

## Surface charge algebra in gauge theories and thermodynamic integrability

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Surface charges and their algebra in interacting Lagrangian gauge field theories are constructed out of the underlying linearized theory using techniques from the variational calculus. In the case of exact solutions and symmetries, the surface charges are interpreted as a Pfaff system. Integrability is governed by Frobenius' theorem and the charges associated with the derived symmetry algebra are shown to vanish. In the asymptotic context, we provide a generalized covariant derivation of the result that the representation of the asymptotic symmetry algebra through charges may be centrally extended. Comparison with Hamiltonian and covariant phase space methods is made. All approaches are shown to agree for exact solutions and symmetries while there are differences in the asymptotic context. © 2008 American Institute of Physics. [DOI: 10.1063/1.2889721]

### I. INTRODUCTION

Surface charges, both in general relativity and in Yang–Mills-type gauge theories, have been extensively studied because of the central role symmetries and conserved charges play in analyzing the dynamics. In the case of general relativity, for instance, these charges describe total energy and angular momentum (see, e.g., Ref. 1), while for gauge theories of Yang–Mills type, electric, color,<sup>2</sup> or even magnetic charges<sup>3</sup> may be encoded in this way.

In the Hamiltonian formulation of general relativity, these surface charges appeared originally in the Arnowitt–Deser–Misner approach (see Ref. 4 and references therein). In this context, the general theory of such charges and their algebra was developed in Refs. 5–9. In the Lagrangian framework, a variety of approaches have been proposed: pseudotensors,<sup>10</sup> Komar integrals,<sup>11</sup> Noether's method applied to linearized field equations,<sup>12–14</sup> a quasilocal approach,<sup>15,16</sup> and covariant phase space methods.<sup>17–24</sup> Recent related work can be found, for instance, in Refs. 25–34.

Two categories of conservation laws associated with local symmetries in gauge theories can be distinguished: exact conservation laws associated with families of symmetric solutions and asymptotic conservation laws where symmetries are defined “close to infinity.” The underlying idea of the framework developed hereafter is that, in both cases, the surface charges and their properties are rooted in the linearized theory.

The starting point for this work is the result<sup>35,36</sup> that exact conserved surface charges are classified by so-called reducibility parameters, i.e., parameters of symmetries of field configurations. For general relativity, for instance, this means that there is no “universal” nontrivial surface charge because no nontrivial Killing vector field exists for a generic metric. Rather, for gravity linearized around a background solution, surface charges are classified by the Killing vectors of the background.<sup>37,38</sup>

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More precisely, equivalence classes of reducibility parameters have been shown<sup>36</sup> to correspond to characteristic cohomology classes in degree  $n-2$ , i.e., cohomology classes of the exterior space-time differential pulled back to the space of solutions. If  $f$  denotes the reducibility parameters, the representatives  $k_f^{n-2}$  of characteristic cohomology are constructed from the Euler–Lagrange derivatives of the Lagrangian.<sup>39</sup> Surface charges are then defined by integration of this representative over a closed  $n-2$  dimensional hypersurface.

These surface charges, which are linear in perturbations around a background field, reproduce the familiar results obtained in general relativity and Yang–Mills theories<sup>1,2</sup> and, in the first order formalism, reduce identically to the Hamiltonian surface term that should be integrated and added to the constraints in order to obtain a well-defined Hamiltonian generator.<sup>5,7</sup>

In the full nonlinear theory, the surface charges of the linearized theory can be reinterpreted as 1-forms in field space, the appropriate mathematical framework being the variational bicomplex associated with a set of Euler–Lagrange equations (see, e.g., Refs. 40–42 and references therein). These surface charge 1-forms can be used both in the exact<sup>22,23,43–45</sup> and in the asymptotic context.<sup>37,39</sup>

In the case of exact solutions and symmetries, surface charges in the full theory are constructed by integrating the surface charge 1-forms of the linearized theory along a path in the space of symmetric configurations. The surface charge then depends only on the choice of solution, reducibility parameter, and on the homology class of the  $n-2$  dimensional hypersurface. For configuration spaces of trivial topology, independence on the path holds if and only if an integrability condition is satisfied. As a new result, we prove the following theorem: under appropriate conditions, the surface charges associated with elements of the derived Lie algebra of the algebra of reducibility parameters vanish, so that the surface charges represent the Abelian quotient algebra of exact symmetries modulo its derived algebra. We also clarify in what sense integrability is governed by Frobenius’ theorem. An important new feature of the present analysis is that we allow the reducibility parameters to vary along the field configurations.

Since the surface charges are defined from the Euler–Lagrange equations and from an on-shell vanishing Noether current, they do not depend on total divergences that are added to the Lagrangian nor on total divergences that may be added to the Noether current. From the outset, our approach thus allows us to control the ambiguities inherent in covariant phase space methods.<sup>23</sup>

In the asymptotic context, we elaborate on the properties of asymptotically conserved charges constructed from the linearized theory. The methods developed hereafter generalize the analysis initiated in Ref. 39 by removing the assumption of “asymptotic linearity” and generalizes also the proposal of Ref. 31 in first order theories to the case of Lagrangians depending on an arbitrary number of derivatives.

More precisely, we derive a Lagrangian version of the Hamiltonian results of Refs. 8 and 9 in the integrable case, a suitably defined covariant Poisson bracket algebra of charges forms a centrally extended representation of the asymptotic symmetry algebra. Besides dealing with ambiguities, it also generalizes to generic gauge systems the original Lagrangian derivation of Ref. 25 in the context of covariant phase space methods for diffeomorphism invariant theories.

The paper is organized as follows. We begin by recalling how central charges appear in the context of Noether charges associated with global symmetries.

Next, we fix our description of irreducible gauge theories and recall that Noether currents associated with gauge symmetries can be chosen to vanish on shell. The surface charge 1-forms are defined and their relation to what we call the invariant presymplectic  $(n-1, 2)$ -form is established. In order to be self-contained, some results of Ref. 39 are rederived, independently of BRST cohomological methods: surface charge 1-forms associated with reducibility parameters are conserved, reducibility parameters form a Lie algebra, the symmetry algebra in the present context, and, on shell, the covariant Poisson algebra of surface charge 1-forms is a representation of the symmetry algebra.

For variations that preserve the symmetries, we then show that the charges associated with commutators of symmetries vanish. The integrability conditions for the surface charge 1-forms are discussed next. In the context of the covariant phase space approach to diffeomorphism invariant

theories, they have been originally discussed for a surface charge 1-form associated with a fixed vector field.<sup>24</sup> Here, we point out that for a given set of gauge fields and gauge parameters, the surface charge 1-forms should be considered as a Pfaff system and that integrability is governed by Frobenius' theorem. This gives the whole subject a thermodynamical flavor, which we emphasize by our notation  $\delta Q_{\vec{v}}[d_V\phi]$  for surface charge 1-forms. In the integrable case, finite charges are defined by integrating the surface charge 1-forms along a path starting from a fixed solution.

In the asymptotic context, we define a space of allowed fields and gauge parameters with respect to a closed surface  $S$ . Asymptotic symmetries at  $S$  are defined as the quotient space of allowed gauge parameters by "proper" gauge parameters associated with vanishing charges. We prove that asymptotic symmetries form a Lie subalgebra of the Lie algebra of all gauge parameters and show that the representation of this algebra by a covariant Poisson bracket for the associated conserved charges may be centrally extended.

In Appendix A, we give elementary definitions, fix notations, and conventions and recall the relevant formulas from the variational bicomplex. In particular, we prove crucial properties of the invariant presymplectic  $(n-1, 2)$ -form associated with the Euler–Lagrange equations of motion. Appendix B is devoted to establishing the covariance of surface charge 1-forms, while the key result needed in order to show how integrability implies the algebra of charges is established in Appendix C. We further motivate our Lagrangian approach in Appendix D by applying it to the case of a first order Hamiltonian action and recovering well-known results of the Hamiltonian approach. Finally, in Appendix E, we apply our approach in the context of pure gravity and highlight the differences with the expressions of covariant phase space methods in the asymptotic context.

## II. GLOBAL SYMMETRIES, NOETHER CHARGES, AND THEIR ALGEBRA

In a Lagrangian field theory, the dynamics is generated from the Lagrangian  $n$ -form  $\mathcal{L} = L d^n x$  through the Euler–Lagrange equations of motion,

$$\frac{\delta L}{\delta \phi^i} = 0. \quad (2.1)$$

A global symmetry  $X$  is required to satisfy the condition  $\delta_X \mathcal{L} = d_H k_X$ . The Noether current  $j_X$  is then defined through the relation

$$X^i \frac{\delta \mathcal{L}}{\delta \phi^i} = d_H j_X, \quad (2.2)$$

a particular solution of which is  $j_X = k_X - I_X^n(\mathcal{L})$ . The operator,

$$I_X^n(\mathcal{L}) = \left( X^i \frac{\partial L}{\partial \phi^i_\mu} + \dots \right) (d^{n-1}x)_\mu,$$

is defined by Eq. (A26) for Lagrangians depending on more than first order derivatives. Applying  $\delta_{X_1}$  to the definition of the Noether current for  $X_2$  and using (A37) together with the facts that  $X_1$  is a global symmetry and that Euler–Lagrange derivatives annihilate  $d_H$ -exact  $n$ -forms, we get

$$d_H \left( \delta_{X_1} j_{X_2} - j_{[X_1, X_2]} - T_{X_1} \left[ X_2, \frac{\delta \mathcal{L}}{\delta \phi} \right] \right) = 0, \quad (2.3)$$

with  $T_{X_1}[X_2, \delta \mathcal{L} / \delta \phi]$  linear and homogeneous in the Euler–Lagrange derivatives of the Lagrangian and defined in (A22). If the expression in the parentheses on the left-hand side (LHS) of (2.3) is  $d_H$ -exact, we get

$$\delta_{x_1} j_{x_2} \approx j_{[x_1, x_2]} + d_H(\cdot), \quad (2.4)$$

where  $\approx 0$  means equal for all solutions of the Euler–Lagrange equations of motion. Upon integration over closed  $n-1$  dimensional surfaces, this yields the usual algebra of Noether charges when evaluated on solutions.

The origin of classical central charges in the context of Noether charges associated with global symmetries is the obstructions for the expression in the parentheses on the LHS of (2.3) to be  $d_H$ -exact, i.e., the cohomology of  $d_H$  in the space of local forms of degree  $n-1$ . This cohomology is isomorphic to the Rham cohomology in degree  $n-1$  of the fiber bundle of fields (local coordinates  $\phi^i$ ) over the base space  $M$  (local coordinates  $x^\mu$ ) (see, e.g., Refs. 40 and 41). The case of classical Hamiltonian mechanics,  $n=1$ ,  $\mathcal{L}=(p\dot{q}-H)dt$ , is discussed, for instance, in Ref. 46. Examples in higher dimensions can be found in Ref. 47.

### III. GAUGE SYMMETRIES, SURFACE CHARGES, AND THEIR ALGEBRA

#### A. Gauge symmetries

In order to describe gauge theories, one needs besides the fields  $\phi^i(x)$ , gauge parameters  $f^\alpha(x)$ . Instead of considering the gauge parameters as additional arbitrary functions of  $x$ , it is useful to extend the jet bundle. Because we want to consider commutation relations involving gauge symmetries, several copies  $f_{a(\mu)}^\alpha$ ,  $a=1, 2, 3, \dots$ , of the jet coordinates associated with gauge parameters are needed.<sup>1</sup> We will denote the whole set of fields as  $\Phi_a^\Delta=(\phi^i, f_a^\alpha)$  and extend the variational bicomplex to this set. More precisely, we continue to denote by  $d_V$  the vertical differential that also involves the  $f_a^\alpha$ , while  $d_V^\phi$  denotes the part that acts on the fields  $\phi^i$  and their derivatives alone.

Let  $\delta_R \phi^i = R_f^i$  be characteristics that depend linearly and homogeneously on the new jet coordinates  $f_{(\mu)}^\alpha$ ,  $R_f^i = R_\alpha^{i(\mu)} f_{(\mu)}^\alpha$ . We assume that these characteristics define a generating set of gauge symmetries of  $\mathcal{L}$ . This means that they define symmetries and that every other symmetry  $Q_f$  that depends linearly and homogeneously on an arbitrary gauge parameter  $f$  is given by  $Q_f^i = R_\alpha^{i(\mu)} \partial_{(\mu)} Z_f^\alpha + M_f^{+i} [\delta L / \delta \phi]$  with  $Z_f^\alpha = Z^{\alpha(v)} f_{(v)}$  and  $M_f^{+i} [\delta L / \delta \phi] = (-\partial)_{(\mu)} (M^{[j(v)i(\mu)]} \partial_{(v)} (\delta L / \delta \phi^j) f)$  (see, e.g., Refs. 36 and 48 for more details). For simplicity, we assume in addition that the generating set is irreducible: if  $R_\alpha^{i(\mu)} \partial_{(\mu)} Z_f^\alpha \approx 0$ , then  $Z_f^\alpha \approx 0$ . Our results can easily be extended to the reducible case (see, e.g., Ref. 49 for a recent application).

For all collections of local functions  $Q_i$ , let us define

$$\forall Q_i, f^\alpha: R_f^i Q_i = R_\alpha^{+i} (Q_i) + \partial_\mu S_\alpha^{\mu i} (f^\alpha, Q_i), \quad (3.1)$$

$$M^{+i} \left[ \frac{\delta L}{\delta \phi^j} \right] Q_i = M^{[j(v)i(\mu)]} \partial_{(v)} \frac{\delta L}{\delta \phi^j} \partial_{(\mu)} Q_i + \partial_\mu M^{\mu j i} \left( \frac{\delta L}{\delta \phi^j}, \frac{\delta L}{\delta \phi^i} \right). \quad (3.2)$$

If  $Q_i = \delta L / \delta \phi^i$ , we get, on account of the Noether identities  $R_\alpha^{+i} (\delta L / \delta \phi^i) = 0$  and the skew symmetry of  $M_\alpha^{[j(v)i(\mu)]}$ , that the Noether current for a gauge symmetry can be chosen to vanish weakly,

$$R_f^i \frac{\delta \mathcal{L}}{\delta \phi^i} = d_H S_f, \quad M^{+i} \left[ \frac{\delta \mathcal{L}}{\delta \phi^i} \right] \frac{\delta \mathcal{L}}{\delta \phi^i} = d_H M, \quad (3.3)$$

where  $S_f = S_\alpha^{\mu i} (\delta L / \delta \phi^j, f^\alpha) (d^{n-1}x)_\mu$  and  $M = M^{\mu j i} (\delta L / \delta \phi^j, \delta L / \delta \phi^i) (d^{n-1}x)_\mu$ .

In the simple case where the gauge transformations depend at most on the derivatives of the gauge parameter to first order,  $R_f^i = R_\alpha^{i\mu} f^\alpha + R_\alpha^{i\mu} \partial_\mu f^\alpha$ , the weakly vanishing Noether current reduces to

$$S_f = R_\alpha^{i\mu} f^\alpha \frac{\delta L}{\delta \phi^i} (d^{n-1}x)_\mu. \quad (3.4)$$

<sup>1</sup>Alternatively, one could make the coordinates  $f_{(\mu)}^\alpha$  Grassmann odd, but we will not do so here.

## B. Surface charge 1-forms

Motivated by the cohomological results of Ref. 39 summarized in the Introduction, we consider the  $(n-2, 1)$ -forms<sup>2</sup>

$$k_f[d_V\phi] = I_{d_V\phi}^{n-1} S_f, \quad (3.5)$$

where the horizontal homotopy operator  $I_{d_V\phi}^{n-1}$  is defined in (A26).

For first order theories and for gauge transformations depending at most on the first derivative of gauge parameters, the forms  $k_f[d_V\phi]$  reduce to those proposed in Refs. 29 and 31,

$$k_f[d_V\phi] = \frac{1}{2} d_V\phi^i \frac{\partial^S}{\partial\phi^i} \left( \frac{\partial}{\partial dx^\nu} S_f \right), \quad (3.6)$$

with  $S_f$  given in (3.4).

The forms  $k_f[d_V\phi]$  are intimately related to the invariant presymplectic  $(n-1, 2)$ -form  $W_{\delta\mathcal{L}/\delta\phi} = -\frac{1}{2} I_{d_V\phi}^n (d_V\phi^i (\delta\mathcal{L}/\delta\phi^i))$ , discussed in details in Appendix A 5. Let  $i_Q = \partial_{(\mu)} Q^i [\partial^S/\partial d_V\phi^i_{(\mu)}]$  denote contraction with  $\delta_Q$  and  $W_{\delta\mathcal{L}/\delta\phi}[d_V\phi, R_f] = -i_{R_f} W_{\delta\mathcal{L}/\delta\phi}$ .

*Lemma 1: The forms  $k_f[d_V\phi]$  satisfy*

$$d_H k_f[d_V\phi] = W_{\delta\mathcal{L}/\delta\phi}[d_V\phi, R_f] - d_V^\phi S_f + T_{R_f} \left[ d_V\phi, \frac{\delta\mathcal{L}}{\delta\phi} \right], \quad (3.7)$$

where  $T_{R_f}[d_V\phi, \delta\mathcal{L}/\delta\phi]$ , defined explicitly in (A22), vanishes on shell.

Indeed, it follows from (3.3) and (A52) that

$$I_{d_V\phi}^n (d_H S_f) = W_{\delta\mathcal{L}/\delta\phi}[d_V\phi, R_f] + T_{R_f} \left[ d_V\phi, \frac{\delta\mathcal{L}}{\delta\phi} \right]. \quad (3.8)$$

The result follows by combining with Eq. (A29).  $\square$

We will consider 1-forms  $d_V^s\phi$  that are tangent to the space of solutions. These 1-forms are to be contracted with characteristics  $Q_s$  such that  $\delta_{Q_s}(\delta\mathcal{L}/\delta\phi^i) \approx 0$ . In particular, they can be contracted with characteristics  $Q_s$  that define symmetries, gauge or global, since  $\delta_{Q_s}\mathcal{L} = d_H(\cdot)$  implies that  $\delta_{Q_s}(\delta\mathcal{L}/\delta\phi^i) \approx 0$  on account of (A18) and (A36). For such 1-forms,

$$d_H k_f[d_V^s\phi] \approx W_{\delta\mathcal{L}/\delta\phi}[d_V^s\phi, R_f]. \quad (3.9)$$

Applying the homotopy operators  $I_f^{n-1}$  defined in (A31) and (3.9), one gets

$$k_f[d_V^s\phi] \approx I_f^{n-1} W_{\delta\mathcal{L}/\delta\phi}[d_V^s\phi, R_f] + d_H(\cdot). \quad (3.10)$$

Note that this relation holds off shell,

$$k_f[d_V\phi] = I_f^{n-1} W_{\delta\mathcal{L}/\delta\phi}[d_V\phi, R_f] + d_H(\cdot), \quad (3.11)$$

if

$$I_f^{n-1} S_f = 0, \quad I_f^{n-1} T_{R_f} \left[ d_V\phi, \frac{\delta\mathcal{L}}{\delta\phi} \right] = 0. \quad (3.12)$$

As we will see below, the relevant components of this condition hold, for instance, for Einstein gravity and in the Hamiltonian framework.

For a given closed  $n-2$  dimensional surface  $S$ , which we typically take to be a sphere inside a hyperplane, the surface charge 1-forms are defined by

<sup>2</sup>For convenience, these forms have been defined with an overall minus sign as compared to the definition used in Ref. 39.

$$\delta Q_f[d_V\phi] = \oint_S k_f[d_V\phi]. \tag{3.13}$$

*Lemma 2: The surface charge 1-forms contracted with gauge transformations are on-shell skew symmetric in the sense that*

$$\delta Q_{f_2}[R_{f_1}] \approx -\delta Q_{f_1}[R_{f_2}]. \tag{3.14}$$

Applying  $i_{R_{f_1}}$  to (3.7) in terms of  $f_2$  and using  $I_{f_1}^{n-1}$ , we get

$$k_{f_2}[R_{f_1}] \approx -I_{f_1}^{n-1}W_{\delta\mathcal{L}/\delta\phi}[R_{f_1}, R_{f_2}] + d_H(\cdot).$$

Comparing with  $i_{R_{f_1}}$  applied to (3.10) in terms of  $f_2$ , this implies

$$k_{f_2}[R_{f_1}] \approx -k_{f_1}[R_{f_2}] + d_H(\cdot), \tag{3.15}$$

from which the lemma follows by integration. □

At a fixed solution  $\phi_s$  to the Euler–Lagrange equations of motion, we consider the space  $\epsilon_{\phi_s}$  of gauge parameters  $f^s$  that satisfy  $R_{f^s}|_{\phi_s} = 0$ . We call such gauge parameters exact reducibility parameters at  $\phi_s$ .

The surface charge 1-forms associated with reducibility parameters are  $d_H$ -closed on shell. More precisely, Eq. (3.7) implies that  $d_H k_{f^s}[d_V^s\phi]|_{\phi_s} = 0$  for 1-forms  $d_V^s\phi$  tangent to the space of solutions. As a consequence of Lemma 1, we then have the following.

*Corollary 3: The surface charge 1-forms  $\delta Q_{f^s}[d_V^s\phi]|_{\phi_s}$  associated with reducibility parameters only depend on the homology class of  $S$ .*

In particular, if  $S$  is the sphere ( $t, r$  constant) in spherical coordinates, for instance,  $\delta Q_{f^s}[d_V^s\phi]|_{\phi_s}$  is  $r$  and  $t$  independent and thus does not depend on any of the coordinates, but only on the solution and the tangent vector in the space of solutions.

*Remarks:*

- (1) Trivial gauge transformations  $\delta\phi^i = M_f^{+i}[\delta\mathcal{L}/\delta\phi]$  can be associated with an  $(n-2, 1)$ -form  $k_f = I_{d_V^s\phi}^{n-1}M_f$  in the same way as (3.5) with  $M_f$  defined in (3.3). Now,  $k_f \approx 0$  since the homotopy operator (A26) can only “destroy” one of the two equations of motion contained in  $M_f$ . Therefore, trivial gauge transformations are associated with on-shell vanishing surface charge 1-forms.
- (2) As briefly recalled in the Introduction, one can, in fact, show under suitable assumptions<sup>35–39</sup> that any other  $(n-2, 1)$ -form that is closed at a given solution  $\phi_s$  when contracted with characteristics tangent to the space of solutions differs from a form  $k_{f^s}[d_V\phi]$  associated with some reducibility  $f^s$  parameters at most by terms that are  $d_H$ -exact or vanish when contracted with characteristics tangent to the space of solutions.
- (3) We will show in Appendix A 5 that

$$-W_{\delta\mathcal{L}/\delta\phi} = \Omega_{\mathcal{L}} + d_H E_{\mathcal{L}}, \quad d_V \Omega_{\mathcal{L}} = 0, \tag{3.16}$$

where  $\Omega_{\mathcal{L}}$  is the standard presymplectic  $(n-1, 2)$ -form used in covariant phase space methods and  $E_{\mathcal{L}}$  is a suitably defined  $(n-2, 2)$ -form. Contracting (3.16) with a gauge transformation  $R_f$ , it follows from (3.10) that, apart from on-shell and  $d_H$ -exact terms,  $k_f[d_V^s\phi]$  differs by the additional term  $E_{\mathcal{L}}[d_V\phi, R_f]$  from similar  $(n-2, 1)$ -forms derived in the context of covariant phase space methods in Refs. 20, 22, and 23.

### C. Algebra

Because we have assumed that  $\delta_{R_f}\phi^i = R_f^i$  provide a generating set of nontrivial gauge symmetries, the commutator algebra of the nontrivial gauge symmetries closes on shell in the sense that

$$\delta_{R_{f_1}} R_{f_2}^i - (1 \leftrightarrow 2) = -R_{[f_1, f_2]}^i + M_{f_1, f_2}^{+i} \left[ \frac{\delta L}{\delta \phi} \right], \quad (3.17)$$

with  $[f_1, f_2]^\gamma = C_{\alpha\beta}^{\gamma(\mu)(\nu)} f_{1(\mu)}^\alpha f_{2(\nu)}^\beta$  for some skew-symmetric functions  $C_{\alpha\beta}^{\gamma(\mu)(\nu)}$  and for some characteristic  $M_{f_1, f_2}^{+i}[\delta L / \delta \phi]$ .

At any solution  $\phi_s(x)$  to the Euler–Lagrange equations of motion, the space of all gauge parameters equipped with the bracket  $[\cdot, \cdot]$  is a Lie algebra.

Indeed, by applying  $\delta_{R_{f_3}}$  to (3.17) and taking cyclic permutations, one gets  $R_{[[f_1, f_2], f_3]} + \text{cyclic}(1, 2, 3) \approx 0$  on account of  $\delta_{R_f}(\delta L / \delta \phi^i) \approx 0$ . Irreducibility then implies the Jacobi identity

$$[[f_1, f_2], f_3]^\gamma + \text{cyclic}(1, 2, 3) \approx 0. \quad (3.18)$$

It then also follows from (3.17) and (3.18) that  $\mathfrak{e}_{\phi_s}$  is a Lie algebra, the Lie algebra of exact reducibility parameters at the particular solution  $\phi_s$ .

*Proposition 4:* When evaluated at a solution  $\phi_s$ , for 1-forms tangent to the space of solutions and for a reducibility parameter  $f^s$  at  $\phi_s$ , the  $(n-2, 1)$ -forms  $k_f[d_V^s \phi]$  are covariant up to  $d_H$ -exact terms,

$$\delta_{R_{f_1}} k_{f_2}^s [d_V^s \phi] \approx -k_{[f_1, f_2]^s} [d_V^s \phi] + d_H(\cdot). \quad (3.19)$$

This proposition is proved in Appendix B. If the Lie bracket of surface charge 1-forms is defined by

$$[\delta Q_{f_1}, \delta Q_{f_2}] = -\delta_{R_{f_1}} \delta Q_{f_2}, \quad (3.20)$$

we thus have shown the following.

*Corollary 5:* At a given solution  $\phi_s$  and for 1-forms  $d_V^s \phi$  tangent to the space of solutions, the Lie algebra of surface charge 1-forms represents the Lie algebra of exact reducibility parameters  $\mathfrak{e}_{\phi_s}$ ,

$$[\delta Q_{f_1^s}, \delta Q_{f_2^s}] [d_V^s \phi]_{\phi_s} = \delta Q_{[f_1^s, f_2^s]} [d_V^s \phi]_{\phi_s}. \quad (3.21)$$

#### D. Exact solutions and symmetries

Suppose one is given a family of exact solutions  $\phi_s \in \mathcal{E}$  admitting ( $\phi_s$ -dependent) reducibility parameters  $f^s \in \mathfrak{e}_{\phi_s}$ , which contains a background solution  $\bar{\phi}$ . Elements of the Lie algebra of exact reducibility parameters  $\mathfrak{e}_{\phi_s}$  at  $\phi_s$  are denoted by  $f^s$ . Let us denote by  $\bar{\phi}$  an element of this family that we single out as the reference solution and let  $\bar{f} \in \mathfrak{e}_{\bar{\phi}}$  be the associated reducibility parameters. We consider 1-forms  $d_V^s f$  that are tangent to the space of reducibility parameters. They are to be contracted with gauge parameters  $q^s$  such that

$$0 = (d_V^s R_f)|_{\phi_s, f^s, Q_s, q_s} = \delta_{Q_s} R_{f^s}|_{\phi_s} + R_{q^s}|_{\phi_s}. \quad (3.22)$$

Definition (3.20) and Corollary 5 imply the following.

*Corollary 6:* For variations preserving the reducibility identities as in (3.22), the surface charge 1-forms vanish for elements of the derived Lie algebra  $\mathfrak{e}'_{\phi_s}$  of exact reducibility parameters at  $\phi_s$ ,

$$\delta Q_{[f_1^s, f_2^s]} [d_V^s \phi]_{\phi_s} = 0. \quad (3.23)$$

In this case, the Lie algebra of surface charge 1-forms represents nontrivially only the Abelian Lie algebra  $\mathfrak{e}_{\phi_s} / \mathfrak{e}'_{\phi_s}$ .

The surface charge  $Q_{\gamma_s, f}$  of  $\Phi_s = (\phi_s, f^s)$  with respect to the fixed background  $\bar{\Phi} = (\bar{\phi}, \bar{f})$  is defined as

$$Q_{\gamma_s, f}[\Phi, \bar{\Phi}] = \int_{\gamma_s} \delta Q_{f^s} [d_V^s \phi'] + N_{\bar{f}}[\bar{\Phi}], \tag{3.24}$$

where integration is done along a path  $\gamma_s$  in the space of exact solutions  $\mathcal{E}$  that joins  $\bar{\phi}$  to  $\phi_s$  for some reducibility parameters that vary continuously along the path from  $\bar{f}$  to  $f^s$ . Note that these charges depend only on the homology class of  $S$  because Eq. (3.7) implies that  $d_H k_{f^s} [d_V^s \phi] |_{\phi_s} = 0$ . Because (3.23) holds in this case, we have the following.

*Corollary 7: If the normalization of the background is chosen to vanish, the surface charges associated with elements of the derived Lie algebra  $\mathfrak{e}'_{\phi_s}$  of reducibility parameters vanish at any  $\phi_s \in \mathcal{E}$ ,*

$$Q_{\gamma_s, [f_1, f_2]} = 0. \tag{3.25}$$

The integrability conditions for the surface charges involve the 2-forms  $\oint_S d_V^s k_{f^s} [d_V^s \phi] |_{\phi_s}$ . Assumption (3.22) together with (3.7) and (3.16) implies that  $d_H d_V^s k_{f^s} [d_V^s \phi] |_{\phi_s} = 0$ , so that the integrability conditions also only depend on the homology class of  $S$ .

We now assume that solutions to the system  $\delta L / \delta \phi = 0, R_j^i = 0$  are described by fields  $\phi_s(x; a)$  depending smoothly on  $p$  parameters,  $a^A, A = 1, \dots, p$  and reducibility parameters  $f^s(x; a, b)$  depending linearly on some additional ones  $b^i, i = 1, \dots, q$ . It follows that  $e_i(x; a) = (\partial / \partial b^i) f^s(x; a, b)$  is a basis of the Lie algebra  $\mathfrak{e}_{\phi_s}, i = 1, \dots, r$ . For each basis element  $e_i(x; a)$ , we consider the 1-forms in parameter space  $\theta_i(a, da) = \oint_S k_{e_i} [d^a \phi_s(x; a)]$ , where  $d^a$  is the exterior derivative in parameter space. We thus have a Pfaff system in parameter space and the question of integrability can be addressed using Frobenius' theorem.

In the completely integrable case, for instance, there exists an invertible matrix  $S_j^i(a)$  such that

$$d^a f_j(a) = S_j^i(a) \theta_i(a, da) = \oint_S k_{S_j^i(a) e_i} [d^a \phi_s(x; a)]. \tag{3.26}$$

In other words, if  $g_j(x; a) = S_j^i(a) e_i(x; a)$ , the surface charges,

$$Q_j[\Phi, \bar{\Phi}] = \int_{\gamma_s} \delta Q_{g_j} [d_V \phi'] + N_{\bar{g}_j}[\bar{\Phi}], \tag{3.27}$$

do not depend on the path  $\gamma_s \in \mathcal{E}$  but only on the final point  $(\phi_s, g_j)$  and the initial point  $(\bar{\phi}, \bar{g}_j)$ .

*Remarks:*

- (1) Because  $d_H k_{f^s} [d_V^s \phi] |_{\phi_s} = 0$  for solutions of the source-free equations of motion, one gets the following generalization of Gauss' law for electromagnetism in the case where the surface  $S$  surrounds several sources  $i$  that can be enclosed by surfaces  $S^i$ ,

$$\delta Q_{f^s} [d_V^s \phi] |_{\phi_s} = \sum_i \oint_{S^i} k_{f^s} [d_V^s \phi] |_{\phi_s}, \tag{3.28}$$

with similar decompositions holding for the charges  $Q_{\gamma_s, f}$  and  $Q_j[\Phi, \bar{\Phi}]$ .

- (2) In the case of exact solutions and exact symmetries thereof, the theory of charges developed above does not depend on asymptotic properties of the fields near some boundary.
- (3) In the case where the surface charge is evaluated at infinity, for instance, a simplification occurs when  $\Phi$  approaches  $\bar{\Phi}$  sufficiently fast at infinity in the sense that the  $(n-2, 1)$ -form can be reduced to

$$k_f [d_V \phi; \phi] |_{S^\infty} = k_{\bar{f}} [d_V \phi; \bar{\phi}] |_{S^\infty}. \tag{3.29}$$

We call this the *asymptotically linear case*. It was treated in detail in Ref. 39. In this case, the charge (3.24) is manifestly path independent and reduces to



$$\mathcal{Q}_f[\Phi, \bar{\Phi}] = \oint_{S^\infty} k_f[\phi - \bar{\phi}; \bar{\phi}] + N_f[\bar{\phi}]. \quad (3.30)$$

#### IV. ASYMPTOTIC ANALYSIS

##### A. Space of admissible fields and gauge parameters

Consider for definiteness the closed surface  $S^{\infty,t}$ , which is obtained as the limit when  $r$  goes to infinity of the surface  $S^{r,t}$ , the intersection of the spacelike hyperplane  $\Sigma_t$  defined by constant  $t$  and the timelike or null hyperplane  $\mathcal{T}_r$  defined by constant  $r$ . The remaining angular coordinates are denoted by  $\theta^A$  and  $y^a = (t, \theta^A)$ . Most considerations below only concern the region of space-time close to  $S^{\infty,t}$ .

We now define the space of allowed (asymptotic) solutions  $\mathcal{F}^\infty$  and for each  $\phi_s \in \mathcal{F}^\infty$ , the space of allowed gauge parameters  $g \in \mathcal{A}_{\phi_s}$ . They are restricted by the following requirements.

- The allowed gauge parameters should be such that the associated gauge transformations leave the allowed field configurations invariant,

$$\delta_{R_g} \phi^i = R_g^i \text{ should be tangent to } \mathcal{F}^\infty. \quad (4.1)$$

It implies that all the relations below should be valid for  $d_V^s \phi^i$  replaced by  $R_g^i$ .

- Integrability of the surface charges,

$$\oint_{S^{r,t}} d_V^s k_g[d_V^s \phi] \approx o(r^0). \quad (4.2)$$

This condition guarantees that the surface charges (3.24) are independent of the path  $\gamma \in \mathcal{F}^\infty$  provided that no global obstructions occurs in  $\mathcal{F}^\infty$ .

- Additional conditions on  $d_V E_{\mathcal{L}}$ ,

$$\oint_{S^{r,t}} i_{R_g} d_V^s E_{\mathcal{L}}[d_V^s \phi, d_V^s \phi] \approx o(r^0), \quad \oint_{S^{r,t}} \delta_{R_g} d_V^s E_{\mathcal{L}}[d_V^s \phi, d_V^s \phi] \approx o(r^0). \quad (4.3)$$

These assumptions are needed below in order to prove that asymptotic symmetries form an algebra. As we will see below, they are automatically fulfilled in the Hamiltonian formalism in Darboux coordinates.

- Asymptotic  $r$ -independence of the charges,

$$\oint_{S^{r,t}} \mathcal{L}_{\partial_r} k_g[d_V^s \phi] \approx o(r^{-1}), \quad (4.4)$$

where  $\mathcal{L}_{\partial_r}$  is defined by (A7). This condition expresses that the surface charge 1-forms (3.13) for  $S = S^{r,t}$  are  $r$ -independent when  $r \rightarrow \infty$ . It implies, in particular, finiteness of the charges.

- Conservation in time of the surface charges for solutions  $\phi_s \in \mathcal{F}^\infty$  and tangent 1-forms  $d_V^s \phi$  to  $\mathcal{F}^\infty$ ,

$$\oint_{S^{r,t}} \mathcal{L}_{\partial_t} k_g[d_V^s \phi] \approx o(r^0). \quad (4.5)$$

##### B. Asymptotic symmetry algebra

As a consequence of the requirements (4.1)–(4.3), we prove in Appendix C the following.

*Proposition 8:* For any field  $\phi_s \in \mathcal{F}^\infty$ , 1-form  $d_V^s \phi$  tangent to  $\mathcal{F}^\infty$  at  $\phi_s$ , and for allowed gauge parameters  $g_1, g_2 \in \mathcal{A}_{\phi_s}$ , the identity

$$\oint_{S^{r,t}} k_{[g_1, g_2]}[d_V^s \phi] \approx \oint_{S^{r,t}} d_V^s k_{g_1}[R_{g_2}] + o(r^0) \tag{4.6}$$

holds.

This allows us to show the following.

*Corollary 9:* The space of allowed gauge parameters  $\mathcal{A}_{\phi_s}$  at  $\phi_s \in \mathcal{F}$  forms a Lie algebra.

Indeed, owing to (3.17), if  $R_{g_1}, R_{g_2}$  are tangent to  $\mathcal{F}$ , then so is  $R_{[g_1, g_2]}$ , and, furthermore, conditions (4.3) hold for  $[g_1, g_2]$  if they hold for  $g_1, g_2$  because of relation (A6) and the last of (A11). Applying  $d_V^s$  to (4.6) implies that the charges associated with the parameters  $[g_1, g_2]$  are integrable. Finally, applying  $\mathcal{L}_{\partial_\mu}$  with  $\mu=t, r$  to (4.6) and using (4.1) shows that conditions (4.4) and (4.5) hold for  $[g_1, g_2]$  if they hold for  $g_1, g_2$ . ■

The subspace of allowed gauge parameters,  $g_P \in \mathcal{A}_{\phi_s}$ , satisfying

$$\oint_{S^{r,t}} k_{g_P}[d_V^s \phi] \approx o(r^0) \tag{4.7}$$

for all  $d_V^s \phi$  tangent to  $\mathcal{F}$  will be called proper gauge parameters at  $\phi_s$ . The associated transformations  $\delta \phi^i = R_{g_P}^i$  will be called proper gauge transformations. On the contrary, gauge parameters (transformations) related to nonvanishing surface charge 1-forms will be called improper gauge parameters (transformations). Improper gauge transformations send field configurations to inequivalent field configurations in the sense that they change the conserved charges.

Proposition 8 also directly implies the following.

*Corolly 10:* Proper gauge transformations at  $\phi_s \in \mathcal{F}$  form an ideal  $\mathcal{N}_{\phi_s}$  of  $\mathcal{A}_{\phi_s}$ .

The quotient space  $\mathcal{A}_{\phi_s} / \mathcal{N}_{\phi_s}$  is therefore a Lie algebra which we call the asymptotic symmetry algebra  $\mathfrak{e}_{\phi_s}^{as}$  at  $\phi_s \in \mathcal{F}$ .

*Remarks:*

- (1) Exact reducibility parameters  $f^s \in \mathfrak{e}_{\phi_s}$  belonging to  $\mathcal{A}_{\phi_s}$  survive in the asymptotic symmetry algebra if there exists at least one solution  $\phi_s \in \mathcal{F}$  and a tangent 1-form  $d_V^s \phi$  such that  $\delta \mathcal{Q}_{f^s}[d_V \phi]|_{\phi_s} \neq 0$ .
- (2) If the relevant components of condition (3.12) hold and if the bracket of gauge transformations closes off shell, i.e., if (3.17) hold with  $M_{f_1, f_2}^{+i}[\delta \mathcal{L} / \delta \phi] = 0$ , the whole discussion can be done off shell. In other words, one can define the space of allowed field configurations  $\phi \in \mathcal{F}$  and of allowed gauge parameters  $g \in \mathcal{A}_\phi$  by imposing the requirements of Sec. IV A with strong instead of weak equalities with all results of the present subsection holding true for  $\mathcal{F}$  and  $\mathcal{A}_\phi$  instead of  $\mathcal{F}^s$  and  $\mathcal{A}_{\phi_s}$ . These conditions hold, for instance, in the case of the Hamiltonian formalism discussed in Appendix D and also for the Einstein gravity discussed in Appendix E.
- (3) A way to avoid assumptions (4.3) is to consider instead of the  $(n-2, 1)$ -forms  $k_f[d_V \phi]$  the forms

$$k'_f[d_V \phi] = k_f[d_V \phi] - E_f[d_V \phi, R_f]. \tag{4.8}$$

Using (3.16), we now have instead of (3.7) and (3.10),

$$d_H k'_f[d_V \phi] = \Omega_{\mathcal{L}}[R_f, d_V \phi] - d_V^s S_f + T_{R_f} \left[ d_V \phi, \frac{\delta \mathcal{L}}{\delta \phi} \right], \tag{4.9}$$

$$k'_f[d_V^s \phi] \approx I_f^{n-1} \Omega_{\mathcal{L}}[R_f, d_V^s \phi] + d_H(\cdot). \tag{4.10}$$

In the proof of Proposition 8 in Appendix C, this amounts to replacing  $W_{\delta \mathcal{L} / \delta \phi}$  by  $-\Omega_{\mathcal{L}}$  and the additional conditions (4.3) are not needed on account of  $d_V \Omega_{\mathcal{L}} = 0$ .

Contrary to  $k_f[d_V\phi]$ , however, the forms  $k'_f[d_V\phi]$  depend on the explicit choice of boundary terms in the Lagrangian. Indeed, if  $\mathcal{L} \rightarrow \mathcal{L} + d_H\mu$ , one has  $E_{\mathcal{L}} \rightarrow E_{\mathcal{L}} + d_V I_{d_V\phi}\mu + \frac{1}{2}d_H I_{d_V\phi}^{n-2} I_{d_V\phi}^{n-1}\mu$  and the resulting change in the  $n-2$  form is given by

$$k'_f[d_V\phi] \rightarrow k'_f[d_V\phi] + \delta_{R_f} I_{d_V\phi}\mu - d_V I_{R_f}\mu - d_H\left(\frac{1}{2}i_{R_f} I_{d_V\phi}^{n-2} I_{d_V\phi}^{n-1}\mu\right). \tag{4.11}$$

**C. Poisson bracket representation**

Applying consecutively  $i_{R_{g_2}}$  and  $i_{R_{g_3}}$  to the integrability conditions (4.2) gives

$$\oint_{S^{\infty,t}} k_{g_1}[R_{[g_2,g_3]}] = \oint_{S^{\infty,t}} (\delta_{R_{g_3}} k_{g_1}[R_{g_2}] - (2 \leftrightarrow 3)). \tag{4.12}$$

Using Proposition 8 on the two terms on the right-hand side (RHS) and the antisymmetry (3.15), we get

$$\oint_{S^{\infty,t}} k_{[g_1,g_2]}[R_{g_3}]|_{\phi^s} + \text{cyclic}(1,2,3) = 0. \tag{4.13}$$

If we define

$$\mathcal{K}_{f_1,f_2} = - \oint_S k_{f_2}[R_{f_1}]|_{\phi^s} = \oint_{S^{\infty,t}} I_{f_2}^{n-1} W_{\delta\mathcal{L}/\delta\phi}[R_{f_1}, R_{f_2}]|_{\phi^s}, \tag{4.14}$$

we have the following.

*Corollary 11:*  $\mathcal{K}_{g_1,g_2}$  defines a Chevalley–Eilenberg 2-cocycle on the Lie algebra  $\mathfrak{e}_{\phi_s}^{as}$ ,

$$\mathcal{K}_{g_1,g_2} + \mathcal{K}_{g_2,g_1} = 0, \tag{4.15}$$

$$\mathcal{K}_{[g_1,g_2],g_3} + \text{cyclic}(1,2,3) = 0. \tag{4.16}$$

The surface charge  $\mathcal{Q}_\gamma[\Phi, \bar{\Phi}]$  of  $\Phi = (\phi, f)$  with respect to the fixed background  $\bar{\Phi} = (\bar{\phi}, \bar{f})$  is defined as

$$\mathcal{Q}_\gamma[\Phi, \bar{\Phi}] := \int_{\gamma_s} \oint_{S^{\infty,t}} k'_{f'}[d_V\phi'] + N_{\bar{f}}[\bar{\phi}], \tag{4.17}$$

where the integration is done along a path  $\gamma_s$  joining  $\bar{\Phi}$  to  $\Phi$  in  $\mathcal{F}^s$ . We assume here that there are no global obstructions in  $\mathcal{F}^s$  for (4.2) to guarantee that the surface charges,

$$\mathcal{Q}[\Phi, \bar{\Phi}] = \int_{\gamma_s} \oint_{S^{\infty,t}} k_{g'}[d_V\phi'] + N_{\bar{g}}[\bar{\phi}], \tag{4.18}$$

do not depend on the path  $\gamma \in \mathcal{F}^s$ . If we denote  $\mathcal{Q}_i \equiv \mathcal{Q}[\Phi_i, \bar{\Phi}_i]$  the charge related to  $\Phi_i = (\phi, g_i)$ , the covariant Poisson bracket of these surface charges is defined by

$$\{\mathcal{Q}_1, \mathcal{Q}_2\}_c := - \delta_{R_{g_1}} \mathcal{Q}_2 = - \oint_{S^{\infty,t}} k_{g_2}[R_{g_1}]. \tag{4.19}$$

For an arbitrary path  $\gamma_s \in \mathcal{F}^s$ , definitions (4.19) and (4.14) lead to

$$\{\mathcal{Q}_1, \mathcal{Q}_2\}_c - \mathcal{K}_{\bar{g}_1, \bar{g}_2}[\bar{\phi}] = - \int_{\gamma_s} \oint_{S^{\infty, t}} d_V^s(k_{g_2'}[R_{g_1'}]), \tag{4.20}$$

$$= \int_{\gamma_s} \oint_{S^{\infty, t}} k_{[g_1, g_2]}[d_V^s \phi] | \phi^s, \tag{4.21}$$

where Proposition 8 has been used in the last line. Defining  $\mathcal{Q}_{[1,2]}$  as associated with  $[g_1, g_2]$ , we thus get the following.

**Theorem 12:** *In  $\mathcal{F}$ , the charge algebra between a fixed background solution  $\bar{\phi}$  and a final solution  $\phi_s$  is determined by*

$$\{\mathcal{Q}_1, \mathcal{Q}_2\}_c = \mathcal{Q}_{[1,2]} + \mathcal{K}_{\bar{g}_1, \bar{g}_2}[\bar{\phi}] - N_{[\bar{g}_1, \bar{g}_2]}[\bar{\phi}], \tag{4.22}$$

where the central charge  $\mathcal{K}_{\bar{g}_1, \bar{g}_2}[\bar{\phi}]$  is a 2-cocycle on the Lie algebra of asymptotic symmetries  $\mathfrak{e}_{\bar{\phi}_s}^{as}$ .

*Remarks:*

- (1) The central charge is nontrivial if one cannot find a normalization  $N_{\bar{g}}[\bar{\phi}]$  of the background such that  $\mathcal{K}_{\bar{g}_1, \bar{g}_2}[\bar{\phi}] = N_{[\bar{g}_1, \bar{g}_2]}[\bar{\phi}]$ .
- (2) The central charge involving an exact reducibility parameter of the background vanishes.
- (3) For a semisimple algebra  $\mathfrak{e}_{\bar{\phi}_s}^{as}$ , the property  $H^2(\mathfrak{e}_{\bar{\phi}_s}^{as}) = 0$  guarantees that the central charge can be absorbed by a suitable normalization of the background, while the property  $H^1(\mathfrak{e}_{\bar{\phi}_s}^{as}) = 0$  implies that this fixes the normalization completely.
- (4) As a consequence of Theorem 12 together with Corollary 10, proper gauge transformations act trivially on the charges,

$$\delta_{R_{g_p}} \mathcal{Q}_i = 0, \tag{4.23}$$

once we assume that the normalizations associated with proper gauge parameters all vanish.

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## APPENDIX A: ELEMENTS FROM THE VARIATIONAL BICOMPLEX

### 1. Elementary definitions and conventions

We assume for notational simplicity that all fields  $\phi^i$  are (Grassmann) even.

Consider  $k$ th order derivatives  $\partial^k \phi^i(x) / \partial x^{\mu_1} \dots \partial x^{\mu_k}$  of a field  $\phi^i(x)$ . The corresponding jet coordinate is denoted by  $\phi_{\mu_1, \dots, \mu_k}^i$ . Because the derivatives are symmetric under permutations of the derivative indices  $\mu_1, \dots, \mu_k$ , these jet coordinates are not independent, but one has  $\phi_{\mu\nu}^i = \phi_{\nu\mu}^i$ , etc. Local functions are smooth functions depending on the coordinates  $x^\mu$  of the base space  $M$ , the fields  $\phi^i$ , and a finite number of the jet coordinates  $\phi_{\mu_1, \dots, \mu_k}^i$ . Horizontal forms involve in addition the differentials  $dx^\mu$  which we treat as anticommuting (Grassmann odd) variables,  $dx^\mu dx^\nu = -dx^\nu dx^\mu$ . We also introduce the notation

$$(d^{n-p}x)_{\mu_1, \dots, \mu_p} := \frac{1}{p!(n-p)!} \epsilon_{\mu_1, \dots, \mu_n} dx^{\mu_{p+1}}, \dots, dx^{\mu_n}, \quad \epsilon_{0, \dots, (n-1)} = 1, \quad (A1)$$

which implies that  $dx^\alpha(d^{n-p-1}x)_{\mu_1, \dots, \mu_{p+1}} = (d^{n-p}x)_{[\mu_1, \dots, \mu_p} \delta^{\alpha}_{\mu_{p+1}]}$ . If the base space is endowed with a metric  $g_{\mu\nu}$  (which can be contained in the set of fields), the Hodge dual of an horizontal  $p$ -form  $\omega^p$  is defined as  $\star\omega^p = \sqrt{|g|} \omega^{\mu_1, \dots, \mu_p} (d^{n-p}x)_{\mu_1, \dots, \mu_p}$ , where indices are raised with the metric. As a consequence,  $\star\star\omega^p = (-)^{p(n-p)+s} \omega^p$ , where  $s$  is the signature of the metric.

As in Refs. 40 and 50, we define derivatives  $\partial^S / \partial \phi^i_{\mu_1, \dots, \mu_k}$  that act on the basic variables through

$$\begin{aligned} \frac{\partial^S \phi^j_{\nu_1, \dots, \nu_k}}{\partial \phi^i_{\mu_1, \dots, \mu_k}} &= \delta^j_i \delta^{\mu_1}_{\nu_1} \dots \delta^{\mu_k}_{\nu_k}, & \frac{\partial^S \phi^j_{\nu_1, \dots, \nu_m}}{\partial \phi^i_{\mu_1, \dots, \mu_k}} &= 0 \quad \text{for } m \neq k, \\ \frac{\partial^S x^\mu}{\partial \phi^i_{\mu_1, \dots, \mu_k}} &= 0, & \frac{\partial^S dx^\mu}{\partial \phi^i_{\mu_1, \dots, \mu_k}} &= 0, \end{aligned} \quad (A2)$$

where the round parentheses denote symmetrization with weight 1,

$$\delta^{\mu_1}_{(\nu_1} \delta^{\mu_2}_{\nu_2)} = \frac{1}{2} (\delta^{\mu_1}_{\nu_1} \delta^{\mu_2}_{\nu_2} + \delta^{\mu_1}_{\nu_2} \delta^{\mu_2}_{\nu_1}), \quad \text{etc.}$$

For instance, the definition gives explicitly (with  $\phi$  any of the  $\phi^i$ )

$$\frac{\partial^S \phi_{11}}{\partial \phi_{11}} = 1, \quad \frac{\partial^S \phi_{12}}{\partial \phi_{12}} = \frac{\partial^S \phi_{21}}{\partial \phi_{12}} = \frac{1}{2}, \quad \frac{\partial^S \phi_{112}}{\partial \phi_{112}} = \frac{1}{3}, \quad \frac{\partial^S \phi_{123}}{\partial \phi_{123}} = \frac{1}{6}.$$

We note that the use of these operators takes automatically care of many combinatorial factors which arise in other conventions, such as those used in Ref. 42.

The vertical differential is defined by

$$d_V = \sum_{k=0} d_V \phi^i_{\mu_1, \dots, \mu_k} \frac{\partial^S}{\partial \phi^i_{\mu_1, \dots, \mu_k}}, \quad (A3)$$

with Grassmann odd generators  $d_V \phi^i_{\mu_1, \dots, \mu_k}$ , so that  $d_V^2 = 0$ . The total derivative is the vector field denoted by  $\partial_\nu$  and acts on local functions according to

$$\partial_\nu = \frac{\partial}{\partial x^\nu} + \sum_{k=0} \phi^i_{\mu_1, \dots, \mu_k} \nu \frac{\partial^S}{\partial \phi^i_{\mu_1, \dots, \mu_k}}. \quad (A4)$$

Here,  $\sum_{k=0}$  means the sum over all  $k$ , from  $k=0$  to infinity, with the summand for  $k=0$  given by  $\phi^i_\nu \partial / \partial \phi^i$ , i.e., by definition  $k=0$  means “no indices  $\mu_i$ .” Furthermore, we are using Einstein’s summation convention over repeated indices, i.e., for each  $k$  there is a summation over all tuples  $(\mu_1, \dots, \mu_k)$ . Hence, for  $k=2$ , the sum over  $\mu_1$  and  $\mu_2$  contains both the tuple  $(\mu_1, \mu_2) = (1, 2)$  and the tuple  $(\mu_1, \mu_2) = (2, 1)$ . These conventions extend to all other sums of similar type.

The horizontal differential on horizontal forms is defined by  $d_H = dx^\nu \partial_\nu$ . It is extended to the vertical generators in such a way that  $\{d_H, d_V\} = 0$ . The derivative of a  $(n-p)$ -form  $k^{(n-p)} = k^{[\mu_1, \dots, \mu_p]} (d^{n-p}x)_{\mu_1, \dots, \mu_p}$  is given by

$$d_H k^{(n-p)} = \partial_\rho k^{[\mu_1, \dots, \mu_{p-1} \rho]} (d^{n-(p-1)}x)_{\mu_1, \dots, \mu_{p-1}}.$$

A vector field of the form  $Q^i \partial / \partial \phi^i$ , for  $Q^i$  a set of local functions, is called an evolutionary vector field with characteristic  $Q^i$ . Its prolongation which acts on local functions is

$$\delta_Q = \sum_{k=0} (\partial_{\mu_1}, \dots, \partial_{\mu_k} Q^i) \frac{\partial^S}{\partial \phi^{\mu_1, \dots, \mu_k}}, \quad (\text{A5})$$

so that  $[\delta_Q, d_H] = 0$ . The Lie bracket of characteristics is defined by  $[Q_1, Q_2]^i = \delta_{Q_1} Q_2^i - \delta_{Q_2} Q_1^i$  and satisfies

$$[\delta_{Q_1}, \delta_{Q_2}] = \delta_{[Q_1, Q_2]}. \quad (\text{A6})$$

One also has  $[\delta_Q, d_H] = 0$ .

An infinitesimal transformation  $v$  is defined by  $x^\mu \rightarrow x^\mu + \epsilon c^\mu$  and  $\phi^i \rightarrow \phi^i + \epsilon b^i$  with  $c^\mu(x, [\phi]), b^i(x, [\phi])$  local functions. If we denote by  $i_c = c^\mu (\partial / \partial x^\mu)$  and

$$\mathcal{L}_c = i_c d_H + d_H i_c = c^\mu \partial_\mu + d_H c^\mu \frac{\partial}{\partial x^\mu}, \quad (\text{A7})$$

the transformation can be extended to act on the horizontal complex as

$$\text{pr } v = \delta_Q + \mathcal{L}_c,$$

where  $Q^i = b^i - \phi_\mu^i c^\mu$ . It satisfies  $[\text{pr } v, d_H] = 0$ . For example, a vector field acting on an  $n$ -form  $Ld^n x$  can be written as

$$\text{pr } v(Ld^n x) = \delta_Q Ld^n x + d_H(c^\mu L(d^{n-1}x)_\mu). \quad (\text{A8})$$

The vector field  $\delta_Q$  can be extended so as to also commute with  $d_V$ : if we continue to denote the extension by  $\delta_Q$ , the defining relation

$$[\delta_Q, d_V] = 0 \quad (\text{A9})$$

implies that  $\delta_Q d_V \phi^i = d_V(Q^i)$ . If  $i_Q = \partial_{(\mu)} Q^i [\partial^S / \partial d_V \phi_{(\mu)}^i]$ , we have

$$\{i_Q, d_V\} = \delta_Q, \quad [i_{Q_1}, \delta_{Q_2}] = i_{[Q_1, Q_2]}. \quad (\text{A10})$$

In the context of gauge theories with  $\Phi_a^\Delta = (\phi^i, f_a^\alpha)$  and  $d_V$  defined in terms of  $\Phi_a^\Delta$ , the relations

$$\{i_{Q_1}, d_V\} = \delta_{Q_1}, \quad [\delta_{Q_1}, d_V] = 0, \quad [i_{Q_1}, \delta_{Q_2}] = i_{[Q_1, Q_2]} \quad (\text{A11})$$

continue to hold when  $Q_1 = R_{f_1}$ ,  $Q_2 = R_{f_2}$ .

The set of multi-indices is simply the set of all tuples  $(\mu_1, \dots, \mu_k)$ , including (for  $k=0$ ) the empty tuple. The tuple with one element is denoted by  $\mu_1$  without round parentheses, while a generic tuple is denoted by  $(\mu)$ . The length, i.e., the number of individual indices, of a multiindex  $(\mu)$  is denoted by  $|\mu|$ . We use Einstein's summation convention also for repeated multiindices as in Ref. 40. For instance, an expression of the type  $(-\partial)_{(\mu)} K^{(\mu)}$  stands for a free sum over all tuples  $(\mu_1, \dots, \mu_k)$  analogous to the one in (A4),

$$(-\partial)_{(\mu)} K^{(\mu)} = \sum_{k=0} (-)^k \partial_{\mu_1}, \dots, \partial_{\mu_k} K^{\mu_1, \dots, \mu_k}.$$

If  $Z = Z^{(\mu)} \partial_{(\mu)}$  is a differential operator, its adjoint is defined by  $Z^+ = (-\partial)_{(v)} [Z^{(v)} \cdot]$  and its ‘‘components’’ are denoted by  $Z^{+(\mu)}$ , i.e.,  $Z^+ = Z^{+(\mu)} \partial_{(\mu)}$ .

More details on the variational bicomplex can be found, for instance, in the textbooks.<sup>40,42,51,52</sup>

## 2. Higher order Lie–Euler operators

Except for a different notation, we follow in this and the next subsection.<sup>40</sup>

Multiple integrations by parts can be done using the following. If for a given collection  $R_i^{(\mu)}$  of local functions the equality

$$\partial_{(\mu)} Q^i P_i^{(\mu)} = \partial_{(\mu)} (Q^i R_i^{(\mu)}) \quad (\text{A12})$$

holds for all local functions  $Q^i$ , then

$$R_i^{(\mu)} = \binom{|\mu| + |\nu|}{|\mu|} (-\partial)_{(\nu)} P_i^{((\mu)(\nu))} \quad (\text{A13})$$

or, equivalently,

$$R_i^{(\mu)} = \sum_{l=0} \binom{k+l}{k} (-)^l \partial_{\nu_1} \dots \partial_{\nu_l} P_i^{\mu_1 \dots \mu_k \nu_1 \dots \nu_l}, \quad (\text{A14})$$

i.e., there is a summation over  $(\nu)$  in (A13) by Einstein's summation convention for repeated multi-indices, and the multi-index  $((\mu)(\nu))$  is the tuple  $(\mu_1, \dots, \mu_k, \nu_1, \dots, \nu_l)$  when  $(\mu)$  and  $(\nu)$  are the tuples  $(\mu_1, \dots, \mu_k)$  and  $(\nu_1, \dots, \nu_l)$ , respectively. Note that the sum contains only finitely many nonvanishing terms whenever  $f$  is a local function: if  $f$  depends only on variables with at most  $M$  "derivatives," i.e., on the  $\phi_{(\rho)}^i$  with  $|\rho| \leq M$ , the only possibly nonvanishing summands are those with  $|\nu| \leq M - |\mu|$  ( $l \leq M - k$ ). Conversely, if (A12) holds for a given collection  $R_i^{(\mu)}$ , then

$$P_i^{(\mu)} = \binom{|\mu| + |\nu|}{|\mu|} \partial_{(\nu)} R_i^{((\mu)(\nu))}. \quad (\text{A15})$$

By definition, when  $P_i^{(\mu)} = \partial^S f / \partial \phi_{(\mu)}^i$ , the higher order Euler–Lagrange derivatives  $\delta f / \delta \phi_{(\mu)}^i$  are given by the associated  $R_i^{(\mu)}$ ,

$$\frac{\delta f}{\delta \phi_{(\mu)}^i} = \binom{|\mu| + |\nu|}{|\mu|} (-\partial)_{(\nu)} \frac{\partial^S f}{\partial \phi_{((\mu)(\nu))}^i}. \quad (\text{A16})$$

As a consequence,

$$\forall Q^i: \delta_Q f = \partial_{(\mu)} \left[ Q^i \frac{\delta f}{\delta \phi_{(\mu)}^i} \right]. \quad (\text{A17})$$

Note also that  $\delta / \delta \phi^i$  is the Euler–Lagrange derivative. The crucial property of these operators is that they "absorb total derivatives,"

$$|\mu| = 0: \frac{\delta(\partial_w f)}{\delta \phi^i} = 0, \quad (\text{A18})$$

$$|\mu| > 0: \frac{\delta(\partial_w f)}{\delta \phi_{(\mu)}^i} = \delta_v^{\mu} \frac{\delta f}{\delta \phi_{(\mu')}^i}, \quad (\mu) = (\mu(\mu')), \quad (\text{A19})$$

where, e.g.,

$$\delta_v^{\mu} \frac{\delta f}{\delta \phi_{(\lambda)}^i} = \frac{1}{2} \left( \delta_v^{\mu} \frac{\delta f}{\delta \phi_{\lambda}^i} + \delta_v^{\lambda} \frac{\delta f}{\delta \phi_{\mu}^i} \right).$$

It may be also deduced that

$$\frac{\delta(\partial_w f)}{\delta \phi_{(\rho(\mu))}^i} = \frac{1}{|\mu| + 1} \delta_v^{\rho} \frac{\delta f}{\delta \phi_{(\mu)}^i} + \frac{|\mu|}{|\mu| + 1} \delta_v^{\mu_1} \frac{\delta f}{\delta \phi_{(\rho \mu_2 \dots \mu_{|\mu|})}^i}. \quad (\text{A20})$$

By considering the particular case where (A12) and (A13) are used in terms of  $Q_2$  with

$$P_i^{(\mu)} \left[ \frac{\delta \omega^n}{\delta \phi} \right] = \frac{\partial^S Q_1^i}{\partial \phi_{(\mu)}^i} \frac{\delta \omega^n}{\delta \phi^j}, \tag{A21}$$

we get  $\delta_{Q_2}(Q_1^j)(\delta \omega^n / \delta \phi^j) = \partial_{(\mu)}(Q_2^i R_i^{(\mu)})[\delta \omega^n / \delta \phi]$ . Splitting the term without derivatives on the RHS from the others and defining

$$\begin{aligned} T_{Q_1} \left[ Q_2, \frac{\delta \omega^n}{\delta \phi} \right] &= \partial_{(\mu)} \left( Q_2^i R_i^{(\mu)\nu} \left[ \frac{\partial}{\partial x^\nu} \frac{\delta \omega^n}{\delta \phi} \right] \right), \\ &= \binom{|\mu| + 1 + |\rho|}{|\mu| + 1} \partial_{(\mu)} \left( Q_2^i (-\partial)_{(\rho)} \left( \frac{\partial^S Q_1^j}{\partial \phi_{(\mu)(\rho)\nu}^j} \frac{\partial}{\partial x^\nu} \frac{\delta \omega^n}{\delta \phi^j} \right) \right) \end{aligned} \tag{A22}$$

gives

$$\delta_{Q_2}(Q_1^j) \frac{\delta \omega^n}{\delta \phi^j} = Q_2^i R_i + d_H T_{Q_1} \left[ Q_2, \frac{\delta \omega^n}{\delta \phi} \right], \quad R_i = (-\partial)_{(\nu)} \left( \frac{\partial^S Q_1^j}{\partial \phi_{(\nu)}^j} \frac{\delta \omega^n}{\delta \phi^j} \right). \tag{A23}$$

We also need the definition

$$\delta_{Q_3} T_{Q_1} \left[ Q_2, \frac{\delta \omega^n}{\delta \phi} \right] = T_{Q_1} \left[ \delta_{Q_3} Q_2, \frac{\delta \omega^n}{\delta \phi} \right] + T_{Q_1} \left[ Q_2, \delta_{Q_3} \frac{\delta \omega^n}{\delta \phi} \right] + T_{\delta_{Q_3} Q_1} \left[ Q_2, \frac{\delta \omega^n}{\delta \phi} \right] - Y_{Q_1, Q_3} \left[ Q_2, \frac{\delta \omega^n}{\delta \phi} \right], \tag{A24}$$

where

$$Y_{Q_1, Q_3} \left[ Q_2, \frac{\delta \omega^n}{\delta \phi} \right] = \binom{|\mu| + |\rho| + 1}{|\mu| + 1} \partial_{(\mu)} \left( Q_2^i (-\partial)_{(\rho)} \left( \frac{\partial}{\partial x^\nu} \frac{\delta \omega^n}{\delta \phi^j} \frac{\partial^S \partial_{(\sigma)} Q_3^k}{\partial \phi_{(\mu)(\rho)\nu}^i} \frac{\partial^S Q_1^j}{\partial \phi_{(\sigma)}^k} \right) \right). \tag{A25}$$

### 3. Horizontal homotopy operators

Define

$$I_{d_V \phi}^p \omega^{p,s} = \frac{|\mu| + 1}{n - p + |\mu| + 1} \partial_{(\mu)} \left( d_V \phi^j \frac{\delta}{\delta \phi_{(\mu)\nu}^j} \frac{\partial \omega^{p,s}}{\partial x^\nu} \right) \tag{A26}$$

for  $\omega^{p,s}$  a  $p, s$ -form. Note that there is a summation over  $(\mu)$  by Einstein's summation convention. The following result (see, e.g., Ref. 40) is the key for showing local exactness of the horizontal part of the variational bicomplex:

$$0 \leq p < n: \delta_Q \omega^{p,s} = I_Q^{p+1}(d_H \omega^{p,s}) + d_H(I_Q^p \omega^{p,s}), \tag{A27}$$

$$p = n: \delta_Q \omega^{n,s} = Q^i \frac{\delta \omega^{n,s}}{\delta \phi^i} + d_H(I_Q^n \omega^{n,s}). \tag{A28}$$

This last relation is sometimes referred to as “the first variational formula.” Similarly,

$$0 \leq p < n: d_V \omega^{p,s} = I_{d_V \phi}^{p+1}(d_H \omega^{p,s}) - d_H(I_{d_V \phi}^p \omega^{p,s}), \tag{A29}$$

$$p = n: d_V \omega^{n,s} = d_V \phi^i \frac{\delta \omega^{n,s}}{\delta \phi^i} - d_H(I_{d_V \phi}^n \omega^{n,s}). \tag{A30}$$

In the context of the extended jet bundle of gauge theories, we will also use the following homotopy operators that only involve the gauge parameters: for local functions  $G^\alpha$ ,



$$I_G^p \omega_f^{p,s} = \frac{|\lambda| + 1}{n - p + |\lambda| + 1} \partial_{(\lambda)} \left( G^\alpha \frac{\delta}{\delta f_{(\lambda)\rho}^\alpha} \frac{\partial \omega_f^{p,s}}{\partial dx^\rho} \right). \quad (\text{A31})$$

When applied to  $p, s$ -forms that are linear and homogeneous in  $f^\alpha$  and its derivatives, we have

$$0 \leq p < n: \omega_G^{p,s} = I_G^{p+1}(\mathbf{d}_H \omega_f^{p,s}) + \mathbf{d}_H(I_G^p \omega_f^{p,s}), \quad (\text{A32})$$

$$p = n: \omega_G^{n,s} = G^\alpha \frac{\delta \omega_f^{n,s}}{\delta f^\alpha} + \mathbf{d}_H(I_G^n \omega_f^{n,s}). \quad (\text{A33})$$

#### 4. Commutation relations

Starting from  $\delta_{Q_1} \delta_{Q_2} \omega^n - \delta_{Q_2} \delta_{Q_1} \omega^n = \delta_{[Q_1, Q_2]} \omega^n$  and using (A28) both on the inner terms of the LHS and on the RHS gives

$$Q_2^i \delta_{Q_1} \frac{\delta \omega^n}{\delta \phi^i} - Q_1^i \delta_{Q_2} \frac{\delta \omega^n}{\delta \phi^i} = \mathbf{d}_H(I_{[Q_1, Q_2]}^n \omega^n - \delta_{Q_1} I_{Q_2}^n \omega^n + \delta_{Q_2} I_{Q_1}^n \omega^n). \quad (\text{A34})$$

Using  $\mathbf{d}_V(\delta_Q \omega) = \delta_Q(\mathbf{d}_V \omega)$ , we get  $\partial_{(\mu)}(\mathbf{d}_V \phi^i(\delta \delta_Q \omega / \delta \phi_{(\mu)}^i)) = \partial_{(\mu)}(\delta_Q(\mathbf{d}_V \phi^i(\delta \omega / \delta \phi_{(\mu)}^i)))$ , which can be written as  $\partial_{(\mu)}(\mathbf{d}_V \phi^i[\delta / \delta \phi_{(\mu)}^i, \delta_Q] \omega) = \partial_{(\mu)}(\mathbf{d}_V Q^i(\delta \omega / \delta \phi_{(\mu)}^i))$ . Applying  $\delta / \delta \mathbf{d}_V \phi_{\mu_1, \dots, \mu_k}^i$  gives

$$\left[ \frac{\delta}{\delta \phi_{\mu_1, \dots, \mu_k}^i}, \delta_Q \right] \omega = \sum_{l \leq k} \binom{l + |\nu|}{l} (-\partial)_{(\nu)} \left( \frac{\partial^S Q^j}{\partial \phi_{(\nu)\mu_1, \dots, \mu_l}^i} \frac{\delta \omega}{\delta \phi_{\mu_{l+1}, \dots, \mu_k}^j} \right). \quad (\text{A35})$$

In particular,

$$\left[ \frac{\delta}{\delta \phi^i}, \delta_Q \right] \omega = (-\partial)_{(\nu)} \left( \frac{\partial^S Q^j}{\partial \phi_{(\nu)}^i} \frac{\delta \omega}{\delta \phi^j} \right). \quad (\text{A36})$$

When combined with (A23), we get

$$Q_2^i \left[ \delta_{Q_1}, \frac{\delta}{\delta \phi^i} \right] \omega^n = -\delta_{Q_2} Q_1^j \frac{\delta \omega^n}{\delta \phi^j} + \mathbf{d}_H T_{Q_1} \left[ Q_2, \frac{\delta \omega^n}{\delta \phi} \right]. \quad (\text{A37})$$

Similarly, applying  $\delta / \delta \mathbf{d}_V \phi_{\mu_1, \dots, \mu_k}^i$  to  $\partial_{(\mu)}(\mathbf{d}_V \phi^i(\delta(\delta_Q \omega) / \delta \phi_{(\mu)}^i)) = \partial_{(\mu)}(\mathbf{d}_V(Q^i(\delta \omega / \delta \phi_{(\mu)}^i)))$  gives

$$\frac{\delta}{\delta \phi_{\mu_1, \dots, \mu_k}^i} (\delta_Q \omega) = \sum_{l \leq k} \frac{\delta}{\delta \phi_{\mu_1, \dots, \mu_l}^i} \left( Q^j \frac{\delta \omega}{\delta \phi_{\mu_{l+1}, \dots, \mu_k}^j} \right). \quad (\text{A38})$$

Applying  $\delta / \delta \mathbf{d}_V \phi_{(\lambda)}^i$  to  $\mathbf{d}_V(\delta \omega^n / \delta \phi^i) = (\delta / \delta \phi^i)(\mathbf{d}_V \phi^i(\delta \omega^n / \delta \phi^i))$ , we also get

$$\frac{\delta}{\delta \phi_{(\lambda)}^i} \frac{\delta \omega^n}{\delta \phi^j} = (-)^{|\lambda|} \frac{\partial^S}{\partial \phi_{(\lambda)}^i} \frac{\delta \omega^n}{\delta \phi^j}. \quad (\text{A39})$$

Starting from  $\mathbf{d}_H([\delta_{Q_1}, I_{Q_2}^n] \omega^n) = \delta_{[Q_1, Q_2]} \omega^n - \delta_{Q_1} Q_2^i (\delta \omega^n / \delta \phi^i) - Q_2^i \delta_{Q_1} (\delta \omega^n / \delta \phi^i) + Q_2^i [\delta(\delta_{Q_1} \omega^n) / \delta \phi^i]$  and using (A37) to compute the last two terms, we find

$$\mathbf{d}_H([\delta_{Q_1}, I_{Q_2}^n] \omega^n) = \mathbf{d}_H(I_{[Q_1, Q_2]}^n \omega^n) - \mathbf{d}_H T_{Q_1} \left[ Q_2, \frac{\delta \omega^n}{\delta \phi} \right]. \quad (\text{A40})$$

Similarly, for  $p < n$ , by evaluating  $\mathbf{d}_H([\delta_{Q_1}, I_{Q_2}^p] \omega^p)$ , one finds

$$\mathbf{d}_H([\delta_{Q_1}, I_{Q_2}^p] \omega^p) = \mathbf{d}_H(I_{[Q_1, Q_2]}^p \omega^p) + (I_{[Q_1, Q_2]}^{p+1} - [\delta_{Q_1}, I_{Q_2}^{p+1}]) (\mathbf{d}_H \omega^p). \quad (\text{A41})$$

By the same type of arguments, one shows

$$d_H(\delta_{Q_1}(I_{Q_2}^n \omega^n) - (1 \leftrightarrow 2)) = d_H\left(I_{[Q_1, Q_2]}^n \omega^n - I_{Q_1}^n(\delta_{Q_2} \omega^n) - T_{Q_1}\left[Q_2, \frac{\delta \omega^n}{\delta \phi}\right] - (1 \leftrightarrow 2)\right), \quad (\text{A42})$$

$$d_H(\delta_{Q_1}(I_{Q_2}^p \omega^p) - (1 \leftrightarrow 2)) = d_H(I_{[Q_1, Q_2]}^p \omega^p) + (I_{[Q_1, Q_2]}^{p+1} - \delta_{Q_1} I_{Q_2}^{p+1} + \delta_{Q_2} I_{Q_1}^{p+1})(d_H \omega^p). \quad (\text{A43})$$

### 5. Properties of the invariant presymplectic $(n-1, 2)$ -form

Let us define the  $(n-1, 2)$ -forms

$$W_{\delta \omega^n / \delta \phi} = -\frac{1}{2} I_{d_V \phi}^n \left( d_V \phi^i \frac{\delta \omega^n}{\delta \phi^i} \right), \quad \Omega_{\omega^n} = d_V I_{d_V \phi}^n \omega^n, \quad (\text{A44})$$

and the  $(n-2, 2)$ -form

$$E_{\omega^n} = \frac{1}{2} I_{d_V \phi}^{n-1} I_{d_V \phi}^n \omega^n. \quad (\text{A45})$$

Using  $[d_V, I_{d_V \phi}^n] = 0$ , (A29) and (A30) imply

$$\frac{1}{2} I_{d_V \phi}^n \left( d_V \phi^i \frac{\delta \omega^n}{\delta \phi^i} \right) = d_V I_{d_V \phi}^n \omega^n + \frac{1}{2} d_H (I_{d_V \phi}^{n-1} I_{d_V \phi}^n \omega^n), \quad (\text{A46})$$

so that

$$-W_{\delta \omega^n / \delta \phi} = \Omega_{\omega^n} + d_H E_{\omega^n}, \quad d_V \Omega_{\omega^n} = 0. \quad (\text{A47})$$

$\Omega_{\omega^n}$  is the presymplectic  $(n-1, 2)$ -form usually used in the context of covariant phase space methods. The “second variational formula,” obtained by applying  $d_V$  to (A30), can be combined with (A47) to give

$$d_V \phi^i d_V \frac{\delta \omega^n}{\delta \phi^i} = d_H \Omega_{\omega^n} = -d_H W_{\delta \omega^n / \delta \phi}. \quad (\text{A48})$$

Our surface charges are related to  $W_{\delta \omega^n / \delta \phi}$ , which is  $d_V$ -closed only up to a  $d_H$ -exact term,

$$d_V W_{\delta \omega^n / \delta \phi} = d_H d_V E_{\omega^n}. \quad (\text{A49})$$

Contrary to  $\Omega_{\omega^n}$ , it involves the Euler–Lagrange derivatives and is thus independent of  $d_H$ -exact  $n$ -forms that are added to  $\omega^n$ . For this reason, we call  $W_{\delta \omega^n / \delta \phi}$  the invariant presymplectic  $(n-1, 2)$ -form.

When  $\omega^n = d_H \omega^{n-1}$ , we have

$$E_{d_H \omega^{n-1}} = d_V I_{d_V \phi}^{n-1} \omega^{n-1} + d_H I_{d_V \phi}^{n-2} I_{d_V \phi}^{n-1} \omega^{n-1}. \quad (\text{A50})$$

The following proposition, proven at the end of this section, is crucial and generalizes corresponding results in Ref. 39 and 31.

*Proposition 13:*

$$W_{\delta \omega^n / \delta \phi}[Q_1, Q_2] = \left( \frac{|\mu| + |\rho| + 1}{|\mu| + 1} \right) \partial_{(\mu)} \left( Q_1^i (-\partial)_{(\rho)} \left( Q_2^j \frac{\partial^S}{\partial \phi_{(\mu)(\rho)\nu}^i} \frac{\partial}{\partial x^\nu} \frac{\delta \omega^n}{\delta \phi^j} \right) \right). \quad (\text{A51})$$

Together with the definition of the homotopy operator (A26), the definition of the higher order Euler–Lagrange derivatives, and definition (A22), we find from (A51) that

$$I_{Q_1}^n \left( Q_2^i \frac{\delta \omega^n}{\delta \phi^i} \right) = W_{\delta \omega^n / \delta \phi} [Q_1, Q_2] + T_{Q_2} \left[ Q_1, \frac{\delta \omega^n}{\delta \phi} \right]. \tag{A52}$$

We can then define

$$\delta_{Q_3} W_{\delta \omega^n / \delta \phi} [Q_1, Q_2] = W_{\delta \omega^n / \delta \phi} [\delta_{Q_3} Q_1, Q_2] + W_{\delta \omega^n / \delta \phi} [Q_1, \delta_{Q_3} Q_2] + Z_{\delta \omega^n / \delta \phi} [Q_1, Q_2, Q_3], \tag{A53}$$

where

$$Z_{\delta \omega^n / \delta \phi} [Q_1, Q_2, Q_3] = \left( \frac{|\mu| + |\rho| + 1}{|\mu| + 1} \right) \partial_{(\mu)} \left( Q_1^i (-\partial)_{(\rho)} \left( Q_2^j \partial_{(\sigma)} Q_3^k \frac{\partial^S}{\partial \phi_{(\sigma)}^k} \frac{\partial^S}{\partial \phi_{((\mu)(\rho)\nu)}^i} \frac{\partial}{\partial dx^\nu} \frac{\delta \omega^n}{\delta \phi^j} \right) \right). \tag{A54}$$

Note also that it follows from Proposition 13 that, in the case of first order theories,  $W_{\delta \mathcal{L} / \delta \phi}$  coincides with the covariant symplectic density  $\hat{\omega}$  considered in Ref. 31,

$$W_{\delta \omega^n / \delta \phi} = \frac{1}{2} d_V \phi^i d_V \phi^j \frac{\partial^S}{\partial \phi_v^i} \left( \frac{\partial}{\partial dx^\nu} \frac{\delta \omega^n}{\delta \phi^j} \right). \tag{A55}$$

Additional relations are obtained by applying  $d_H$  to (A52) and using (A37), which gives

$$\begin{aligned} & d_H \left( W_{\delta \omega^n / \delta \phi} [Q_1, Q_2] + T_{Q_2} \left[ Q_1, \frac{\delta \omega^n}{\delta \phi} \right] - T_{Q_1} \left[ Q_2, \frac{\delta \omega^n}{\delta \phi} \right] \right) \\ &= d_H (-I_{[Q_1, Q_2]}^n \omega^n + I_{Q_1}^n (\delta_{Q_2} \omega^n) - I_{Q_2}^n (\delta_{Q_1} \omega^n)). \end{aligned} \tag{A56}$$

Starting from  $[\delta_{Q_1}, \delta_{Q_2}] \omega^n = \delta_{[Q_1, Q_2]} \omega^n$  and using (A28) on the outer terms of the LHS, (A52) and (A37) gives

$$Q_1^i \delta_{Q_2} \frac{\delta \omega^n}{\delta \phi^i} - Q_2^i \delta_{Q_1} \frac{\delta \omega^n}{\delta \phi^i} = d_H (I_{[Q_1, Q_2]}^n \omega^n - 2W_{\delta \omega^n / \delta \phi} [Q_1, Q_2] - \delta_{Q_1} I_{Q_2}^n \omega^n + \delta_{Q_2} I_{Q_1}^n \omega^n). \tag{A57}$$

Adding to (A34) gives, in particular,

$$d_H W_{\delta \omega^n / \delta \phi} [Q_1, Q_2] = d_H (I_{[Q_1, Q_2]}^n \omega^n - \delta_{Q_1} I_{Q_2}^n \omega^n + \delta_{Q_2} I_{Q_1}^n \omega^n). \tag{A58}$$

*Proof of Proposition 13:* Let  $R[Q_1, Q_2]$  be the RHS of (A51). By applying  $i_{Q_2} i_{Q_1}$  to  $W_{\delta \omega^n / \delta \phi}$ , it follows that Proposition 13 amounts to showing

$$R[Q_1, Q_2] = -R[Q_2, Q_1]. \tag{A59}$$

Splitting the derivatives ( $\mu$ ) in those acting on  $Q_1$ , denoted by ( $\alpha$ ), and in those acting on the remaining expression, denoted by ( $\mu'$ ) and regrouping the indices  $((\mu')(\rho)) \equiv (\sigma)$ , we get

$$\begin{aligned} R[Q_1, Q_2] &= \sum_{|\alpha| \geq 0} \sum_{|\sigma| \geq |\mu'| \geq 0} \binom{|\sigma| + |\alpha| + 1}{|\mu'| + |\alpha| + 1} \binom{|\mu'| + |\alpha|}{|\alpha|} (-)^{|\mu'|} \\ & \quad \partial_{(\alpha)} Q_1^i (-\partial)_{(\sigma)} \left( Q_2^j \frac{\partial^S}{\partial \phi_{((\sigma)(\alpha)\nu)}^i} \frac{\partial}{\partial dx^\nu} \frac{\delta \omega^n}{\delta \phi^j} \right). \end{aligned} \tag{A60}$$

We now evaluate  $\sum_{|\sigma| \geq |\mu'| \geq 0}$  as  $\sum_{|\sigma| \geq 0} \sum_{|\mu'|=0}^{|\sigma|}$  and use the fact that

$$\sum_{|\mu'|=0}^{|\sigma|} \binom{|\sigma|+|\alpha|+1}{|\mu'|+|\alpha|+1} \binom{|\mu'|+|\alpha|}{|\alpha|} (-)^{|\mu'|} = 1, \quad (\text{A61})$$

for all  $|\alpha|, |\sigma|$ , so that

$$R[Q_1, Q_2] = \partial_{(\alpha)} Q_1^i (-\partial)_{(\sigma)} \left( Q_2^j \frac{\partial^S}{\partial \phi_{(\alpha)(\beta)(\sigma)\nu}^i} \frac{\partial}{\partial dx^\nu} \frac{\delta \omega^n}{\delta \phi^j} \right). \quad (\text{A62})$$

Expanding the  $\sigma$  derivatives,

$$R[Q_1, Q_2] = \partial_{(\alpha)} Q_1^i \partial_{(\beta)} Q_2^j C_{ij}^{(\alpha)(\beta)}, \quad (\text{A63})$$

where

$$C_{ij}^{(\alpha)(\beta)} = (-)^{|\beta|} \binom{|\rho|+|\beta|}{|\beta|} (-\partial)_{(\rho)} \frac{\partial^S}{\partial \phi_{(\alpha)(\beta)(\rho)\nu}^i} \frac{\delta}{\delta \phi^j} \frac{\partial}{\partial dx^\nu} \omega^n. \quad (\text{A64})$$

Antisymmetry (A59) amounts to prove that

$$C_{ij}^{(\alpha)(\beta)} = -C_{ji}^{(\beta)(\alpha)}. \quad (\text{A65})$$

From Eq. (A39), we get

$$C_{ij}^{(\alpha)(\beta)} = -(-)^{|\alpha|} \binom{|\rho|+|\beta|}{|\beta|} \partial_{(\rho)} \frac{\delta}{\partial \phi_{(\alpha)(\beta)(\rho)\nu}^i} \frac{\delta}{\delta \phi^j} \frac{\partial}{\partial dx^\nu} \omega^n. \quad (\text{A66})$$

Using the definition of higher order Lie operators (A16), we get

$$C_{ij}^{(\alpha)(\beta)} = - \sum_{|\sigma'| \geq |\rho| \geq 0} (-)^{|\alpha|+|\sigma'|+|\rho|} \binom{|\rho|+|\beta|}{|\beta|} \binom{|\alpha|+|\beta|+|\sigma'|+1}{|\alpha|+|\beta|+|\rho|+1} \partial_{(\sigma')} \frac{\partial^S}{\phi_{(\alpha)(\beta)(\sigma')\nu}^i} \frac{\delta}{\delta \phi^j} \frac{\partial}{\partial dx^\nu} \omega^n. \quad (\text{A67})$$

Evaluating  $\sum_{|\sigma'| \geq |\rho| \geq 0}$  as  $\sum_{|\sigma'| \geq 0} \sum_{|\rho|=0}^{|\sigma'|}$  and using the equality

$$\sum_{|\rho|=0}^{|\sigma'|} (-)^{|\rho|} \binom{|\rho|+|\beta|}{|\beta|} \binom{|\alpha|+|\beta|+|\sigma'|+1}{|\alpha|+|\beta|+|\rho|+1} = \binom{|\sigma'|+|\alpha|}{|\alpha|}, \quad (\text{A68})$$

we finally obtain

$$C_{ij}^{(\alpha)(\beta)} = -(-)^{|\alpha|} \binom{|\sigma'|+|\alpha|}{|\alpha|} (-\partial)_{(\sigma')} \frac{\partial^S}{\partial \phi_{(\alpha)(\beta)(\sigma')\nu}^i} \frac{\delta}{\delta \phi^j} \frac{\partial}{\partial dx^\nu} \omega^n. \quad (\text{A69})$$

Comparing with (A64), we have (A65) as it should.  $\square$

## APPENDIX B: ALGEBRA OF SURFACE CHARGE 1-FORMS

For compactness, let us define a generalized gauge transformation through

$$\delta_{f_1}^T \Phi_2^\Delta = (R_{f_1}^i, [f_1, f_2]^\alpha).$$

According to the same reasoning that led to (2.3), combined with (3.17) and the definition (3.3) of Noether currents for gauge symmetries, we get

$$d_H \left( \delta_{f_1}^T S_{f_2} - M_{f_1, f_2} - T_{R_{f_1}} \left[ R_{f_2}, \frac{\delta \mathcal{L}}{\delta \phi} \right] \right) = 0. \quad (\text{B1})$$

Applying the contracting homotopy with respect to the gauge parameters  $f_1^\alpha$  now gives

$$\delta_{f_1}^T S_{f_2} = M_{f_1, f_2} + T_{R_{f_1}} \left[ R_{f_2}, \frac{\delta \mathcal{L}}{\delta \phi} \right] + d_H N_{f_1, f_2}, \quad (\text{B2})$$

where

$$N_{f_1, f_2} \left[ \frac{\delta \mathcal{L}}{\delta \phi} \right] = I_{f_1}^{n-1} \left( \delta_{f_1}^T S_{f_2} - M_{f_1, f_2} - T_{R_{f_1}} \left[ R_{f_2}, \frac{\delta \mathcal{L}}{\delta \phi} \right] \right). \quad (\text{B3})$$

By applying  $1 = \{I_{f_1}, d_H\}$  to  $\delta_{f_1}^T k_{f_2}$  and using  $d_H k_{f_2} = -d_V^\phi S_{f_2} + I_{d_V \phi}^n (d_H S_{f_2})$ , we get

$$\delta_{f_1}^T k_{f_2} = I_{f_1}^{n-1} \left( -\delta_{f_1}^T d_V^\phi S_{f_2} + \delta_{f_1}^T (I_{d_V \phi}^n (d_H S_{f_2})) \right) + d_H(\cdot). \quad (\text{B4})$$

Using the properties of the homotopy operators, the expression inside the parentheses of RHS of (B4) becomes

$$-\delta_{f_1}^T d_V^\phi S_{f_2} + \delta_{f_1}^T (I_{d_V \phi}^n (d_H S_{f_2})) = -[\delta_{f_1}^T, d_V^\phi] S_{f_2} + [\delta_{f_1}^T, I_{d_V \phi}^n] (d_H S_{f_2}) + d_H I_{d_V \phi}^{n-1} (\delta_{f_1}^T S_{f_2}). \quad (\text{B5})$$

From Eq. (B2), we get

$$\delta_{f_1}^T k_{f_2} = I_{f_1}^{n-1} \left( -\delta_{R_{d_V \phi}} S_{f_2} + [\delta_{f_1}^T, I_{d_V \phi}^n] (d_H S_{f_2}) \right) + d_V^\phi N_{f_1, f_2} + I_{d_V \phi}^{n-1} \left( M_{f_1, f_2} + T_{R_{f_1}} \left[ R_{f_2}, \frac{\delta \mathcal{L}}{\delta \phi} \right] \right) + d_H(\cdot). \quad (\text{B6})$$

Using (3.8), (A24), and (A53) for the direct computation of  $[\delta_{f_1}^T, I_{d_V \phi}^n] (d_H S_{f_2})$  gives

$$\begin{aligned} [\delta_{f_1}^T, I_{d_V \phi}^n] (d_H S_{f_2}) &= W_{\delta \mathcal{L} / \delta \phi} [R_{f_2}, d_V^\phi R_{f_1}] + T_{R_{f_2}} \left[ d_V^\phi R_{f_1}, \frac{\delta \mathcal{L}}{\delta \phi} \right] - Y_{R_{f_2}, R_{f_1}} \left[ d_V \phi, \frac{\delta \mathcal{L}}{\delta \phi} \right] \\ &\quad + Z_{\delta \mathcal{L} / \delta \phi} [R_{f_2}, d_V \phi, R_{f_1}] - W_{\delta_{R_{f_1}} (\delta \mathcal{L} / \delta \phi)} [R_{f_2}, d_V \phi]. \end{aligned} \quad (\text{B7})$$

If

$$\begin{aligned} \mathcal{T}_{f_1, f_2} [d_V \phi] &:= \left[ I_{f_1}^{n-1} \left( -\delta_{R_{d_V \phi}} S_{f_2} + W_{\delta \mathcal{L} / \delta \phi} [d_V^\phi R_{f_1}, R_{f_2}] + T_{R_{f_2}} \left[ d_V^\phi R_{f_1}, \frac{\delta \mathcal{L}}{\delta \phi} \right] - Y_{R_{f_2}, R_{f_1}} \left[ d_V \phi, \frac{\delta \mathcal{L}}{\delta \phi} \right] \right. \right. \\ &\quad \left. \left. + Z_{\delta \mathcal{L} / \delta \phi} [d_V \phi, R_{f_2}, R_{f_1}] - W_{\delta_{R_{f_1}} (\delta \mathcal{L} / \delta \phi)} [d_V \phi, R_{f_2}] \right) \right. \\ &\quad \left. + d_V^\phi N_{f_1, f_2} + I_{d_V \phi}^{n-1} \left( M_{f_1, f_2} + T_{R_{f_1}} \left[ R_{f_2}, \frac{\delta \mathcal{L}}{\delta \phi} \right] \right) \right], \end{aligned} \quad (\text{B8})$$

we finally have

$$\delta_{R_{f_1}} k_{f_2} [d_V \phi] = -k_{[f_1, f_2]} [d_V \phi] + \mathcal{T}_{f_1, f_2} [d_V \phi] + d_H(\cdot). \quad (\text{B9})$$

Note that  $\mathcal{T}_{f_1, f_2} [d_V \phi] = 0$  if (i)  $\phi^s$  is a solution to the Euler–Lagrange equations of motion, (ii)  $R_{f_2}|_{\phi^s} = 0$ , and (iii)  $d_V \phi$  is tangent to the space of solutions at  $\phi^s$ . This proves Proposition 4.  $\square$

### APPENDIX C: INTEGRABILITY IMPLIES ALGEBRA

Owing to (3.10)

$$d_V^s k_f[d_V^s \phi] \approx I_{d_V^s}^{n-1}(W[d_V^s \phi, R_f]) - I_f^{n-1}(d_V^s W[d_V^s \phi, R_f]) + d_H(\cdot). \quad (C1)$$

After application of  $i_{R_{g_1}}$ , we find that the integrability condition (4.2) for the charge associated with  $g_2$  reads

$$\oint_{S^{r,t}} (I_{d_V^s}^{n-1}(W[R_{g_2}, R_{g_1}]) + I_{g_2}^{n-1}(\delta_{R_{g_1}} W[R_{g_2}, d_V^s \phi] - d_V^s(W[R_{g_2}, R_{g_1}])) \approx o(r^0). \quad (C2)$$

Notice that it follows from (3.9) that

$$k_{[f_1, f_2]}[d_V^s \phi] \approx I_{f_2}^{n-1}(W[d_V^s \phi, R_{[f_1, f_2]}]) + d_H(\cdot) \quad (C3)$$

and

$$I_{f_1}^{n-1}(\delta_{R_{f_2}} W[d_V^s \phi, R_{f_1}]) \approx I_{f_2}^{n-1}(\delta_{R_{f_2}} W[d_V^s \phi, R_{f_1}]) - I_{d_V^s}^{n-1}(W[R_{f_2}, R_{f_1}]) + d_H(\cdot). \quad (C4)$$

Applying  $I_{f_2}^{n-1}$  to (A49) contracted with  $R_{f_1}, R_{f_2}$  and using (C4), we get

$$\begin{aligned} I_{f_2}^{n-1}(d_V^s W[R_{f_2}, R_{f_1}] - \delta_{R_{f_1}} W[R_{f_2}, d_V^s \phi] - W[R_{[f_1, f_2]}, d_V^s \phi]) + I_{f_1}^{n-1}(\delta_{R_{f_2}} W[R_{f_1}, d_V^s \phi]) + I_{d_V^s}^{n-1}(W[R_{f_1}, R_{f_2}]) \\ \approx i_{R_{f_1}} i_{R_{f_2}} d_V^s E + d_H(\cdot). \end{aligned} \quad (C5)$$

Integrating over  $S^{r,t}$  and using gauge parameters  $g_1, g_2$  so that (C2) can be used to treat the second and fourth terms on the RHS, we find

$$\oint_{S^{r,t}} (-I_{d_V^s}^{n-1}(W[R_{g_1}, R_{g_2}]) + I_{g_1}^{n-1}(d_V^s W[R_{g_1}, R_{g_2}])) \approx \oint_{S^{r,t}} (I_{g_2}^{n-1} W[R_{[g_1, g_2]}, d_V^s \phi]) + o(r^0), \quad (C6)$$

if and only if  $\int_{S^{r,t}} i_{R_{g_2}} i_{R_{g_1}} d_V^s E_{\mathcal{L}} \approx o(r^0)$ , which holds as a consequence of assumption (4.3). Using (3.10) on the LHS and (C3) on the RHS, we get (4.6) and have proved Proposition 8. ■

## APPENDIX D: HAMILTONIAN FORMALISM

In this appendix, we will analyze how the results discussed so far appear in the particular case of a Hamiltonian action where the surface  $S$  is a closed surface inside the spacelike hyperplane  $\Sigma_t$  defined by constant  $t$ .

We follow closely the conventions and use results of Ref. 48 for the Hamiltonian formalism. The Hamiltonian action is first order in time derivatives and given by

$$S_H[z, \lambda] = \int \mathcal{L}_H = \int dt d^{n-1}x (\dot{z}^A a_A - h - \lambda^a \gamma_a), \quad (D1)$$

where we assume that we have Darboux coordinates  $z^A = (\phi^\alpha, \pi_\alpha)$  and  $\dot{z}^A a_A = \dot{\phi}^\alpha \pi_\alpha$ . It follows that  $\sigma_{AB} = \partial_A a_B - \partial_B a_A$  is the constant symplectic matrix with  $\sigma^{AB} \sigma_{BC} = \delta_C^A$  and  $d^{n-1}x \equiv (d^{n-1}x)_0$ . We assume for simplicity that the constraints  $\gamma_a$  are first class, irreducible, and time independent. In the following, we shall use a local ‘‘Poisson’’ bracket with spatial Euler–Lagrange derivatives for spatial  $(n-1)$ -forms  $\hat{g} = g d^{n-1}x$ ,

$$\{\hat{g}_1, \hat{g}_2\} = \frac{\delta g_1}{\delta z^A} \sigma^{AB} \frac{\delta g_2}{\delta z^B} d^{n-1}x. \quad (D2)$$

If  $\tilde{d}_H$  denotes the spatial exterior derivative, this bracket defines a Lie bracket in the space  $H^{n-1}(\tilde{d}_H)$ , i.e., in the space of equivalence classes of local functions modulo spatial divergences (see, e.g., Ref. 53).

Similarly, the Hamiltonian vector fields associated with an  $(n-1)$ -form  $\hat{h} = h d^{n-1}x$ ,

$$\tilde{\delta}_h^{\cdot}(\cdot) = \frac{\partial^S}{\partial z_{(i)}^A}(\cdot) \sigma^{AB} \partial_{(i)} \frac{\delta h}{\delta z^B} = \{\cdot, \hat{h}\}_{\text{alt}}, \tag{D3}$$

$$\tilde{\delta}_h^{\cdot}(\cdot) = \partial_{(i)} \frac{\delta h}{\delta z^B} \sigma^{BA} \frac{\partial^S}{\partial z_{(i)}^A}(\cdot) = \{\hat{h}, \cdot\}_{\text{alt}}, \tag{D4}$$

only depend on the class  $[\hat{h}] \in H^{n-1}(\tilde{d}_H)$ . Here,  $(i)$  is a multi-index denoting the spatial derivatives, over which we freely sum. The combinatorial factor needed to take the symmetry properties of the derivatives into account is included in  $\partial^S / \partial z_{(i)}^A$ . If we denote by  $\hat{\gamma}_a = \gamma_a d^{n-1}x$  and  $\hat{h}_E = \hat{h} + \lambda^a \hat{\gamma}_a$ , an irreducible generating set of gauge transformations for (D1) is given by

$$\delta_f z^A = \{z^A, \hat{\gamma}_a f^a\}_{\text{alt}}, \tag{D5}$$

$$\delta_f \lambda^a = \frac{Df^a}{Dt} + \{f^a, \hat{h}_E\}_{\text{alt}} + C_{bc}^a(f^b, \lambda^c) - \mathcal{V}_b^a(f^b), \tag{D6}$$

where the arbitrary gauge parameters  $f^a$  may depend on  $x^\mu$ , the Lagrange multipliers and their derivatives, as well as the canonical variables and their spatial derivatives, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \dot{\lambda}^a \frac{\partial}{\partial \lambda^a} + \ddot{\lambda}^a \frac{\partial}{\partial \dot{\lambda}^a} + \dots, \tag{D7}$$

$$\{\gamma_a, \hat{\gamma}_b \lambda^b\}_{\text{alt}} = C_{ab}^{+c}(\gamma_c, \lambda^b), \tag{D8}$$

$$\{\gamma_a, \hat{h}\}_{\text{alt}} = -\mathcal{V}_a^{+b}(\gamma_b). \tag{D9}$$

For later use, we define  $[f_1, f_2]_H^a = C_{bc}^a(f_1^b, f_2^c)$  and assume the  $f$ 's to be independent jet coordinates.

Let us denote the set of fields collectively by  $\phi^i = \{z^A, \lambda^a\}$ . In order to construct the surface charges, we first have to compute the current  $S_f$  defined according to (3.1) and (3.3),

$$\begin{aligned} R_\alpha^i(f^\alpha) \frac{\delta \mathcal{L}_H}{\delta \phi^i} &= \left[ \sigma^{AB} \frac{\delta(\gamma_a f^a)}{\delta z^B} \left( \sigma_{AC} \dot{z}^C - \frac{\delta h}{\delta z^A} - \frac{\delta \lambda^b \gamma_b}{\delta z^A} \right) + \left( \frac{Df^a}{Dt} + \{f^a, \hat{h}_E\}_{\text{alt}} + C_{bc}^a(f^b, \lambda^c) - \mathcal{V}_b^a(f^b) \right) \right. \\ &\quad \left. \times (-\gamma_a) \right] d^n x = \left[ -\frac{d}{dt}(\gamma_a f^a) + \partial_k \left( V_B^k \left[ \dot{z}^B - \sigma^{BA} \frac{\delta h_E}{\delta z^A}, \gamma_a f^a \right] + J_b^{ka}(\gamma_a, f^b) \right) \right] d^n x. \end{aligned} \tag{D10}$$

Here, the current  $J_b^{ka}(\gamma_a, f^b)$  is determined in terms of the Hamiltonian structure operators through the formula

$$\partial_k J_b^{ka}(\gamma_a, f^b) = \gamma_a \mathcal{V}_b^a(f^b) - f^b \mathcal{V}_b^{+a}(\gamma_a) - \gamma_c C_{ab}^c(f^a, \lambda^b) + f^a C_{ab}^{+c}(\gamma_c, \lambda^b), \tag{D11}$$

while  $V_A^i(Q^A, g) = \partial_{(j)}(Q^A(\delta g / \delta z_{(j)}^i))$ . We have

$$\partial_{(k)} Q^A \frac{\delta g}{\delta z_{(k)}^A} = Q^A \frac{\delta g}{\delta z^A} + \partial_i V_A^i(Q^A, g), \tag{D12}$$

if the function  $g$  does not involve time derivatives of  $z^A$ . In other words,  $V_A^i(Q^A, g)$  coincides with the components of the  $(n-2)$ -form  $I_Q^{n-1}(g d^{n-1}x)$ , as defined in (A26) and (A28) with  $\phi^i$  replaced by  $z^A$  and  $n$  replaced by  $n-1$ , i.e., for spatial forms with no time derivatives on  $z^A$ .

The weakly vanishing Noether current  $S_f^\mu$  is thus given by

$$S_f^0 = -\gamma_a f^a, \tag{D13}$$

$$S_f^k = V_B^k \left[ \dot{z}^B - \sigma^{BA} \frac{\delta h_E}{\delta z^A}, \gamma_a f^a \right] + j_b^{ka} (\gamma_a, f^b). \tag{D14}$$

Note that  $k_f^{[0i]}[d_V \phi, \phi]$ , which is the relevant part of the  $(n-2)$ -form  $k_f$  at constant time, only involves the canonical variables  $d_{Vz^A}, z^A$  and the gauge parameters  $f^a$ , but not the Lagrange multipliers  $\lambda^a$  nor their variations,  $d_V \lambda^a$ . This is so because  $S_f^0$  does not involve  $\lambda^a$ , while the terms in  $S_f^k$  with time derivatives only involve time derivatives of  $z^a$  and no Lagrange multipliers,

$$k_f^{[0i]}[d_V \phi, \phi] = k_f^{[0i]}[d_{Vz}, z]. \tag{D15}$$

More precisely, it follows from (A49), (D13), and (D14) and that

$$k_f^{[0i]}[d_{Vz}, z] = \frac{|k|+1}{|k|+2} \partial_{(k)} \left[ d_{Vz^A} \frac{\delta(-\gamma_a f^a)}{\delta z_{(k)i}^A} - d_{Vz^A} \frac{\delta V_B^i [z^B, \gamma_a f^a]}{\delta z_{(k)0}^A} \right]. \tag{D16}$$

Equation (A19) then allows one to show that  $\delta V_B^i [z^B, \gamma_a f^a] / \delta z_{(k)0}^A = (1/|k|+1) [\delta(\gamma_a f^a) / \delta z_{(k)i}^A]$  so that the terms nicely combine to give

$$k_f^{[0i]}[d_{Vz}, z] = -V_A^i [d_{Vz^A}, \gamma_a f^a]. \tag{D17}$$

Let  $d\sigma_i = 2(d^{n-2}x)_{0i}$ . For  $S$ , a closed surface inside the hyperplane  $\Sigma_t$  defined by constant  $t$ , the surface charge 1-form is given by

$$\delta Q_f [d_{Vz}] = \int_S k_f^{[0i]} [d_{Vz}] d\sigma_i. \tag{D18}$$

Taking into account (D12), we thus recover the following result from the Hamiltonian approach.<sup>5</sup>

*Proposition 14:* In the context of the Hamiltonian formalism, the surface charge 1-forms at constant time do not depend on the Lagrange multipliers and are given by the boundary terms that arise when converting the variation of minus the constraints smeared with gauge parameters into an Euler–Lagrange derivative contracted with the undifferentiated variation of the canonical variables,

$$-d_V^z (\gamma_a f^a) = -d_{Vz^A} \frac{\delta \gamma_a f^a}{\delta z^A} + \partial_i k_f^{[0i]} [d_{Vz}, z]. \tag{D19}$$

In addition, because of the simple way time derivatives enter into the Hamiltonian action  $\mathcal{L}_H$ , we have for all  $Q_1^i, Q_2^i$ ,

$$W_{\delta \mathcal{L}_H / \delta \phi}^0 [Q_1, Q_2] = -\sigma_{AB} Q_1^A Q_2^B, \tag{D20}$$

$$T_{R_f}^0 \left[ d_V \phi, \frac{\delta \mathcal{L}_H}{\delta \phi} \right] = 0, \quad E_{\mathcal{L}_H}^{0i} [d_V \phi, d_V \phi] = 0. \tag{D21}$$

Note that the last relation follows from our assumption that we are using Darboux coordinates. As a consequence of the first relation, we then also have

$$W_{\delta \mathcal{L}_H / \delta \phi}^0 [d_V \phi, R_f] d^{n-1}x = -d_{Vz^A} \frac{\delta(\hat{\gamma}_a f^a)}{\delta z^A}, \tag{D22}$$

$$W_{\delta \mathcal{L}_H / \delta \phi}^0 [R_{f_1}, R_{f_2}] d^{n-1}x = \{ \hat{\gamma}_a f_1^a, \hat{\gamma}_b f_2^b \}. \tag{D23}$$



Let us now analyze how we can recover the results of Ref. 8 from the present perspective. Suppose that  $\Sigma_t$  has a unique boundary  $S^{\infty,t}$ , assume that the fields and gauge parameters belong to the spaces  $\mathcal{F}$  and  $\mathcal{A}_{\mathcal{F}}$  defined “off shell” (cf. Remark 2 of Sec. IV B), and that

$$T_{R_g}^0 \left[ d_V \phi, \frac{\delta \mathcal{L}}{\delta \phi} \right] = 0. \quad (\text{D24})$$

We can define the functionals

$$G[\Phi, \bar{\Phi}] = - \int_{\Sigma} S_g + \mathcal{Q}[\Phi, \bar{\Phi}], \quad (\text{D25})$$

i.e., the Noether charges associated with the weakly vanishing Noether current  $(n-1)$ -forms  $S_g$ , “improved” by the boundary term  $\mathcal{Q}[\Phi, \bar{\Phi}]$ . In the Hamiltonian formalism,  $G[\Phi, \bar{\Phi}] = \int \hat{\gamma}_a g^a + \mathcal{Q}[\Phi, \bar{\Phi}]$ . It then follows from Stokes’ theorem and (3.7) that

$$d_V^\phi G = - \int_{\Sigma_t} W_{\delta \mathcal{L} / \delta \phi} [d_V \phi, R_g]. \quad (\text{D26})$$

By analogy with the Hamiltonian analysis,<sup>5</sup> in which we get

$$d_V^\phi G = \int_{\Sigma_t} d_V z^A \frac{\delta(\hat{\gamma}_a g^a)}{\delta z^A},$$

we say that the functional  $G$  is differentiable. Because  $R_g$  is by assumption tangent to  $\mathcal{F}$ , we say furthermore that  $G$  is a differentiable generator.

We can define the covariant Poisson bracket of the functionals  $G_i \equiv G[\Phi_i, \bar{\Phi}_i]$  by

$$\{G_1, G_2\}_c = - \delta_{R_{g_1}} G_2. \quad (\text{D27})$$

By applying  $i_{R_{g_1}}$  to (D26) in terms of  $g_2$ ,

$$\{G_1, G_2\}_c = \int_{\Sigma_t} W_{\delta \mathcal{L} / \delta \phi} [R_{g_1}, R_{g_2}]. \quad (\text{D28})$$

In the Hamiltonian formalism, it coincides with the usual one,

$$\{G_1, G_2\}_c = \left\{ \int_{\Sigma_t} \hat{\gamma}_a g_1^a, G_2 \right\}_{\text{alt}} = \int_{\Sigma_t} \{ \hat{\gamma}_a g_1^a, \hat{\gamma}_b g_2^b \}.$$

If we now assume that the integrability conditions for the gauge parameters  $g$  are satisfied without the need to vary the parameters  $g$ , as usually done in the Hamiltonian formalism,

$$\oint_{S^{\infty,t}} k_{d_V g} [d_V \phi] = 0, \quad (\text{D29})$$

it follows that  $\{G_1, G_2\}_c$  is differentiable,

$$d_V^\phi \{G_1, G_2\}_c = \int_{\Sigma_t} W_{\delta \mathcal{L} / \delta \phi} [d_V \phi, R_{[g_1, g_2]}]. \quad (\text{D30})$$

Indeed, applying  $d_V^\phi$  to (3.7) and using (D24), we get

$$\int_{\Sigma_t} d_V^\phi W_{\delta\mathcal{L}/\delta\phi}[d_V\phi, R_g] = - \int_{S^{\infty,t}} d_V^\phi k_g[d_V\phi].$$

Using integrability (4.2) and (D29), and the off-shell assumptions, the expression on the RHS vanishes so that

$$\int_{\Sigma_t} d_V^\phi W_{\delta\mathcal{L}/\delta\phi}[d_V\phi, R_g] = 0. \quad (\text{D31})$$

Using (A10) and (A47), we have

$$\begin{aligned} d_V^\phi\{G_1, G_2\}_c &= \int_{\Sigma_t} d_V^\phi W_{\delta\mathcal{L}/\delta\phi}[R_{g_1}, R_{g_2}] = \int_{\Sigma_t} (W_{\delta\mathcal{L}/\delta\phi}[d_V\phi, R_{[g_1, g_2]}] + i_{R_{g_1}} d_V^\phi i_{R_{g_2}} W_{\delta\mathcal{L}/\delta\phi} \\ &\quad - i_{R_{g_2}} d_V^\phi i_{R_{g_1}} W_{\delta\mathcal{L}/\delta\phi} - d_H i_{R_{g_1}} i_{R_{g_2}} d_V^\phi E_{\mathcal{L}}), \end{aligned}$$

where the second and third terms on the RHS vanish as a consequence of (D31), while the last one vanishes on account of the off-shell version of (4.3).

This is the theorem proved in Ref. 8 in the Hamiltonian framework, where

$$d_V^\phi\{G_1, G_2\}_c = \int d_V z^A \frac{\delta \hat{\gamma}_A[g_1, g_2]_H^a}{\delta z^A}.$$

## APPENDIX E: GENERAL RELATIVITY

We start from the Einstein–Hilbert action with cosmological constant  $\Lambda$ ,

$$S[g] = \int \mathcal{L}^{EH} = \int d^n x \frac{\sqrt{|g|}}{16\pi G} (R - 2\Lambda). \quad (\text{E1})$$

A generating set of gauge transformations is given by

$$\delta_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho}. \quad (\text{E2})$$

Reducibility parameters at  $g$  are thus given by Killing vectors of  $g$ . The weakly vanishing Noether current (3.3) is given by

$$S_\xi^\mu \left[ \frac{\delta \mathcal{L}^{EH}}{\delta g} \right] = 2 \frac{\delta \mathcal{L}^{EH}}{\delta g_{\mu\nu}} \xi_\nu = \frac{\sqrt{|g|}}{8\pi G} (-G^{\mu\nu} - \Lambda g^{\mu\nu}) \xi_\nu. \quad (\text{E3})$$

An explicit expression for  $k_\xi = I_{d_V g}^{n-1} S_\xi$  using (A26) has been originally derived in Ref. 39<sup>3</sup> and compared to other proposals in the literature. We point out here that  $k_\xi = I_{d_V g}^{n-1} S_\xi$  can also be written in the compact form

$$k_\xi[d_V g] = \frac{2}{3} (d^{n-2}x)_{\mu\nu} P^{\mu\delta\nu\gamma\alpha\beta} (2D_\gamma d_V g_{\alpha\beta} \xi_\delta - d_V g_{\alpha\beta} D_\gamma \xi_\delta), \quad (\text{E4})$$

where

<sup>3</sup>Note, however, that we have changed conventions, which are here taken to be those of Misner-Thorne-Wheeler,<sup>54</sup> and introduced an overall minus sign in the definition.

$$P^{\mu\nu\alpha\beta\gamma\delta} = \frac{\partial^S}{\partial g_{\gamma\delta,\alpha\beta}} \left( \frac{\delta L^{EH}}{\delta g_{\mu\nu}} \right), \tag{E5}$$

$$= \frac{\sqrt{-g}}{32\pi G} (g^{\mu\nu} g^{\gamma\alpha} g^{\beta\delta} + g^{\mu(\gamma} g^{\delta)\nu} g^{\alpha\beta} + g^{\mu(\alpha} g^{\beta)\nu} g^{\gamma\delta} - g^{\mu\nu} g^{\gamma\delta} g^{\alpha\beta} - g^{\mu(\gamma} g^{\delta)(\alpha} g^{\beta)\nu}) - g^{\mu(\alpha} g^{\beta)(\gamma} g^{\delta)\nu}). \tag{E6}$$

The tensor density  $P^{\mu\nu\alpha\beta\gamma\delta}$  is related to the supermetric defined in Ref. 55, which has the symmetries of the Riemann tensor, through

$$\frac{2}{n-2} g_{\mu\nu} P^{\mu\nu\alpha\beta\gamma\delta} = G^{\alpha\beta\gamma\delta}, \tag{E7}$$

where

$$G^{\alpha\beta\gamma\delta} = \frac{\partial^S L^{EH}}{\partial g_{\gamma\delta,\alpha\beta}}, \tag{E8}$$

$$= \frac{\sqrt{-g}}{16\pi G} \left( \frac{1}{2} g^{\alpha\gamma} g^{\beta\delta} + \frac{1}{2} g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\beta} g^{\gamma\delta} \right). \tag{E9}$$

The tensor density  $P^{\mu\nu\alpha\beta\gamma\delta}$  itself is symmetric in the pair of indices  $\mu\nu$ ,  $\alpha\beta$ , and  $\gamma\delta$  and the total symmetrization of any three indices is zero. The symmetries of these tensors are thus summarized by the Young tableaux

$$G^{\alpha\beta\gamma\delta} \sim \begin{array}{|c|c|} \hline \alpha & \beta \\ \hline \gamma & \delta \\ \hline \end{array}, \quad P^{\mu\nu\alpha\beta\gamma\delta} \sim \begin{array}{|c|c|} \hline \mu & \nu \\ \hline \alpha & \beta \\ \hline \gamma & \delta \\ \hline \end{array}. \tag{E10}$$

Note that in the context of covariant phase space methods,<sup>24</sup> a similar tensor density  $P^{\mu\nu\alpha\beta\gamma\delta}$  is defined, which lacks, however, the above symmetry properties. In order to compare further with the formulas obtained in that approach, it is convenient to rewrite  $k_{\xi}[d_V g]$  as

$$k_{\xi}[d_V \phi] = -d_V k_{\mathcal{L}^{EH},\xi}^K + k_{\mathcal{L}^{EH},d_V \xi}^K + i_{\xi} I_{d_V \phi}^n \mathcal{L}^{EH} - E_{\mathcal{L}^{EH}}[\mathcal{L}_{\xi} \phi, d_V \phi], \tag{E11}$$

where

$$k_{\mathcal{L}^{EH},\xi}^K = \frac{\sqrt{-g}}{16\pi G} (D^{\mu} \xi^{\nu} - D^{\nu} \xi^{\mu})(d^{n-2}x)_{\mu\nu} \tag{E12}$$

is the Komar integral,

$$I_{d_V g}^n \mathcal{L}^{EH}[d_V g] = \frac{\sqrt{-g}}{16\pi G} (g^{\mu\alpha} D^{\beta} d_V g_{\alpha\beta} - g^{\alpha\beta} D^{\mu} d_V g_{\alpha\beta})(d^{n-1}x)_{\mu}, \tag{E13}$$

and the additional term

$$E_{\mathcal{L}^{EH}}[\mathcal{L}_{\xi} g, d_V g] = \frac{\sqrt{-g}}{16\pi G} \left( \frac{1}{2} g^{\mu\alpha} d_V g_{\alpha\beta} (D^{\beta} \xi^{\nu} + D^{\nu} \xi^{\beta}) - (\mu \leftrightarrow \nu) \right) (d^{n-2}x)_{\mu\nu} \tag{E14}$$

vanishes for exact Killing vectors of  $g$  but not necessarily for asymptotic ones.

Explicitly, the quantities involved in  $d_H k_{\xi}[d_V g]$  in (3.7) are given by

$$W_{\delta\mathcal{L}^{EH}/\delta\phi}[d_V g, \mathcal{L}_{\xi g}] = P^{\mu\delta\beta\gamma\epsilon\zeta}(d_V g_{\beta\gamma}\nabla_{\delta}\mathcal{L}_{\xi g\epsilon\zeta} - \mathcal{L}_{\xi g\beta\gamma}\nabla_{\delta}d_V g_{\epsilon\zeta})(d^{n-1}x)_{\mu},$$

$$T_{\mathcal{L}_{\xi g}}\left[d_V g, \frac{\delta\mathcal{L}^{EH}}{\delta g}\right] = d_V g_{\alpha\beta}\frac{\delta\mathcal{L}^{EH}}{\delta g_{\alpha\beta}}\xi^{\mu}(d^{n-1}x)_{\mu}.$$
(E15)

It follows that all conditions are satisfied to proceed with an off-shell analysis in the asymptotic context (cf. remark 2 of Sec. IV B).

- <sup>1</sup>L. F. Abbott and S. Deser, *Nucl. Phys. B* **195**, 76 (1982).
- <sup>2</sup>L. F. Abbott and S. Deser, *Phys. Lett.* **116B**, 259 (1982).
- <sup>3</sup>G. Barnich and A. Gomberoff, e-print arXiv:hep-th/0705.0632.
- <sup>4</sup>R. Arnowitt, S. Deser, and C. Misner, *Gravitation, An Introduction to Current Research* (Wiley, New York, 1962), Chap. 7, pp. 227–265.
- <sup>5</sup>T. Regge and C. Teitelboim, *Ann. Phys. (N.Y.)* **88**, 286 (1974).
- <sup>6</sup>R. Benguria, P. Cordero, and C. Teitelboim, *Nucl. Phys. B* **122**, 61 (1977).
- <sup>7</sup>M. Henneaux and C. Teitelboim, *Commun. Math. Phys.* **98**, 391 (1985).
- <sup>8</sup>J. D. Brown and M. Henneaux, *J. Math. Phys.* **27**, 489 (1986).
- <sup>9</sup>J. D. Brown and M. Henneaux, *Commun. Math. Phys.* **104**, 207 (1986).
- <sup>10</sup>L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, London, 1962).
- <sup>11</sup>A. Komar, *Phys. Rev.* **113**, 934 (1959).
- <sup>12</sup>J. Katz, J. Bicak, and D. Lynden-Bell, *Phys. Rev. D* **55**, 5957 (1997).
- <sup>13</sup>S. Deser and B. Tekin, *Phys. Rev. Lett.* **89**, 101101 (2002), e-print arXiv:hep-th/0205318.
- <sup>14</sup>S. Deser and B. Tekin, *Phys. Rev. D* **67**, 084009 (2003), e-print arXiv:hep-th/0212292.
- <sup>15</sup>J. D. Brown, N. James, and J. York, *Phys. Rev. D* **47**, 1407 (1993).
- <sup>16</sup>J. D. Brown, S. R. Lau, J. York, and W. James, e-print arXiv:gr-qc/0010024.
- <sup>17</sup>J. Kijowski and W. Szczyrba, *Commun. Math. Phys.* **46**, 183 (1976).
- <sup>18</sup>C. Crnkovic and E. Witten, *Three Hundred Years of Gravitation* (Cambridge University Press, Cambridge, 1987), pp. 676–684.
- <sup>19</sup>A. Ashtekar, L. Bombelli, and O. Reula, *Mechanics, Analysis and Geometry: 200 Years After Lagrange* (Elsevier, New York, 1991), pp. 417–450.
- <sup>20</sup>J. Lee and R. M. Wald, *J. Math. Phys.* **31**, 725 (1990).
- <sup>21</sup>R. M. Wald, *J. Math. Phys.* **31**, 2378 (1990).
- <sup>22</sup>R. M. Wald, *Phys. Rev. D* **48**, R3427 (1993), e-print arXiv:gr-qc/9307038.
- <sup>23</sup>V. Iyer and R. M. Wald, *Phys. Rev. D* **50**, 846 (1994), e-print arXiv:gr-qc/9403028.
- <sup>24</sup>R. M. Wald and A. Zoupas, *Phys. Rev. D* **61**, 084027 (2000), e-print arXiv:gr-qc/9911095.
- <sup>25</sup>J.-I. Koga, *Phys. Rev. D* **64**, 124012 (2001), e-print arXiv:gr-qc/0107096.
- <sup>26</sup>S. C. Anco and R. S. Tung, *J. Math. Phys.* **43**, 5531 (2002), e-print arXiv:gr-qc/0109013.
- <sup>27</sup>S. C. Anco and R. S. Tung, *J. Math. Phys.* **43**, 3984 (2002), e-print arXiv:gr-qc/0109014.
- <sup>28</sup>B. Julia and S. Silva, *Class. Quantum Grav.* **15**, 2173 (1998), e-print arXiv:gr-qc/9804029.
- <sup>29</sup>S. Silva, *Nucl. Phys. B* **558**, 391 (1999), e-print arXiv:hep-th/9809109.
- <sup>30</sup>B. Julia and S. Silva, *Class. Quantum Grav.* **17**, 4733 (2000), e-print arXiv:gr-qc/0005127.
- <sup>31</sup>B. Julia and S. Silva, e-print arXiv:hep-th/0205072.
- <sup>32</sup>M. Blagojevic and M. Vasilic, *Class. Quantum Grav.* **22**, 3891 (2005), e-print arXiv:hep-th/0410111.
- <sup>33</sup>S. Hollands, A. Ishibashi, and D. Marolf, *Class. Quantum Grav.* **22**, 2881 (2005), e-print arXiv:hep-th/0503045.
- <sup>34</sup>I. Papadimitriou and K. Skenderis, *J. High Energy Phys.* **08**, 004 (2005), e-print arXiv:hep-th/0505190.
- <sup>35</sup>G. Barnich, F. Brandt, and M. Henneaux, *Commun. Math. Phys.* **174**, 57 (1995), e-print arXiv:hep-th/9405109.
- <sup>36</sup>G. Barnich, F. Brandt, and M. Henneaux, *Phys. Rep.* **338**, 439 (2000), e-print arXiv:hep-th/0002245.
- <sup>37</sup>I. M. Anderson and C. G. Torre, *Phys. Rev. Lett.* **77**, 4109 (1996), e-print arXiv:hep-th/9608008.
- <sup>38</sup>G. Barnich, S. Leclercq, and P. Spindel, *Lett. Math. Phys.* **68**, 175 (2004), e-print arXiv:gr-qc/0404006.
- <sup>39</sup>G. Barnich and F. Brandt, *Nucl. Phys. B* **633**, 3 (2002), e-print arXiv:hep-th/0111246.
- <sup>40</sup>I. Anderson, “The variational bicomplex,” Formal Geometry and Mathematical Physics, Department of Mathematics, Utah State University technical report, 1989 (unpublished).
- <sup>41</sup>I. Anderson, *Contemp. Math.* **132**, 51 (1992).
- <sup>42</sup>P. Olver, *Applications of Lie Groups to Differential Equations*, 2nd ed. (Springer, New York, 1993).
- <sup>43</sup>G. Barnich, *Class. Quantum Grav.* **20**, 3685 (2003), e-print arXiv:hep-th/0301039.
- <sup>44</sup>G. Barnich and G. Compère, *Phys. Rev. D* **71**, 044016 (2005), e-print arXiv:gr-qc/0412029.
- <sup>45</sup>G. Barnich and G. Compère, *Phys. Rev. Lett.* **95**, 031302 (2005), e-print arXiv:hep-th/0501102.
- <sup>46</sup>V. Arnold, *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics, 2nd ed. (Springer, New York, 1989).
- <sup>47</sup>J. A. de Azcarraga, J. P. Gauntlett, J. M. Izquierdo, and P. K. Townsend, *Phys. Rev. Lett.* **63**, 2443 (1989).
- <sup>48</sup>M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, NJ, 1992).
- <sup>49</sup>G. Compère, *Phys. Rev. D* **75**, 124020 (2007), e-print arXiv:hep-th/0703004.
- <sup>50</sup>T. De Donder, *Théorie Invariantive du Calcul des Variations*, Institut Belge de Recherches Radioscientifiques Vol. IV (Gauthier-Villars, Paris, 1935).
- <sup>51</sup>D. J. Saunders, *The Geometry of Jet bundles* (Cambridge University Press, Cambridge, 1989).

- <sup>52</sup>L. Dickey, *Soliton Equations and Hamiltonian Systems*, Advanced Series in Mathematical Physics Vol. 12 (World Scientific, Singapore, 1991).
- <sup>53</sup>G. Barnich and M. Henneaux, *J. Math. Phys.* **37**, 5273 (1996), e-print arXiv:hep-th/9601124.
- <sup>54</sup>C. Misner, K. Thorne, and J. Wheeler, *Gravitation* (Freeman, New York, 1973).
- <sup>55</sup>B. S. DeWitt, *Phys. Rev.* **160**, 1113 (1967).