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Surface Integral Formulae for Geomagnetic Studies

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Surface integral formulae are derived expressing any one of certain field quantities, namely current functions, magnetic potentials and normal components of magnetic fields, in terms of any one other, for current systems flowing in concentric spherical surfaces. In all, 36 such formulae are obtained, which should prove useful in many geomagnetic studies, especially in geomagnetic induction problems.

1. Introduction

Many geomagnetic studies are concerned with time-varying magnetic fields at the Earth's surface, and their relation to electric current systems in the upper atmosphere and in the Earth. There are, for example, many recent (as well as many earlier) studies of the Solar and Lunar diurnal variation fields, and of the ionospheric and earth current systems associated with them. These studies have led to the development of various methods and formulae for separating the surface field into parts of external and internal origin, and for determining the ionospheric currents that correspond to the external part. Further, the relationship found between the fields of internal and external origin is of the form that would be expected if the earth currents were due simply to electromagnetic induction by the moving and varying ionospheric current systems. A study of this relationship can lead to information about the conductivity of the Earth at various depths. This requires the solution of various mathematical problems on the electromagnetic induction of currents in concentric spheres and spherical shells of non-uniform conductivity, and involves the evaluation of self and mutual induction effects in these conductors. One of the important problems is to estimate the influence of currents induced in the relatively highly conducting oceans, and the screening effects of these currents on the conducting layers of the earth below.

In the earlier studies the current systems and magnetic fields were expressed in terms of series of spherical harmonics and many valuable results were obtained in this way. When, however, it is desired to study detailed features of certain fields, such as the great enhancement of the *Sq* field near the dip equator, or, again, when one attempts to solve an induction problem which has, as one of the conductors, a thin shell with abrupt changes in conductivity, as at the surface of the Earth, then this method leads to difficulties owing to the slow convergence of the spherical harmonic expansions required. In the solution of such induction problems, the numerical work involved in solving the infinite sets of simultaneous equations, that are found for the coefficients of the required spherical harmonic expansions, is often prohibitive, and the solution obtained by truncating the series at a point that would make the numerical work possible very inaccurate. (See for example, the discussion by Ashour (1965a) of a problem considered by Rikitake & Yokoyama (1955)). Hence, alternative methods of treating these problems have been sought.

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A direct method of interpolating the Sq field to a regularly spaced system of mesh points on the Earth was developed by Price & Wilkins (1963), who also developed a numerical method, based on a surface integral formula given by Vestine, to separate the field into parts of external and internal origin.

It is the purpose of this paper to derive a number of similar surface integral formulae relating to currents flowing in concentric spherical surfaces, and their magnetic fields. Each formula expresses any one of several field quantities as a surface integral involving one (only) of the other field quantities. The particular formulae considered are those found useful for geomagnetic studies of the kind already mentioned. For the purpose of illustration and as an aid to the presentation, we shall occasionally refer to the self and mutual induction effects of currents flowing in two concentric non-uniform thin shells of radii a and b with $a > b$. We intend in a later paper to discuss the electromagnetic induction of currents in two such shells, but it should be noted that the formulae have other applications apart from their use in the solution of electromagnetic induction problems.

2. Relations between field quantities in terms of spherical harmonics

Although some of the formulae obtained by us could be derived directly from general theorems in potential theory, it seems easier and more instructive to obtain them by integration from the expressions in spherical harmonics for the various field quantities involved, using the orthogonal properties of spherical harmonic functions over a sphere. The field quantities of particular importance in the geomagnetic applications are: (1) the streamline function for the currents in each shell; (2) the potential inside and outside each shell of the magnetic fields of the currents in that shell; and (3) the normal components of the magnetic field of the currents in each shell both at the surface of that shell and at the surface of the other shell.

The streamline function for a system of steady or quasi-steady non-divergent currents flowing in a spherical shell of radius $r = a$, being finite and single valued, may be represented as a series of spherical harmonic terms

$$\Psi = \sum_{n,m} \left(\frac{2n+1}{4\pi}\right) a_n^m S_n^m (\Theta, \Lambda) \tag{1}$$

where the a_n^m's are complex coefficients, which are in general functions of time, (r, Θ, Λ) are spherical polar co-ordinates, and S_n^m is a surface harmonic of degree n satisfying

$$\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\,\frac{\partial S_n^m}{\partial\Theta}\right) + \frac{1}{\sin^2\Theta}\frac{\partial^2 S_n^m}{\partial\Lambda^2} + n(n+1)\,S_n^m = 0 \tag{2}$$

S_n^m is of the form $e^{im\Lambda} P_n^m(\cos\Theta)$, and the factor $(2n+1)/4\pi$ is introduced into equation (1) to simplify expressions for the magnetic field potential.

The potential, Ω, of the magnetic field of the current system Ψ, inside and outside the shell of radius a, can then be written

$$\Omega_1 = -\sum_{n,m} (n+1)\left(\frac{r}{a}\right)^n a_n^m S_n^m \quad r < a \tag{3}$$

and

$$\Omega_0 = \sum_{n,m} n\left(\frac{a}{r}\right)^{n+1} a_n^m S_n^m \quad r > a \tag{4}$$

respectively, and the normal component, $\partial\Omega/\partial r$, of this magnetic field, being continuous at $r = a$, is then

$$\left(\frac{\partial\Omega}{\partial r}\right)_{r=a} = -\sum_{n,m} n(n+1)\frac{1}{a} a_n^m S_n^m. \qquad (5)$$

If the potential of the magnetic field just inside and just outside the shell is denoted by Ω_- and Ω_+ respectively, we have

$$\Omega_- = (\Omega_1)_{r=a} = -\sum_{n,m} (n+1)\, a_n^m S_n^m \qquad (6)$$

and

$$\Omega_+ = (\Omega_0)_{r=a} = \sum_{n,m} n\, a_n^m S_n^m. \qquad (7)$$

The above expressions will be used in Section 3 to obtain twelve surface integral formulae giving directly any one of the quantities Ψ, Ω_-, Ω_+ and $(\partial\Omega/\partial r)_{r=a}$ in terms of any one other.

If we now assume that a second concentric conducting shell of radius $b < a$ is present, and that the currents in the outer shell are varying with time, then the varying magnetic field of these currents will induce a system of currents, having current function Ψ', say, in this inner conductor, the system Ψ' clearly depending on the conductivity of this conductor. The inner shell may or may not be uniformly conducting, and in some geomagnetic applications might usefully be replaced by a sphere having a spherically symmetrical distribution of conductivity. The current function Ψ' would not then exist, but the induced currents can be calculated (see e.g., Lahiri & Price 1939).

From equation (3) the potential and normal component of the inducing field due to Ψ, at the surface $r = b$, are

$$(\Omega_1)_{r=b} = -\sum_{n,m} (n+1)\left(\frac{b}{a}\right)^n a_n^m S_n^m \qquad (8)$$

$$\left(\frac{\partial\Omega_1}{\partial r}\right)_{r=b} = -\sum_{n,m} n(n+1)\frac{1}{a}\left(\frac{b}{a}\right)^{n-1} a_n^m S_n^m. \qquad (9)$$

Solution of the induction equation for currents flowing in the conductor radius b, with the inducing field given by equations (8) and (9), would give a current system Ψ' which may be written

$$\Psi' = \sum_{n,m} \left(\frac{2n+1}{4\pi}\right) b_n^m S_n^m \qquad (10)$$

where the coefficients b_n^m have to be determined in the solution.

Analogy with (4) shows that the magnetic potential of the current system Ψ' outside the shell of radius b, may be written

$$\Omega_0' = \sum_{n,m} n\left(\frac{b}{r}\right)^{n+1} b_n^m S_n^m \qquad (11)$$

and that the normal component is

$$\frac{\partial\Omega_0'}{\partial r} = -\sum_{n,m} n(n+1)\frac{1}{b}\left(\frac{b}{r}\right)^{n+2} b_n^m S_n^m. \qquad (12)$$

The potential and normal component may be extrapolated to the surface of the shell $r = a$, giving

$$(\Omega_0')_{r=a} = \sum_{n,m} n\left(\frac{b}{a}\right)^{n+1} b_n^m S_n^m \tag{13}$$

$$\left(\frac{\partial\Omega_0'}{\partial r}\right)_{r=a} = -\sum_{n,m} n(n+1)\frac{1}{b}\left(\frac{b}{a}\right)^{n+2} b_n^m S_n^m. \tag{14}$$

The magnetic field represented by equations (11) and (12) is the magnetic field of the current system Ψ' induced in the inner conductor by the current system Ψ in the outer shell. This varying magnetic field will itself induce currents in the shell $r = a$, which will contribute to Ψ, so that both Ψ and Ψ' are affected by mutual induction between the two conductors. In order to determine this mutual induction effect, surface integral formulae will be obtained for the potential and normal component of the magnetic field at the surface of either shell in terms of each one of the quantities Ψ, Ω_-, Ω_+ and $\partial\Omega/\partial r$ associated with a system of currents flowing in the other shell.

In general, determination of the current system flowing in the inner shell due to currents flowing in the outer shell, will require the solution of a linear set of equations connecting the coefficients b_n^m and a_n^m. The complexity of the linear set of equations is governed by the conductivity of the inner shell (or sphere). When the inner conductor is a non-uniformly conducting thin shell, the induced current system may be determined by the theory given by Price (1949). When the conductivity of the inner shell is uniform, application of the resulting simplified boundary equation enables the linear set of equations to be written

$$b_n^m = \alpha_n^m a_n^m. \tag{15}$$

For finite conductivity, the α_n^m's are easily derived and are complex. In the especially simple case of infinite conductivity, whether the inner conductor be a shell or solid sphere, the α_n^m's are real and equation (15) becomes

$$b_n^m = -\left(\frac{b}{a}\right)^n a_n^m. \tag{16}$$

In this simplest case, using equation (16), the potential and normal component, say Ω^* and $\partial\Omega^*/\partial r$, at $r = a$, of the magnetic field of the currents in the shell $r = b$, due to mutual induction between two conductors radii a and b $(a > b)$, arising from a current function ψ defined by equation (1), may be written

$$(\Omega^*)_{r=a} = -\sum_{n,m} n\left(\frac{b}{a}\right)^{2n+1} a_n^m S_n^m \tag{17}$$

$$\left(\frac{\partial\Omega^*}{\partial r}\right)_{r=a} = \sum_{n,m} n(n+1)\frac{1}{a}\left(\frac{b}{a}\right)^{2n+1} a_n^m S_n^m. \tag{18}$$

Asterisks denote the mutual induction effect in the special case of a perfectly conducting inner conductor, and again surface integral formulae will be obtained relating $(\Omega^*)_{r=a}$ and $(\partial\Omega^*/\partial r)_{r=a}$ with each of the quantities Ψ, Ω_-, Ω_+ and $(\partial\Omega/\partial r)_{r=a}$ associated with current flow in $r = a$.

3. Surface integral formulae relating to a single shell

We shall first illustrate one method of obtaining the surface integral formulae by considering the particular problem of finding the potential Ω_+ of the magnetic field just outside a spherical shell, due to currents flowing in it, in terms of the normal component $\partial\Omega/\partial r$ assumed known everywhere over the surface. The spherical harmonic expansions for the potential and the normal component are given by (7) and (5), from which it will be seen that the series for Ω_+ can be obtained from that for $(\partial\Omega/\partial r)$ at $r = a$ by replacing each coefficient $-n(n+1)\,a_n^m/a$ in (5) by na_n^m. To obtain an operator which will make this transformation we can use some of the orthogonal properties of the spherical harmonics, in particular

$$\left.\begin{aligned}\iint P_n(\cos\Theta)\, S_{n'}^{m'}(\Theta,\Lambda)\, dS &= 0, \qquad m' \neq 0;\ \text{all } n \text{ and } n' \\ &= 0, \qquad n' \neq n;\ \text{all } m \\ &= \frac{4\pi a^2}{2n+1}, \quad n' = n,\ m = 0\end{aligned}\right\} \tag{19}$$

the integral being over the sphere of radius a.

We now consider the function defined by

$$K(\Theta) = \sum_{n=0}^{\infty} \frac{2n+1}{n+1} P_n(\cos\Theta) \tag{20}$$

and evaluate the integral

$$\iint K(\Theta)\left(\frac{\partial\Omega}{\partial r}\right)_a dS \tag{21}$$

over the sphere.

On substituting for $K(\Theta)$ from (20) and for $(\partial\Omega/\partial r)$ from (5), this integral reduces, in virtue of the orthogonal properties (19), to

$$\iint \sum_{1}^{\infty} -\frac{n(2n+1)}{a} a_n^0 \{P_n(\cos\Theta)\}^2\, dS = -4\pi a \sum_{1}^{\infty} na_n^0. \tag{22}$$

But from (7) we have, since $S_n^m(0,\Lambda) = 0$ unless $m = 0$, $= 1$ when $m = 0$,

$$(\Omega_+)_A = \sum_{1}^{\infty} na_n^0 \tag{23}$$

where A is the point $(a, 0, \Lambda)$.

Hence

$$(\Omega_+)_A = -\frac{1}{4\pi a}\iint K(\Theta)\left(\frac{\partial\Omega}{\partial r}\right)_a dS. \tag{24}$$

To evaluate $K(\Theta)$ we can write (20) in the form

$$K(\Theta) = \sum_{0}^{\infty} 2P_n - \sum_{0}^{\infty} \frac{1}{n+1} P_n$$

and summing these series we obtain

$$K(\Theta) = \frac{1}{\sin\frac{1}{2}\Theta} - \log\left(\frac{1+\sin\frac{1}{2}\Theta}{\sin\frac{1}{2}\Theta}\right). \tag{25}$$

Substituting in (24) we obtain the formula

$$(\Omega_+)_A = -\frac{1}{4\pi a}\int\int\left(\frac{\partial\Omega}{\partial r}\right)_a\left\{\frac{1}{\sin\frac{1}{2}\Theta} - \log\left(\frac{1+\sin\frac{1}{2}\Theta}{\sin\frac{1}{2}\Theta}\right)\right\} dS. \tag{26}$$

Although $K(\Theta)$ has a singularity at $\Theta = 0$, the integral in (26) is convergent because dS could be taken as $a^2 \sin\Theta\, d\Theta\, d\Lambda$, and the factor $\sin\Theta$ removes the singularity from the integrand. Note that if the field is given relative to some fixed co-ordinate system (r, θ, λ), a typical point $A(a, \theta_A, \lambda_A)$ may be taken as the pole of the above co-ordinate system (r, Θ, Λ). We then have

$$\cos\Theta = \cos\theta\cos\theta_A + \sin\theta\sin\theta_A\cos(\lambda - \lambda_A),$$

and (26) may now be written in terms of θ, λ, and thus affords a method of calculating Ω_+ at any point A from a knowledge of $\partial\Omega/\partial r$ over the surface.

The above formula (26) can be checked by obtaining it directly from known theorems in Potential Theory. A known result (Jeffreys & Jeffreys 1950, p. 219) is that for any continuously differentiable function Ω satisfying Laplace's equation and tending to zero at infinity,

$$4\pi\Omega(P) = \int\int \Omega_+ \frac{r^2-a^2}{aR^3} dS \tag{27}$$

where P is any point outside the sphere of radius a, r is its distance from the centre of the sphere, and R its distance from the surface element dS.

Since Ω_0 satisfies Laplace's equation it easily follows that $r(\partial\Omega/\partial r)$ also satisfies this equation, and we may apply (27), giving

$$4\pi r\frac{\partial\Omega}{\partial r} = \int\int a\left(\frac{\partial\Omega}{\partial r}\right)_a \frac{r^2-a^2}{aR^3} dS. \tag{28}$$

Dividing by r and integrating along the radius vector from a point A on the surface to infinity, we have

$$-4\pi(\Omega_+)_A = \int\int\int_{r=a}^{\infty} \left(\frac{\partial\Omega}{\partial r}\right)_a \frac{r^2-a^2}{rR^3} dr\, dS \tag{29}$$

where

$$R^2 = r^2 - 2ar\cos\Theta + a^2. \tag{30}$$

By elementary integration we find after some reduction that

$$\int \frac{r^2-a^2}{rR^3} dr = \frac{1}{a}\log\left(\frac{R+a-r\cos\Theta}{r}\right) - \frac{2}{R} \tag{31}$$

and on inserting this in (29) with the appropriate limits, and noting that

$$R = 2a\sin\tfrac{1}{2}\Theta \tag{32}$$

when $r = a$, we get

$$-4\pi(\Omega_+)_A = \frac{1}{a}\int\int\left(\frac{\partial\Omega}{\partial r}\right)_a\left\{\frac{1}{\sin\frac{1}{2}\Theta} - \log\left(\frac{1+\sin\frac{1}{2}\Theta}{\sin\frac{1}{2}\Theta}\right)\right\} dS$$

which is identical with (26), as required. Although we shall use a different method, all the formulae derived here may be derived by using such Green's functions.

We now obtain a general integral formula for obtaining any one of the quantities Ω_+, Ω_-, Ψ and $\partial\Omega/\partial r$ in terms of any one of the others. To do this we first consider a formal expansion for the kernel $K(\Theta)$ corresponding to (20), and we show that this expansion, though not always convergent, defines a generalized function which makes the required surface integral convergent.

It is obvious from the expressions in spherical harmonics for the various field quantities that if the expansion for any one of them, W_1 say, is

$$W_1 = \sum_{n,m} a_n^m S_n^m \tag{33}$$

the expansion for any other, W_2 say, will be of the form

$$W_2 = \sum_{n,m} k_n a_n^m S_n^m \tag{34}$$

where k_n is a simple rational function of n, the numerator and denominator being in general of degree not greater than 2. Thus in the case just considered k_n was $-1/(n+1)$. The procedure found suitable in this special case clearly shows that we can get a formal expression for the transformation by taking

$$K(\Theta) = \sum_{n=0}^{\infty} (2n+1) k_n P_n. \qquad \dagger(35)$$

To obtain W_2 in terms of W_1 we then evaluate the integral

$$\iint K(\Theta) W_1 \, dS \tag{36}$$

over the sphere, and, making use of the orthogonal properties (19) of the spherical harmonics, we have that this integral has the value

$$4\pi a^2 \sum_{1}^{\infty} k_n a_n^0. \tag{37}$$

But from (34) we have at $\Theta = 0$, i.e. at the point A,

$$(W_2)_A = \sum_{n=1}^{\infty} k_n a_n^0$$

and therefore

$$(W_2)_A = \frac{1}{4\pi a^2} \iint K(\Theta) W_1 \, dS \tag{38}$$

which is the required formula.

For any particular pair of functions W_1 and W_2, we have to 'sum' the series (35) for $K(\Theta)$. This series is not always convergent, but it can generally be summed by Euler's method. There is always a singularity at $\Theta = 0$, and if this is of order greater than 2, it is necessary to consider the complete integrand $K(\Theta) W_1$.

An important special case is when $k_n = 1$. Denoting the special value of $K(\Theta)$ in this case by $I(\Theta)$ we have formally

$$I(\Theta) = \sum_{n=0}^{\infty} (2n+1) P_n(\cos \Theta) \tag{39}$$

† The term $n = 0$ is omitted from the summation when k_n contains the factor n^{-1}.

and the integral on the right of (38) reduces to $(W_1)_A$, i.e.

$$(W_1)_A = \frac{1}{4\pi a^2} \iint I(\Theta)\, W_1\, dS \tag{40}$$

This shows that $I(\Theta)/4\pi a^2$ is a Dirac delta function. When it is multiplied with W_1 and integrated over the spherical surface it picks out the value of W_1 at the point A.

The expansion (39) for $I(\Theta)$ is not convergent but we can obtain an Euler sum of this expansion by considering it as

$$I(\Theta) = \lim_{r \to a-} \Sigma(2n+1) \frac{r^n}{a^n} P_n(\cos\Theta). \tag{41}$$

Now the generating function for the Legendre polynomials is

$$\frac{1}{R} = \frac{1}{(a^2+r^2-2ar\cos\Theta)^{\frac{1}{2}}} = \sum_{n=0}^{\infty} \frac{r^n}{a^{n+1}} P_n(\cos\Theta) \tag{42}$$

and it is easily shown that

$$2r\frac{\partial}{\partial r}\left(\frac{1}{R}\right) + \frac{1}{R} = \frac{a^2-r^2}{R^3}. \tag{43}$$

Hence

$$\sum_{n=0}^{\infty} (2n+1)\left(\frac{r}{a}\right)^n P_n(\cos\Theta) = \left(\frac{a}{R}\right)^3 \left(\frac{a^2-r^2}{r^2}\right) \tag{44}$$

so that

$$I(\Theta) = \lim_{r \to a-} \sum_{n=0}^{\infty} (2n+1)\left(\frac{r}{a}\right)^n P_n(\cos\Theta) = \lim_{r \to a-} \left(\frac{a}{R}\right)^3 \left(\frac{a^2-r^2}{r^2}\right). \tag{45}$$

Thus $I(\Theta)$ has the property that it is infinite at $\Theta = 0$, and zero elsewhere, agreeing with the requirement already indicated in equation (40).

Another important case is when $k_n = n$, and the corresponding expansion for $K(\Theta)$ can again be summed using the relation (44).

Consider

$$G(r) = \iint \frac{1}{R^3}\{W_1 - (W_1)_A\}\, dS. \tag{46}$$

Since

$$\lim_{r \to a} R = 2a \sin\tfrac{1}{2}\Theta$$

then

$$\lim_{r \to a} G(r) = \iint \frac{W_1 - (W_1)_A}{8a^3 \sin^3 \frac{1}{2}\Theta}\, dS \tag{47}$$

if this integral exists.

Using equation (44), this limit may be expressed in an alternative form,

$$\lim_{r \to a} G(r) = \lim_{r \to a} \iint \left\{ \sum_{n=1}^{\infty} \sum_{m=0}^{n} a_n^m (S_n^m - 1) \right\} \left\{ \frac{1}{a^3} \cdot \frac{r^2}{a^2-r^2} \cdot \sum_{n=0}^{\infty} (2n+1)\left(\frac{r}{a}\right)^n P_n(\cos\Theta) \right\} dS \tag{48}$$

since

$$S_n^m(0, \Lambda) = 0 \quad m \neq 0 \\ = 1 \quad m = 0 \Bigg\} \text{all } n$$

and using the orthogonal properties of surface harmonics, this may be reduced to

$$\lim_{r\to a} G(r) = \lim_{r\to a} \frac{4\pi}{a} \cdot \frac{r^2}{a^2-r^2} \cdot \sum_{n=1}^{\infty} a_n^0 \left\{ \left(\frac{r}{a}\right)^n - 1 \right\}$$

$$= \lim_{r\to a} \frac{4\pi}{a} \cdot \frac{r^2}{a^2-r^2} \cdot \left\{ \sum_{n=1}^{\infty} a_n^0 (r-a)\, a^{-n} (r^{n-1} + ar^{n-2} + \ldots a^{n-1}) \right\}$$

$$= -\frac{2\pi}{a} \sum_{n=1}^{\infty} n a_n^0 . \tag{49}$$

Hence

$$\iint \frac{W_1 - (W_1)_A}{8a^3 \sin^3 \frac{1}{2}\Theta} dS = -\frac{2\pi}{a} \sum_{n=1}^{\infty} n a_n^0 . \tag{50}$$

Now consider the generalized function

$$I'(\Theta) = \sum_{n=0}^{\infty} n(2n+1)\, P_n(\cos \Theta) \tag{51}$$

then

$$\iint I'(\Theta)\, W_1\, dS = \sum_{n=1}^{\infty} n a_n^0 \,.\, 4\pi a^2 \tag{52}$$

and from equation (50)

$$\iint I'(\Theta)\, W_1\, dS = -2a^3 \iint \frac{W_1 - (W_1)_A}{8a^3 \sin^3 \frac{1}{2}\Theta} dS \tag{53}$$

which defines the operator associated with the kernel $I'(\Theta)$ given by (51).

We may now state the following rule for interpreting the kernel $K(\Theta)$, when it is obtained from the spherical harmonic expansions of W_1 and W_2 in the form of a formal expansion in Legendre functions P_n. The formal expansion is expressed as a linear combination of series of certain forms (i)–(v), which are then replaced by their Euler sums or other operator kernels as follows:—

(i) $\sum_{n=1}^{\infty} n(2n+1)\, P_n = I'(\Theta)$,

(ii) $\sum_{n=0}^{\infty} (2n+1)\, P_n = I(\Theta)$,

(iii) $\sum_{n=0}^{\infty} P_n = \dfrac{1}{2 \sin \frac{1}{2}\Theta}$,

(iv) $\sum_{n=0}^{\infty} \dfrac{1}{n+1} P_n = \log \left\{ 1 + \dfrac{1}{\sin \frac{1}{2}\Theta} \right\}$,

(v) $\sum_{n=1}^{\infty} \dfrac{1}{n} P_n = -\log \{(\sin \frac{1}{2}\Theta)(1 + \sin \frac{1}{2}\Theta)\}$.

The last three results can be obtained as Euler summations directly from the generating function for the Legendre polynomials. We have

$$\frac{1}{(r^2+a^2-2ar\cos\Theta)^{\frac{1}{2}}}=\frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n P_n(\cos\Theta),\ r<a \tag{54a}$$

$$=\frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^{n+1} P_n(\cos\Theta),\ r>a. \tag{54b}$$

Letting $r \to a$ in either (54a) or (54b) gives the Euler sum (iii). Integrating (54a) from 0 to r and then letting $r \to a-$ gives the result (iv), and integrating (54b) from r to infinity and letting $r \to a+$ gives (v). The results (i) and (ii) are different in character from the others in that $I(\Theta)$ and $I'(\Theta)$ are generalized functions.

As an illustrative example, consider the derivation of Ψ in terms of Ω_-. Equation (35) requires that

$$K(\Theta)=-\frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(2n+1)^2}{n+1}P_n(\cos\Theta)$$

$$=-\frac{1}{4\pi}\left\{2\sum_{n=0}^{\infty}(2n+1)P_n(\cos\Theta)+\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-2\right)P_n(\cos\Theta)\right\}$$

$$=-\frac{1}{4\pi}\left\{2I(\Theta)+\left[\log\left(1+\frac{1}{\sin\frac{1}{2}\Theta}\right)-\frac{1}{\sin\frac{1}{2}\Theta}\right]\right..$$

With this interpretation of $K(\Theta)$, equation (38) becomes

$$(\Psi)_A=\frac{1}{4\pi a^2}\iint_S -\frac{1}{4\pi}\left\{2I(\Theta)-\left[\frac{1}{\sin\frac{1}{2}\Theta}-\log\left(1+\frac{1}{\sin\frac{1}{2}\Theta}\right)\right]\right\}\Omega_-\,dS.$$

Hence from equation (40)

$$(\Psi)_A=-\frac{1}{2\pi}(\Omega_-)_A+\frac{1}{16\pi^2a^2}\iint_S\left\{\left(\frac{1}{\sin\frac{1}{2}\Theta}\right)-\log\left(1+\frac{1}{\sin\frac{1}{2}\Theta}\right)\right\}\Omega_-\,dS$$

and this is the required surface integral formula.

The following is a list of the 12 surface integral formulae obtained in this way relating any two of the functions $\Psi, \partial\Omega/\partial r, \Omega_-$ and Ω_+ defined in Section (2).

$$(\Omega_+)_A=-(\Omega_-)_A+\frac{1}{4\pi a^2}\iint_S\left\{\frac{1}{\sin\frac{1}{2}\Theta}-\log\left(1+\frac{1}{\sin\frac{1}{2}\Theta}\right)\right\}\Omega_-\,dS \tag{55}$$

$$(\Omega_+)_A=2\pi(\Psi)_A-\frac{1}{4a^2}\iint_S\frac{1}{\sin\frac{1}{2}\Theta}\Psi\,dS \tag{56}$$

$$(\Omega_+)_A=\frac{1}{4\pi a}\iint_S\left\{\log\left(1+\frac{1}{\sin\frac{1}{2}\Theta}\right)-\frac{1}{\sin\frac{1}{2}\Theta}\right\}\left(\frac{\partial\Omega}{\partial r}\right)_a dS \tag{57}$$

$$(\Omega_-)_A=-(\Omega_+)_A-\frac{1}{4\pi a^2}\iint_S\left\{\frac{1}{\sin\frac{1}{2}\Theta}-\log\left[(1+\sin\tfrac{1}{2}\Theta)\sin\tfrac{1}{2}\Theta\right]\right\}\Omega_+\,dS \tag{58}$$

$$(\Omega_-)_A = -2\pi(\Psi)_A - \frac{1}{4a^2}\iint_S \frac{1}{\sin\frac{1}{2}\Theta}\Psi\,dS \tag{59}$$

$$(\Omega_-)_A = \frac{1}{4\pi a}\iint_S \left\{\frac{1}{\sin\frac{1}{2}\Theta} - \log\left[(1+\sin\tfrac{1}{2}\Theta)\sin\tfrac{1}{2}\Theta\right]\right\}\left(\frac{\partial\Omega}{\partial r}\right)_a dS \tag{60}$$

$$(\Psi)_A = \frac{1}{2\pi}(\Omega_+)_A$$

$$+\frac{1}{16\pi^2 a^2}\iint_S \left\{\frac{1}{\sin\frac{1}{2}\Theta} - \log\left[(1+\sin\tfrac{1}{2}\Theta)\sin\tfrac{1}{2}\Theta\right]\right\}\Omega_+\,dS \tag{61}$$

$$(\Psi)_A = -\frac{1}{2\pi}(\Omega_-)_A + \frac{1}{16\pi^2 a^2}\iint_S \left\{\frac{1}{\sin\frac{1}{2}\Theta} - \log\left(1+\frac{1}{\sin\frac{1}{2}\Theta}\right)\right\}\Omega_-\,dS \tag{62}$$

$$(\Psi)_A = \frac{1}{8\pi^2 a}\iint_S \left\{\log(1+\sin\tfrac{1}{2}\Theta) - \frac{1}{\sin\frac{1}{2}\Theta}\right\}\left(\frac{\partial\Omega}{\partial r}\right)_a dS \tag{63}$$

$$\left(\frac{\partial\Omega}{\partial r}\right)_A = \frac{1}{2\pi}\iint_S \frac{\Omega_+ - (\Omega_+)_A}{8a^3\sin^3\frac{1}{2}\Theta}\,dS - \frac{1}{a}(\Omega_+)_A \tag{64}$$

$$\left(\frac{\partial\Omega}{\partial r}\right)_A = -\frac{1}{2\pi}\iint_S \frac{\Omega_- - (\Omega_-)_A}{8a^3\sin^3\frac{1}{2}\Theta}\,dS \tag{65}$$

$$\left(\frac{\partial\Omega}{\partial r}\right)_A = \iint_S \frac{\Psi(1+\sin^2\frac{1}{2}\Theta) - (\Psi)_A}{8a^3\sin^3\frac{1}{2}\Theta}\,dS - \frac{\pi}{a}(\Psi)_A. \tag{66}$$

4. Relations between fields and currents on two concentric shells

Suppose now a current system Ψ exists in a shell of radius a, and we require to express either the potential Ω_b or its normal derivative $(\partial\Omega/\partial r)_b$ on a concentric shell of radius $b(< a)$, in terms of any one of the quantities $\Psi, \Omega_+, \Omega_-, \partial\Omega/\partial r$ on the shell of radius a.

The potential Ω_1 of the field inside the shell of radius a, due to the current system Ψ given by (1) in the shell has the value

$$\Omega_1 = -\sum_{n=0}^{\infty}(n+1)\left(\frac{r}{a}\right)^n a_n^m S_n^m. \tag{67}$$

Hence

$$\Omega_b = -\sum_{n=0}^{\infty}(n+1)\left(\frac{b}{a}\right)^n a_n^m S_n^m \tag{68}$$

and

$$\left(\frac{\partial\Omega}{\partial r}\right)_b = -\sum_{n=1}^{\infty} n(n+1)\frac{b^{n-1}}{a^n} a_n^m S_n^m. \tag{69}$$

Writing any of the above four quantities $\Psi, \Omega_+, \ldots$ on the shell $r = a$ in the spherical harmonic form for W_1, given by (33), and either of the series (68) and (69) in the form for W_2 given by (34), we see that k_n will now be a function of a and b as well as of n.

We now consider the kernel function

$$K(\Theta) = \sum_{n=0}^{\infty} (2n+1)\, k_n(a,b)\, P_n \tag{70}$$

and evaluate the integral

$$\iint K(\Theta)\, W_1\, dS$$

over the sphere $r = a$. Using the orthogonal properties (19) of the surface harmonics, we obtain

$$\iint K(\Theta)\, W_1\, dS = 4\pi a^2 \sum_{n=0}^{\infty} k_n(a,b)\, a_n{}^0$$

$$= 4\pi a^2 (W_2)_B \tag{71}$$

where B is the point on $r = b$ when $\Theta = 0$.

We thus have a formula for determining W_2 at a point B on $r = b$ in terms of the field quantity W_1 given on $r = a$. As an example, let W_2 correspond to $(\partial\Omega/\partial r)_b$ and W_1 correspond to Ψ. Then equations (69) for $(\partial\Omega/\partial r)_b$ and (1) for Ψ show that

$$k_n = -\frac{n(n+1)4\pi}{2n+1}\,\frac{b^{n-1}}{a^n}$$

and therefore

$$K(\Theta) = -\frac{4\pi}{a}\,\Sigma n(n+1)\left(\frac{b}{a}\right)^{n-1} P_n(\cos\Theta). \tag{72}$$

This series for $K(\Theta)$ is convergent for $b < a$, and using (54a) the sum is found to be

$$K(\Theta) = 4\pi a^2 \left\{\frac{ab\cos^2\Theta - 2(a^2+b^2)\cos\Theta + 3ab}{(a^2+b^2-2ab\cos\Theta)^{\frac{5}{2}}}\right\}. \tag{73}$$

Therefore the required surface integral is, from equation (71),

$$\left(\frac{\partial\Omega}{\partial r}\right)_B = \iint_S \left\{\frac{ab\cos^2\Theta - 2(a^2+b^2)\cos\Theta + 3ab}{(a^2+b^2-2ab\cos\Theta)^{\frac{5}{2}}}\right\} \Psi\, dS \tag{74}$$

where S is the surface $r = a$.

Similarly Ω and $\partial\Omega/\partial r$ on $r = b$ may be obtained in surface integral form in terms of each of the functions $\Psi, \partial\Omega/\partial r, \Omega_-$ and Ω_+ defined on $r = a$ by suitable substitutions in (71). This results in the following eight surface integral formulae:—

$$\left(\frac{\partial\Omega}{\partial r}\right)_B = \iint_S \frac{1}{R^5}\{ab\cos^2\Theta - 2(a^2+b^2)\cos\Theta + 3ab\}\,\Psi\, dS \tag{75}$$

where $R = (a^2+b^2-2ab\cos\Theta)^{\frac{1}{2}}$ and S is the surface $r = a$,

$$\left(\frac{\partial\Omega}{\partial r}\right)_B = -\frac{1}{4\pi b}\iint_S \frac{1}{R^5}\{b(a^2+3b^2)\cos\Theta + a(a^2-5b^2)\}\,\Omega_+\, dS \tag{76}$$

$$\left(\frac{\partial \Omega}{\partial r}\right)_B = \frac{1}{4\pi a} \iint\limits_S \frac{1}{R^5} \{a(b^2+3a^2)\cos\Theta + b(b^2-5a^2)\}\, \Omega_-\, dS \tag{77}$$

$$\left(\frac{\partial \Omega}{\partial r}\right)_B = \frac{1}{4\pi b} \iint\limits_S \frac{1}{R^5} (a^2-b^2) \left(\frac{\partial \Omega}{\partial r}\right)_a dS \tag{78}$$

$$(\Omega)_B = -\iint\limits_S \frac{1}{R^3} (a - b\cos\Theta)\, \Psi\, dS \tag{79}$$

$$(\Omega)_B = -\frac{1}{4\pi a} \iint\limits_S \left\{\frac{1}{R^3}(b^2+3a^2-4ab\cos\Theta) - \frac{1}{a}\log\left[\frac{a-b\cos\Theta+R}{2a}\right]\right\} \Omega_+\, dS \tag{80}$$

$$(\Omega)_B = \frac{1}{4\pi a} \iint\limits_S \frac{1}{R^3} (a^2-b^2)\, \Omega_-\, dS \tag{81}$$

$$(\Omega)_B = \frac{1}{4\pi a} \iint\limits_S \left\{\frac{2a}{R} - \log\left[\frac{a-b\cos\Theta+R}{2a}\right]\right\} \left(\frac{\partial \Omega}{\partial r}\right)_a dS. \tag{82}$$

It will be noted that since R does not now become zero anywhere on the surface, the integrand has no singularity in any of the above eight integrals.

Analogous to the above, a current system Ψ' on $r = b$ will give rise to a magnetic field at $r = a$, the potential Ω' and normal component $\partial\Omega'/\partial r$ of which can be obtained in series form from equations (13) and (14). Eight further surface integral formulae may therefore be derived for Ω' and $\partial\Omega'/\partial r$ on $r = a$ in terms of each of Ψ' and the associated Ω_-', Ω_+' and $\partial\Omega'/\partial r$ on $r = b$, defined similarly to Ω_-, Ω_+ and $\partial\Omega/\partial r$ in (5)–(7). These are found to be:—

$$\left(\frac{\partial \Omega'}{\partial r}\right)_A = \iint\limits_S \frac{1}{R^5} \{ab\cos^2\Theta - 2(a^2+b^2)\cos\Theta + 3ab\}\, \Psi'\, dS \tag{83}$$

where S is the surface $r = b$ in (83)–(90) inclusive.

$$\left(\frac{\partial \Omega'}{\partial r}\right)_A = -\frac{1}{4\pi b} \iint\limits_S \frac{1}{R^5} \{b(a^2+3b^2)\cos\Theta + a(a^2-5b^2)\}\, \Omega_+'\, dS \tag{84}$$

$$\left(\frac{\partial \Omega'}{\partial r}\right)_A = \frac{1}{4\pi a} \iint\limits_S \frac{1}{R^5} \{a(b^2+3a^2)\cos\Theta + b(b^2-5a^2)\}\, \Omega_-'\, dS \tag{85}$$

$$\left(\frac{\partial \Omega'}{\partial r}\right)_A = \frac{1}{4\pi a} \iint\limits_S \frac{1}{R^3} (a^2-b^2) \left(\frac{\partial \Omega'}{\partial r}\right)_b dS \tag{86}$$

$$(\Omega')_A = \iint\limits_S \frac{1}{R^3} \{b - a\cos\Theta\}\, \Psi'\, dS \tag{87}$$

$$(\Omega')_A = \frac{1}{4\pi b} \iint\limits_S \frac{1}{R^3} (a^2-b^2)\, \Omega_+'\, dS \tag{88}$$

$$(\Omega')_A = \frac{1}{4\pi b}\iint_S \left\{\frac{1}{R^3}(a^2+3b^2-4ab\cos\Theta) - \frac{1}{b}\log\left[\frac{b-a\cos\Theta+R}{2a\sin^2\frac{1}{2}\Theta}\right]\right\}\Omega_-'\,dS \qquad (89)$$

$$(\Omega')_A = \frac{1}{4\pi b}\iint_S \left\{\log\left[\frac{b-a\cos\Theta+R}{2a\sin^2\frac{1}{2}\Theta}\right] - \frac{1}{R^3}b(a^2-b^2)\right\}\left(\frac{\partial\Omega'}{\partial r}\right)_b dS. \qquad (90)$$

In some geomagnetic studies, e.g. in estimating the effects on magnetic variations of induced currents in the oceans and in the Earth's mantle, it is useful to consider an induction problem in which a non-uniform shell ($r = a$) surrounds a spherical conductor ($r = b$). It is sometimes possible, for a certain range of frequencies of the variations, to treat the inner conductor as effectively of infinite conductivity. In this case the potential Ω^* and normal component $\partial\Omega^*/\partial r$ at $r = a$ of the magnetic field of the system of currents induced in the perfectly conducting inner conductor $r = b$, by a system of varying currents flowing in $r = a$, may be expressed directly in terms of each of the quantities Ω_-, Ω_+ and $\partial\Omega/\Omega r$, and the streamline function Ψ, corresponding to the system of currents flowing in $r = a$. The eight surface integral formulae in this special case are:—

$$\left(\frac{\partial\Omega^*}{\partial r}\right)_A = -b^3\iint_S \frac{1}{R_1^5}\{a^2b^2\cos^2\Theta - 2(a^4+b^4)\cos\Theta + 3a^2b^2\}\Psi\,dS \qquad (91)$$

where $R_1 = (a^4+b^4-2a^2b^2\cos\Theta)^{\frac{1}{2}}$ and S is the surface $r = a$.

$$\left(\frac{\partial\Omega^*}{\partial r}\right)_A = \frac{b}{4\pi}\iint_S \frac{1}{R_1^5}\{b^2(a^4+3b^4)\cos\Theta + a^2(a^4-5b^4)\}\Omega_+\,dS \qquad (92)$$

$$\left(\frac{\partial\Omega^*}{\partial r}\right)_A = -\frac{b^3}{4\pi a^2}\iint_S \frac{1}{R_1^5}\{a^2(b^4+3a^4)\cos\Theta + b^2(b^4-5a^4)\}\Omega_-\,dS \qquad (93)$$

$$\left(\frac{\partial\Omega^*}{\partial r}\right)_A = -\frac{b}{4\pi a}\iint_S \frac{1}{R_1^3}\{a^4-b^4\}\left(\frac{\partial\Omega}{\partial r}\right)_a dS \qquad (94)$$

$$(\Omega^*)_A = -\frac{b^3}{a}\iint_S \frac{1}{R_1^3}(b^2-a^2\cos\Theta)\Psi\,dS \qquad (95)$$

$$(\Omega^*)_A = -\frac{b}{4\pi a}\iint_S \frac{1}{R_1^3}(a^4-b^4)\Omega_+\,dS \qquad (96)$$

$$(\Omega^*)_A = -\frac{b}{4\pi a}\iint_S \left\{\frac{1}{R_1^3}(a^4+3b^4-4a^2b^2\cos\Theta) - \frac{1}{b^2}\log\left[\frac{b^2-a^2\cos\Theta+R_1}{2a^2\sin^2\frac{1}{2}\Theta}\right]\right\}\Omega_-\,dS \qquad (97)$$

$$(\Omega^*)_A = \frac{1}{4\pi b} \iint_S \left\{ \frac{b^2}{R_1} - \log \left[\frac{b^2 - a^2 \cos \Theta + R_1}{2a^2 \sin^2 \frac{1}{2}\Theta} \right] \right\} \left(\frac{\partial \Omega}{\partial r} \right)_a dS. \tag{98}$$

In these eight formulae there are again no singularities in any of the integrands.

5. Concluding remarks

A number of the surface integral formulae derived in this paper have been programmed for evaluation computer. In particular, (63), (66) and (91) have been programmed and used in the discussion of geomagnetic induction problem relating to currents induced in the oceans and the Earth's mantle, and it is hoped to publish the results in a later paper. Formula (62) is being used by Price & Stone (1970) to determine representative current systems for the magnetic daily variation field, by automatic analysis of observational data fed into a computer.

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