k-1}$. Let $R = \bigcup_{k=1}^{\infty} R_k$. Then $R_k \uparrow R$ and we claim R is a rooting set for F'.

First, check that $[R]_{F'} = F'$. To see this, note that for each h'_k, the range of r_k is equal to the range of h'_k so $\mathbf{h}'_k \subseteq [\mathbf{r}_k]_{F'}$. Now

$$[R_k]_{F'} = [(\mathbf{r}_k \setminus [R_{k-1}]_{F'}) \cup R_{k-1}]_{F'} \supseteq [(\mathbf{r}_k \setminus [R_{k-1}]_{F'}] \cup [R_{k-1}]_{F'} \supseteq \mathbf{r}_k$$

so $[R_k]_{F'} \supseteq [\mathbf{r}^k]_{F'} \supseteq \mathbf{h}'_k$. Hence $F' \supseteq [R]_{F'} \supseteq \cup \mathbf{L}' = F'$.

Second, check that elements of R have disjoint orbits. Start with $R_1 = \mathbf{r}_1 \subseteq \mathbf{f}'_1$. Let $\mathbf{x}, \mathbf{y} \in \mathbf{r}_1$ and suppose $[\mathbf{x}]_{F'} = [\mathbf{y}]_{F'}$. Let $\mathbf{x} = \mathbf{x}_1, \ldots, \mathbf{x}_n = \mathbf{y}$ be a minimal length axial path from $\mathbf{x}$ to $\mathbf{y}$. We will show that $n = 1$ and therefore $\mathbf{x} = \mathbf{y}$. Suppose $n \geq 2$. There are two cases: $\mathbf{x}_2 \in \mathbf{g}'_2$ or $\mathbf{x}_2 \in \mathbf{f}'_1$. If $\mathbf{x}_2 \in \mathbf{g}'_2$, then Lemma 13 implies $\mathbf{y} \notin \mathbf{f}'_1$ which is impossible. If $\mathbf{x}_2 \in \mathbf{f}'_1$, then either $n > 2$ and $\mathbf{x}_3 \in \mathbf{g}'_2$ or $n = 2$ and $\mathbf{x}_2 = \mathbf{y}$. If $n > 2$ and $\mathbf{x}_3 \in \mathbf{g}'_2$, then Lemma 13 again implies $\mathbf{y} \notin \mathbf{f}'_1$ so is $n > 2$ is impossible. If $n = 2$ and $\mathbf{x}_2 = \mathbf{y}$ then $\pi_2\mathbf{x} = \pi_2\mathbf{y}$, which implies $\mathbf{x} = \mathbf{y}$ because r_1 is bijective. Hence $n = 2$ is impossible as well and therefore $n = 1$ and $\mathbf{x} = \mathbf{y}$. Elements of R_1 therefore have disjoint orbits. Now assume elements of R_{k-1} have disjoint orbits and consider $\mathbf{x}, \mathbf{y} \in R_k$. Suppose $[\mathbf{x}]_{F'} = [\mathbf{y}]_{F'}$. If $\mathbf{x} \in R_{k-1}$ and $\mathbf{y} \in R_{k-1}$ then $\mathbf{x} = \mathbf{y}$ because elements of R_{k-1} have disjoint orbits by assumption. The case $\mathbf{x} \in \mathbf{r}_k \setminus [R_{k-1}]_{F'}$ and $\mathbf{y} \in R_{k-1}$ is clearly impossible, so suppose $\mathbf{x}, \mathbf{y} \in \mathbf{r}_k \setminus [R_{k-1}]_{F'}$. The paragraphs above show that $\mathbf{R}_j \uparrow$ and $\mathbf{h}'_{k-1} \subseteq [R_{k-1}]_{F'}$. Hence $\bigcup_{j \leq k-1} \mathbf{h}'_j \subseteq [R_{k-1}]_{F'}$. Therefore $[\mathbf{x}]_{F'} = [\mathbf{y}]_{F'} \subseteq \bigcup_{j \geq k} \mathbf{h}'_j$. So, replacing F' with $\bigcup_{j \geq k} \mathbf{h}'_j$, the situation is equivalent to the case $\mathbf{x}, \mathbf{y} \in r_1$ and the same argument implies $\mathbf{x} = \mathbf{y}$. R is therefore a rooting set.

Now we have Borel rooting set R which is not necessarily an F_σ set. To get an F_σ essential rooting set define μ' on Borel sets of R by

$$\mu'(C) = \mu([C]_{F'}).$$

Then μ' is a Borel probability measure so there exists an F_σ set $R' \subseteq R$ with $1 = \mu'(R) = \mu'(R') = \mu([R']_{F'})$ and R' is the desired F_σ essential rooting set. □

Lemma 15 Let $\mu \in DSM$ have analytic support, A, with an essential analytic rooting set. Then μ is supported on a limb numbering system contained in A with measurable limbs and analytic rooting set.

Proof

Let F be the analytic support of μ and A the essential analytic rooting set. Define limb numbering system $\mathbf{L} = \{\mathbf{f}_{2k-1}, \mathbf{g}_{2k}\}_{k=1}^{\infty}$ as in Lemma 7. Consider the following sets:

$$\begin{aligned} A_1 &= \pi_2^{-1}\pi_2(A) \cap F, \\ A_2 &= \pi_1^{-1}\pi_1(A_1) \cap F \\ &\vdots \\ A_{2k-1} &= \pi_2^{-1}\pi_2(A_{2k-2}) \cap F, \\ A_{2k} &= \pi_1^{-1}\pi_1(A_{2k-1}) \cap F. \end{aligned}$$

Now A is analytic so $\pi_2(A)$ is analytic. Because π_2 is Borel measurable $\pi_2^{-1}\pi_2(A)$ is also analytic (Cohn 1980, Chapter 8). Hence, A_1 is analytic. The same argument with induction shows that A_j is analytic for each $j \geq 1$. Next, note that

$$A_1 = V_1 \cup H_2,$$
$$A_2 = (V_1 \cup H_2) \cup (V_2 \cup H_3)$$
$$\vdots$$
$$A_j = \bigcup_{k=1}^{j+1} (V_k \cup H_{k+1}).$$

Therefore the graph of each function from $\mathbf{L}$ can be written as the intersection of an analytic set and the complement of an analytic set. Hence the graph of each function from $\mathbf{L}$ is a measurable subset of $[0, 1]^2$.

Next, one can check that $[A]_F = [A]_{[A]_F}$ so A is an analytic rooting set (not just an essential rooting set) for $[A]_F \bigcup_{k=1}^{\infty}(\mathbf{f}_{2k-1} \cup \mathbf{g}_{2k})$. □

Lemma 16 Let $\mu \in DSM$ have F_σ support, A, with essential F_σ rooting set. Then μ is supported on a limb numbering system $\mathbf{L} = \{\mathbf{f}_{2k-1}, \mathbf{g}_{2k}\}_{k=1}^{\infty}$ with Borel limbs (that is, $\mathbf{f}_{2k-1}$ and $\mathbf{g}_{2k}$ are Borel sets for all k) and Borel rooting set. Further, $\cup \mathbf{L} \subseteq A$.

Proof
Repeat the proof of Lemma 15 replacing 'analytic' with 'F_σ' and note that A_j is an F_σ set for each j. □

Theorem 17 Let $\mu \in DSM$. The following are equivalent:

(i) μ has analytic support with essential analytic rooting set.
(ii) μ has F_σ support with essential F_σ rooting set.
(iii) μ is supported on a limb numbering system with measurable limbs.
(iv) μ is supported on a limb numbering system with Borel limbs and Borel rooting set.

Proof
By Lemma 14, (iii) implies (ii). It is trivial that (ii) implies (i) and (i) implies (iii) by Lemma 15, so (i)–(iii) are equivalent. Lemma 16 shows that (ii) implies (iv) and clearly (iv) implies (i). □

Definition A $\mu \in DSM$ satisfying the conditions of Theorem 17 is said to have *Borel rooted Borel limbing system support.*

5. Uniqueness of doubly stochastic measures with Borel rooted Borel limbing system support

In this section we address the uniqueness of doubly stochastic measures with Borel rooted Borel limbing system support. We start with some definitions.

Definition Let $GDSM$ denote the set of signed measures on $[0, 1]^2$ with uniform marginals. Also, let $CDSM$ denote the set of complex-valued measures on $[0, 1]^2$ with uniform marginals.

Definition A measure $\mu \in DSM$ is said to be in $EDSM_0$ if and only if for every $\nu \in DSM$, $\nu \ll \mu$ implies $\nu = \mu$. A measure $\mu \in DSM$ is said to be in $EDSM_1$ if and only if for every $\nu \in GDSM$, $\nu \ll \mu$ implies $\nu = \mu$. Likewise, a measure $\mu \in GDSM$ is in $EGDSM_1$ if and only if for every $\nu \in GDSM$, $\nu \ll \mu$ implies $\nu = \mu$.

Definition Let $A \subseteq [0, 1]^2$ be a universally measurable set. A is called a *set of uniqueness for DSM* if and only if there exists a unique $\mu \in DSM$ supported on A. A is called a *set of uniqueness for GDSM* if and only if there exists a unique $\nu \in GDSM$ supported on A.

Note that this definition of a set of uniqueness requires the existence of a $\mu \in DSM$. This differs from the definition of a set of marginal uniqueness given in Beneš and Štěpán (1987), which does not include this requirement.

Lemma 18 The following hold:

(i) $ESDM \subseteq DSM \subseteq GDSM \subseteq CDSM$.
(ii) $EDSM_1 \subseteq EDSM_0 \subseteq EDSM$.
(iii) $EDSM_1 \subseteq EGDSM_1$.
(iv) Let A be a set of uniqueness for DSM and let $\mu \in DSM$ be supported on A. Then $\mu \in EDSM_0$.
(v) Let A be a set of uniqueness for $GDSM$, and let $\mu \in GDSM$ be supported on A. Then $\mu \in EGDSM_1$.
(vi) A measure $\mu \in DSM$ is in $EDSM_1$ if and only if, for every $\nu \in CDSM$, $\nu \ll \mu$ implies $\nu = \mu$.

The verification of these statements is straightforward. Note that result (iv) shows that measures in $EDSM_1$ are the same as those described in Theorem 1 of Losert (1982), where a characterization of such measures is given.

Definition Let $A \subseteq [0, 1]^2$ and $GDSM(A)$ be the set of measures in $GDSM$ supported on A. Let D be a collection of Borel functions from A to the reals. D is a *$GDSM(A)$-determining class* if and only if for $\nu, \mu \in GDSM(A)$, $\int f\,\mathrm{d}\nu = \int f\,\mathrm{d}\mu$ for all $f \in D$ implies $\nu = \mu$.

Lemma 19 Suppose $\mu \in GDSM$ is supported on Borel $A \subseteq [0, 1]^2$. Let $L_{\mathrm{Marg},\infty}(A)$ denote the set of bounded measurable $G\colon [0, 1]^2 \to \mathbb{R}$ such that there exist Borel functions $f, g\colon [0, 1] \to \mathbb{R}$ with $G(x, y) = f(x) + g(y)$ for all $(x, y) \in A$. If $L_{\mathrm{Marg},\infty}(A)$ is a $GDSM(A)$-determining class, then A is a set of uniqueness for $GDSM$ and $\mu \in EGDSM_1$.

Proof
Let $G(x, y) = f(x) + g(y)$ be an element of $L_{\mathrm{Marg},\infty}(A)$. The idea is to show that $\int G\,\mathrm{d}\nu$ only depends on the marginals of ν, hence $\int G\,\mathrm{d}\nu = \int G\,\mathrm{d}\mu$. This would clearly be true if f and g were integrable with respect to λ, but this is not necessarily the case so a truncation argument is required.

For each positive integer n, let $B_n = f^{-1}(-n,n)$, $C_n = g^{-1}(-n,n)$,

$$f_n(x) = I_{B_n}(x)f(x) + nI_{f^{-1}[n,\infty)}(x) - nI_{f^{-1}(-\infty,-n]}(x)$$

and

$$g_n(y) = I_{C_n}(y)g(y) + nI_{g^{-1}[n,\infty)}(y) - nI_{g^{-1}(-\infty,-n]}(y).$$

Now approximate the integral of G with integrals of f_n and g_n,

$$\int |G(x,y) - f_n(x) - g_n(y)|\,\mathrm{d}\nu = \int_{B_n\times C_n} |G(x,y) - f_n(x) - g_n(y)|\,\mathrm{d}\nu + \int_{(B_n\times C_n)^c} |G(x,y) - f_n(x) - g_n(y)|\,\mathrm{d}\nu.$$

Now $|G(x,y) - f_n(x) - g_n(y)| = 0$ on $B_n \times C_n$, so we only need to look at the integral over $(B_n \times C_n)^c$. Let $\|G\|_u = \sup_{(x,y)\in[0,1]^2} |G(x,y)|$. Then

$$|G(x,y) - f_n(x) - g_n(y)| = |G(x,y)| \le \|G\|_u \qquad \text{on } B_n^c \times C_n^c.$$

$$|G(x,y) - f_n(x) - g_n(y)| = |g(y) - g_n(y)| \le \|G\|_u \qquad \text{on } B_n \times C_n^c$$

$$|G(x,y) - f_n(x) - g_n(y)| = |f(x) - f_n(x)| \le \|G\|_u \qquad \text{on } B_n^c \times C_n.$$

Now, as $n \to \infty$, $\lambda(B_n) \to 1$ and $\lambda(C_n) \to 1$ hence $\nu(B_n \times C_n) \to 1$. Thus,

$$\int |G(x,y) - f_n(x) - g_n(y)|\,\mathrm{d}\nu \le \nu((B_n \times C_n)^c)\|G\|_u \to 0 \tag{4}$$

and $\int G\,\mathrm{d}\nu$ can be approximated by integrals of $f_n(x)$ and $g_n(y)$. Finally, we have

$$\int_{[0,1]^2} f_n(x) + g_n(y)\,\mathrm{d}\nu = \int_{[0,1]} f_n(x)\,\mathrm{d}\lambda + \int_{[0,1]} g_n(y)\,\mathrm{d}\lambda = \int_{[0,1]^2} f_n(x) + g_n(y)\,\mathrm{d}\mu \tag{5}$$

and ν and μ have equal integrals on $L_{\mathrm{Marg},\infty}(A)$. □

Theorem 20 Let $\mu \in DSM$ have Borel rooted Borel limbing system support; then μ is supported on a set of uniqueness for DSM and $\mu \in EDSM_1$. A similar statement holds for $\nu \in GDSM$ having Borel rooted Borel limbing system support.

Proof
Let A be the Borel rooted Borel limbing system support of μ. Then by Corollary 11, $L_{\mathrm{Marg},\infty}(A)$ equals the collection of bounded measurable functions on $[0,1]^2$ and the result follows from Lemma 19. □

6. Further results concerning Borel rooted Borel limbing system support

Theorem 20 can also be proved by using the uniform marginals of μ to solve explicitly for μ. Results of this argument are useful in their own right and lead to a generalized example of a limb numbering system which is the support of a measure in $GDSM$. To present these results we examine the structure of Borel rooted Borel limbing system supports in more detail, with the goal of finding a direct proof of Theorem 20.

To begin, let $\mathbf{L} = \{\mathbf{f}_{2k-1}, \mathbf{g}_{2k}\}_{k=1}^{\infty}$ be a limb numbering system with Borel limbs. As done previously, let h_n denote a generic function from $\mathbf{L}$: $h_n = f_n$ when n is odd and $h_n = g_n$ when n is even. Also, let $\{C_{2k+1}\}_{k=0}^{\infty}$ and $\{C_{2k}\}_{k=0}^{\infty}$ be the corresponding partitions of $[0,1]$ with h_k: $C_k \to C_{k-1}$: that is, f_{2k+1}: $C_{2k+1} \to C_{2k}$ and g_{2k}: $C_{2k} \to C_{2k-1}$.

Let $\nu \in GDSM$ be supported on $\cup\mathbf{L}$. It will be useful to consider the x and y marginals of the ν mass restricted to h_n. For this purpose, let $\nu_{2k}(\cdot) = \nu(\pi_2^{-1}(\cdot) \cap \mathbf{g}_{2k})$ and $\nu_{2k+1}(\cdot) = \nu(\pi_1^{-1}(\cdot) \cap \mathbf{f}_{2k+1})$. Note that the ν_n's uniquely determine ν. To see this, suppose that $B = [a,b] \times [c,d] \cap C_{2k+1} \times C_{2k}$. On B, ν is supported on $\mathbf{f}_{2k+1}$ so $\nu(B) = \nu_{2k+1}(f_{2k+1}^{-1}([c,d]) \cap [a,b])$.

Lemma 21 The following hold for the above-described limb numbering system:

(i) If $\cup\mathbf{L}$ is the support of a $\mu \in DSM$ then $\sum_{k=0}^{n} \lambda(C_{2k}) \geq \sum_{k=1}^{n} \lambda(C_{2k-1})$ and $\sum_{k=0}^{n} \lambda(C_{2k+1}) \geq \sum_{k=0}^{n} \lambda(C_{2k})$.

(ii) For each $n \geq 0$ let '$\lambda|_{C_n}$' denote λ restricted to C_n, that is, $\lambda|_{C_n}(B) = \lambda(B \cap C_n)$. Then $\lambda|_{C_0} = \nu_1 \circ f_1^{-1}$ and for each $n \geq 1$, $\lambda|_{C_n} = \nu_n + \nu_{n+1} \circ h_{n+1}^{-1}$.

Proof

Note that $\lambda(C_0) = \mu([0,1] \times C_0) = \mu(C_1 \times C_0) = \mu(\mathbf{f}_1)$. Likewise, $\lambda(C_1) = \mu(\mathbf{f}_1) + \mu(\mathbf{g}_2)$. In general,

$$\lambda(C_{2k-1}) = \mu(\mathbf{f}_{2k-1}) + \mu(\mathbf{g}_{2k})$$

and

$$\lambda(C_{2k}) = \mu(\mathbf{g}_{2k}) + \mu(\mathbf{f}_{2k+1}).$$

A simple induction argument then shows that

$$\lambda(C_{2k}) = \mu(\mathbf{f}_{2k+1}) + \sum_{n=0}^{2k-1} (-1)^{n+1} \lambda(C_n) \tag{6}$$

and

$$\lambda(C_{2k-1}) = \mu(\mathbf{g}_{2k}) + \sum_{n=0}^{2k-2} (-1)^{n} \lambda(C_n). \tag{7}$$

Because $\mu \in DSM$, $\mu(\mathbf{f}_{2k-1})$ and $\mu(\mathbf{g}_{2k})$ are non-negative, and (i) follows.

Part (ii) is simply a result of the definition of ν_n and the fact that ν has Lebesgue marginals. To show this for $\lambda|_{C_0}$, let $B \subseteq C_0$ and check that $(C_1 \times B) \cap \mathbf{f}_1 = \pi_1^{-1} f_1^{-1}(B) \cap \mathbf{f}_1$. So

$$\lambda|_{C_0}(B) = \lambda(B) = \nu([0,1] \times B) = \nu(C_1 \times B) \cap \mathbf{f}_1) = \nu(\pi_1^{-1} f_1^{-1}(B) \cap \mathbf{f}_1) = \nu_1 \circ f_1^{-1}(B).$$

To prove the result for $\lambda|_{C_n}$, assume $n = 2k - 1$, let $B \subseteq C_{2k-1}$, and check that $B \times [0,1] \cap \mathbf{g}_{2k} = \pi_2^{-1} g_{2k}^{-1}(B) \cap \mathbf{g}_{2k}$. Then

$$\begin{aligned}\lambda|_{C_n}(B) &= \lambda(B) = \nu(B \times [0,1]) = \nu(B \times [0,1] \cap \mathbf{f}_{2k-1}) + \nu(B \times [0,1] \cap \mathbf{g}_{2k}) \\ &= \nu(\pi_1^{-1}(B) \cap \mathbf{f}_{2k-1}) + \nu(\pi_2 g_{2k}^{-1}(B) \cap \mathbf{g}_{2k}) = \nu_{2k-1}(B) + \nu_{2k}(g_{2k}^{-1}(B)).\end{aligned}$$

The case for $n = 2k$ follows with a similar argument. □

As shown next, part (ii) of Lemma 21 can be used to prove that the ν_n's and hence ν are uniquely determined by the limb numbering system **L**.

Lemma 22 Let $m > n$. The following formula holds for ν_n:

$$\nu_n = \lambda|_{C_n} + \sum_{k=n+1}^{m} (-1)^{k-n} \lambda|_{C_k} \circ h_k^{-1} \circ h_{k-1}^{-1} \circ \ldots \circ h_{n+1}^{-1}$$
$$+ (-1)^{m-n+1} \nu_{m+1} \circ h_{m+1}^{-1} \circ h_m^{-1} \circ \ldots \circ h_{n+1}^{-1}. \quad (8)$$

Further, as $m \to \infty$

$$\nu_{m+1} \circ h_{m+1}^{-1} \circ h_m^{-1} \circ \ldots \circ h_{n+1}^{-1} \to 0$$

in total variation norm. Hence ν_n and therefore ν are uniquely determined by the limb numbering system **L** via

$$\nu_n = \lambda|_{C_n} + \sum_{k=n+1}^{\infty} (-1)^{k-n} \lambda|_{C_k} \circ h_k^{-1} \circ h_{k-1}^{-1} \circ \ldots \circ h_{n+1}^{-1} \quad (9)$$

and $\cup$ **L** is a set of uniqueness for *GDSM*.

Proof
Use part (ii) of Lemma 21 to write

$$\nu_n = \lambda|_{C_n} - \nu_{n+1} \circ h_{n+1}^{-1}. \quad (10)$$

Now use (ii) of Lemma 21 again to solve for ν_{n+1} in terms of $\lambda|_{C_{n+1}}$ and $\nu_{n+2} \circ h_{n+2}^{-1}$, and substitute into (10) to get

$$\nu_n = \lambda|_{C_n} - (\lambda|_{C_{n+1}} - \nu_{n+2} \circ h_{n+2}^{-1}) \circ h_{n+1}^{-1}.$$

This proves the lemma for $m = n + 1$. A simple induction with similar substitution proves (8).

To prove that $\nu_{m+1} \circ h_{m+1}^{-1} \circ h_m^{-1} \circ \ldots \circ h_{n+1}^{-1}$ goes to 0 in total variation norm, it suffices to show that ν_{m+1} goes to 0 in total variation norm. To this end, choose arbitrary $B \subseteq [0, 1]$ and suppose $m + 1 = 2k - 1$. Then

$$|\nu_{2k-1}|(B) \leq |\nu_{2k-1}|(C_{2k-1}).$$

Let $\epsilon > 0$. Then there exists partition E_i of C_{2k-1} such that

$$|\nu_{2k-1}|(C_{2k-1}) < \sum_{i=1}^{m} |\nu_{2k-1}(E_i)| + \epsilon = \sum_{i=1}^{m} |\nu(\pi_1^{-1}(E_i) \cap \mathbf{f}_{2k-1}| + \epsilon$$
$$\leq \sum_{i=1}^{m} |\nu|(\pi_1^{-1}(E_i) \cap \mathbf{f}_{2k-1}) + \epsilon = |\nu|(C_{2k-1} \times C_{2k-2}) + \epsilon.$$

So for all k

$$|\nu_{2k-1}|(B) \leq |\nu|(C_{2k-1} \times C_{2k-2}).$$

But

$$\sum_{k=1}^{\infty} |\nu|(C_{2k-1} \times C_{2k-2}) \leq |\nu|([0,1]^2) < \infty$$

so

$$|\nu|(C_{2k-1} \times C_{2k-2}) \to 0$$

and $\nu_{2k-1} \to 0$ in total variation norm. Once again a similar argument works for $m = 2k$ and ν_{2k}. □

7. Some examples of doubly stochastic measures

EXAMPLE 1. CONSTRUCTION OF A GENETIC LIMBING SYSTEM

The results of Lemmas 21 and 22 can be used to construct a limb numbering system in a generic way which supports a measure in *GDSM*. To do this, we start with two partitions, $\{C_{2k+1}\}_{k=0}^{\infty}$ and $\{C_{2k}\}_{k=0}^{\infty}$, of $[0,1]$ that satisfy condition (i) of Lemma 21. To avoid complication, we also require that $\lambda(C_n) > 0$ for all $n \geq 0$. Next we take a collection of Borel functions $\{\phi_i\}_{i=1}^{\infty}$, ϕ_i: $[0,1] \to [0,1]$. To eliminate the possibility of atoms with positive mass in the marginals we require that $\lambda(\phi_i^{-1}(\{x\})) = 0$ for all $x \in [0,1]$. Finally, we squeeze each ϕ_i into an h_i: $C_i \to C_{i-1}$. If the squeeze is done correctly Lemmas 21 and 22 can be used to check that the resulting limb numbering system supports a measure $\nu \in GDSM$. This procedure is discussed in detail below.

First, define constants

$$\beta_1 = \frac{\lambda(C_0)}{\lambda(C_1)}$$

and, for $n \geq 1$,

$$\beta_{2n} = \frac{\sum_{i=0}^{n-1} \lambda(C_{2i+1}) - \sum_{i=0}^{n-1} \lambda(C_{2i})}{\lambda(C_{2n})}$$

$$\beta_{2n+1} = \frac{\sum_{i=0}^{n} \lambda(C_{2i}) - \sum_{i=0}^{n-1} \lambda(C_{2i+1})}{\lambda(C_{2n+1})}.$$

It is easy to check that for all $i \geq 1$, $0 \leq \beta_i < 1$ and

$$\lambda(C_{i+1})\beta_{i+1} = (1 - \beta_i)\lambda(C_i). \tag{11}$$

Next, define S_i: $[0,1] \to [0,1]$ by

$$S_i(z) = \frac{\lambda(C_i \cap [0,z])}{\lambda(C_i)}.$$

Also, let $T_i = S_i|_{C_i}$. Next, for each $i > 0$ define ψ_i: $[0,1] \to [0,1]$ by

$$\psi_i(x) = \lambda \circ \phi_i^{-1}[0,x].$$

Lastly, let I denote the identity function on $[0,1]$, $I(x) = x$, and define G_i: $[0,1] \to [0,1]$ by

$$G_0(x) = \psi_1(x)$$

and, for $i \geq 1$,

$$G_i(x) = \beta_i I(x) + (1 - \beta_i)\psi_{i+1}(x).$$

To squeeze the ϕ_i's we set

$$h_i = T_{i-1}^{-1} \circ G_{i-1} \circ \phi_i \circ G_i^{-1} \circ T_i. \tag{12}$$

Clearly h_i: $C_i \to C_{i-1}$. To show that this squeeze gives a limb numbering system which supports a measure in $GDSM$ requires some work, starting with the following lemma.

Lemma 23 Let $A \subseteq [0,1]$ be a Borel subset of $[0,1]$. Then:

(i) ψ_{i+1}, T_i and G_i are all Borel measurable functions for $i \geq 0$.
(ii) For $i \geq 0$, $T_i(A)$, $G_i(A)$, and $\psi_{i+1}(A)$ are all Borel sets and $\lambda \circ \psi_{i+1}$, $\lambda \circ S_i$ are measures.
(iii) G_i is injective for $i \geq 1$, and T_i is injective a.s. for $i \geq 0$. Also, $\lambda(S_i(A)) = \lambda(T_i(A)) = \lambda(A \cap C_i)/\lambda(C_i)$ so T_i is surjective a.s. for $i \geq 0$.
(iv) $\lambda|_{C_i} \circ T_i^{-1}(A) = \lambda(C_i)\lambda(A)$.
(v) $\lambda \circ \lambda(C_i)[(\beta_i I + (1-\beta_i)\psi_{i+1}](A) = \lambda \circ \lambda(C_i)\beta_i I(A) + \lambda \circ \lambda(C_i)(1-\beta_i)\psi_{i+1}(A)$.
(vi) $\lambda(\psi_i(A)) = \lambda(\phi_i^{-1}(A))$.
(vii) $\lambda(\phi_1^{-1}\psi_1^{-1}(A)) = \lambda(A)$.
(viii) $\lambda \circ \phi_{n+1}^{-1} \circ G_n^{-1} \circ S_n[0,x] = \lambda \circ \phi_{n+1}^{-1} \circ G_n^{-1} \circ T_n[0,x]$.

Proof
Statements (i) and (ii) are true for any non-decreasing function f: $[0,1] \to [0,1]$. To show this, suppose f is such a function. Clearly $f^{-1}(-\infty, a)$ is Borel for any $a \in \mathbb{R}$. To check that $f(A)$ is Borel, notice that for each $y \in [0,1]$, $f^{-1}(y)$ is empty, a singleton set, or an interval contained in $[0,1]$. Hence there is a countable set $\{y_i\}$ such that $f^{-1}(y_i)$ is an interval. Now set $B = \bigcup_i f^{-1}(y_i)$. Then $C = [0,1] \setminus B$ is Borel and $f|_C$ is injective. By a well-known result (Theorem 3.9 in Parthasarathy 1967, p. 21), an injective image of a Borel set under a measurable map is Borel. Hence

$$f(A) = f(A \cap B) \cup f(A \cap C)$$

with $f(A \cap B)$ countable so $f(A)$ is Borel. To show that $\lambda \circ f$ is a measure, notice that $\lambda \circ f(A) = \lambda \circ f(A \cap C)$ with $f|_C$ being injective.

To prove (iii), first note that G_i is obviously injective for $i \geq 1$. Also, T_i is injective at points of density of C_i (for a definition, see Hewitt and Stromberg 1975, p. 274) so T_i is injective λ a.s. To prove the statement about T_i, first notice that S_i is continuous and increasing so $S_i((a,b)) = (S_i(a), S_i(b))\lambda$ a.s. Hence, $\lambda(S_i((a,b)) = \{\lambda(C_i)\}^{-1}\lambda((a,b) \cap C_i)$. By (ii), $\lambda \circ S_i(\cdot)$ is a measure as is $\{\lambda(C_i)\}^{-1}\lambda(\cdot \cap C_i)$ so we can conclude that

$$\lambda(S_i(B)) = \frac{1}{\lambda(C_i)}\lambda(B \cap C_i)$$

for any Borel set $B \subseteq [0,1]$. Hence

$$\frac{1}{\lambda(C_i)}\lambda(A \cap C_i) = \lambda(S_i(A)) = \lambda(S_i(A \cap C_i)) + \lambda(S_i(A \cap C_i^c) = \lambda(S_i(A \cap C_i)) = \lambda(T_i(A)).$$

Part (iv) can be verified in the case where A is an interval. Because both sides of (iv) are measures the result follows for general A.

Next consider statement (v). By (ii) both sides of (v) are measures so we only need prove (v) for $A = [0,a]$. Notice that the requirement $\lambda(\phi_i^{-1}(\{x\})) = 0$ makes ψ_{i-1} continuous, so checking (v) for $A = [0,a]$ is a straightforward calculation.

To prove (vi), recall that by (ii) $\lambda \circ \psi_i(\cdot)$ is a measure. Clearly $\lambda \circ \phi_i^{-1}(\cdot)$ is a measure. Now check that $\lambda \circ \psi_i$ and $\lambda \circ \phi_i^{-1}$ agree on sets of the form $A = [0,a]$. Part (vii) can be established in the same way.

To prove (viii), note that (iii) implies $\lambda(S_n[0,x] \setminus T_n[0,x]) = 0$. So (viii) can be established by showing $\lambda \circ \phi_{n+1}^{-1} \circ G_n^{-1} \ll \lambda$. To prove this, note that G_n is an injective continuous increasing map from $[0,1]$ onto $[0,1]$. Hence

$$\begin{aligned}\lambda[0,x] &= \lambda[0, G_n G_n^{-1}(x)] = G_n G_n^{-1}(x)\\ &= \beta_n \lambda[0, G_n^{-1}(x)] + (1-\beta_n)\lambda \circ \phi_{n+1}^{-1}[0, G_n^{-1}(x)]\\ &= \beta_n \lambda \circ G_n^{-1}[0,x] + (1-\beta_n)\lambda \circ \phi_{n+1}^{-1} \circ G_n^{-1}[0,x].\end{aligned}$$

Now $\lambda \circ G_n^{-1}$ and $\lambda \circ \phi_{n+1}^{-1} \circ G_n^{-1}$ are measures, so

$$\lambda(A) = \beta_n \lambda \circ G_n^{-1}(A) + (1-\beta_n)\lambda \circ \phi_{n+1}^{-1} \circ G_n^{-1}(A)$$

for all Borel sets A. Hence $\lambda \circ \phi_{n+1}^{-1} \circ G_n^{-1} \ll \lambda$. □

Note that parts (i)–(iii) of Lemma 23 show that h_i is single-valued a.s. and measurable. Next, we can define a measure ν_i using the limb numbering system $\mathbf{L} = \{\mathbf{h}_i\}_{i=1}^{\infty}$ and equation (9). Note that (9) is an absolutely convergent sum so that ν_i is a well-defined finite signed measure on $[0,1]$ which is supported on C_i. We can then define ν supported on $\cup\mathbf{L}$ using the ν_i's in the obvious way; for $i = 2k+1$ and $B = B_1 \times B_2 \cap h_{2k+1}$ set $\nu(B) = \nu_{2k+1}(h_{2k+1}^{-1}(B_2) \cap B_1)$, and for $i = 2k$ and $B = B_1 \times B_2 \cap h_{2k}$ set $\nu(B) = \nu_{2k}(h_{2k}^{-1}(B_1) \cap B_2)$. So defined, ν is a σ-finite signed measure supported on $\cup\mathbf{L} = \bigcup_{i=1}^{\infty} \mathbf{h}_i$.

Now to check that the marginals of ν are Lebesgue measure, suppose $A \subseteq C_n$. Set $B = A \times [0,1]$ if n is odd and $B = [0,1] \times A$ if n is even. Then, for $n > 0$,

$$\begin{aligned}\nu(B) &= \nu(B \cap \mathbf{h}_n) + \nu(B \cap \mathbf{h}_{n+1})\\ &= \nu_n(A) + \nu_{n+1}(h_{n+1}^{-1}(A)) = \lambda|_{C_n}(A)\end{aligned}$$

by (10) which holds for ν_n defined by (9).

Lastly, we need to check the case $n = 0$. Here all the mass which cancels using (9) piles up, so a more delicate calculation is required.

First, the following equation needs to be established:

$$\nu_n(A) = \lambda|_{C_n}(A) = \lambda(C_{n-1})\beta_{n-1}\lambda \circ \phi_{n+1}^{-1} \circ G_n^{-1} \circ T_n(A). \tag{13}$$

To prove (13), chose Borel $A \subseteq [0,1]$ and use (9) to write

$$\nu_n(A) = \lambda|_{C_n}(A) + \sum_{k=n+1}^{\infty} (-1)^{k-n} \lambda|_{C_k} \circ h_k^{-1} \circ \ldots \circ h_{n+1}^{-1}(A)$$

$$= \lambda|_{C_n}(A) + \sum_{k=n+1}^{\infty} (-1)^{k-n} \lambda|_{C_k} \circ T_k^{-1} \circ G_k \circ \phi_k^{-1} \circ \phi_{k-1}^{-1} \circ \ldots \circ \phi_{n+1}^{-1} \circ G_n^{-1} \circ T_n(A).$$

By parts (iv) and (v) of Lemma 23 this yields

$$\nu_n(A) = \lambda|_{C_n}(A) + \sum_{k=n+1}^{\infty} (-1)^{k-n} \lambda(C_k)\lambda \circ (\beta_k I + (1-\beta_k)\psi_{k+1}) \circ \phi_k^{-1} \circ \ldots \circ \phi_{n+1}^{-1} \circ G_n^{-1} \circ T_n(A)$$

$$= \lambda|_{C_n}(A) - \lambda(C_{n+1})\beta_{n+1}\lambda \circ \phi_{n+1}^{-1} \circ G_n^{-1} \circ T_n(A) + \sum_{k=n+1}^{\infty} (-1)^{k-n} (\lambda(C_k)(1-\beta_k)\lambda \circ \psi_{k+1}$$

$$- \lambda(C_{k+1})\beta_k \lambda \circ \phi_{k+1}^{-1}) \circ \ldots \circ \phi_{n+1}^{-1} \circ G_n^{-1} \circ T_n(A).$$

By (11) the terms in the infinite sum are zero, which establishes (13).

Now choose Borel $A \subseteq C_0$ and set $B + [0,1] \times A$. Using (13), we then have

$$\nu(B) = \nu_1(h_1^{-1}(A))$$

$$= \lambda|_{C_1} \circ h_1^{-1}(A) - \lambda(C_2)\beta_2\lambda \circ \phi_2^{-1} \circ G_1^{-1} \circ T_1 \circ h_1^{-1}(A)$$

$$= \lambda|_{C_1} \circ h_1^{-1}(A) - \lambda(C_2)\beta_2\lambda \circ \phi_2^{-1} \circ \phi_1^{-1} \circ G_0^{-1} \circ T_0(A)$$

$$= (\lambda|_{C_1} \circ T_1^{-1} \circ G_1 - \lambda(C_2)\beta_2\lambda \circ \phi_2^{-1}) \circ \phi_1^{-1} \circ G_0^{-1} \circ T_0(A).$$

By parts (iv)–(vii) of Lemma 23 this gives

$$\nu(B) = (\lambda(C_1)\lambda \circ G_1 - \lambda(C_2)\beta_2\lambda \circ \phi_2^{-1}) \circ \phi_1^{-1} \circ G_0^{-1} \circ T_0(A)$$

$$= (\lambda(C_1)\beta_1\lambda + \lambda(C_1)(1-\beta_1)\lambda \circ \psi_2 - \lambda(C_2)\beta_2\lambda \circ \phi_2^{-1}) \circ \phi_1^{-1} \circ G_0^{-1} \circ T_0(A)$$

$$= \lambda(C_1)\beta_1\lambda \circ \phi_{-1}^{1} \circ G_0^{-1} \circ T_0(A)$$

$$= \lambda(C_0)\lambda \circ T_0(A) = \lambda(A \cap C_0) = \lambda|_{C_0}(A).$$

Finally, we need to determine if ν is a positive measure ($\nu \in DSM$) or ν is a signed measure ($\nu \in GDSM \setminus DSM$). Using part (viii) of Lemma 23 and equation (13) we have

$$\nu_n([0,x]) = \lambda|_{C_n}([0,x]) - \lambda(C_{n+1})\beta_{n+1}\lambda \circ \phi_{n+1}^{-1} \circ G_n^{-1} \circ S_n([0,x])$$

$$= \lambda(C_n)S_n(x) - \lambda(C_{n+1})\beta_{n+1}\psi_{n+1} \circ G_n^{-1} \circ S_n(x).$$

Now we claim that this is an increasing function of x. To see this, set

$$\xi(x) = \xi = G_n^{-1} \circ S_n(x).$$

Clearly ξ is an increasing function of x. So

$$\begin{aligned}\nu_n([0,x]) &= \lambda(C_n)S_n(x) - \lambda(C_{n+1})\beta_{n+1}\psi_{n+1} \circ G_n^{-1} \circ S_n(x)\\ &= (\beta_n\xi + (1-\beta_n)\psi_{n+1}(\xi))\lambda(C_n) - \lambda(C_{n+1})\beta_{n+1}(1-\beta_n)\psi_{n+1}(\xi)\\ &= \lambda(C_n)\beta_n\xi + \beta_n\lambda(C_{n+1})\beta_{n+1}\psi_{n+1}(\xi)\end{aligned}$$

which is an increasing function of ξ. This makes ν_n a positive measure and hence ν a positive measure. To summarize the result:

Theorem 24 Let $\{C_{2k+1}\}_{k=0}^{\infty}$ and $\{C_{2k}\}_{k=0}^{\infty}$ be partitions of $[0,1]$ that satisfy condition (i) of Lemma 21. Let $\{\phi_i\}_{i=1}^{\infty}$ be a collection of Borel functions ϕ_i: $[0,1] \to [0,1]$, with $\lambda(\phi_i^{-1}(\{x\})) = 0$ for all $x \in [0,1]$. Define h_i as in equation (12), ν_n as in equation (9),

$$\mathbf{f}_{2n-1} = \{(x, h_{2n-1}(x)) \colon x \in C_{2n-1}\}$$

and

$$\mathbf{g}_{2n} = \{(h_{2n}(x), x) \colon x \in C_{2n}\}.$$

Let B_1 and B_2 be arbitrary Borel subsets of $[0,1]$. Define measure ν on $[0,1]^2$ by the following: for $i = 2k+1$ and $B = B_1 \times B_2 \cap h_{2k+1}$ set $\nu(B) = \nu_{2k+1}(h_{2k+1}^{-1}(B_2) \cap B_1)$, and for $i = 2k$ and $B = B_1 \times B_2 \cap h_{2k}$ set $\nu(B) = \nu_{2k}(h_{2k}^{-1}(B_1) \cap B_2)$. Then ν is the unique element of *DSM* supported on limb numbering system $\mathbf{L} = \{\mathbf{f}_{2k-1}, \mathbf{g}_{2k}\}_{k=1}^{\infty}$.

Proof
See above discussion. □

Example 1A *Sets of uniqueness for DSM*
Concrete examples of the generic construction in Theorem 24 are plotted in Figs 1 and 2. These limb numbering systems were constructed by setting $\phi_i(x) = f(x)$ for each i. In Fig. 1, $f(x) = \sqrt{x}$. In Fig. 2, $f(x) = (x - \frac{1}{2})^2$. The two partitions, $\{C_{2k+1}\}_{k=0}^{\infty}$ and $\{C_{2k}\}_{k=0}^{\infty}$, were taken to be the same in each of the figures. To describe the partitions, let $c_i = \lambda(C_i)$ for $i \geq 0$,

$$\alpha_{2n} = \sum_{k=0}^{n} c_{2k} - \sum_{k=1}^{n} c_{2k-1} \tag{14a}$$

and

$$\alpha_{2n+1} = \sum_{k=0}^{n} c_{2k+1} - \sum_{k=0}^{n} c_{2k}. \tag{14b}$$

Then $\alpha_j \geq 0$,

$$\sum_{j=1}^{\infty} \alpha_j = 1 - c_0,$$

$$c_1 = \alpha_1 + c_0$$

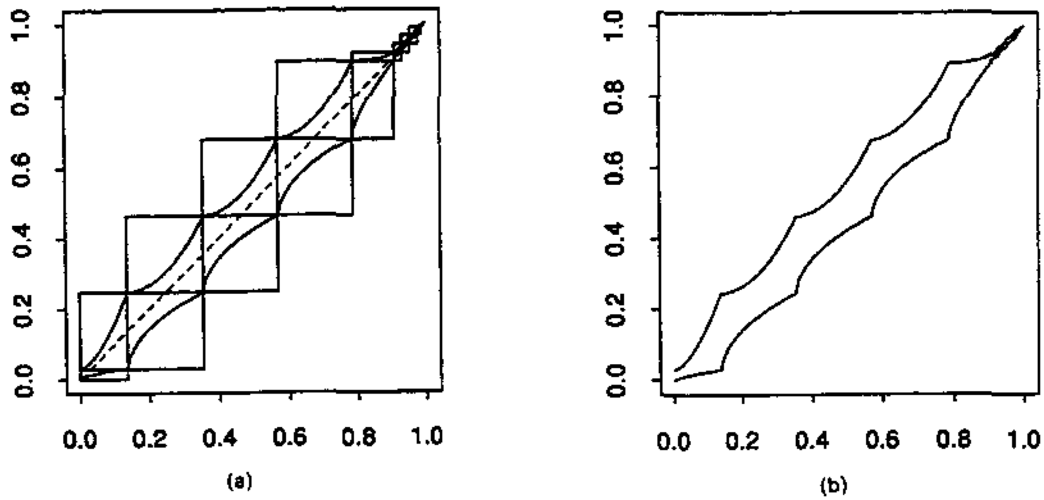

Figure 1. Example set of uniqueness for DSM using $\phi_i(x) = \sqrt{x}$. (a) shows the limbing numbering system as well as the diagonal and boxes outlining the sets $C_{2n+1} \times C_{2n}$. In (b) only the limb numbering system support is shown

and, for $k \geq 2$,

$$c_k = \alpha_{k-1} + \alpha_k.$$

So for positive α_j's summing to $1 - c_0$ we set

$$C_0 = [0, c_0), \tag{15a}$$

$$C_1 = [0, \alpha_1 + c_0) \tag{15b}$$

and

$$C_k = \left[c_0 + \sum_{j=1}^{k-2} \alpha_j, c_0 + \sum_{j=1}^{k} \alpha_j\right). \tag{15c}$$

The example in Fig. 1 is based on partitions defined by $c_0 = 0.03$ and $\alpha_{8m+k} = r^{m+1}$, which leads to the requirement that $r = (1 - c_0)/(9 - c_0)$. In Figs 1a and 2a the diagonal $y = x$ is shown as well as the boxes outlining the sets $C_{2n+1} \times C_{2n}$ and $C_{2n-1} \times C_{2n+2}$. In Figs 1b and 2b just the limb numbering system support $\{h_i\}$ is plotted. Both figures are sets of uniqueness for DSM and $GDSM$. Note also that the functions in Example 2 are not monotone or bijective as are examples in Seethoff and Shiflett (1978) and Kamiński *et al.* (1990).

Example 1B *Sets of uniqueness for $GDSM \setminus DSM$*

To construct a Borel rooted Borel limbing system which supports a measure in $GDSM \setminus DSM$, one can adjust the partitions $\{C_{2k-1}\}_{k=0}^{\infty}$ and $\{C_{2k}\}_{k=0}^{\infty}$ so that they no longer satisfy condition (i) of

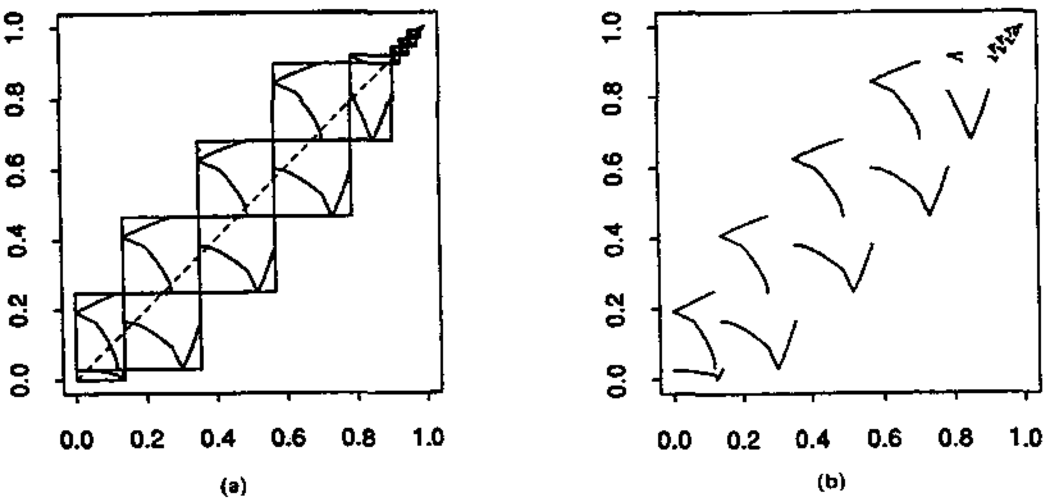

Figure 2. Example set of uniqueness for *DSM* using $\phi_i(x) = (x - \frac{1}{2})^2$. (a) shows the limbing numbering system as well as the diagonal and boxes outlining the sets $C_{2n+1} \times C_{2n}$. In (b) only the limb numbering system support is shown

Lemma 21. In this case we no longer have $0 \le \beta_i < 1$, so some care must be used in applying the above arguments. However, Lemma 23 will still hold if G_i is strictly increasing for all i and we can make G_i strictly increasing by carefully choosing the ϕ_i's.

With the ϕ_i's chosen to satisfy Lemma 23, the above arguments go through and we have a measure ν on a Borel limb numbering system which is a set of uniqueness for *GDSM* but not for *DSM*. Figure 3 is an example of such a set. To describe Fig. 3, first consider the partitions $\{C_{2k+1}\}_{k=0}^{\infty}$ and $\{C_{2k}\}_{k=0}^{\infty}$. Fix $r \in (-1, 0)$. Consider α_k's as defined in (14). Set

$$\alpha_{2n} = r^{2n+3}$$

and

$$\alpha_{2n+1} = r^{2n+2}.$$

Then $c_k = \alpha_{k-1} + \alpha_k$ is positive. Now we require

$$0 < \sum_{j=1}^{\infty} \alpha_j = \frac{r^2(r^3+1)}{1-r^2} = 1 - c_0 < 1. \tag{16}$$

Given that (16) holds, define C_k as in (15). Next, check that

$$\beta_1 = \frac{c_0}{c_0 + r^2} > 0$$

and, for $n \ge 1$,

$$\beta_{2n} = \frac{1}{r^3 + 1} > 0$$

and

$$\beta_{2n+1} = \frac{r}{r+1} < 0.$$

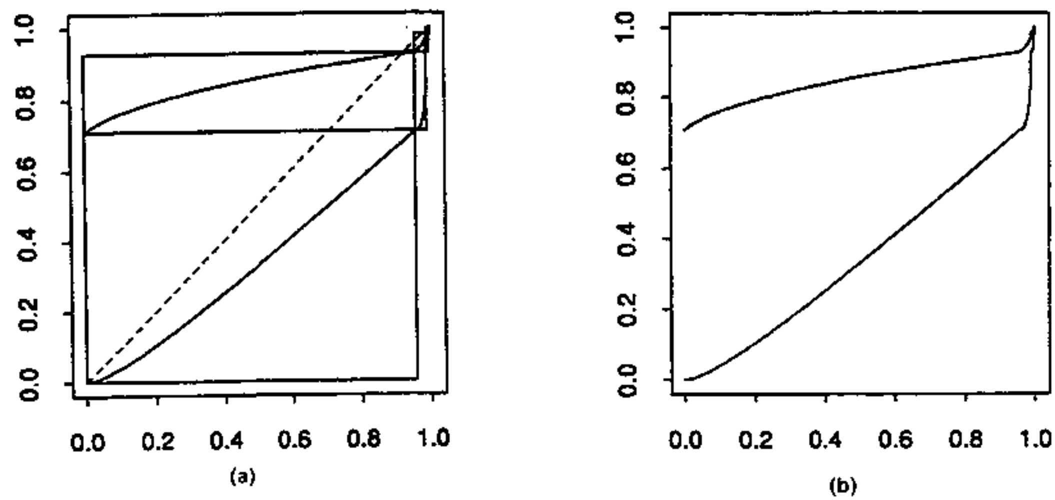

Figure 3. Example set of uniqueness for *GDSM* but not for *DSM*, as explained in the text. (a) shows the limbing numbering system as well as the diagonal and boxes outlining the sets $C_{2n+1} \times C_{2n}$. In (b) only the limb numbering system support is shown

Finally, set $\phi_{2n+1}(x) = x$ and $\phi_{2n}(x) = x^2$. Then $G_{2n}(x) = x$ and $G_{2n+1}(x) = \beta_{2n-1}x + (1 - \beta_{2n-1})\sqrt{x}$. Choosing $r = -0.5$ then makes G_k increasing for all k and satisfies (15) as well. Figure 3 shows the resulting limb numbering system which is a set of uniqueness for *GDSM* but not for *DSM*.

Example 2 *$\mu \in EDSM_1$ which does not have Borel rooted Borel limiting system support*
There are elements of $EDSM_1$ which do not have Borel rooted Borel limbing system support. One such example can be constructed by considering a generalization of limb numbering systems called an *axial function forest*.

Definition Let $\{C_n\}_{n=-\infty}^{\infty}$ be a collection of Borel subsets of $[0, 1]$ so that both $\{C_{2n}\}_{n=-\infty}^{\infty}$ and $\{C_{2n-1}\}_{n=-\infty}^{\infty}$ are partitions of $[0, 1]$. For each n, let h_n: $C_n \to C_{n-1}$ be a Borel function, and let

$$\mathbf{f}_{2n-1} = \{(x, h_{2n-1}(x)) \colon x \in C_{2n-1}\}$$

and

$$\mathbf{g}_{2n} = \{(h_{2n}(x), x) \colon x \in C_{2n}\}.$$

Then $\mathbf{L} = \{\mathbf{f}_{2n-1}, \mathbf{g}_{2n}\}_{n=-\infty}^{\infty}$ is called a *Borel axial function forest*.

A result similar to Theorem 20 holds for Borel axial function forests.

Theorem 25 Let $\mathbf{L}$ be a Borel axial function forest which is the support of $\mu \in DSM$ (or $\mu \in GDSM$). Then $\mathbf{L}$ is a set of uniqueness for *DSM* (for *GDSM*).

Proof

Let $\mathbf{L} = \{\mathbf{f}_{2k-1}, \mathbf{g}_{2k}\}_{k=-\infty}^{\infty}$ as in the above definition. Set $A = \bigcup_{k=-\infty}^{\infty}(\mathbf{f}_{2k-1} \cup \mathbf{g}_{2k})$ and, for $n \in \mathbb{Z}$, $A_n = \bigcup_{k \geq n}(\mathbf{f}_{2k-1} \cup \mathbf{g}_{2k})$. Let $G \in L_1(A)$ and $G_n(x, y) = G(x, y) I_{A_n}(x, y)$. Now $\mathbf{L}_n = \{\mathbf{f}_{2k-1}, \mathbf{g}_{2k}\}_{k \geq n}$ is a limb numbering system, so by Corollary 11 there exist measurable f and g with $G_n(x, y) = f(x) + g(y)$ on A_n. Extend f and g to all of the domain $[0, 1]$ by defining them to be 0 on $\bigcup_{k<n} C_{2k-1}$ and $\bigcup_{k<n} C_{2k}$, respectively. Because $g|_{C_{2n-2}} = 0$, $G_n(x, y) = f(x) + g(y)$ on all of A. Hence, $G_n \in L_{\mathrm{Marg},\infty}(A)$. By dominated convergence, $\int G_n \,\mathrm{d}\mu \to \int G \,\mathrm{d}\mu$ as $n \to -\infty$, so $L_{\mathrm{Marg},\infty}(A)$ is a $GDSM(A)$-determining class and A is a set of uniqueness by Lemma 19. □

One instance of a Borel axial function forest can be constructed in the same fashion as a limb numbering system except for one change: first take two Borel partitions of $[0, 1]$, $\{C_{2n}\}_{n=0}^{\infty}$, and $\{C_{2n-1}\}_{n=0}^{\infty}$, and define Borel h_n: $C_n \to C_{n+1}$ (instead of h_n: $C_n \to C_{n-1}$ as in the limb numbering system case). Changing the index of C_n to C_{-n} for all $n \geq 0$ then gives a Borel axial function forest. This idea can be used to generate an axial function forest which supports a measure in DSM and does not have a Borel rooting set.

To construct this example, let

$$0 = b_0 < a_1 < b_1 < a_2 < \dots$$

with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 1$. Let $C_0 = (0, a_1]$, $C_1 = (0, b_1]$, $C_2 = (a_1, a_2]$, $C_3 = (b_1, b_2]$, etc. Further, for each $n \geq 0$, let $C_{n,1}$ equal the right half-interval of C_n and let $C_{n,0}$ equal the left half-interval of C_n. For each $n \geq 0$ and $i \in \{0, 1\}$, define $h_{n,i}$: $C_{n,i} \to C_{n+1}$ to be the unique linear map with positive slope from $C_{n,i}$ to C_{n+1}. For n even and $i \in \{0, 1\}$ set

$$\mathbf{k}_{n,i} = \{(h_{n,i}(x), x): x \in C_{n,i}\},$$

and for n odd and $i \in \{0, 1\}$ set

$$\mathbf{k}_{n,i} = \{(x, h_{n,i}(x)): x \in C_{n,i}\}.$$

For each n, let $\mathbf{k}_n = \mathbf{k}_{n,0} \cup \mathbf{k}_{n,1}$ and set $F = \bigcup_{n\geq 0} \mathbf{k}_n$. Define a probability measure, μ, on F by spreading uniformly on $\mathbf{k}_{2n}$ a mass of $a_{n+1} - b_n$ and spreading uniformly on $\mathbf{k}_{2n-1}$ a mass of $b_n - a_n$. Clearly $\mu \in DSM$. Figure 4 shows the set F given by sequences

$$a_n = \sum_{k=1}^{2n-1} \alpha_k$$

$$b_n = \sum_{k=1}^{2n} \alpha_k.$$

with $\alpha_{2k-1} = 3^{-k}$, and $\alpha_{2k} = 3^{-k}$ for $k \geq 1$. In Fig. 4a the diagonal $y = x$ is shown as well as the boxes outlining the sets $C_{2n-1} \times C_{2n}$ and $C_{2n-1} \times C_{2n+2}$.

Because the measure μ is supported on a Borel axial function forest, Theorem 25 implies that $\mu \in EDSM_1$. We show that μ does not have a Borel rooting set by establishing two facts:

(i) If $A \subsetneq \mathbf{k}_{n,i}$ with $\mu(A) = 0$ then $[A]_F = 0$.

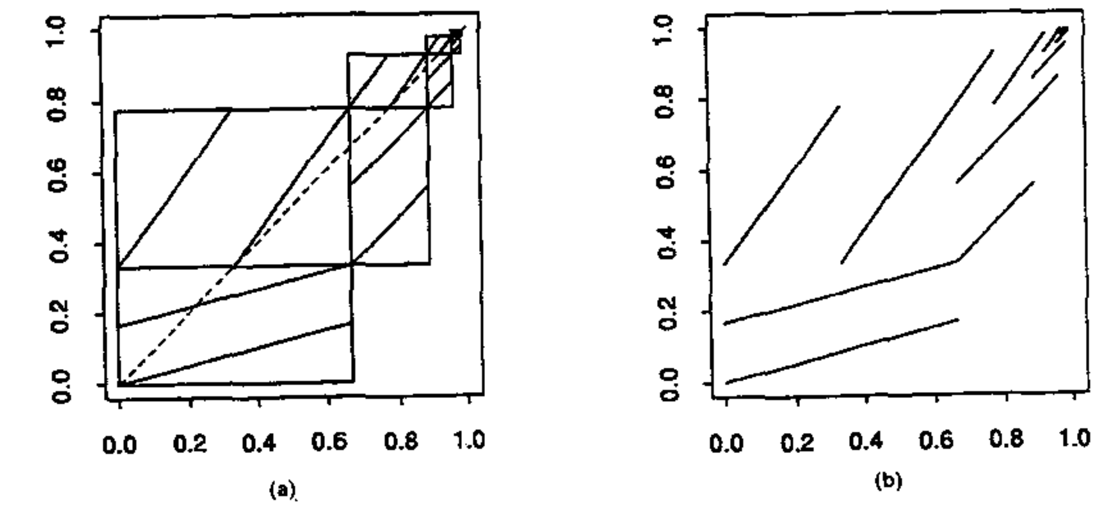

Figure 4. Example of a Borel axial function forest which does not have a Borel rooting set and is a set of uniqueness for *DSM*. (a) shows the axial function forest as well as the diagonal and boxes outlining the sets $C_{2n+1} \times C_{2n}$. In (b) only the axial function forest is shown

(ii) If $A \subseteq \mathbf{k}_{n,i}$ is measurable and a partial rooting set for F (elements of A have distinct orbits) then $\mu(A) = 0$.

Now any partial rooting set A, which is Borel, can be partitioned into $A_{n,i} = A \cap \mathbf{k}_{n,i}$. Fact (ii) then implies that $\mu(A_{n,i}) = 0$, while fact (i) gives $\mu([A]_F) = \mu(\bigcup_{n,i}[A_{n,i}]_F) = 0$. So A cannot be a rooting set for F.

To prove facts (i) and (ii), introduce the following functions: for each $n \geq 1$, $i, j \in \{0, 1\}$, let $G_{n,i,j}\colon \mathbf{k}_{n,i} \to \mathbf{k}_{n-1,j}$ be the unique linear map so that if $\mathbf{w} = G_{n,i,j}(\mathbf{z})$ then $\mathbf{z}, \mathbf{w}$ is an axial path. Set $\mathbf{G} = \{G_{n,i,j}, G^{-1}_{n,i,j} \mid n \geq 1, i, j \in \{0, 1\}\}$. Now $G_{n,i,j}$ and $G^{-1}_{n+1,j,i}$ send sets of measure zero to sets of measure zero. To prove fact (i), we can write $[A]_F = \cup A_i$, where $A_i = H_1 \circ H_2 \circ \ldots H_m(A)$ for some collection of H_j's in $\mathbf{G}$.

Fact (ii) is a little more subtle. Start by defining for each $\mathbf{i} = \{i_n\}_{n=1}^{m} \in \{0, 1\}^m$,

$$B_{\mathbf{i}} = G_{n_0+1,i_1,i_0} \circ G_{n_0+2,i_2,i_1} \circ \ldots \circ G_{n_0+m,i_m,i_{m-1}}(\mathbf{k}_{n_0+m,i_m}).$$

A simple induction argument on m shows that $\bigcup_{\mathbf{i} \in \{0,1\}^m} B_{\mathbf{i}} = \mathbf{k}_{n_0,i_0}$. Also, $G^{-1}_{n,i,j}$ is single-valued for all n, i and j, so for each $\mathbf{z} \in \mathbf{k}_{n_0,i_0}$ there is a unique $\mathbf{i} \in \{0, 1\}^m$ with $\mathbf{z} \in B_{\mathbf{i}}$. Hence, the $B_{\mathbf{i}}$'s are a partition of $\mathbf{k}_{n_0,i_0}$.

Next, for each $\mathbf{i}, \mathbf{j} \in \{0, 1\}^m$ with $j_1 = i_0$, set

$$A_{\mathbf{i},\mathbf{j}} = G_{n_0+1,j_2,j_1} \circ G_{n_0+2,j_3,j_2} \circ \ldots \circ G^{-1}_{n_0+m,i_m,i_{m-1}} \circ G^{-1}_{n_0+m-1,i_{m-1},i_{m-2}} \circ \ldots \circ G^{-1}_{n_0+1,i_1,i_0}(A \cap B_{\mathbf{i}}).$$

Now $\mu(A_{\mathbf{i},\mathbf{j}}) = \mu(A \cap B_{\mathbf{i}})$ because the maps $G_{n,i,j}$ are linear and each stretch by a $G^{-1}_{n_0+k,i_k,i_{k-1}}$ is undone by a $G^{-1}_{n_0+k,j_k,j_{k-1}}$. Also, the $A_{\mathbf{i},\mathbf{j}}$'s are disjoint. To see disjointness, suppose $\mathbf{z} \in A_{\mathbf{i},\mathbf{j}} \cap A_{\mathbf{m},\mathbf{l}}$. Then $A_{\mathbf{i},\mathbf{j}} \subseteq B_{\mathbf{j}}$ and $A_{\mathbf{m},\mathbf{l}} \subseteq B_{\mathbf{l}}$ so $\mathbf{j} = \mathbf{l}$. Now, there exists a $\mathbf{v} \in A \cap B_{\mathbf{i}}$ and $\mathbf{w} \in A \cap B_{\mathbf{m}}$ such that $\mathbf{z} \in [\mathbf{v}]_F \cap [\mathbf{w}]_F$. But A is a partial rooting set so $\mathbf{v} = \mathbf{w}$ and $\mathbf{i} = \mathbf{m}$.

To prove fact (ii), write

$$2^{2m-1}\mu(A) = \sum_{\mathbf{i},\mathbf{j}} \mu(A_{\mathbf{i},\mathbf{j}}) \leq \mu(\mathbf{k}_{n_0,i_0})$$

for all $m \geq 1$, so $\mu(A) = 0$. Thus, A cannot be a rooting set for F.

References

Beneš, V. and Štěpán, J. (1987) The support of extremal probability measures with given marginals. In M.L. Puri, P. Révész and W. Wertz (eds), *Mathematical Statistics and Probability Theory*, Vol. A, pp. 33–41. Dordrecht: D. Reidel Publishing Company.

Cohn, D. (1980) *Measure Theory*. Boston: Birkhäuser.

Denny, J.L. (1980) The support of discrete extremal measures with given marginals. *Michigan Math. J.*, **27**, 59–64.

Douglas, R.D. (1964) On extremal measures and subspace density. *Michigan Math. J.*, **11**, 243–246.

Halmos, P.R. (1950) *Measure Theory*. New York: D. Van Nostrand Company Inc.

Hewitt, E. and Stromberg, K. (1975) *Real and Abstract Analysis*. New York: Springer-Verlag.

Kamiński, A., Mikusiński, P., Sherwood, H. and Taylor, M.D. (1990) Doubly stochastic measures, topology and latticework hairpins. *J. Math. Anal. Appl.*, **152**, 252–268.

Lindenstrauss, J. (1965) A remark on doubly stochastic measures. *Amer. Math. Monthly*, **72**, 379–382.

Losert, V. (1982) Counter-examples to some conjectures about doubly stochastic measures. *Pacific J. Math.*, **99**, 387–397.

Parthasarathy, K.R. (1967) *Probability Measures on Metric Spaces*. New York: Academic Press.

Parthasarathy, T. (1972) *Selection Theorems and their Applications*, Lecture Notes in Math., **263**. Berlin: Springer-Verlag.

Seethoff, T.L. and Shiflett, R.C. (1978) Doubly stochastic measures with prescribed support. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **41**, 283–288.

Štěpán, J. (1979) Probability measures with given expectations. In P. Mandl and Marie Hušková (eds), *Proceedings of the Second Prague Symposium on Asymptotic Statistics*. Amsterdam: North-Holland Publishing Company.

Received February 1994 and revised December 1994