

# SURFACE AREA MEASURES OF LOG-CONCAVE FUNCTIONS

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ABSTRACT. This paper's origins are in two papers: One by Colesanti and Fragalà studying the surface area measure of a log-concave function, and one by Cordero-Erausquin and Klartag regarding the moment measure of a convex function. These notions are the same, and in this paper we continue studying the same construction as well as its generalization.

In the first half the paper we prove a first variation formula for the integral of log-concave functions under minimal and optimal conditions. We also explain why this result is a common generalization of two known theorems from the above papers.

In the second half we extend the definition of the functional surface area measure to the  $L^p$ -setting, generalizing a classic definition of Lutwak. In this generalized setting we prove a functional Minkowski existence theorem for even measures. This is a partial extension of a theorem of Cordero-Erausquin and Klartag that handled the case  $p = 1$  for not necessarily even measures.

## 1. FUNCTIONAL SURFACE AREA MEASURES

This paper has two main parts, so it also has two introductory sections. In this section we give the necessary background concerning surface area measures and their functional extensions. We conclude it by stating our first main theorem, which is proved in Sections 2 and 3. In Section 4 we introduce the Minkowski problem and its known functional analogues, and state our second main theorem. This second theorem is proved in Section 5.

We begin this section by recalling the classical definition of the surface area measure of a convex body. For us, a *convex body* is a convex and compact set  $K \subseteq \mathbb{R}^n$  with non-empty interior. The *support function*  $h_K : \mathbb{R}^n \rightarrow \mathbb{R}$  of  $K$  is defined by

$$h_K(y) = \max_{x \in K} \langle x, y \rangle,$$

where  $\langle \cdot, \cdot \rangle$  denotes the standard inner product on  $\mathbb{R}^n$ . For points  $x \in \mathbb{R}^n$  we will write  $|x| = \sqrt{\langle x, x \rangle}$  for the Euclidean norm, while for convex bodies we will write  $|K|$  for their (Lebesgue) volume. The use of the same notation for both should not cause any confusion.

If  $K, L \subseteq \mathbb{R}^n$  are convex bodies and  $t \geq 0$  we write

$$K + tL = \{x + ty : x \in K, y \in L\}$$

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for the Minkowski addition. A fundamental fact in convex geometry is that for every convex body  $K$  there exists a Borel measure  $S_K$  on the unit sphere  $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$  such that

$$(1.1) \quad \lim_{t \rightarrow 0^+} \frac{|K + tL| - |K|}{t} = \int_{\mathbb{S}^{n-1}} h_L dS_K$$

for every convex body  $L$ . The measure  $S_K$  is called the surface area measure of the body  $K$ . For a proof of this fact, as well as alternative equivalent definitions of  $S_K$  and general background in convex geometry, we refer the reader to [20].

Over the last two decades it became more and more apparent that important problems in convexity and asymptotic analysis can be attacked by embedding the class of convex bodies into appropriate classes of functions or measures on  $\mathbb{R}^n$ . The idea is to think of such analytic objects as “generalized convex bodies”, and study geometric constructions such as volume, addition and support functions on these larger classes. The motivation behind this idea is twofold. First, even if one is ultimately only interested in convex geometry and convex bodies, working in such a larger class can be extremely useful as it allows the use of various analytic and probabilistic tools. Second, since we are now working with functions and measures, the geometrically inspired theorems we obtain can be of interest in analysis. In fact, the main theorem we prove in Section 5 can be viewed as a theorem about the existence of generalized solutions to a certain PDE, as we will see.

Since this fundamental idea of “functional convexity” is so widespread nowadays, it is impossible to pick a manageable list of representative papers to cite. Instead we settle for referring the reader to Section 9.5 of [20], to the survey [15] and to the references therein. Unfortunately these only cover slightly older results and not the massive explosion of the field over the last decade.

In this paper we will study functional surface area measures. We will work with the standard class of log-concave functions:

**Definition 1.1.** A function  $f : \mathbb{R}^n \rightarrow [0, \infty)$  is called log-concave if for every  $x, y \in \mathbb{R}^n$  and every  $0 \leq \lambda \leq 1$ ,

$$f((1 - \lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^\lambda.$$

For every convex body  $K$ , the indicator function  $\mathbf{1}_K$  is log-concave. This gives a natural embedding of the class of convex bodies into the class of log-concave functions. We will always assume that our log-concave functions are upper semi-continuous, which is analogous to assuming the body  $K$  is closed. We denote the class of all upper semi-continuous log-concave functions  $f : \mathbb{R}^n \rightarrow [0, \infty)$  by  $\text{LC}_n$ . Every  $f \in \text{LC}_n$  is of the form  $f = e^{-\varphi}$  for a lower semi-continuous convex function  $\varphi : \mathbb{R}^n \rightarrow (-\infty, \infty]$ . We denote the class of such convex functions by  $\text{Cvx}_n$ .

In order to have a functional analogue of (1.1) we first need to understand a few things: what is the “volume” of log-concave functions, how to add such

functions, what is the support function  $h_f$  of a log-concave function  $f$ , and most importantly what is the surface area measure  $S_f$  of  $f$ .

The easiest of the four is the volume. Since  $|K| = \int \mathbf{1}_K$ , it makes sense to define the “volume” of  $f$  to be its Lebesgue integral  $\int f$  (unless explicitly stated otherwise, our integrals will always be on  $\mathbb{R}^n$  with respect to the Lebesgue measure). As for addition and support functions, recall that for convex bodies the two operations are intimately connected by the relation  $h_{K+tL} = h_K + th_L$ . If one wants to keep this relation for log-concave functions, and have other natural properties such as monotonicity, it turns out that there is essentially only one possible definition:

**Theorem 1.2** ([19]). *Assume we are given a map  $\mathcal{T} : \text{LC}_n \rightarrow \text{Cvx}_n$  and a map  $\oplus : \text{Cvx}_n \times \text{Cvx}_n \rightarrow \text{Cvx}_n$  such that*

- (1)  $f \leq g$  if and only if  $\mathcal{T}f \leq \mathcal{T}g$ .
- (2)  $\mathcal{T}\mathbf{1}_K = h_K$ .
- (3)  $\mathcal{T}(f \oplus g) = \mathcal{T}f + \mathcal{T}g$ .

Then:

- (1) There exists  $C > 0$  such that

$$(\mathcal{T}f)(x) = \frac{1}{C} \cdot (-\log f)^*(Cx).$$

- (2) We have

$$(f \oplus g)(x) = \sup_{y \in \mathbb{R}^n} f(y)g(x-y).$$

The  $*$  that appears in the theorem is the *Legendre transform* map  $*$  :  $\text{Cvx}_n \rightarrow \text{Cvx}_n$ , defined by

$$(1.2) \quad \varphi^*(y) = \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - \varphi(x)).$$

Therefore, for a log-concave function  $f = e^{-\varphi}$  we define its support function  $h_f : \mathbb{R}^n \rightarrow (-\infty, \infty]$  by  $h_f = \varphi^*$ . We also define the addition of log-concave functions to be

$$(f \star g)(x) = \sup_{y \in \mathbb{R}^n} f(y)g(x-y).$$

This addition is also known as the sup-convolution, or Asplund sum, and as an addition on log-concave functions was it first considered in [11]. Like the support function  $h_f$ , it is by now a standard definition in convex geometry. For  $t > 0$  and  $f \in \text{LC}_n$  we also define  $(t \cdot f)(x) = f\left(\frac{x}{t}\right)^t$ . This dilation operation is consistent with the sup-convolution in the sense that  $2 \cdot f = f \star f$ .

With these standard definitions in our disposal, we may now define:

**Definition 1.3.** For  $f, g \in \text{LC}_n$  we define

$$\delta(f, g) = \lim_{t \rightarrow 0^+} \frac{\int (f \star (t \cdot g)) - \int f}{t}.$$

In other words,  $\delta(f, g)$  is the directional derivative of the integral at the point  $f$  and in the direction  $g$ . The first to seriously study this quantity were Colesanti and Fragalà, who proved in [6] that  $\delta(f, g)$  is well defined whenever  $\int f > 0$ . More importantly, they were able to prove a functional version of (1.1):

**Definition 1.4.** For  $f = e^{-\varphi} \in \text{LC}_n$ , we define its surface area measure  $S_f$  as the push-forward of the measure  $f dx$  under the map  $\nabla\varphi$ . More explicitly, for every Borel subset  $A \subseteq \mathbb{R}^n$  we define

$$S_f(A) = \int_{\{x: \nabla\varphi(x) \in A\}} f dx.$$

Equivalently, if  $f = e^{-\varphi}$  then  $S_f$  is the unique Borel measure on  $\mathbb{R}^n$  such that

$$(1.3) \quad \int_{\mathbb{R}^n} \rho(y) dS_f(y) = \int_{\mathbb{R}^n} \rho(\nabla\varphi(x)) f(x) dx$$

for all functions  $\rho$  for which the left hand side is well defined (it could be  $\pm\infty$ ). Note that this definition does not require any regularity assumptions from  $f$ . Indeed, since  $\varphi$  is a convex function it is differentiable almost everywhere on the set  $\text{dom } \varphi = \{x \in \mathbb{R}^n : \varphi(x) < \infty\}$  – see e.g. [17] for this fact as well as other basic analytic properties of convex functions. Therefore  $\nabla\varphi$  exists  $f dx$ -a.e., which means the push-forward is well defined.

Definition 1.4 can seem a bit strange at first. For example, we note that  $S_f(\mathbb{R}^n) = \int f$ , which is the “volume” of  $f$  and not its “surface area”. This is unlike the classical case of convex bodies, where we have  $S_K(\mathbb{S}^{n-1}) = |\partial K|$ , the surface area of  $K$ . However, at least for sufficiently regular functions  $f$  it turns out that  $S_f$  is indeed the correct definition for a surface area measure, because of the following theorem of Colesanti and Fragalà:

**Theorem 1.5** ([6]). *Fix  $f, g \in \text{LC}_n$ . Assume that  $\varphi = -\log f$  and  $\psi = -\log g$  belong to the class*

$$\left\{ \rho \in \text{Cvx}_{\mathbb{R}^n} : \begin{array}{l} \rho \text{ is finite and } C^2\text{-smooth, } \nabla^2 \rho(x) \succ 0 \\ \text{for all } x \in \mathbb{R}^n, \text{ and } \lim_{|x| \rightarrow \infty} \frac{\rho(x)}{|x|} = \infty \end{array} \right\}.$$

*Assume further that  $h_f - ch_g$  is a convex function for sufficiently small  $c > 0$ . Then*

$$(1.4) \quad \delta(f, g) = \int_{\mathbb{R}^n} h_g dS_f.$$

Theorem 1.5 was one of the main motivations behind this paper. Comparing it with (1.1) explains why  $S_f$  should indeed be considered as the surface area measure of the function  $f$ . At the same time, the assumptions of the theorem are definitely not optimal. For example, it was already proved in [18] that when  $f(x) = e^{-|x|^2/2}$ , the first variation formula (1.4) holds for all  $g \in \text{LC}_n$  with no technical assumptions.

However, there is no doubt that some assumptions are needed, as (1.4) cannot hold for all functions  $f, g \in \text{LC}_n$ . For example, if  $f = \mathbf{1}_K$  for some convex body  $K$  then  $S_f = |K| \cdot \delta_0$ , so (1.4) cannot hold even if  $g$  is also the indicator of a convex body. We therefore see that the theory developed in this paper is a functional analogue of the classical theory for convex bodies, but does not formally extend it. We remark that in [6] the authors did define a generalization of the measure  $S_f$  to the case where  $f$  is supported on some smooth convex body  $K$  and satisfies some technical assumptions, and proved a generalization of Theorem 1.5 for such functions  $f$ . Unfortunately the indicator function  $\mathbf{1}_K$  does not satisfy these technical assumptions, so even this more general theorem does not recover the case of convex bodies. It's an interesting open problem to find extensions of Definition 1.4 and Theorem 1.5 that will hold for all  $f \in \text{LC}_n$ , including both smooth functions and indicators of convex bodies. In this paper, however, we will keep Definition 1.4 as our definition of  $S_f$ , and prove a necessary and sufficient condition on  $f$  for (1.4) to hold.

The correct condition to impose on  $f$  appeared in the work of Cordero-Erausquin and Klartag ([7]). In this work the authors study the moment measure of a convex function  $\varphi$ . While this term comes from the very different field of toric Kähler manifolds, the moment measure of  $\varphi$  is precisely the same measure as the surface area measure  $S_{e^{-\varphi}}$ . We refer the reader to [7] and the previous works cited therein (especially [3]) for more information about the connection between this measure and complex geometry. The results of [7] and of Section 5 below can also be considered as establishing the existence of generalized solutions to certain Monge–Ampère differential equations. This shows again how functional results that are motivated by convex geometry can have applications to very different areas of mathematics.

In any case, the crucial definition from [7] is the following:

**Definition 1.6.** Fix  $f \in \text{LC}_n$  with  $0 < \int f < \infty$ . We say that  $f$  is essentially continuous if

$$\mathcal{H}^{n-1}(\{x \in \mathbb{R}^n : f \text{ is not continuous at } x\}) = 0,$$

where  $\mathcal{H}^{n-1}$  denotes the  $(n-1)$ -dimensional Hausdorff measure (see, e.g. Chapter 2 of [8] for definition of the Hausdorff measure).

To explain this definition, let us write  $K = \overline{\{x : f(x) \neq 0\}}$  for the support of  $f$ . Since  $f = e^{-\varphi}$  for a convex function  $\varphi$  and convex functions are continuous on the interior of their domain,  $f$  is continuous everywhere outside of  $\partial K$ . Moreover, since  $f$  is upper semi-continuous, it is easy to see that  $f$  is continuous at a boundary point  $x_0 \in K$  if and only if  $f(x_0) = 0$ . Therefore  $f$  is essentially continuous if and only if  $f \equiv 0$   $\mathcal{H}^{n-1}$ -a.e. on  $\partial K$ .

The following theorem is an adaptation of Theorem 8 from [7] to our notation:

**Theorem 1.7** ([7]). *Fix  $f, g \in \text{LC}_n$  with  $0 < \int f, \int g < \infty$ . Assume that  $f$  is essentially continuous. Then*

$$\log \int f - \log \int g \geq \frac{1}{\int f} \cdot \int (h_f - h_g) dS_f.$$

The relationship between Theorems 1.5 and 1.7 may not be immediately clear, and as far as we know did not previously appear in the literature. To understand it, let us define  $F : \text{Cvx}_n \rightarrow [-\infty, \infty]$  by

$$F(\psi) = -\log \int e^{-\psi}.$$

In other words for every  $f \in \text{LC}_n$  we have  $F(h_f) = -\log \int f$ , where we are using the fact that  $\psi^{**} = \psi$  for all  $\psi \in \text{Cvx}_n$ . Theorem 1.5 is a theorem about the differential of  $F$  at some point  $h_f$ . Indeed, under its technical assumptions we have by the chain rule

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0^+} F(h_f + th_g) &= \frac{d}{dt} \Big|_{t=0^+} F(h_{f \star (t \cdot g)}) = - \frac{d}{dt} \Big|_{t=0^+} \left[ \log \int (f \star (t \cdot g)) \right] \\ &= - \frac{1}{\int f} \cdot \frac{d}{dt} \Big|_{t=0^+} \int (f \star (t \cdot g)) = - \frac{1}{\int f} \cdot \int h_g dS_f. \end{aligned}$$

In other words, the linear map  $L_f(h_g) = -\frac{1}{\int f} \int h_g dS_f$  is the differential of  $F$  at the point  $h_f$ . On the other hand, Theorem 1.7 is a theorem about the *subdifferential* of  $F$ . Indeed, the Prékopa–Leindler inequality ([16], [12]) states that for every  $f, g \in \text{LC}_n$  with  $0 < \int f, \int g < \infty$  and  $0 < t < 1$  one has

$$\int ((1-t) \cdot f) \star (t \cdot g) \geq \left( \int f \right)^{1-t} \left( \int g \right)^t,$$

or

$$F((1-t)h_f + th_g) \leq (1-t)F(h_f) + tF(h_g).$$

Therefore  $F$  is convex on the appropriate domain. The conclusion of Theorem 1.7 may be written as

$$F(h_g) - F(h_f) \geq L_f(h_g - h_f),$$

which means that  $L_f$  belongs to the *subdifferential*  $\partial F(h_f)$ . This conclusion is weaker than the conclusion of Theorem 1.5, but the assumptions are much weaker as well. In fact, Cordero-Erausquin and Klartag show that essential continuity of  $f$  is necessary for Theorem 1.7 to hold.

The first major goal of this paper is to prove a common generalization of both Theorems – We will obtain the stronger conclusion of Theorem 1.5 under the optimal assumptions of Theorem 1.7. More concretely, we will prove the following result:

**Theorem 1.8.** *Fix  $f, g \in \text{LC}_n$  with  $0 < \int f, \int g < \infty$ . If  $f$  is essentially continuous, then*

$$(1.5) \quad \delta(f, g) = \int h_g dS_f.$$

Moreover, (1.5) holding for  $g = \mathbf{1}_{B_2^n}$  is **equivalent** to  $f$  being essentially continuous.

We immediately remark that it is not necessarily true that both sides of (1.5) are finite, and this equality can take the form  $+\infty = +\infty$ . As an example it is enough to take  $f(x) = e^{-|x|^2/2}$ , so  $S_f = \int f dx$ , and the function  $g \in LC_n$  which satisfies  $h_g(x) = e^{|x|^2}$ .

Theorem 1.8 is proved in Section 3, after some preliminary technicalities are proved in Section 2. Other than a general desire to state theorems under the minimal and most elegant conditions, we believe that our proof also explains in a very transparent way exactly *why* essential continuity is the natural condition here. The proof also hints about possible extensions of the theorem to the non essentially continuous case.

## 2. FIRST VARIATION OF THE LEGENDRE TRANSFORM

This section is fairly short and technical, and is dedicated to a proof of the following result:

**Proposition 2.1.** *Let  $\psi, \alpha : \mathbb{R}^n \rightarrow (-\infty, \infty]$  be lower semi-continuous functions. Assume that  $\alpha$  is bounded from below and that  $\alpha(0), \psi(0) < \infty$ . Write  $\varphi = \psi^*$ . Then at every point  $x_0 \in \mathbb{R}^n$  where  $\varphi$  is differentiable we have*

$$(2.1) \quad \left. \frac{d}{dt} \right|_{t=0^+} (\psi + t\alpha)^*(x_0) = -\alpha(\nabla\varphi(x_0)).$$

Conceptually, Proposition 2.1 is well-known. For example, it is similar to Lemma 4.11 of [6]. In fact, if  $\{\psi_t\}$  is any family of convex functions, then it is well-known that under sufficient regularity assumptions we have

$$(2.2) \quad \left. \frac{d}{dt} \right|_{t=0} \psi_t^*(x_0) = - \left. \frac{d}{dt} \right|_{t=0} \psi_t(\nabla\psi_0^*(x_0))$$

(This result is folklore, but see e.g. Proposition 5.1 of [2] for one rigorous formulation). The thorny issue here is the words “sufficient regularity assumptions”: for the proof of Theorem 1.8 we will need Proposition 2.1 as stated, with no extra smoothness or boundness assumptions. In fact, we do not even assume that  $\psi$  and  $\alpha$  are convex – The Legendre transform of any function  $\varphi : \mathbb{R}^n \rightarrow (-\infty, \infty]$ , convex or not, can be defined using formula (1.2). This will not be important for the proof of Theorem 1.8, but will be useful in the second half of this paper.

As we were unable to find Proposition 2.1 in the literature with our minimal assumptions, we give a full proof in this section. We do mention that Lemma 2.7 of [3] is fairly close to our proposition, and the proofs will have some similarities as well. We begin with a lemma:

**Lemma 2.2.** *Let  $\psi : \mathbb{R}^n \rightarrow (-\infty, \infty]$  be a lower semi-continuous function and let  $\varphi = \psi^*$ . Assume that for some fixed  $x_0 \in \mathbb{R}^n$  the function  $\varphi$  is differentiable at  $x_0$ . Then:*

- (1)  $\lim_{|y| \rightarrow \infty} (\langle y, x_0 \rangle - \psi(y)) = -\infty$ .  
(2) *The supremum in the definition of  $\psi^*(x_0) = \varphi(x_0)$  is attained at the unique point  $y_0 = \nabla\varphi(x_0)$ .*

*Proof.* For (1) we do not need that  $\varphi$  is differentiable at  $x_0$ , but only that it is finite in a neighborhood of  $x_0$ . Since  $\varphi$  is convex, it follows that there exists  $\varepsilon > 0$  and  $M > 0$  such that  $\varphi \leq M$  on  $\overline{B}(x_0, \varepsilon)$ .

For any  $0 \neq y \in \mathbb{R}^n$  we have

$$M \geq \varphi\left(x_0 + \varepsilon \frac{y}{|y|}\right) \geq \left\langle y, x_0 + \varepsilon \frac{y}{|y|} \right\rangle - \psi(y) = (\langle y, x_0 \rangle - \psi(y)) + \varepsilon |y|.$$

Hence

$$\langle y, x_0 \rangle - \psi(y) \leq M - \varepsilon |y| \xrightarrow{|y| \rightarrow \infty} -\infty,$$

so (1) is proved.

Now we prove (2). Since the function

$$y \mapsto \langle y, x_0 \rangle - \psi(y)$$

is upper semi-continuous and tends to  $-\infty$  as  $|y| \rightarrow \infty$ , it must attain a maximum at some point  $y_0$ . We will show that necessarily  $y_0 = \nabla\varphi(x_0)$ , which will also imply that the maximizer  $y_0$  is unique.

Indeed, for every  $v \in \mathbb{R}^n$  and every small  $t > 0$  we have

$$\begin{aligned} \varphi(x_0 + tv) &\geq \langle y_0, x_0 + tv \rangle - \psi(y_0) = \langle y_0, x_0 + tv \rangle - (\langle y_0, x_0 \rangle - \varphi(x_0)) \\ &= \varphi(x_0) + t \langle y_0, v \rangle. \end{aligned}$$

Hence

$$\langle y_0, v \rangle \leq \frac{\varphi(x_0 + tv) - \varphi(x_0)}{t} \xrightarrow{t \rightarrow 0^+} \langle \nabla\varphi(x_0), v \rangle.$$

By replacing  $v$  with  $-v$  we have  $\langle y_0, v \rangle = \langle \nabla\varphi(x_0), v \rangle$  for all  $v \in \mathbb{R}^n$ , so indeed  $y_0 = \nabla\varphi(x_0)$  and (2) is proved.  $\square$

We can now prove Proposition 2.1:

*Proof of Proposition 2.1.* Choose  $M > 0$  such that  $\alpha \geq -M$ .

By definition we have

$$\varphi_t(x_0) = \sup_{y \in \mathbb{R}^n} (\langle x_0, y \rangle - \psi_t(y)) = \sup_{y \in \mathbb{R}^n} \underbrace{(\langle x_0, y \rangle - \psi(y) - t\alpha(y))}_{=: G_t(y)}.$$

According to Lemma 2.2(1) we know that  $\lim_{|y| \rightarrow \infty} G_0(y) = -\infty$ . Since  $G_t \leq G_0 + tM$  we also have  $\lim_{|y| \rightarrow \infty} G_t(y) = -\infty$ . Since  $\psi$  and  $\alpha$  are lower semi-continuous it follows that the  $\sup_y G_t(y)$  is attained at some point  $y_t$ . We claim that the set  $\{y_t\}_{0 \leq t \leq 1}$  is bounded. Indeed, for  $0 \leq t \leq 1$  we have

$$\begin{aligned} G_0(y_t) &\geq G_t(y_t) - tM = \sup_{y \in \mathbb{R}^n} G_t(y) - tM \\ &\geq \sup_{y \in \mathbb{R}^n} (\langle x_0, y \rangle - \psi(y) - \alpha(y) - M) \geq -\psi(0) - \alpha(0) - M > -\infty, \end{aligned}$$



Since  $\lim_{|y| \rightarrow \infty} G_0(y) = -\infty$ ,  $\{y_t\}_{0 \leq t \leq 1}$  is indeed bounded. For  $t = 0$  we know from Lemma 2.2(2) that  $G_0(y)$  is maximized at the *unique* point  $y_0 = \nabla \varphi_0(x_0)$ .

On the one hand, we have

$$\varphi_t(x_0) \geq G_t(y_0) = G_0(y_0) - t\alpha(y_0) = \varphi_0(x_0) - t\alpha(y_0),$$

so we have the bound

$$\liminf_{t \rightarrow 0^+} \frac{\varphi_t(x_0) - \varphi_0(x_0)}{t} \geq -\alpha(y_0).$$

On the other hand, we have

$$\varphi_0(x_0) \geq G_0(y_t) = G_t(y_t) + t\alpha(y_t) = \varphi_t(x_0) + t\alpha(y_t)$$

so

$$\limsup_{t \rightarrow 0^+} \frac{\varphi_t(x_0) - \varphi_0(x_0)}{t} \leq -\liminf_{t \rightarrow 0^+} \alpha(y_t).$$

Therefore, to finish the proof it is enough to show that  $\liminf_{t \rightarrow 0^+} \alpha(y_t) \geq \alpha(y_0)$ . Since  $\alpha$  is lower semi-continuous, it is enough to show that  $y_t \xrightarrow{t \rightarrow 0^+} y_0$ . Assume by contradiction this is not the case. Since  $\{y_t\}_{0 \leq t \leq 1}$  is bounded we can find a converging sequence  $t_i \rightarrow 0$  such that  $y_{t_i} \rightarrow y^* \neq y_0$ . Since  $G_0$  is upper semi-continuous it follows that  $G_0(y^*) \geq \limsup_{i \rightarrow \infty} G_0(y_{t_i})$ . But since  $y_{t_i}$  maximizes  $G_{t_i}$  we have

$$G_0(y_{t_i}) + t_i M \geq G_{t_i}(y_{t_i}) \geq G_{t_i}(y_0) = G_0(y_0) - t_i \alpha(y_0).$$

Since  $t_i \rightarrow 0$  as  $i \rightarrow \infty$  it follows that

$$G_0(y^*) \geq \limsup_{i \rightarrow \infty} G_0(y_{t_i}) \geq G_0(y_0),$$

contradicting the fact that  $y_0$  is the *unique* maximizer of  $G_0$ . Hence  $y_t \rightarrow y_0$  and (2.1) is proved.  $\square$

### 3. ESSENTIALLY CONTINUITY AND THE VARIATION FORMULA

We now begin our proof of Theorem 1.8. The following fact about essentially continuous log-concave functions was proved in [7]:

**Proposition 3.1.** *For every  $f \in \text{LC}_n$  with  $0 < \int f < \infty$  we have  $\int |\nabla f| < \infty$ . If  $f$  is essentially continuous then  $\int \nabla f = 0$ .*

The following result is the main place essential continuity is used in our proof. It may be of independent interest:

**Theorem 3.2.** *Fix  $f \in \text{LC}_n$  with  $0 < \int f < \infty$  and let  $K = \overline{\{x : f(x) \neq 0\}}$  denote its support. Then*

$$\int_0^\infty \mathcal{H}^{n-1}(\{x : f(x) = t\}) dt = \int_{\mathbb{R}^n} |\nabla f| dx + \int_{\partial K} f d\mathcal{H}^{n-1}.$$

In particular,  $f$  is essentially continuous if and only if we have the classic co-area formula

$$(3.1) \quad \int_0^\infty \mathcal{H}^{n-1}(\{x : f(x) = t\}) dt = \int_{\mathbb{R}^n} |\nabla f| dx.$$

*Proof.* We will use the co-area formula for BV functions – see e.g. [8] for the statement and the necessary definitions. By translating  $f$  we may assume without loss of generality that 0 is in the interior of  $K$ . Let  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^1$  vector field with compact support. Choose a ball  $B$  such that  $\text{support}(\Phi) \subseteq B$ . For every  $\lambda < 1$  the set  $\lambda K \cap B$  is convex, hence a Lipschitz domain. Since convex functions are locally Lipschitz on the interior of their support, it follows that  $f$  is Lipschitz on  $\lambda K \cap B$ . Hence  $f\Phi$  is also Lipschitz and we may apply the divergence theorem:

$$\begin{aligned} \int_{\partial(\lambda K \cap B)} \langle f\Phi, n_{\lambda K \cap B} \rangle d\mathcal{H}^{n-1} &= \int_{\lambda K \cap B} \text{div}(f\Phi) = \int_{\lambda K} \text{div}(f\Phi) \\ &= \int_{\lambda K} \langle \nabla f, \Phi \rangle + \int_{\lambda K} f \text{div} \Phi. \end{aligned}$$

Here of course  $n_{\lambda K \cap B}$  denotes the outer unit normal to the set  $\lambda K \cap B$ , which exists  $\mathcal{H}^{n-1}$ -almost everywhere.

Since  $\Phi \equiv 0$  on  $\partial B$  we also have

$$\begin{aligned} \int_{\partial(\lambda K \cap B)} \langle f\Phi, n_{\lambda K \cap B} \rangle d\mathcal{H}^{n-1} &= \int_{\partial(\lambda K)} f(y) \langle \Phi(y), n_{\lambda K}(y) \rangle d\mathcal{H}^{n-1}(y) \\ &= \lambda^{n-1} \int_{\partial K} f(\lambda x) \langle \Phi(\lambda x), n_{\lambda K}(\lambda x) \rangle d\mathcal{H}^{n-1}(x) \\ &= \lambda^{n-1} \int_{\partial K} f(\lambda x) \langle \Phi(\lambda x), n_K(x) \rangle d\mathcal{H}^{n-1}(x). \end{aligned}$$

Letting  $\lambda \rightarrow 1^-$  and using the dominated convergence theorem we obtain

$$\int_{\partial K} f \langle \Phi, n_K \rangle d\mathcal{H}^{n-1} = \int_K \langle \nabla f, \Phi \rangle + \int_K f \text{div} \Phi,$$

and since  $f$  is supported on  $K$  we may also write

$$\int_{\mathbb{R}^n} f \text{div} \Phi = - \int_{\mathbb{R}^n} \langle \nabla f, \Phi \rangle + \int_{\partial K} f \langle \Phi, n_K \rangle d\mathcal{H}^{n-1}.$$

By definition, this means that  $f$  is a function of locally bounded variation, and its variation measure  $\|Df\|$  satisfies

$$(3.2) \quad d\|Df\| = |\nabla f| dx + f \cdot d\mathcal{H}^{n-1}|_{\partial K}$$

(see Section 5.1 of [8]). In particular, we may apply the co-area formula (Section 5.5 of [8]) and conclude that

$$\int_0^\infty \mathcal{H}^{n-1}(\{x : f(x) = t\}) dt = \|Df\|(\mathbb{R}^n) = \int |\nabla f| dx + \int_{\partial K} f d\mathcal{H}^{n-1},$$

which is what we wanted to prove.

Finally, for the “in particular” part of the theorem, we see from the last equation that (3.1) holds if and only if  $\int_{\partial K} f d\mathcal{H}^{n-1} = 0$ . This holds if and only if  $f \equiv 0$   $\mathcal{H}^{n-1}$ -a.e. on  $\partial K$ , which exactly means that  $f$  is essentially continuous.  $\square$

*Remark 3.3.* Equation (3.2) actually shows that  $f$  is essentially continuous if and only if its variation measure  $\|Df\|$  is absolutely continuous with respect to the Lebesgue measure. This is equivalent to  $f$  belonging to the Sobolev space  $W_{loc}^{1,1}(\mathbb{R}^n)$  – see again Section 5.1 of [8]. By (3.2) and Proposition 3.1 we know that in this case

$$\|Df\|(\mathbb{R}^n) = \int |\nabla f| < \infty,$$

so we actually obtain the following characterization: a function  $f \in \text{LC}_n$  with  $0 < \int f < \infty$  is essentially continuous if and only if  $f \in W^{1,1}(\mathbb{R}^n)$ . We will not need this characterization in this paper.

We can already prove the “moreover” part of Theorem 1.8. In fact we will show something slightly more general:

**Proposition 3.4.** *Fix  $f \in \text{LC}_n$  with  $0 < \int f < \infty$ . Write  $g = \lambda \cdot \mathbf{1}_{mB_2^n}$  for some  $\lambda, m > 0$ . Then*

$$(3.3) \quad \delta(f, g) = \int h_g dS_f$$

*if and only if  $f$  is essentially continuous.*

*Proof.* We first observe that multiplying  $g$  by a constant cannot change the validity of (3.3). Indeed, define  $\tilde{g} = e^c \cdot g$  for some  $c \in \mathbb{R}$ . Then on the left hand side we obtain

$$\begin{aligned} \delta(f, \tilde{g}) &= \frac{d}{dt} \Big|_{t=0^+} \int (f \star (t \cdot \tilde{g})) = \frac{d}{dt} \Big|_{t=0^+} \left[ e^{tc} \cdot \int (f \star (t \cdot g)) \right] \\ &= c \cdot \int f + \frac{d}{dt} \Big|_{t=0^+} \int (f \star (t \cdot g)) = c \int f + \delta(f, g), \end{aligned}$$

While on the right hand side we obtain

$$\int h_{\tilde{g}} dS_f = \int (h_g + c) dS_f = c \int dS_f + \int h_g dS_f = c \int f + \int h_g dS_f.$$

Since both sides changed by the same additive term, the validity of (3.3) did not change. Hence we may assume without loss of generality that  $\lambda = 1$ , i.e.  $g = \mathbf{1}_{mB_2^n}$ .

Let us compute both sides of (3.3). Write  $f_t = f \star (t \cdot g)$ , so by definition

$$f_t(x) = \sup_{y \in \mathbb{R}^n} f(x - y) \mathbf{1}_{mB_2^n} \left( \frac{y}{t} \right)^t = \sup_{y \in tmB_2^n} f(x - y).$$

It follows that if we set  $K_s = \{x : f(x) \geq s\}$  then

$$\{x : f_t(x) \geq s\} = K_s + tmB_2^n,$$

so by the layer cake representation we have

$$\frac{\int f_t - \int f}{t} = \frac{\int_0^\infty |K_s + tB_2^n| ds - \int_0^\infty |K_s| ds}{t} = \int_0^\infty \frac{|K_s + tmB_2^n| - |K_s|}{t} ds.$$

For every  $s > 0$  we have

$$\frac{|K_s + tmB_2^n| - |K_s|}{t} = m \frac{|K_s + tmB_2^n| - |K_s|}{tm} \xrightarrow{t \rightarrow 0^+} m \cdot |\partial K_s|.$$

Moreover, Minkowski's polynomiality theorem (see e.g. Theorem 5.1.7 of [20]) implies that for every fixed  $s > 0$  the left hand side is a polynomial in  $t$  with non-negative coefficients, and hence monotone in  $t$ . Therefore we may apply the monotone convergence theorem and deduce that

$$\lim_{t \rightarrow 0^+} \frac{\int f_t - \int f}{t} = m \int_0^\infty |\partial K_s| ds.$$

On the other hand, we have  $h_g(y) = m|y|$ , so

$$\int h_g dS_f = m \int |\nabla(-\log f)| f = m \int |\nabla f|.$$

Therefore in the case  $g = \mathbf{1}_{mB_2^n}$ , formula (3.3) reduces to the co-area formula (3.1). By Theorem 3.2, it holds if and only if  $f$  is essentially continuous.  $\square$

Proposition 3.4 explains the role of essential continuity in the subject, but the full proof of Theorem 1.8 is more technically involved. We will need Proposition 2.1 from Section 2, as well as two more results. The first is contained e.g. in Lemma 3.2 of [1]:

**Proposition 3.5.** *Let  $f, f_1, f_2, \dots : \mathbb{R}^n \rightarrow [0, \infty)$  be log-concave functions such that  $f_i \xrightarrow{i \rightarrow \infty} f$  pointwise. Then  $\int f_i \rightarrow \int f$ .*

The second is a very simple measure theoretic lemma:

**Lemma 3.6.** *Let  $\{u_t\}_{t>0}, \{v_t\}_{t>0}, \{w_t\}_{t>0}$  be families of integrable functions  $u_t, v_t, w_t : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $u_t \xrightarrow{t \rightarrow 0^+} u$ ,  $v_t \xrightarrow{t \rightarrow 0^+} v$  and  $w_t \xrightarrow{t \rightarrow 0^+} w$  almost everywhere. Assume that:*

- (1)  $u_t \leq v_t \leq w_t$  for all  $t > 0$ .
- (2)  $\int w_t \xrightarrow{t \rightarrow 0^+} \int w < \infty$ .
- (3)  $\int u_t \xrightarrow{t \rightarrow 0^+} \int u > -\infty$ .

*Then we also have  $\int v_t \xrightarrow{t \rightarrow 0^+} \int v$ .*

*Proof.* Applying Fatou's lemma to  $w_t - v_t$  we have

$$\int w - \limsup_{t \rightarrow 0^+} \int v_t = \liminf_{t \rightarrow 0^+} \int (w_t - v_t) \geq \int (w - v) = \int w - \int v,$$

so  $\limsup_{t \rightarrow 0^+} \int v_t \leq \int v$ . Similarly we may apply Fatou's lemma to  $v_t - u_t$  and obtain

$$\liminf_{t \rightarrow 0^+} \int v_t - \int u = \liminf_{t \rightarrow 0^+} \int (v_t - u_t) \geq \int (v - u) = \int v - \int u,$$

so  $\liminf_{t \rightarrow 0^+} \int v_t \geq \int v$ . The claim follows.  $\square$

We are now ready to prove Theorem 1.8. We will need a bit of notation for the proof. First, we write  $f = e^{-\varphi}$  and  $g = e^{-\beta}$ . We also set  $\psi = h_f = \varphi^*$  and  $\alpha = h_g = \beta^*$ . Finally we define  $\psi_t = \psi + t\alpha$ ,  $\varphi_t = \psi_t^*$  and  $f_t = e^{-\varphi_t} = f \star (t \cdot g)$ .

We first prove the theorem under the extra assumption  $\alpha$  grows very slowly:

**Lemma 3.7.** *Under the assumptions of Theorem 1.8 assume further that*

$$-m \leq h_g(y) \leq m|y| + c$$

for some  $m, c > 0$ . Then

$$\delta(f, g) = \int h_g dS_f.$$

*Remark 3.8.* Unlike the more general Theorem 1.8, the equality in the lemma is always an equality of finite quantities. Indeed,

$$\int h_g dS_f = \int h_g (\nabla \varphi) e^{-\varphi} \leq \int (m|\nabla \varphi| + c) e^{-\varphi} = m \int |\nabla (e^{-\varphi})| + c \int e^{-\varphi},$$

which is finite by Proposition 3.1.

*Proof.* Write  $\tilde{g} = e^c \cdot \mathbf{1}_{mB_2^n}$ , and observe that  $h_{\tilde{g}}(y) = m|y| + c$ . Define

$$\tilde{f}_t(x) = (f \star (t \cdot \tilde{g}))(x) = e^{tc} \cdot \max_{z: |z-x| \leq mt} f(z).$$

Since  $h_g \leq h_{\tilde{g}}$  we also have  $g \leq \tilde{g}$ , so for all  $t > 0$  we have  $f_t \leq \tilde{f}_t$ . On the other hand, we also have

$$f_t = e^{-(\psi+t\alpha)^*} \geq e^{-(\psi-tm)^*} = e^{-(\varphi+tm)} = e^{-tm} f.$$

Therefore if we define  $u_t = \frac{e^{-tm} f - f}{t}$ ,  $v_t = \frac{f_t - f}{t}$  and  $w_t = \frac{\tilde{f}_t - f}{t}$  then  $u_t \leq v_t \leq w_t$  for all  $t > 0$ .

We claim that

$$(3.4) \quad v_t \rightarrow \alpha(\nabla \varphi) f$$

almost everywhere, where we interpret the right hand side to be 0 whenever  $f = 0$ .

Indeed, this will follow from Proposition 2.1. More precisely, let  $K = \overline{\{x : f(x) \neq 0\}}$  denote the support of  $f$ . As a convex function  $\varphi$  is differentiable almost everywhere on  $K$ , so at almost every  $x \in K$  we may apply Proposition 2.1 and deduce that

$$\left. \frac{d}{dt} \right|_{t=0^+} \varphi_t(x) = -\alpha(\nabla \varphi(x)).$$

By the chain rule we then have

$$\frac{d}{dt} \Big|_{t=0^+} f_t(x) = \frac{d}{dt} \Big|_{t=0^+} e^{-\varphi t}(x) = \alpha(\nabla\varphi(x)) \cdot f(x)$$

like we wanted. On the other hand, for every  $x \notin K$  we have  $d(x, K) = \delta > 0$ , and then for every  $t < \frac{\delta}{m}$  we have  $\tilde{f}_t(x) = 0$ . Hence  $f_t(x) = 0$  as well for all small enough  $t > 0$ , so we obviously have  $\lim_{t \rightarrow 0^+} v_t(x) = 0$ . This concludes the proof of (3.4).

The same proof with  $g$  replaced by  $\tilde{g}$  shows that  $w_t \rightarrow (m|\nabla\varphi| + c)f$  almost everywhere, and basic calculus implies that  $u_t \rightarrow -mf$ . Let us call these three pointwise limits  $v$ ,  $w$  and  $u$ . It is trivial that

$$\int u_t \xrightarrow{t \rightarrow 0^+} -m \int f = \int u > -\infty.$$

Since  $f$  is essentially continuous we may apply Proposition 3.4 and deduce that

$$\lim_{t \rightarrow 0^+} \int w_t = \frac{d}{dt} \Big|_{t=0^+} \int \tilde{f}_t = \int h_{\tilde{g}} dS_f = \int w.$$

Moreover, remark 3.8 explains why  $\int w < \infty$ .

Hence we may apply Lemma 3.6 and deduce that

$$\lim_{t \rightarrow 0^+} \int v_t = \int \lim_{t \rightarrow 0^+} v_t = \int \alpha(\nabla\varphi) f = \int h_g dS_f,$$

completing the proof.  $\square$

Now we can finally prove Theorem 1.8 in its full generality:

*Proof of Theorem 1.8.* First we claim that translating  $g$  does not change the validity of the theorem. Indeed, the left hand side clearly doesn't change when we replace  $g$  by  $\tilde{g}(x) = g(x - v)$ . For the right hand side we have  $h_{\tilde{g}}(x) = h_g(x) + \langle x, v \rangle$ , so by Proposition 9 we have

$$\begin{aligned} \int h_{\tilde{g}} dS_f &= \int h_g dS_f + \int \langle x, v \rangle dS_f = \int h_g dS_f + \int \langle \nabla\varphi, v \rangle f \\ &= \int h_g dS_f - \left\langle \int \nabla f, v \right\rangle = \int h_g dS_f. \end{aligned}$$

Hence we may translate and assume that  $\max g = g(0) > 0$ , which means that  $\min \beta = \beta(0) < \infty$ . This implies that  $\alpha \geq -\beta(0) > -\infty$  is bounded from below.

For every integer  $i > 0$  we define

$$g_i(x) = \begin{cases} g(x) & |x| \leq i \\ 0 & \text{otherwise.} \end{cases}$$

Define  $f_{t,i} = f \star_t g_i$  and  $\alpha_i = h_{g_i}$ . We claim that  $\alpha_i(x) \nearrow \alpha(x)$  and  $f_{t,i}(x) \nearrow f_t(x)$  as  $i \rightarrow \infty$ , where  $x \in \mathbb{R}^n$  and  $t > 0$  are fixed.

Let us show that  $f_{t,i}(x) \nearrow f_t(x)$ . Since  $g_i$  is increasing in  $i$  and  $g_i \leq g$  it follows that  $f_{t,i}$  is also increasing in  $i$  and  $f_{t,i} \leq f$ . Therefore we only have to prove that

$$\sup_{i \geq 1} f_{t,i}(x) \geq f_t(x).$$

Indeed, for every  $\varepsilon > 0$  there exists  $y_0 \in \mathbb{R}^n$  such that

$$f_t(x) = \sup_{y \in \mathbb{R}^n} f(x-y)g\left(\frac{y}{t}\right) \leq f(x-y_0)g\left(\frac{y_0}{t}\right) + \varepsilon.$$

Therefore for every  $i_0 > |y_0/t|$  we have

$$\begin{aligned} \sup_{i \geq 1} f_{t,i}(x) &\geq f_{t,i_0}(x) = \sup_{y \in \mathbb{R}^n} f(x-y)g_{i_0}\left(\frac{y}{t}\right) \geq f(x-y_0)g_{i_0}\left(\frac{y_0}{t}\right) \\ &= f(x-y_0)g\left(\frac{y_0}{t}\right) \geq f_t(x) - \varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  was arbitrary the claim is proved. The proof that  $\alpha_i(x) \rightarrow \alpha(x)$  is similar.

Note that

$$\alpha_i(y) = \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - \beta_i(x)) = \sup_{|x| \leq i} (\langle x, y \rangle - \beta(x)) \leq i|y| - \beta(0),$$

and that  $\alpha_i(y) \geq -\beta_i(0) = -\beta(0)$ . Hence we may apply Lemma 3.7 and conclude that

$$\delta(f, g_i) = \lim_{t \rightarrow 0^+} \frac{\int f_{t,i} - \int f}{t} = \int h_{g_i} dS_f.$$

In particular, as was explained in Remark 3.8 these expressions are finite.

Since  $\alpha_i \nearrow \alpha$  we have by monotone convergence  $\int h_{g_i} dS_f \nearrow \int h_g dS_f$ . For the left hand side, define  $\rho_i(t) = \int f_{t,i}$  and  $\rho(t) = \int f_t$ , and set  $\rho_i(0) = \rho(0) = \int f$ . By Proposition 3.5 we have  $\rho_i(t) \xrightarrow{i \rightarrow \infty} \rho(t)$  for all  $t > 0$ .

For every  $i$  the function

$$\varphi_{i,t}(x) = (\psi + t\alpha_i)^*(x) = \sup_{y \in \mathbb{R}^n} [\langle x, y \rangle - \psi(y) - t\alpha_i(y)]$$

is jointly convex in  $(t, x) \in \mathbb{R}^{n+1}$  as the supremum of linear functions. The Prékopa-Leindler inequality then implies that  $\rho_i(t) = \int e^{-\varphi_{i,t}(x)} dx$  is log-concave as well. Similarly  $\rho$  is log-concave. Hence

$$\begin{aligned} (\log \rho)'_+(0) &= \lim_{t \rightarrow 0^+} \frac{\log \rho(t) - \log \rho(0)}{t} = \sup_{t > 0} \frac{\log \rho(t) - \log \rho(0)}{t} \\ &= \sup_{t > 0} \sup_i \frac{\log \rho_i(t) - \log \rho_i(0)}{t} = \sup_i \sup_{t > 0} \frac{\log \rho_i(t) - \log \rho_i(0)}{t} \\ &= \sup_i (\log \rho_i)'_+(0) = \sup_i \frac{(\rho_i)'_+(0)}{\int f} = \sup_i \frac{\int \alpha_i dS_f}{\int f} = \frac{\int \alpha dS_f}{\int f}. \end{aligned}$$

On the other hand, we also have  $(\log \rho)'_+(0) = \frac{\rho'_+(0)}{\int f}$ . One has to be careful here, since we do not know if  $\rho$  is even continuous at  $t = 0$ , let

alone differentiable. Therefore we interpret this equality to mean that if  $(\log \rho)'_+(0) = +\infty$  then  $\rho'_+(0) = +\infty$  as well. Under this convention we see that indeed

$$\lim_{t \rightarrow 0^+} \frac{\int f_t - \int f}{t} = \rho'_+(0) = \int \alpha dS_f,$$

and the proof is complete.  $\square$

#### 4. MINKOWSKI'S THEOREM AND $L^p$ -SURFACE AREA MEASURES

It is now time to discuss a classic problem we avoided so far: What measures are surface area measures? In the classic case of convex bodies, the answer is known as Minkowski's existence theorem:

**Theorem 4.1.** *Let  $\mu$  be a finite Borel measure on  $\mathbb{S}^{n-1}$ . Then  $\mu = S_K$  for some convex body  $K$  if and only if it satisfies the following two conditions:*

- (1)  $\mu$  is centered, i.e.  $\int_{\mathbb{S}^{n-1}} x d\mu(x) = 0$ .
- (2)  $\mu$  is not supported on any great sub-sphere of  $\mathbb{S}^{n-1}$ .

In this classical setting, the uniqueness is also well known – if  $S_K = S_L$  for some convex bodies  $K$  and  $L$  then necessarily  $K = L + v$  for some  $v \in \mathbb{R}^n$ . For a proof of these facts see e.g. Sections 8.1 and 8.2 of [20].

In [7] Cordero-Erausquin and Klartag proved a functional version of Theorem 4.1. In our notation their result reads as follows:

**Theorem 4.2** ([7]). *Let  $\mu$  be a finite Borel measure on  $\mathbb{R}^n$ . Then  $\mu = S_f$  for an essentially continuous  $f \in \text{LC}_n$  with  $0 < \int f < \infty$  if and only if  $\mu$  satisfies the following two conditions:*

- (1)  $\mu$  is centered (so in particular  $\int_{\mathbb{R}^n} |x| d\mu(x) < \infty$ ).
- (2)  $\mu$  is not supported on any lower dimensional linear subspace of  $\mathbb{R}^n$ .

Moreover,  $f$  is uniquely determined up to translations.

Besides its geometric content this theorem can also be viewed analytically, as an existence and uniqueness theorem for generalized solutions of a Monge-Ampère type differential equation. Indeed, assume that  $\frac{d\mu}{dx} = g$  for a smooth function  $g$ , and that the solution  $f = e^{-\varphi}$  to the equation  $S_f = \mu$  is smooth as well. Then using (1.3) and the classic change of variables formula we see that

$$g(\nabla\varphi(x)) \cdot \det(\nabla^2\varphi(x)) = e^{-\varphi(x)}$$

for all  $x \in \mathbb{R}^n$ . This point of view is further explained in [7]. We also remark that the “uniqueness” part of the theorem was also proved in [6] under some technical conditions.

For the rest of this paper we will be mostly interested in an extension of the surface area measure, known as the  $L^p$ -surface area measure. Given two convex bodies  $K, L$  containing the origin and a number  $p \geq 1$ , the  $L^p$ -combination  $K +_p(t \cdot L)$  of  $K$  and  $L$  is defined via its support function by

$$(4.1) \quad h_{K+_p t \cdot L} = (h_K^p + t \cdot h_L^p)^{1/p}.$$



Note that since  $0 \in K, L$  we know that  $h_K, h_L \geq 0$ , so the right hand side is well-defined. Moreover, using the fact that the  $L^p$  norm is indeed a norm it is not hard to check that the right hand side defines a convex, 1-homogeneous function, and hence the body  $K +_p t \cdot L$  exists. For  $p = 1$  the  $L^p$ -addition  $+_1$  coincides with the usual Minkowski addition.  $L^p$  additions of convex bodies were first defined by Firey ([9]), and the Brunn-Minkowski theory of such bodies was developed by Lutwak ([13], [14]). In particular, in [13] Lutwak proved an extension of (1.1) for this case: For every pair convex bodies  $K, L$  containing the origin we have

$$(4.2) \quad \lim_{t \rightarrow 0^+} \frac{|K +_p t \cdot L| - |K|}{t} = \frac{1}{p} \int_{\mathbb{S}^{n-1}} h_L^p h_K^{1-p} dS_K.$$

We refer to the measure  $h_K^{1-p} dS_K$  as the  $L^p$ -surface areas measure of  $K$ , and denote it by  $S_{K,p}$ . In the same paper Lutwak proved an extension of Minkowski's existence theorem for  $p \geq 1$ :

**Theorem 4.3** ([13]). *Fix  $p \geq 1$ , and let  $\mu$  be an **even** finite Borel measure on  $\mathbb{S}^{n-1}$  which is not supported on any great sub-sphere. Then:*

- (1) *If  $p \neq n$  there exists an origin-symmetric convex body  $K$  (i.e.  $K = -K$ ) such that  $S_{K,p} = \mu$ .*
- (2) *For  $p = n$  there exists an origin-symmetric convex body  $K$  such that  $S_{K,n} = c \cdot \mu$  for some  $c > 0$ .*

*Moreover, the body  $K$  is unique.*

In order to explain some peculiarities about the statement of Theorem 4.3, it is useful to say a few words about its proof. In the proof one finds the body  $K$  by minimizing the functional

$$\Phi(L) = |L|^{-\frac{p}{n}} \int_{\mathbb{S}^{n-1}} h_L^p d\mu$$

over the class of origin symmetric convex bodies. If the minimum is attained at some body  $K$ , then it turns out that the first order optimality condition " $\nabla \Phi(K) = 0$ " implies that  $S_K = c \cdot \mu$  for some  $c > 0$ . If  $p \neq n$  we can dilate  $K$  to have exactly  $S_K = \mu$ , by noticing that  $S_{\lambda K,p} = \lambda^{n-p} S_{K,p}$ . Obviously this scaling idea cannot work when  $p = n$ , as in this case  $S_{\lambda K,n} = S_{K,n}$ . This explains why the case  $p = n$  is special.

This very rough sketch of the proof can also explain why  $\mu$  is assumed to be even, an assumption that was unnecessary in the case  $p = 1$ . Indeed, the use of the first order optimality condition " $\nabla \Phi(K) = 0$ " requires the minimizer  $K$  to be an interior point of the domain of  $\Phi$ . When  $p > 1$ ,  $\Phi$  can only be defined on convex bodies containing the origin, to make  $h_L^p$  well defined. Without the assumption that  $L$  is origin symmetric the minimum of  $\Phi$  may be obtained at some body  $K$  containing 0 at its boundary, which will make the argument impossible. Variants of Theorem 4.3 are known for non-even measures (see [5] and [10]), but we will not require them here.

In recent years there has been a lot of interest in the  $L^p$  theory, and in particular in  $L^p$ -surface area measures, for  $0 < p < 1$ . When  $p < 1$  one cannot define  $K +_p t \cdot L$  using (4.1), as the right hand side is not necessarily a convex function. Instead, for any function  $\rho : \mathbb{S}^{n-1} \rightarrow (0, \infty)$ , convex or not, one defines the Alexandrov body (or Wulff shape) of  $\rho$  as

$$A[\rho] = \{x \in \mathbb{R}^n : \langle x, \theta \rangle \leq \rho(\theta) \text{ for all } \theta \in \mathbb{S}^{n-1}\}.$$

In other words,  $A[\rho]$  is the largest convex body with  $h_{A[\rho]} \leq \rho$ . In particular for every convex body  $K$  we have  $A[h_K] = K$ . Then one can define

$$K +_p t \cdot L = A \left[ \left( h_K^p + t h_L^p \right)^{1/p} \right]$$

for any  $p > 0$ . Using the saw called Alexandrov Lemma, one can verify that (4.2) remains true for  $0 < p < 1$ . Furthermore, as Schneider observes in [20] (see Theorem 9.2.1), the existence part of Theorem 4.3 continues to hold in this case, with the same proof. However, for  $p < 1$ , the uniqueness problem is highly non-trivial. In fact it was proved by Böröczky, Lutwak, Yang and Zhang ([4]) that this uniqueness problem is equivalent to the so called  $L^p$ -Brunn-Minkowski conjecture, a major open problem in convex geometry. While this relation was one of our original motivation to study  $L^p$ -surface area measures it will not play any role in the sequel, so we will not give any further details.

We will be interested in functional  $L^p$ -addition and functional  $L^p$ -surface area measures. The definitions are fairly straightforward:

**Definition 4.4.** Let  $\psi : \mathbb{R}^n \rightarrow (-\infty, \infty]$  be a lower semi-continuous function (which may or may not be convex). The Alexandrov Function of  $\psi$  is  $f = A[\psi] = e^{-\psi^*}$ .

Note that  $h_f = \psi^{**}$ , so  $f$  is the largest log-concave function with  $h_f \leq \psi$  in analogy to the classical theory. We then define:

**Definition 4.5.** Fix  $p > 0$  and fix functions  $f, g \in \text{LC}_n$  with  $h_f, h_g \geq 0$ . The  $L^p$ -combination  $f \star_p t \cdot g$  is defined by

$$f \star_p t \cdot g = A \left[ \left( h_f^p + t h_g^p \right)^{1/p} \right].$$

Let us compute the first variation of  $f \star_p t \cdot g$ . Unlike the case  $p = 1$  we will not do it rigorously under minimal assumptions, but use (2.2) to

derive the answer formally assuming sufficient regularity:

$$\begin{aligned}
\frac{d}{dt} \Big|_{t=0^+} \int (f \star_p t \cdot g) &= \int \frac{d}{dt} \Big|_{t=0^+} (f \star_p t \cdot g) \\
&= \int \frac{d}{dt} \Big|_{t=0^+} \exp \left( - \left[ (h_f^p + th_g^p)^{1/p} \right]^* \right) \\
&= - \int \left( e^{-\varphi} \cdot \frac{d}{dt} \Big|_{t=0^+} \left[ (h_f^p + th_g^p)^{1/p} \right]^* \right) \\
&= \int \left( e^{-\varphi} \cdot \left( \frac{d}{dt} \Big|_{t=0^+} \left[ (h_f^p + th_g^p)^{1/p} \right] \circ \nabla \varphi \right) \right) \\
&= \frac{1}{p} \int e^{-\varphi} \cdot h_f^{1-p} (\nabla \varphi) \cdot h_g^p (\nabla \varphi) = \frac{1}{p} \int h_g^p h_f^{1-p} dS_f.
\end{aligned}$$

As expected, the result is completely analogous to the case of convex bodies, so we make the following definition:

**Definition 4.6.** For  $f \in \text{LC}_n$  with  $0 < \int f < \infty$  and  $0 < p < 1$  we define the  $L^p$ -surface area measure of  $f$  to be  $S_{f,p} = h_f^{1-p} dS_f$ .

*Remark 4.7.* For  $S_{f,p}$  to be well defined and not identically equal to  $+\infty$  we should verify that  $h_f(x) < \infty$  for  $S_f$ -almost every  $x$ . This is true since at every point  $x \in \mathbb{R}^n$  where  $f(x) > 0$  and  $\varphi(x) = -\log f(x)$  is differentiable we have by Lemma 2.2

$$\varphi(x) = \langle x, \nabla \varphi(x) \rangle - h_f(\nabla \varphi(x)),$$

so in particular  $h_f(\nabla \varphi(x)) < \infty$ . Hence

$$S_f(\{y : h_f(y) = \infty\}) = \int \mathbf{1}_{\{h_f(\nabla \varphi(x)) = \infty\}} f(x) dx = 0,$$

so  $S_f$ -almost everywhere we have  $h_f < \infty$ .

The second major goal of this paper is to prove a Minkowski existence theorem for functional  $L^p$ -surface area measures. More concretely we will prove the following:

**Theorem 4.8.** Fix  $0 < p < 1$ . Let  $\mu$  be an *even* finite Borel measure on  $\mathbb{R}^n$ . Assume that:

- (1)  $\int |x| d\mu < \infty$  (and then of course  $\mu$  is centered, as it is even)
- (2)  $\mu$  is not supported on any hyperplane

Then there exists  $c > 0$  and an even function  $f \in \text{LC}_n$  with  $h_f \geq 0$  such that  $S_{f,p} = c \cdot \mu$ .

If again we assume that  $\frac{d\mu}{dx} = g$  for some smooth function  $g$  and that the solution  $f = e^{-\varphi}$  to  $S_{f,p} = c \cdot \mu$  is also smooth, then  $\varphi$  solves the Monge-Ampère type differential equation

$$c \cdot g(\nabla \varphi(x)) \cdot \det(\nabla^2 \varphi(x)) = (\varphi^*(x))^{1-p} e^{-\varphi(x)}.$$

Note that we claim nothing about the uniqueness of  $f$ . As was explained above this is a much more delicate issue that we will not address here. Also note that we only prove the result for even measures  $\mu$ , and we can only deduce that  $S_{f,p}$  coincides with the measure  $\mu$  up to a constant  $c > 0$ . This is very similar to the case  $p = n$  of Theorem 4.3, and happens for essentially the same reasons.

We will prove Theorem 4.8 in the next section.

## 5. A FUNCTIONAL $L^p$ MINKOWSKI'S EXISTENCE THEOREM

In this section we prove Theorem 4.8. Not surprisingly, we will find the function  $f$  we are looking for by solving a certain optimization problem. Unlike the proofs of Theorems 4.2 and 4.3 however, it will be much more convenient to work with a *constrained* optimization problem. First we will need a result guaranteeing the existence of a minimizer:

**Proposition 5.1.** *Assume  $\mu$  satisfies the assumptions of Theorem 4.8. Fix  $0 < p < 1$ , and consider the minimization problem*

$$\min \left\{ \int \psi^p d\mu : \begin{array}{l} \psi : \mathbb{R}^n \rightarrow [0, \infty] \text{ is even,} \\ \text{measurable and } \int e^{-\psi^*} \geq a \end{array} \right\}.$$

*If  $a > 0$  is large enough then the minimum is attained for a lower semi-continuous convex function  $\psi_0$ . Moreover,  $\psi_0(0) > 0$  and  $\int e^{-\psi_0^*} = a$ .*

In order to prove this proposition we will need two lemmas, which are both variants of lemmas from [7]. First let us state our version of Lemma 15 from this paper:

**Lemma 5.2.** *Assume  $\mu$  satisfies the assumptions of Theorem 4.8. Let  $\psi : \mathbb{R}^n \rightarrow [0, \infty]$  be an even lower semi-continuous convex function with  $\psi(0) = 0$ . Write  $\varphi = \psi^*$  and fix  $0 < p < 1$ . Then*

$$\int \psi^p d\mu \geq c_\mu \left( \int e^{-\varphi} \right)^{\frac{p}{n}} - C_\mu$$

*for  $c_\mu, C_\mu > 0$  that depend on  $\mu$  and  $p$  but not on  $\psi$ .*

For completeness we provide a proof of the lemma. The fact that we only deal with even functions and measures makes the proof shorter than the corresponding proof in [7]:

*Proof.* If  $\int \psi^p d\mu = \infty$  there is nothing to prove, so we may assume that  $\int \psi^p d\mu < \infty$ . Therefore  $\psi$  is finite on the support of  $\mu$ , and since  $\psi$  is convex it is also finite on its convex hull  $K = \text{conv}(\text{supp}(\mu))$ . By our assumptions on  $\mu$  the body  $K$  is an origin symmetric convex body with non-empty interior, so it must contain 0 in its interior. In particular  $\psi$  is bounded in a neighborhood of 0. If  $|\psi(y)| \leq M$  for  $|y| \leq \delta$  then

$$\varphi(x) = \sup_{y \in \mathbb{R}^n} [\langle x, y \rangle - \psi(y)] \geq \left\langle x, \frac{\delta x}{|x|} \right\rangle - \psi \left( \frac{\delta x}{|x|} \right) \geq \delta |x| - M,$$

So in particular  $\int e^{-\varphi} < \infty$ . If  $\int e^{-\varphi} = 0$  again there is nothing to prove, so we may assume  $0 < \int e^{-\varphi} < \infty$ . By the Blaschke-Santaló inequality (see [1]) it follows that

$$\int e^{-\psi} \cdot \int e^{-\varphi} \leq (2\pi)^{\frac{n}{2}}.$$

Next we define

$$K = \{y \in \mathbb{R}^n : \psi(y) \leq 1\}.$$

We also define  $r = \min_{\theta \in \mathbb{S}^{n-1}} h_K(\theta)$ , and let  $\theta_0 \in \mathbb{S}^{n-1}$  be the direction in which this minimum is attained. It follows that

$$\frac{(2\pi)^{n/2}}{\int e^{-\varphi}} \geq \int_{\mathbb{R}^n} e^{-\psi} \geq \int_K e^{-\psi} \geq \frac{1}{e} |K| \geq \frac{1}{e} |rB^n| = c_n \cdot r^n,$$

so  $r \leq C_n \cdot (\int e^{-\varphi})^{-1/n}$ . Here and everywhere else  $c_n, C_n > 0$  are some constants that depend only on the dimension  $n$ .

For every  $y \in K$  we have  $r = h_K(\theta_0) \geq |\langle y, \theta_0 \rangle|$ . Equivalently, if  $|\langle y, \theta_0 \rangle| > r$  then  $x \notin K$ , so  $\psi(y) > 1$ . It follows that if  $2r \leq |\langle y, \theta_0 \rangle|$  then

$$1 < \psi\left(\frac{2r}{|\langle y, \theta_0 \rangle|} y\right) = \psi\left(\left(1 - \frac{2r}{|\langle y, \theta_0 \rangle|}\right) \cdot 0 + \frac{2r}{|\langle y, \theta_0 \rangle|} \cdot y\right) \leq \frac{2r}{|\langle y, \theta_0 \rangle|} \psi(y),$$

so  $\psi(y) + 1 \geq \psi(y) \geq \frac{|\langle y, \theta_0 \rangle|}{2r}$ . Obviously if  $|\langle y, \theta_0 \rangle| < 2r$  this inequality still holds trivially, so it holds for every  $y \in \mathbb{R}^n$ . Hence

$$\int (\psi^p + 1) d\mu \geq \int (\psi + 1)^p d\mu \geq \frac{1}{(2r)^p} \int |\langle y, \theta_0 \rangle|^p d\mu.$$

The function  $\theta \mapsto \int |\langle y, \theta \rangle|^p d\mu(y)$  is continuous on  $\mathbb{S}^{n-1}$  by the dominated convergence theorem, and is strictly positive since  $\mu$  is not supported on any hyperplane. Hence it has a positive minimum which we may denote by  $\tilde{c}_\mu$ . It follows that

$$\int (\psi^p + 1) d\mu \geq \frac{\tilde{c}_\mu}{(2r)^p} \geq c_\mu \left( \int e^{-\varphi} \right)^{\frac{p}{n}},$$

completing the proof.  $\square$

The second lemma we will need is a variant of Lemma 17 from [7]:

**Lemma 5.3.** *Assume  $\mu$  satisfies the assumptions of Theorem 4.8. Let  $\{\psi_i\}_{i=1}^\infty$  be non-negative, even, lower semi-continuous convex functions such that*

$$\sup_i \int \psi_i^p d\mu < \infty.$$

*Then there exists a subsequence  $\{\psi_{i_j}\}_{j=1}^\infty$  and an even lower semi-continuous convex function  $\psi : \mathbb{R}^n \rightarrow [0, \infty]$  such that*

$$\int \psi^p d\mu \leq \liminf_{j \rightarrow \infty} \int \psi_{i_j}^p d\mu \quad \text{and} \quad \int e^{-\psi^*} \geq \limsup_{j \rightarrow \infty} \int e^{-\psi_{i_j}^*}.$$

In fact, Lemma 17 of [7] is exactly the same statement in the case  $p = 1$ , and with the assumption of evenness replaced with the assumption  $\psi_i(0) = 0$ . The proofs are identical, as writing the extra power  $p$  everywhere doesn't affect the argument in any way. The assumption  $\psi_i(0) = 0$  is only used in [7] to know that  $\psi_i(\lambda x)$  is increasing in  $\lambda$ . This is trivial when  $\psi_i$  is even, so this assumption may be omitted. Since the proofs are otherwise identical we omit a proof of Lemma 5.3.

Using these two lemma we can prove Proposition 5.1:

*Proof of Proposition 5.1.* Without loss of generality assume that  $\mu$  is a probability measure. Consider the function  $\tilde{\psi}(y) = \log a + c_n |y|$ , where the constant  $c_n$  is chosen such that  $|c_n B_2^n| = 1$ . Then  $\tilde{\psi}^*(x) = -\log a + \mathbf{1}_{c_n B_2^n}^\infty$ , so  $\int e^{-\tilde{\psi}^*} = a$ . For  $a \geq e$  we have  $\tilde{\psi} \geq 1$ , so

$$\int \tilde{\psi}^p d\mu \leq \int \tilde{\psi} d\mu = \log a + c_n \int |y| d\mu(y) = \log a + C_\mu.$$

In particular we see that

$$m = \inf \left\{ \int \psi^p d\mu : \begin{array}{l} \psi : \mathbb{R}^n \rightarrow [0, \infty] \text{ is even,} \\ \text{measurable and } \int e^{-\psi^*} \geq a \end{array} \right\} < \infty.$$

Next we choose a sequence  $\{\psi_i\}_{i=1}^\infty$  of even, measurable functions with  $\int e^{-\psi_i^*} \geq a$  and such that  $\int \psi_i^p d\mu \rightarrow m$ . By replacing each  $\psi_i$  with its second Legendre conjugate  $\psi_i^{**}$  we may assume the functions  $\{\psi_i\}$  are all lower semi-continuous and convex. Obviously  $\int \psi_i^p d\mu < m + 1$  for all but finitely many values of  $i$ . Hence we can apply Lemma 5.3 and find a subsequence  $\{\psi_{i_j}\}$  and an even lower semi-continuous convex function  $\psi : \mathbb{R}^n \rightarrow [0, \infty]$  such that

$$\int \psi^p d\mu \leq \liminf_{j \rightarrow \infty} \int \psi_{i_j}^p d\mu = m$$

and

$$\int e^{-\psi^*} \geq \limsup_{j \rightarrow \infty} \int e^{-\psi_{i_j}^*} \geq a.$$

It follows that  $\int \psi^p d\mu = m$  and therefore  $\psi$  is the minimizer we were looking for.

Assume by contradiction that  $\psi(0) = 0$ . Then by Lemma 5.2 we have

$$\int \psi^p d\mu \geq c_\mu \left( \int e^{-\psi^*} \right)^{\frac{p}{n}} - C_\mu \geq c_\mu a^{\frac{p}{n}} - C_\mu.$$

Therefore for large enough  $a > 0$  we have  $\int \psi^p d\mu > \int \tilde{\psi}^p d\mu$ , which is a contradiction to the minimality of  $\psi$ . Hence  $\psi(0) > 0$  for  $a > 0$  large enough.

Finally, assume by contradiction that  $\int e^{-\psi^*} > a$ . Since  $\psi(0) > 0$  the function  $\psi_\varepsilon = \psi - \varepsilon$  is non-negative for small enough  $\varepsilon > 0$ . Since  $\int e^{-\psi_\varepsilon^*} = e^{-\varepsilon} \int e^{-\psi^*}$  we see that  $\psi_\varepsilon$  is in our domain for small enough  $\varepsilon > 0$ . But this

is impossible since  $\psi$  is a minimizer and  $\int \psi_\varepsilon^p d\mu < \int \psi^p d\mu$ . This complete the proof.  $\square$

Now that we have our minimizer, Theorem 4.8 will follow by writing the first order optimality condition. In order to do this, we will need the following result, which is a much simpler variant of Theorem 1.8:

**Proposition 5.4.** *Fix  $f \in \text{LC}_n$  with  $0 < \int f < \infty$  and set  $\psi = h_f$ . Let  $v : \mathbb{R}^n \rightarrow \mathbb{R}$  be bounded and continuous. Then*

$$\frac{d}{dt} \Big|_{t=0} \int e^{-(\psi+tv)^*} = \int v dS_f.$$

*Proof.* Fix a point  $x_0$  where  $\varphi = -\log f$  is differentiable. By Proposition 2.1 we know that

$$\frac{d}{dt} \Big|_{t=0^+} (\psi + tv)^*(x_0) = -v(\nabla\varphi(x_0)).$$

(Recall that we very explicitly did not assume in Proposition 2.1 that the function  $v$  is convex). Applying the same proposition to  $-v$  instead of  $v$  we see that

$$\frac{d}{dt} \Big|_{t=0^-} (\psi + tv)^*(x_0) = - \left( \frac{d}{dt} \Big|_{t=0^+} (\psi - tv)^*(x_0) \right) = -v(\nabla\varphi(x_0)),$$

so the two sided derivative exists. Therefore if we write  $f_t = e^{-(\psi+tv)^*}$  then by the chain rule we have  $\frac{d}{dt} \Big|_{t=0} f_t(x_0) = v(\nabla\varphi(x_0)) f$ .

Choose  $M > 0$  such that  $|v| \leq M$ . Then the functions  $f_t = e^{-(\psi+tv)^*}$  satisfy  $e^{-tM} f \leq f_t \leq e^{tM} f$ . In particular all functions  $f_t$  have the same support which we denote by  $K$ . Moreover for  $|t| \leq 1$  we have

$$\begin{aligned} \left| \frac{f_t - f}{t} \right| &\leq \max \left\{ \left| \frac{e^{tM} f - f}{t} \right|, \left| \frac{e^{-tM} f - f}{t} \right| \right\} \\ &= f \cdot \max \left\{ \left| \frac{e^{tM} - 1}{t} \right|, \left| \frac{e^{-tM} - 1}{t} \right| \right\} \leq (e^M - 1) \cdot f \end{aligned}$$

which is an integrable function. Hence by dominated convergence we have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \int e^{-(\psi+tv)^*} &= \frac{d}{dt} \Big|_{t=0} \int_K f_t = \int_K \left( \frac{d}{dt} \Big|_{t=0} f_t \right) = \int_K v(\nabla\varphi) f \\ &= \int v(\nabla\varphi) f = \int v dS_f. \end{aligned}$$

$\square$

We can now complete the proof of Theorem 4.8. In theory, since we are working with a constrained optimization problem, the first order optimality condition should involve Lagrange multipliers. Luckily our functions are simple enough that we may compute this optimality condition directly and

we do not have to worry about the theory of Lagrange multipliers on an infinite dimensional space:

*Proof of Theorem 4.8.* Fix  $a > 0$  large enough and Let  $\psi$  be the minimizer from Proposition 5.1. Write  $f = e^{-\varphi} = e^{-\psi^*}$  and define  $K = \{x : \psi(x) < \infty\}$ . Since  $\int \psi^p d\mu < \infty$  the measure  $\mu$  is supported on  $K$ . From Remark 4.7 the measure  $S_f$  is also supported on  $K$ .

Fix an even, bounded and continuous function  $v : \mathbb{R}^n \rightarrow \mathbb{R}$  whose support is contained in the *interior* of  $K$ . Define

$$\psi_t = \psi + tv\psi^{1-p} + \log a - \log \int e^{-(\psi+tv\psi^{1-p})^*}.$$

Note that

$$\psi_t^* = (\psi + tv\psi^{1-p})^* - \log a + \log \int e^{-(\psi+tv\psi^{1-p})^*}$$

so that  $\int e^{-\psi_t^*} = a$ . Moreover  $v\psi^{1-p}$  is a bounded function, since  $v$  is bounded on  $\mathbb{R}^n$  and  $\psi$  is bounded on the support of  $v$ . Let us write  $|v\psi^{1-p}| \leq M$ . We then also have

$$\log \int e^{-(\psi+tv\psi^{1-p})^*} \leq \log \int e^{-(\psi+|t|M)^*} = \log \left( e^{|t|M} \int e^{-\psi^*} \right) = \log a + |t|M$$

and similarly  $\log \int e^{-(\psi+tv\psi^{1-p})^*} \geq \log a - |t|M$ . Hence  $|\psi_t - \psi| \leq 2|t|M$ . In particular, since  $\min \psi = \psi(0) > 0$ , there exists  $\delta > 0$  such that  $\psi_t \geq \delta$  if  $|t|$  is small enough.

By Proposition 5.4 we have

$$\left. \frac{d}{dt} \right|_{t=0} \psi_t = v\psi^{1-p} - \frac{1}{a} \cdot \int v\psi^{1-p} dS_f.$$

The function  $x \mapsto x^p$  is  $\frac{p}{\delta^{1-p}}$ -Lipschitz on the interval  $[\delta, \infty)$ . Since for small enough  $|t|$  we have  $\psi, \psi_t \geq \delta$  we obtain

$$(5.1) \quad \left| \frac{\psi_t^p - \psi^p}{t} \right| \leq \frac{p}{\delta^{1-p}} \left| \frac{\psi_t - \psi}{t} \right| \leq \frac{2Mp}{\delta^{1-p}}.$$

From the fact that  $\psi$  is a minimizer it follows that  $\int \psi_t^p d\mu \geq \int \psi^p d\mu$  for all  $|t|$  small enough. Because of (5.1) we may apply dominated convergence and conclude that

$$\begin{aligned} 0 &= \left. \frac{d}{dt} \right|_{t=0} \int \psi_t^p d\mu = \int \left( \left. \frac{d}{dt} \right|_{t=0} \psi_t^p \right) d\mu \\ &= \int \left( p\psi^{p-1} \cdot \left( v\psi^{1-p} - \frac{1}{a} \int v\psi^{1-p} dS_f \right) \right) d\mu \\ &= p \int v d\mu - \frac{1}{a} \int v\psi^{1-p} dS_f \cdot \int p\psi^{p-1} d\mu. \end{aligned}$$

We see that

$$\int v d\mu = c \cdot \int v\psi^{1-p} dS_f = c \cdot \int v dS_{f,p}$$



for some constant  $c > 0$  that depends on  $\mu$  and  $\psi$  but not on  $v$ . Since  $\mu$  and  $S_{f,p}$  are even and supported on  $K$ , and since this equality holds for *every* even, bounded and continuous function  $v$  whose support is contained in the interior of  $K$ , it follows that  $\mu = c \cdot S_{f,p}$ . This completes the proof.  $\square$

## REFERENCES

- [1] Shiri Artstein-Avidan, Bo'az Klartag, and Vitali Milman. The Santaló point of a function, and a functional form of the Santaló inequality. *Mathematika*, 51(1-2):33–48, feb 2010.
- [2] Shiri Artstein-Avidan and Yanir A. Rubinstein. Differential analysis of polarity: Polar Hamilton-Jacobi, conservation laws, and Monge Ampère equations. *Journal d'Analyse Mathématique*, 132(1):133–156, jun 2017.
- [3] Robert J. Berman and Bo Berndtsson. Real Monge-Ampère equations and Kähler-Ricci solitons on toric log Fano varieties. *Annales de la faculté des sciences de Toulouse Mathématiques*, 22(4):649–711, 2013.
- [4] Károly J. Böröczky, Erwin Lutwak, Deane Yang, and Gaoyong Zhang. The log-Brunn-Minkowski inequality. *Advances in Mathematics*, 231(3-4):1974–1997, oct 2012.
- [5] Kai Seng Chou and Xu Jia Wang. The  $L_p$ -Minkowski problem and the Minkowski problem in centroaffine geometry. *Advances in Mathematics*, 205(1):33–83, 2006.
- [6] Andrea Colesanti and Ilaria Fragalà. The first variation of the total mass of log-concave functions and related inequalities. *Advances in Mathematics*, 244:708–749, sep 2013.
- [7] Dario Cordero-Erausquin and Bo'az Klartag. Moment measures. *Journal of Functional Analysis*, 268(12):3834–3866, 2015.
- [8] Lawrence C. Evans and Ronald F. Gariepy. *Measure Theory and Fine Properties of Functions*. CRC Press, New York, NY, 1992.
- [9] William J. Firey.  $p$ -means of convex bodies. *Mathematica Scandinavica*, 10:17–24, 1962.
- [10] Daniel Hug, Erwin Lutwak, Deane Yang, and Gaoyong Zhang. On the  $L_p$  Minkowski Problem for Polytopes. *Discrete & Computational Geometry*, 33(4):699–715, apr 2005.
- [11] Bo'az Klartag and Vitali Milman. Geometry of log-concave functions and measures. *Geometriae Dedicata*, 112(1):169–182, apr 2005.
- [12] László Leindler. On a Certain Converse of Hölder's Inequality II. *Acta Scientiarum Mathematicarum*, 33(3-4), 1972.
- [13] Erwin Lutwak. The Brunn-Minkowski-Firey theory I: Mixed volumes and the Minkowski problem. *Journal of Differential Geometry*, 38(1):131–150, 1993.
- [14] Erwin Lutwak. The Brunn-Minkowski-Firey theory II: Affine and geominimal surface areas. *Advances in Mathematics*, 118(2):244–294, 1996.
- [15] Vitali Milman. Geometrization of probability. In Mikhail Kapranov, Sergiy Kolyada, Yuri Ivanovich Manin, Pieter Moree, and Leonid Potyagailo, editors, *Geometry and Dynamics of Groups and Spaces*, volume 265 of *Progress in Mathematics*, pages 647–667. Birkhäuser, Basel, 2008.
- [16] András Prékopa. Logarithmic concave measures with application to stochastic programming. *Acta Scientiarum Mathematicarum*, 32(3-4):301–316, 1971.
- [17] R. Tyrrell Rockafellar. *Convex analysis*. Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1970.
- [18] Liran Rotem. On the mean width of log-concave functions. In Bo'az Klartag, Shahar Mendelson, and Vitali Milman, editors, *Geometric Aspects of Functional Analysis, Israel Seminar 2006-2010*, volume 2050 of *Lecture Notes in Mathematics*, pages 355–372. Springer, Berlin, Heidelberg, 2012.
- [19] Liran Rotem. Support functions and mean width for  $\alpha$ -concave functions. *Advances in Mathematics*, 243:168–186, aug 2013.

- [20] Rolf Schneider. *Convex Bodies: The Brunn-Minkowski Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, second edition, 2014.

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