Suprema of chains of operators

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Abstract

Let $\mathcal{E}$ be a real Banach space of operators ordered by a cone K. We give a sufficient condition for that each chain which is bounded above has a supremum. This condition is satisfied in several classical cases, as for the Loewner ordering on the space of all symmetric operators on a Hilbert space, for example.

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1 Introduction

Let E be a real Banach space ordered by a cone K. A cone K is a closed convex subset of E such that $\lambda K \subseteq K$ ($\lambda \geq 0$), and $K \cap (-K) = \{0\}$. As usual $x \leq y :\iff y - x \in K$. For $x \leq y$ let $[x, y]$ denote the order interval of all z with $x \leq z \leq y$. A chain $\mathfrak{C} \subseteq E$ is a nonempty and totally ordered subset of E and a chain $\mathfrak{C}$ is called order bounded above, if

$$\exists y \in E \; \forall x \in \mathfrak{C} : \; x \leq y.$$

We say that the order on E (or K for short) has condition (C) if $\sup \mathfrak{C}$ exists for each chain $\mathfrak{C} \subseteq E$ which is order bounded above. Condition (C)

is an assumption for various fixed point theorems and existence results for differential equations in abstract spaces, see [6], [7], [8], [11], [12], [13] and the references given there. Moreover condition (C) is valid if K is a regular cone (that is, each increasing and order bounded sequence is convergent), see [2, Lemma 2] or [7, Lemma 1], and from condition (C) it follows that K is normal (that is $0 \leq x \leq y$ implies $\|x\| \leq \gamma \|y\|$ for some constant $\gamma \geq 1$), see [1, Lemma 2]. Several examples for nonregular cones with property (C) are known [7], an important example is the Banach space of all bounded sequences $l^\infty(\mathbb{N}, \mathbb{R})$ ordered by the cone of all nonnegative sequences. In this note we will prove property (C) for a certain class of operator cones, including the cone of all positive semidefinite operators on the space of all symmetric operators on a Hilbert space, for example.

2 Ordered operator spaces

Let E be a real or complex Banach space, let $\mathcal{L}(E)$ denote the Banach algebra of all continuous linear operators on E endowed with the operator norm, and in the sequel let $\mathcal{E} \subseteq \mathcal{L}(E)$ be always a real Banach space with respect to this norm. Moreover let $\mathcal{E}$ be ordered by a cone $K \subseteq \mathcal{E}$. Let us say that K has property (P) if it satisfies the following conditions.

1.) If $(A_n)_{n \in \mathbb{N}}$ is an increasing sequence in $\mathcal{E}$ such that

$$A_n \leq B \quad (n \in \mathbb{N})$$

for some $B \in \mathcal{E}$, then $(A_n x)_{n \in \mathbb{N}}$ is convergent in E for each $x \in E$.

2.) If $A \in \mathcal{L}(E)$ and if

$$\forall x_1, \ldots, x_m \in E \; \forall \varepsilon > 0 \; \exists C \in K \; \forall j = 1, \ldots, m: \; \|Ax_j - Cx_j\| \leq \varepsilon,$$

then $A \in K$.

Examples:

1. Let $E = H$ be a complex Hilbert space with inner product $(\cdot, \cdot)$, and let $\mathcal{E} = \mathcal{L}_s(H)$ denote the real Banach space of all linear and symmetric (hence continuous [10, Ch. VIII]) operators on H, endowed with the operator norm, and ordered by the cone

$$K = \{A \in \mathcal{L}_s(H) : \; (Ax, x) \geq 0 \;\; (x \in H)\}.$$

The order defined by K is called Loewner ordering. Let I denote the identity operator, and note the following property of this ordering:

$$\|A\| \leq c \iff -cI \leq A \leq cI.$$

In particular the unit ball in $\mathcal{L}_s(H)$ is the order interval $[-I, I]$. It is clear that K is regular if and only if $\dim H < \infty$. Moreover $\mathcal{L}_s(H)$ is not a lattice with respect to this ordering unless $\dim H = 1$ [3, Ch. 1, Ex. 4]. However, K has property (P): Condition 1. of property (P) follows from Vigier's Theorem [10, Ch. VIII, Theorem 1], and condition 2. can easily be checked by means of the Cauchy Schwarz inequality.

2. Let E be any real Banach space ordered by a regular cone K_0 which is total (that is $\overline{K_0 - K_0} = E$). Then, the set of all monotone operators

$$K = \{A \in \mathcal{L}(E) : A(K_0) \subseteq K_0\}$$

is a cone in $\mathcal{E} = \mathcal{L}(E)$, and K has property (P): Condition 1. of property (P) follows trivially from the regularity of K_0, and condition 2. holds since K_0 is a closed set.

Using the method for regular cones from [7] we now prove:

Theorem 1 *Property (P) implies property (C).*

Proof: Let $\mathfrak{C} \subseteq \mathcal{E}$ be a chain and $B \in \mathcal{E}$ such that

$$A \leq B \quad (A \in \mathfrak{C}).$$

For each $C \in \mathfrak{C}$ let

$$\mathfrak{C}(C) := \{A \in \mathfrak{C} : A \geq C\},$$

and note that $\mathfrak{C}(C)$ is a chain for each $C \in \mathfrak{C}$.

Step 1. Let $x_1, \ldots x_m \in E$ be fixed. We prove that to each $\varepsilon > 0$ there exists $C_\varepsilon \in \mathfrak{C}$ such that

$$\|Ax_j - C_\varepsilon x_j\| < \varepsilon \quad (A \in \mathfrak{C}(C_\varepsilon),\ j = 1, \ldots, m).$$

Assume by contradiction that

$$\forall C \in \mathfrak{C}\ \exists A_C \in \mathfrak{C}(C)\ \exists j \in \{1, \ldots, m\} :\ \|A_C x_j - C x_j\| \geq \varepsilon.$$

Choose any $C_1 \in \mathfrak{C}$. By setting

$$C_{n+1} = A_{C_n} \quad (n \in \mathbb{N})$$

we recursively obtain a sequence $(C_n)_{n\in\mathbb{N}}$ in $\mathfrak{C}$ and a sequence $(x_{j_n})_{n\in\mathbb{N}}$ in $\{x_1, \ldots, x_m\}$ with the following properties:

a.) $C_{n+1} \in \mathfrak{C}(C_n)$ $(n \in \mathbb{N})$, thus $(C_n)_{n\in\mathbb{N}}$ is increasing and bounded by B.

b.) $\|C_{n+1}x_{j_n} - C_n x_{j_n}\| \geq \varepsilon$ $(n \in \mathbb{N})$.

According to property (P) we know that $(C_n)_{n\in\mathbb{N}}$ is strongly convergent, and since $\{x_1, \ldots, x_m\}$ is finite $(x_{j_n})_{n\in\mathbb{N}}$ has a constant subsequence $(x_{j_{n_k}})_{k\in\mathbb{N}}$ with constant x_l, say. Thus

$$\varepsilon \leq \|C_{n_k+1}x_l - C_{n_k}x_l\| \to 0 \quad (k \to \infty),$$

a contradiction.

Step 2. Fix $x \in E$. According to step 1. there is a sequence $(C_n)_{n\in\mathbb{N}}$ with $C_{n+1} \in \mathfrak{C}(C_n)$ $(n \in \mathbb{N})$ such that

$$\|Ax - C_n x\| < \frac{1}{n} \quad (A \in \mathfrak{C}(C_n),\ n \in \mathbb{N}).$$

Again $(C_n)_{n\in\mathbb{N}}$ is increasing and bounded by B. Thus

$$s(x) := \lim_{n\to\infty} C_n x$$

exists. Next, we prove that $s(x)$ is independent on the choice of the operators C_n in the construction above. Let $(D_n)_{n\in\mathbb{N}}$ be a sequence with $D_{n+1} \in \mathfrak{C}(D_n)$ $(n \in \mathbb{N})$ such that

$$\|Ax - D_n x\| < \frac{1}{n} \quad (A \in \mathfrak{C}(D_n),\ n \in \mathbb{N}),$$

and let $h(x) := \lim_{n\to\infty} D_n x$. Since $\mathfrak{C}$ is a chain C_n and D_n are comparable for each $n \in \mathbb{N}$. Without loss of generality assume that there is a subsequence (n_k) of (n) such that

$$C_{n_k} \leq D_{n_k} \quad (k \in \mathbb{N}),$$

that is $D_{n_k} \in \mathfrak{C}(C_{n_k})$ $(k \in \mathbb{N})$, and therefore

$$\|D_{n_k}x - C_{n_k}x\| < \frac{1}{n_k} \quad (k \in \mathbb{N}).$$

Hence $s(x) = h(x)$, and we have defined a function $s : E \to E$.

Step 3. If $\{x_1, \ldots x_m\}$ is a finite subset of E we can choose according to step 1. a sequence $(C_n)_{n\in\mathbb{N}}$ with $C_{n+1} \in \mathfrak{C}(C_n)$ $(n \in \mathbb{N})$ such that

$$\|Ax_j - C_n x_j\| < \frac{1}{n} \quad (A \in \mathfrak{C}(C_n),\ j = 1, \ldots, m,\ n \in \mathbb{N}).$$

According to Step 2. we have

$$s(x_j) = \lim_{n\to\infty} C_n x_j \quad (j = 1, \ldots, m).$$

Let $\alpha \in \mathbb{R}$ or $\mathbb{C}$ (the scalar field of E) and $x, y \in E$ be fixed, and apply the observation above to the finite set $\{x, y, \alpha x + y\}$. We find

$$s(\alpha x + y) = \lim_{n\to\infty} C_n(\alpha x + y) = \lim_{n\to\infty} (\alpha C_n x + C_n y) = \alpha s(x) + s(y).$$

Thus $s : E \to E$ is linear, and we therefore denote it by S ($Sx = s(x)$).

Next, to prove that $S \in \mathcal{L}(E)$ we apply the Closed Graph Theorem: Let $(x_j)_{j=1}^{\infty}$ be a convergent sequence in E with limit x_0 such that $(Sx_j)_{j=1}^{\infty}$ is convergent with limit y_0, say. According to step 1. we can choose a sequence $(C_n)_{n\in\mathbb{N}}$ with $C_{n+1} \in \mathfrak{C}(C_n)$ such that

$$\|Ax_j - C_n x_j\| < \frac{1}{n} \quad (A \in \mathfrak{C}(C_n),\ j = 0, \dots, n,\ n \in \mathbb{N}).$$

In particular

$$\|C_l x_j - C_n x_j\| < \frac{1}{n} \quad (j = 0, \dots, n,\ l, n \in \mathbb{N},\ l > n),$$

hence (as $l \to \infty$)

$$\|Sx_j - C_n x_j\| \le \frac{1}{n} \quad (j = 0, \dots, n,\ n \in \mathbb{N}).$$

The sequence $(C_n)_{n\in\mathbb{N}}$ is strongly convergent, hence bounded in norm (by μ, say), according to the Banach Steinhaus Theorem. We obtain

$$\|Sx_0 - y_0\| \le \|Sx_0 - C_n x_0\| + \|C_n x_0 - C_n x_j\| + \|C_n x_j - Sx_j\| + \|Sx_j - y_0\|$$

$$\le \|Sx_0 - C_n x_0\| + \mu\|x_0 - x_j\| + \|C_n x_j - Sx_j\| + \|Sx_j - y_0\|.$$

If $\varepsilon > 0$ we first choose $j \in \mathbb{N}$ such that

$$\mu\|x_0 - x_j\| + \|Sx_j - y_0\| < \varepsilon,$$

and then $n \in \mathbb{N}$ such that

$$\|Sx_0 - C_n x_0\| + \|C_n x_j - Sx_j\| < \varepsilon.$$

Then $\|Sx_0 - y_0\| < 2\varepsilon$, and we conclude $Sx_0 = y_0$. Thus $S \in \mathcal{L}(E)$.

Step 4. We finally prove that S is the supremum of $\mathfrak{C}$.

Let $C \in \mathfrak{C}$ be fixed, and let $x_1, \dots, x_m \in E$.

Again, we can choose $(C_n)_{n\in\mathbb{N}}$ with $C_{n+1} \in \mathfrak{C}(C_n)$ $(n \in \mathbb{N})$ such that

$$\|Ax_j - C_n x_j\| < \frac{1}{n} \quad (A \in \mathfrak{C}(C_n),\ j = 1, \dots, m,\ n \in \mathbb{N}),$$

and by applying step 1. to the chain $\mathfrak{C}(C)$ we can arrange in addition

$$C_n \geq C \quad (n \in \mathbb{N}).$$

As in step 3. we conclude

$$\|(S-C)x_j - (C_n - C)x_j\| = \|Sx_j - C_n x_j\| \leq \frac{1}{n} \quad (j = 1, \ldots, m, \ n \in \mathbb{N}).$$

Since $C_n - C \in K$ we conclude $S \geq C$ according to property (P) (in particular $S \in \mathcal{E}$). Thus S is an upper bound of $\mathfrak{C}$.

Next, let $V \in \mathcal{E}$ with $A \leq V$ $(A \in \mathfrak{C})$. Let $x_1, \ldots, x_m \in E$, and let $(C_n)_{n \in \mathbb{N}}$ be as above. Then

$$\|(V-S)x_j - (V - C_n)x_j\| \leq \frac{1}{n} \quad (j = 1, \ldots, m, \ n \in \mathbb{N}).$$

Again property (P) implies $S \leq V$. Summing up $S = \sup \mathfrak{C}$. ■

Remark. From the construction in step 3. of the proof it can easily be seen that if E is separable, then there is an increasing sequence in $\mathfrak{C}$ which is strongly convergent to $\sup \mathfrak{C}$ (choose for $(x_j)_{j=1}^{\infty}$ a dense sequence in E).

3 An Application

Let $\mathcal{E}$ be ordered by a cone K with property (P), let $A, B \in \mathcal{E}$ with $A \leq B$. If $\mathfrak{C} \subseteq [A, B]$ is a chain then $\sup \mathfrak{C}$ exists according to Theorem 1, and clearly $\sup \mathfrak{C} \in [A, B]$. Recall the following version of Tarski's Fixed Point Theorem [9]:

Theorem 2 *Let Ω be an ordered set such that $\min \Omega$ exists, and such that each chain in Ω has a supremum. Let $f : \Omega \to \Omega$ be increasing. Then $\min\{x \in \Omega : f(x) = x\}$ exists.*

Now, let $f : [A, B] \to [A, B]$ be increasing. According to Theorem 2 we have a minimal fixed point $X_{\min} \in [A, B]$ of f. The function $f_- : [-B, -A] \to [-B, -A]$ defined as $f_-(X) = -f(-X)$ is increasing too, and the negative of its minimal fixed point is a maximal fixed point $X_{\max} \in [A, B]$ of f. Thus

$$\min\{X \in [A, B] : f(X) = X\} \quad \text{and} \quad \max\{X \in [A, B] : f(X) = X\}$$

exist.

Example: Let H be a Hilbert space, and let $\mathcal{E}$ and K be as in our first example. Let $A_1, \ldots, A_m \in K$, $x_1, \ldots, x_m \in H$, and let $\varphi_1, \ldots \varphi_m : \mathbb{R} \to [0, \infty)$ be increasing and bounded. Let $f : K \to K$ be defined by

$$f(X) = \sum_{k=1}^{m} \varphi_k((Xx_k, x_k))\sqrt{X + A_k}.$$

Then f is increasing (note that $X \to \sqrt{X + A_k}$ is increasing according to the Loewner-Heinz inequality [5]), and, since

$$0 \leq f(X) \leq \left(\sum_{k=1}^{m} (\sup \varphi_k(\mathbb{R}))\sqrt{\|X\| + \|A_k\|} \right) I \quad (X \geq 0),$$

we may choose $\lambda \in (0, \infty)$ sufficiently big, such that

$$f([0, \lambda I]) \subseteq [0, \lambda I].$$

Thus, the equation

$$\sum_{k=1}^{m} \varphi_k((Xx_k, x_k))\sqrt{X + A_k} = X.$$

has in $[0, \lambda I]$ a greatest and a smallest solution.

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