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Surface measures in Carnot-Carathéodory spaces

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Abstract. In the framework of Carnot-Carathéodory spaces we study Minkowski content and perimeter, we prove some coarea formulas, and finally we prove some variational approximations of the perimeter.

1. Introduction

In this paper we deal with some problems concerning Geometric Measure Theory and Calculus of Variations in metric spaces. Although such a theory is in embryonic stage contained in Federer's book [22], its systematic study has been carried out only since a few years (see also Mattila's recent book [43] and references therein). An interesting proposal of a general Geometric Measure Theory in metric spaces was given in [20] and the program has been developed in [4] and [5]. A general report on problems and techniques of Analysis and Geometry on metric spaces can be found in [19], [35], [54].

In our framework we actually consider a special class of metric spaces, the Carnot-Carathéodory (C-C) spaces. More precisely, given a family $X = (X_1, ..., X_m)$ of Lipschitz vector fields $X_j(x) = \sum_{i=1}^n a_{ij}(x)\partial_i$ ($j = 1, ..., m$) with $a_{ij} \in Lip(\mathbb{R}^n)$ ($j = 1, ..., m$, $i = 1, ..., n$), we call *subunit* a Lipschitz continuous curve $\gamma : [0, T] \longrightarrow \mathbb{R}^n$ such that

(1.1)
$$\dot\gamma(t) = \sum_{j=1}^m h_j(t) X_j(\gamma(t)), \quad \text{and} \quad \sum_{j=1}^m h_j^2(t) \le 1 \quad \text{for a.e. } t \in [0, T],$$

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with $h_1, ..., h_m$ measurable coefficients. Then define the C-C dinstance between the points $x, y \in \mathbb{R}^n$ as

$$d(x,y) = \inf\{T \geq 0 : \text{there exists a subunit path } \gamma : [0,T] \to \mathbb{R}^n \text{ such that } \gamma(0) = x \text{ and } \gamma(T) = y\}. \tag{1.2}$$

If the above set is empty put $d(x,y) = +\infty$. If d is finite on $\mathbb{R}^n$, i.e. $d(x,y) < \infty$ for every $x, y \in \mathbb{R}^n$, it turns out to be a metric on $\mathbb{R}^n$ and the metric space $(\mathbb{R}^n, d)$ is called C-C space (see, for instance, [33]). In particular we shall always assume that

(H1) the metric d is finite and the identity map $\mathrm{Id} : (\mathbb{R}^n, d) \to (\mathbb{R}^n, |\cdot|)$ is a homeomorphism.

Such spaces have been much studied in the last years with applications ranging from degenerate elliptic equations to optimal control theory and differential geometry (see, for instance, [37], [53], [49], [26], the recent book [9] and references therein). On the other hand even some topics of Geometric Measure Theory have been studied in the setting of these spaces. In particular the space of functions with *bounded variation* with respect to X has been introduced in [12], [15], [27], [30]. Namely, given an open set $\Omega \subset \mathbb{R}^n$, the space $BV_X(\Omega)$ is the set of functions $f \in L^1(\Omega)$ such that

$$\begin{aligned} &\|Xf\|(\Omega) \\ &:= \sup\Big\{ \int_\Omega f(x) \sum_{j=1}^m \sum_{i=1}^n \partial_i(a_{ij}(x)\varphi_j(x))\, dx : \varphi = (\varphi_1, ..., \varphi_m) \in \\ &\in C_0^1(\Omega; \mathbb{R}^m) \text{ and } |\varphi(x)| \leq 1 \ \forall x \in \Omega \Big\} < \infty. \end{aligned}$$

Many interesting properties of BV_X functions have been investigated and in particular *isoperimetric type inequalities* have been proved for the $X-$perimeter $\|\partial E\|_X$ associated to a measurable set $E \subset \mathbb{R}^n$ of *locally finite perimeter*, that is

$$\|\partial E\|_X(\Omega) := \|X\chi_E\|(\Omega) < \infty \tag{1.3}$$

for every open bounded set $\Omega \subset \mathbb{R}^n$ (see [51], [25] and [30]). Moreover, the existence of minimizing sets of locally finite perimeter has been obtained in [30], generalizing similar De Giorgi's results for the Euclidean perimeter measure $\|\partial E\| := \|\partial E\|_X$ with $X = \nabla = (\partial_1, ..., \partial_n)$ (see [32]).

More recently, introducing a suitable intrinsic notion of rectifiability in the Heisenberg group, a counterpart of De Giorgi's result on the structure of sets of finite Euclidean perimeter (see [32]) has been obtained in [29] (see Sect. 5 and see also [2] and [52]). Finally area formulas have been proved in the setting of Carnot groups (see [56] and [41]).

In this work we tackle the problem of investigating the relations among some classical measures of the Geometric Measure Theory concentrated on surface type sets in the setting of C-C spaces, such as $X-$perimeter and Minkowski content. To this aim we are able to prove for some C-C dinstances the following *eikonal type equation* (see Theorem 3.1):

(H2) Let $K \subset \mathbb{R}^n$ be a closed set. If $d_K(x) := \inf_{y\in K} d(x,y)$ then $Xd_K(x) = (X_1 d_K(x), ..., X_m d_K(x)) \in \mathbb{R}^m$ exists and $|Xd_K(x)| = 1$ for a.e. $x \in \mathbb{R}^n \setminus K$.

Actually, we stress that only assumption (H1) and (H2) are needed in order that our main results turn to be true in a general C-C space. In particular, assuming (H1) we will get that the $X-$perimeter $\|\partial E\|_X(\mathbb{R}^n)$ coincides with the Minkowski content of ∂E

$$M(\partial E)(\mathbb{R}^n) := \lim_{r\to 0^+} \frac{|\{x \in \mathbb{R}^n : d_{\partial E}(x) < r\}|}{2r}$$

for all bounded set $E \subset \mathbb{R}^n$ with C^∞ boundary (see Theorem 5.1). The proof relies on a *Riemannian type approximation* of the metric d (see Remark 5.2), and as a preliminary step we also prove a *coarea formula* of the following type

$$\int_{\mathbb{R}^n} u(x)|Xf(x)|dx = \int_{-\infty}^{+\infty} \Big(\int_{\{f=t\}} u\, d\mu_t \Big)\, dt, \tag{1.4}$$

where u is an integrable function and f is a Lipschitz continuous function with respect to the C-C metric d, $E_t = \{x \in \mathbb{R}^n : f(x) > t\}$ and $\mu_t = \|\partial E_t\|_X$ (see Theorem 4.2). This coarea formula extends previous ones already known in the literature (see [27], [30] and [29]). Moreover let us note that, although only assumption (H1) is involved in the proof of formula (1.4), looking at the weight $|Xf|$ one can immediately guess the key role played by hypothesis (H2) in integration over level sets of dinstance functions.

Our main result is the variational approximation of the perimeter $\|\partial E\|_X$ by means of degenerate elliptic functionals of the type $F_\varepsilon : L^1(\Omega) \to [0,+\infty]$

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega (\varepsilon|Xu|^2 + \frac{1}{\varepsilon}W(u))dx & \text{if } u \in H^1_X(\Omega) \\ +\infty & \text{if } u \in L^1(\Omega) \setminus H^1_X(\Omega) \end{cases} \tag{1.5}$$

where $W(u) = u^2(1-u)^2$ and $H^1_X(\Omega)$ is the set of the functions $u \in L^2(\Omega)$ such that $X_j u \in L^2(\Omega)$, $j = 1, ..., m$, exist in distributional sense. More precisely for every bounded, regular open set $\Omega \subset \mathbb{R}^n$ we get the existence of the $\Gamma-$limit functional $F := \Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} F_\varepsilon : L^1(\Omega) \to [0,+\infty]$ with

$$F(u) = \begin{cases} 2\alpha\|\partial E\|_X(\Omega) & \text{if } u = \chi_E \in BV_X(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

provided assumptions (H1) and (H2) hold, with $\alpha = \int_0^1 \sqrt{W(s)}ds$ (see Definition 6.1 and Theorem 6.5). In the case $X = \nabla = (\partial_1, ..., \partial_n)$ this result is proved in [47] for the classical Euclidean perimeter. The same variational approximation was used in [46] to establish a connection between the classical model and the Cahn-Hilliard model of phase transition (see [8], [1] and references therein). More recently the general non degenerate anisotropic case was also studied, namely when one considers the functionals $F_\varepsilon : L^1(\Omega) \to [0,+\infty]$

(1.6)
$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega (\varepsilon\langle BDu, Du\rangle + \frac{1}{\varepsilon}W(u))dx & \text{if } u \in H^1(\Omega) \\ +\infty & \text{if } u \in L^1(\Omega) \setminus H^1(\Omega) \end{cases}$$

where $H^1(\Omega)$ is the classical Sobolev space and $B(x)$ is a $n \times n$ symmetric matrix of bounded measurable functions on Ω such that

$$\langle B(x)\xi, \xi\rangle \geq \lambda_0|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^n, \tag{1.7}$$

for a given $\lambda_0 > 0$ (see for instance [13] and [10]). However, we explicitly point out that the case (1.6) is not comprehensive of the one (1.5). In fact in this case, if $A(x) = (a_{ij}(x))$ is the matrix of the coefficients of the vector fields, then $B(x) = A(x)A(x)^{\mathrm{T}}$ would be the appropriate matrix in (1.6), but it could not satisfy (1.7) (see Remark 6.6).

Moreover we are able to prove that the $X-$perimeter (1.3) is the $\Gamma-$limit of a suitable family of *Riemannian type perimeters* for a general family of Lipschitz vector fields without assumptions (H1) and (H2) (see Theorem 6.4 and Remark 5.2). These arguments show that the definition of $X-$perimeter is from the variational point of view as stable as the Euclidean perimeter.

Finally let us give a short abstract of the paper. In Sect. 2 we establish our notations and recall some known results about C-C spaces. In Sect. 3 we prove (H2) for some C-C metrics by means of an extension of a well known differentiability theorem due to Pansu. In Sect. 4 we prove coarea formula (1.4) and get some applications. In Sect. 5 we study the relation between $X-$perimeter and Minkowski content, proving that they coincide for regular surfaces. Finally, in Sect. 6 we perform the study of $X-$perimeter's approximations by means of $\Gamma-$convergence.

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2. Notations and preliminary results

In this section we recall some well known results about C-C spaces that will be used in the sequel. Consider the vector fields $X_1, ..., X_m \in Lip(\mathbb{R}^n; \mathbb{R}^n)$. We shall as usual identify vector fields and differential operators. If

$$X_j(x) = \sum_{i=1}^{n} a_{ij}(x)\partial_i, \quad j = 1, ..., m,$$

define the matrix

$$A = \mathrm{col}[X_1, ..., X_m] = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}. \tag{2.1}$$

We shall denote by X_j^* the operator formally adjoint to X_j in $L^2(\mathbb{R}^n)$, that is the operator which for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} \varphi(x) X_j \psi(x)\, dx = \int_{\mathbb{R}^n} \psi(x) X_j^* \varphi(x)\, dx.$$

We first introduce some functional spaces associated with the vector fields. If f is a scalar function and φ is a $m-$vector valued function, define the $X-$*gradient* and $X-$*divergence*

$$Xf := (X_1 f, ..., X_m f), \quad \mathrm{div}_X(\varphi) := \sum_{j=1}^{m} X_j^* \varphi_j.$$

If $\Omega \subset \mathbb{R}^n$ is an open set, the anisotropic Sobolev space $H_X^{1,p}(\Omega)$, $1 \le p \le \infty$, is the set of functions $f \in L^p(\Omega)$ such that the derivatives $X_j f$, $j = 1, ..., m$, exist in the sense of distributions as $L^p(\Omega)$ functions. We shall write $H_X^1(\Omega) := H_X^{1,2}(\Omega)$. The space $H_{X,loc}^{1,p}(\Omega)$ is the set of functions belonging to $H_X^{1,p}(U)$ for each open set U compactly contained in Ω ($U \subset\subset \Omega$).

We now introduce functions of bounded $X-$variation and recall some of their properties (see [27] and [30]). Let $\Omega \subset \mathbb{R}^n$ be an open set and let

$$F(\Omega; \mathbb{R}^m) := \{\varphi \in C_0^1(\Omega; \mathbb{R}^m) : |\varphi(x)| \le 1 \ \ \forall x \in \Omega\}.$$

The space $BV_X(\Omega)$ is the set of functions $f \in L^1(\Omega)$ such that

$$\|Xf\|(\Omega) := \sup_{\varphi \in F(\Omega;\mathbb{R}^m)} \int_\Omega f(x)\mathrm{div}_X(\varphi)(x)\,dx < \infty. \tag{2.2}$$

The space $BV_{X,loc}(\Omega)$ is the set of functions belonging to $BV_X(U)$ for each open set $U \subset\subset \Omega$. A measurable set $E \subset \mathbb{R}^n$ is of locally finite $X-$perimeter in Ω (or a $X-$Caccioppoli set) if $\chi_E \in BV_{X,loc}(\Omega)$, namely if

$$\|\partial E\|_X(U) := \|X\chi_E\|(U) < \infty \tag{2.3}$$

for every open set $U \subset\subset \Omega$. By means of Riesz representation Theorem one can prove that if $f \in BV_{X,loc}(\Omega)$ then $\|Xf\|$ is a Radon measures on Ω (see [22, 2.2.5]).

Proposition 2.1. *Let $f, f_k \in L^1(\Omega)$, $k \in \mathbb{N}$, be such that $f_k \to f$ in $L^1(\Omega)$, then*

$$\liminf_{k\to\infty} \|Xf_k\|(\Omega) \geq \|Xf\|(\Omega).$$

Proposition 2.2. *If E is a $X-$Caccioppoli set with C^1 boundary the perimeter has the following representation*

$$\|\partial E\|_X(\Omega) = \int_{\partial E\cap\Omega} |C(x)n(x)|d\mathcal{H}^{n-1}, \tag{2.4}$$

where $n(x)$ is the Euclidean normal to ∂E at x and $C = A^{\mathrm{T}}$ (recall (2.1)).

Theorem 2.3. *Let $f \in BV_X(\Omega)$ and write $\mu = \|Xf\|$. There exists a $\mu-$measurable function $\sigma_f : \Omega \to \mathbb{R}^m$ such that $|\sigma_f| = 1$ $\mu-$a.e. and*

$$\int_\Omega f(x)\mathrm{div}_X(\varphi)(x)\,dx = -\int_\Omega \langle\varphi(x), \sigma_f(x)\rangle d\mu,$$

for all $\varphi \in F(\Omega;\mathbb{R}^m)$.

When $f = \chi_E$ in Theorem 2.3, then the vector

$$\nu_E(x) := -\sigma_{\chi_E}(x) \tag{2.5}$$

will be called $X-$*generalized inner normal* of E .

The representation of the perimeter (2.4) is in [27, Remark 2.3.3]. Theorem 2.3 is a direct consequence of Riesz Theorem (see for example [32] or [21]).

We now turn back to C-C metrics (recall definitions (1.1) and (1.2)). We already noticed that it is not always possible to connect two points by a subunit path. An interesting condition that implies the $X-$connectivity is the so-called Chow-Hörmander condition

$$\text{rank}\,\mathcal{L}(X_1,...,X_m)(x) = n \quad \text{for every } x \in \mathbb{R}^n. \tag{2.6}$$

$\mathcal{L}(X_1,...,X_m)$ is the Lie algebra generated by $X_1,...,X_m \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ (see [37]). C-C metrics induced by vector fields satisfying (2.6) satisfy (H1). Precisely, if $\Omega \subset \mathbb{R}^n$ is a bounded open set and if $k \in \mathbb{N}$ is the minimum length of the commutators necessary to guarantee (2.6) relatively to Ω then there exists $C > 0$ such that

$$\frac{1}{C}|x-y| \le d(x,y) \le C|x-y|^{\frac{1}{k}}$$

for all $x, y \in \Omega$ (see for example [39]).

Metric balls induced by vector fields satisfing the Chow-Hörmander condition are well behaved with respect to Lebesgue measure. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $r_0 > 0$. Then there exists a constant $\delta > 0$ such that for all $x \in \Omega$ and for all $0 < r < r_0$ the following *doubling property* holds (see [49])

$$|B(x,2r)| \le \delta |B(x,r)|. \tag{2.7}$$

Here and in the sequel $|E|$ will always stand for $\mathcal{L}^n(E)$, where $\mathcal{L}^n$ is the $n-$dimensional Lebesgue measure on $\mathbb{R}^n$. $B(x,r)$ is a C-C open ball centered at x with radius r. The constant δ is a doubling constant and $Q := \log_2 \delta \ge n$ is a *local homogeneous dimension* of Ω relatively to the vector fields $X_1,...,X_m$.

A subunit path $\gamma : [0,T] \to \mathbb{R}^n$ is a *geodesic* if $d(\gamma(0),\gamma(T)) = T$. If $\mathbb{R}^n$ is $X-$connected then it has the *geodesic segment property*, i.e. for every two points of $\mathbb{R}^n$ there exists a geodesic connecting them. The following proposition can be proved by a classical compactness method due to Hilbert (see for example [34, Theorem 1.10]).

Proposition 2.4. *Let $(\mathbb{R}^n, d)$ be $X-$connected and complete. For every $x, y \in \mathbb{R}^n$ there exists a Lipschitz continuous subunit curve $\gamma : [0, d(x,y)] \longrightarrow \mathbb{R}^n$ such that $\gamma(0) = x$, $\gamma(d(x,y)) = y$ and $t = d(\gamma(t), x)$.*

We state now a deep result which will be needed. Consider a C-C space $(\mathbb{R}^n,d)$. A function $f : (\mathbb{R}^n,d) \to \mathbb{R}$ is $L-$Lipschitz if

$$|f(x) - f(y)| \le L d(x,y) \tag{2.8}$$

for all $x, y \in \mathbb{R}^n$. In this case we shall write $f \in Lip(\mathbb{R}^n,d)$. The infimum of the constants L such that (2.8) holds will be denoted by $Lip(f)$. Lipschitz functions are differentiable a.e. along the fields X_j (see [28] and [31]).

Theorem 2.5. *Let $(\mathbb{R}^n, d)$ be a Carnot-Carathéodory space associated with a family of vector fields $X_1, ..., X_m \in Lip(\mathbb{R}^n; \mathbb{R}^n)$, and suppose that (H1) holds. Then, for every L–Lipschitz function f the derivatives $X_j f(x)$, $j = 1, ..., m$, exist and $|Xf(x)| \le L$ for a.e. $x \in \mathbb{R}^n$.*

Next Morrey's type estimate is proved in [40]. Here and in the sequel we write $\fint_B u = \frac{1}{|B|}\int_B u$.

Lemma 2.6. *Let $X_1, ..., X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ satisfy the Chow-Hörmander condition (2.6) and let $\Omega \subset \mathbb{R}^n$ be a bounded open set with homogeneous dimension $Q \ge n$ (recall (2.7)). If $p > Q$ there exists $C = C(\Omega, X, Q, p) > 0$ such that for every ball $B = B(x_0, r) \subset\subset \Omega$*

$$|f(x) - f(y)| \le Cr\Big(\fint_B |Xf(z)|^p\, dz\Big)^{\frac{1}{p}} \quad \text{for a.e. } x, y \in B \tag{2.9}$$

for all $f \in H^{1,p}_X(B)$.

Finally, we recall the definition of *Carnot group* (see also [36] and [50]). Let $(\mathbb{R}^n, \cdot)$ be a nilpotent Lie group whose Lie algebra $\mathfrak{g}$ admits a *stratification*, i.e. there exist $V_1, ..., V_k$ linear subspaces of $\mathfrak{g}$ such that

$$\mathfrak{g} = V_1 \oplus ... \oplus V_k, \quad [V_1, V_i] = V_{i+1}, \quad V_k \ne \{0\}, \quad V_i = \{0\} \text{ if } i > k,$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the elements $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. It is well known that, identifying $\mathfrak{g}$ with $\mathbb{R}^n$ via the exponential map, it is possible to induce on $\mathbb{R}^n$ a family of automorphisms of the group, called *dilations*, $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ $(\lambda > 0)$ such that

$$\delta_\lambda(x_1, ..., x_n) = (\lambda^{\alpha_1} x_1, ..., \lambda^{\alpha_n} x_n),$$

where $1 = \alpha_1 = ... = \alpha_m < \alpha_{m+1} \le ... \le \alpha_n$ are integers and $m = \dim(V_1)$ (see [23, Chapter 1]). The integer $Q = \sum_{j=1}^n \alpha_j = \sum_{j=1}^k j \dim(V_j)$ is the *homogeneous dimension* of the group, which turns out to be the Hausdorff dimension of $(\mathbb{R}^n, d)$ (see [44]). The group law can be written in the form

$$x \cdot y = P(x, y) = x + y + Q(x, y), \quad x, y \in \mathbb{R}^n$$

where $P, Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ have polinomial components and $Q_1 = ... = Q_m = 0$ (see [55, Chapter 12, Sect. 5]). Note that the inverse of an element $x \in \mathbb{R}^n$ has the form $x^{-1} = (-x_1, ..., -x_m, i(x))$ with $i(x) \in \mathbb{R}^{n-m}$.

Let $X_1, ..., X_m$ form a basis of V_1 and let d be the C-C metric induced on $\mathbb{R}^n$ by them. The structure $\mathbb{G} = (\mathbb{R}^n, \cdot, \delta_\lambda, d)$ is said to be a Carnot group. Let us point out that C-C metrics induced by different bases of V_1 are equivalent (see [50, Sect. 1.3]).

The $n-$dimensional Lebesgue measure in $\mathbb{R}^n$ is the Haar measure of the group $\mathbb{G}$. This means that if $E \subset \mathbb{R}^n$ is measurable, then $|x \cdot E| = |E|$ for all $x \in \mathbb{G}$. Moreover, if $\lambda > 0$ then $|\delta_\lambda(E)| = \lambda^Q |E|$.

The C-C metric d is well behaved with respect to left translations and dilations. In fact one can prove that

$$d(z \cdot x, z \cdot y) = d(x, y), \quad d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x, y)$$

for $x, y, z \in \mathbb{G}$ and $\lambda > 0$.

The vector fields X_j have polinomial coefficients and can be assumed to be of the form

$$X_j(x) = \partial_j + \sum_{i=j+1}^{n} a_{ij}(x)\partial_i, \quad X_j(0) = \partial_j, \quad j = 1, ..., m,$$

where a_{ij} are polinomials such that $a_{ij}(\delta_\lambda(x)) = \lambda^{\alpha_i - \alpha_j} a_{ij}(x)$ (see [55, p.621]). Since $\alpha_1 = ... = \alpha_m = 1$ it follows that a_{ij} is homogeneous of degree zero for every $i = 1, ..., m$. Thus the condition $X_j(0) = \partial_j$ for $j = 1, ..., m$ implies that $a_{ij} = 0$ if $i = 1, ..., m$. Consequently, we can also write

$$X_j(x) = \partial_j + \sum_{i=m+1}^{n} a_{ij}(x)\partial_i, \quad X_j(0) = \partial_j, \quad j = 1, ..., m. \tag{2.10}$$

By the same argument $a_{ij}(x) = a_{ij}(x_1, ..., x_{i-1})$ and thus $X_j = -X_j^*$. We refer to (2.10) as the *canonical generating vector fields* of the group.

Lipschitz maps between Carnot groups are differentiable in a suitable sense. This is the content of a well known result due to Pansu (see [50] and see also [16] for more general metric spaces). Here we consider as target space the Carnot group $\mathbb{R}$.

Theorem 2.7. *Let $f : \mathbb{G} \to \mathbb{R}$ be a Lipschitz map. Then for a.e. $x \in \mathbb{G}$ there exists a $\mathbb{G}-$linear differential $Df(x) : \mathbb{G} \to \mathbb{R}$, i.e. $Df(x)$ is a group homomorphism and $Df(x)(\delta_\lambda(\xi)) = \lambda Df(x)(\xi)$ for all $\lambda > 0$ and $\xi \in \mathbb{G}$, and*

$$\lim_{y \to x} \frac{f(y) - f(x) - Df(x)(x^{-1} \cdot y)}{d(x, y)} = 0. \tag{2.11}$$

Note that Pansu's differential is explicitely given by the "directional" derivatives

$$Df(x)(\xi) = \lim_{t \to 0^+} \frac{f(x \cdot \delta_t(\xi)) - f(x)}{t}, \tag{2.12}$$

and the convergence is uniform in ξ.

3. Rademacher-Pansu type theorem and Eikonal equation for some Carnot-Carathéodory metrics

Let $(\mathbb{R}^n, d)$ be a C-C space induced by the vector fields $X_1, ..., X_m \in Lip(\mathbb{R}^n; \mathbb{R}^n)$ and suppose that the metric d satisfies (H1). If $K \subset \mathbb{R}^n$ is a closed set define $d_K(x) := \inf_{y \in K} d(x, y)$. Notice that if $(y_k)_{k \in \mathbb{N}}$ is a minimizing sequence for a fixed x, we can suppose that $y_k \in K \cap \overline{B(x, R)}$ for R large enough and for all $k \in \mathbb{N}$. Since $K \cap \overline{B(x, R)}$ is compact there exists a convergent subsequence. Thus $d_K(x) = \inf_{y \in K} d(x, y) = \min_{y \in K} d(x, y)$.

Since the function $d_K : (\mathbb{R}^n, d) \to \mathbb{R}$ is $1-$Lipschitz, Theorem 2.5 implies that the derivatives $X_j d_K$, $j = 1, ..., m$, exist a.e. and $|X d_K(x)| \le 1$ for a.e. $x \in \mathbb{R}^n$. We shall prove for some C-C metrics d that the dinstance function d_K actually verifies an eikonal equation with respect to the $X-$gradient

$$|X d_K(x)| = 1, \quad \text{a.e. } x \in \mathbb{R}^n \setminus K. \tag{3.1}$$

It is well known that Euclidean and Riemannian metrics satisfy (3.1) (see for instance [6] and [42]). We are able to prove (3.1) in the following three cases.

Case A. $X_1, ..., X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$, $m < n$, *satisfy the Chow-Hörmander condition (2.6) and are of the form*

$$X_j = \partial_j + \sum_{i=m+1}^{n} a_{ij}(x) \partial_i, \quad j = 1, ..., m, \tag{3.2}$$

where $a_{ij} \in C^\infty(\mathbb{R}^n)$.

Case B. $X_1, ..., X_n \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ *are of the form*

$$X_1 = \partial_1, \quad X_2 = p_2(x_1)\partial_2, \quad ... \quad X_n = p_n(x_1, ..., x_{n-1})\partial_n, \tag{3.3}$$

where $p_j \in C^\infty(\mathbb{R}^{j-1})$, $j = 2, ..., n$, *are functions vanishing on a set of null* $(j-1)-$*dimensional Lebesgue measure.*

Case C. $X_1, ..., X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ *and* $\text{span}\{X_1(x), ..., X_m(x)\} = \mathbb{R}^n$ *for every* $x \in \mathbb{R}^n$.

Vector fields in Case A may be called "of Carnot type". This expression is motivated by the analogy with the canonical generating vector fields of a Carnot group (see (2.10)). For vector fields of Carnot type we are able to prove a differentiation theorem that generalizes Pansu differentiation Theorem 2.7. We give two simple examples of vector fields of Carnot type.

Example A1: Heisenberg group. The most important example of case A are Carnot groups whose definition was recalled in the previous section. Here we introduce the most simple group of this class: the Heisenberg group (see [55] for a general introduction). In $\mathbb{R}^{2n+1}$ consider the vector fields

$$X_j = \partial_{x_j} + 2y_j\partial_t \text{ and } Y_j = \partial_{y_j} - 2x_j\partial_t, \quad j = 1, ..., n, \tag{3.4}$$

where $(z,t) = (x,y,t) \in \mathbb{C}^n \times \mathbb{R} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. Since $[X_j, Y_j] = -4\partial_t$ the Chow-Hörmander condition (2.6) is satisfied and the C-C dinstance d satisfies (H1). The group law is

$$(\zeta, \tau) \cdot (z,t) := (\zeta + z, \tau + t + 2\,\mathrm{Im}(\zeta\bar{z})).$$

Dilations are given by $\delta_\lambda(z,t) := (\lambda z, \lambda^2 t)$.

The structure $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \cdot, \delta_\lambda, d)$ is the Heisenberg group, whose homogenous dimension is $Q = 2n+2$. The Lie algebra of $\mathbb{H}^n$ is generated by the canonical vector fields X_j, Y_j. The intrinsic $X-$gradient is $X = \nabla_{\mathbb{H}} = (X_1, ..., X_n, Y_1, ..., Y_n)$.

Example A2. Consider in $\mathbb{R}^3$ the vector fields

$$X = \partial_x + y^2\partial_z, \;\; Y = \partial_y.$$

Since $[X,Y] = -2y\partial_z$ and $[Y,[X,Y]] = -2\partial_z$ the Chow-Hörmander condition (2.6) is verified. Thus the C-C metric d induced by X and Y satisfies (H1). The C-C space $(\mathbb{R}^3, d)$ is not a group and it has not a uniform homogeneous dimension. Still we are in Case A.

Case B, which is inspired by [26], generalizes the Grushin vector fields $X = \partial_x$ and $Y = x\partial_y$ in $\mathbb{R}^2$. Notice that every couple of points in $\mathbb{R}^n$ can be connected by polygonals that are piecewise integral curves of the vector fields $X_1, ..., X_n$. Moreover the C-C dinstance induced by them satisfies (H1) (see for instance [26]).

We now state the main result of this section.

Theorem 3.1. *Let $(\mathbb{R}^n, d)$ be a C-C complete space induced by $X_1, ..., X_m$, and suppose that the vector fields satisfy one of the cases A, B or C. Let $K \subset \mathbb{R}^n$ be a closed set and let d_K be the dinstance from K. Then*

$$|X d_K(x)| = 1 \tag{3.5}$$

for a.e. $x \in \mathbb{R}^n \setminus K$.

Our task is to prove equation (3.5) but first we need some preliminary lemmas. We begin by a Rademacher-Pansu type Theorem (recall 2.11).

Theorem 3.2. *Let* $X_1, ..., X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ *be as in Case A and let* $f \in Lip(\mathbb{R}^n, d)$*. Then for a.e.* $x \in \mathbb{R}^n$

$$\lim_{y\to x} \frac{f(y) - f(x) - \sum_{j=1}^m X_j f(x)(y_j - x_j)}{d(x,y)} = 0. \tag{3.6}$$

Proof. The proof follows an idea of Calderón (see [14] and for example [21, Theorem 6.1]). The derivatives $X_j f(x)$, $j = 1, ..., m$, exist a.e. by Theorem 2.5. Moreover $|Xf| \in L^p_{loc}(\mathbb{R}^n)$ for all $p \geq 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with homogeneous dimension Q and fix $p > Q$. By Lebesgue differentiation Theorem in spaces of homogeneous type we have for a.e. $x \in \Omega$

$$\lim_{r\to 0^+} \fint_{B(x,r)} |Xf(x) - Xf(y)|^p \, dy = 0. \tag{3.7}$$

Fix $x \in \Omega$ such that $|Xf(x)| < \infty$ and (3.7) holds, define

$$g(y) = f(y) - \sum_{j=1}^m X_j f(x)(x_j - y_j),$$

and notice that $X_j g(y) = X_j f(y) - X_j f(x)$ (recall (3.2)). By Morrey's inequality (2.9) we get

$$|g(y) - g(x)| \leq Cr\Big(\fint_{B(x,r)} |Xg(z)|^p \, dz\Big)^{\frac{1}{p}} \quad \text{for all } y \in B,$$

and choose $r = 2d(x,y)$ to find

$$\frac{|f(y) - f(x) - \sum_{j=1}^m X_j f(x)(x_j - y_j)|}{d(x,y)} \leq 2C\Big(\fint_{B(x,2d(x,y))} |X_j f(z) - X_j f(x)|^p \, dz\Big)^{\frac{1}{p}}.$$

The last term tends to zero as $d(x,y) \to 0$ owing to (3.7). □

Remark 3.3. If $\mathbb{G} = (\mathbb{R}^n, \cdot, \delta_\lambda, d)$ is a Carnot group, and $f : \mathbb{G} \to \mathbb{R}$ is a Lipschitz map, then Pansu's differential $Df(x) : \mathbb{G} \to \mathbb{R}$ defined in (2.11) has the explicit representation

$$Df(x)(\xi) = \sum_{j=1}^m \xi_j X_j f(x), \tag{3.8}$$

for a.e. $x \in \mathbb{G}$ and for all $\xi \in \mathbb{G}$.

Thus Theorem 3.2 truly generalizes Pansu Theorem 2.7 to vector fields of Carnot type. We prove (3.8). Consider first the case $f \in C_0^\infty(\mathbb{R}^n)$. Let $P(x,y) = x \cdot y$ be the group law, and notice that if $j = 1, ..., m$ then

$$X_j f(x) = \langle \nabla f(x), \frac{\partial}{\partial y_j} P(x,y)\Big|_{y=0} \rangle.$$

Now recall the characterization of Pansu differential (2.12)

$$Df(x)(\xi) = \lim_{t \to 0} \frac{f(x \cdot \delta_t(\xi)) - f(x)}{t} = \frac{d}{dt} f(x \cdot \delta_t(\xi))\Big|_{t=0}.$$

Compute

$$\begin{aligned} \frac{d}{dt} P(x, \delta_t(\xi))\Big|_{t=0} &= \langle \nabla_y P(x,y)\Big|_{y=0}, \frac{d}{dt}\delta_t(\xi)\Big|_{t=0} \rangle \\ &= \langle \nabla_y P(x,y)\Big|_{y=0}, (\xi_1, ..., \xi_m, 0, ..., 0) \rangle. \end{aligned}$$

In fact, the coordinate $(\frac{d}{dt}\delta_t(\xi))_i$ contains t^α with $\alpha \geq 1$ if $i = m+1, ..., n$. Finally we get

$$\begin{aligned} Df(x)(\xi) &= \langle \nabla f(x), \nabla_y P(x,y)\Big|_{y=0} (\xi_1, ..., \xi_m, 0, ..., 0) \rangle \\ &= \sum_{j=1}^m \xi_j X_j f(x). \end{aligned}$$

If $f \in Lip(\mathbb{G})$, take $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$. By the dominated convergence theorem we can write

$$\begin{aligned} \int_{\mathbb{R}^n} Df(x)(\xi)\varphi(x)\, dx &= \int_{\mathbb{R}^n} \lim_{t \to 0^+} \frac{f(x \cdot \delta_t(\xi)) - f(x)}{t} \varphi(x)\, dx \\ &= \lim_{t \to 0^+} \int_{\mathbb{R}^n} \frac{f(x \cdot \delta_t(\xi)) - f(x)}{t} \varphi(x)\, dx. \end{aligned}$$

In fact, if $L = Lip(f)$, since the metric d is left-invariant and homogeneous

$$\Big| \frac{f(x \cdot \delta_t(\xi)) - f(x)}{t} \Big| \leq \frac{Ld(x \cdot \delta_t(\xi), x)}{t} = L \frac{d(\delta_t(\xi), 0)}{t} = Ld(\xi, 0).$$

Now recall that Lebesgue measure is left and right invariant and perform a change of variable to find

$$\int_{\mathbb{R}^n} \frac{f(x \cdot \delta_t(\xi)) - f(x)}{t} \varphi(x)\, dx = \int_{\mathbb{R}^n} f(x) \frac{\varphi(x \cdot (\delta_t(\xi))^{-1}) - \varphi(x)}{t}\, dx.$$

Recalling that $(\delta_t(\xi))^{-1} = \delta_t(\xi^{-1})$ and $\xi^{-1} = (-\xi_1, ..., -\xi_m, i(\xi))$ we infer as above that

$$\lim_{t\to 0} \frac{\varphi(x \cdot (\delta_t(\xi))^{-1}) - \varphi(x)}{t} = -\sum_{j=1}^{m} \xi_j X_j \varphi(x),$$

and integrating by parts we get

$$\begin{aligned}\int_{\mathbb{R}^n} Df(x)(\xi)\varphi(x)\,dx &= -\int_{\mathbb{R}^n} f(x) \sum_{j=1}^{m} \xi_j X_j \varphi(x)\,dx \\ &= \int_{\mathbb{R}^n} \varphi(x) \sum_{j=1}^{m} \xi_j X_j f(x)\,dx,\end{aligned}$$

as every X_j is self-adjoint. □

Lemma 3.4. *Let $(\mathbb{R}^n, d)$ be a C-C space corresponding to case B. For a.e. $x \in \mathbb{R}^n$ every geodesic starting from x is of class C^1 in a neighborhood of x.*

Proof. See for example [24]. Let $Z_j = \{y \in \mathbb{R}^{j-1} : p_j(y) = 0\}$ if $j = 2, ..., n$ and set $A = \bigcup_{j=2}^{n} Z_j \times \mathbb{R}^{n-j+1}$. Then $|A| = 0$. Choose $x \in \mathbb{R}^n \setminus A$ and take $r > 0$ such that $B(x, r) \subset \mathbb{R}^n \setminus A$ (A is closed).

If $X_1, ..., X_n \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ the metric d is actually a Riemannian metric on $\mathbb{R}^n \setminus A$. Thus geodesics are locally regular (of class C^1) in $\mathbb{R}^n \setminus A$. □

Proof of Theorem 3.1. Case A. Since d_K is the lower envelope of 1-Lipschitz functions, then d_K is $1-$Lipschitz. By Theorem 2.5 the derivatives $X_j d_K(x)$, $j = 1, ..., m$, exist for a.e. $x \in \mathbb{R}^n$ and

$$|X d_K(x)| \le 1 \tag{3.9}$$

for a.e. $x \in \mathbb{R}^n$.

Fix $x \in \mathbb{R}^n \setminus K$ such that (3.9) holds. Since the function $\xi \to d(x, \xi)$ is continuous and coercive, it has minimum in K. Thus there exists $\bar{\xi} \in K$ such that $d(x, \bar{\xi}) = d_K(x) := T > 0$. There exists a geodesic $\gamma \in Lip([0, T]; \mathbb{R}^n)$ such that $\gamma(0) = x$ and $\gamma(T) = \bar{\xi}$. By Proposition 2.4 we have

$$d_K(\gamma(t)) = d(\gamma(t), \bar{\xi}) = T - t$$

for all $t \in [0, T]$.

Since the function $f = d_K$ is Lipschitz, by Theorem 3.2 we have for a.e. $x \in \mathbb{R}^n$

$$\lim_{y\to x} \frac{f(y) - f(x) - \sum_{j=1}^{m} X_j f(x)(y_j - x_j)}{d(x, y)} = 0. \tag{3.10}$$

Fix $x \in \mathbb{R}^n \setminus K$ such that (3.9) and (3.10) hold. As $t = d(\gamma(t), \gamma(0))$, we can write

(3.11)
$$\frac{f(\gamma(t)) - f(\gamma(0))}{d(\gamma(t), \gamma(0))} = \frac{d_K(\gamma(t)) - d_K(\gamma(0))}{t} = \frac{(T-t) - T}{t} = -1,$$

and from (3.10) it follows

$$\frac{f(\gamma(t)) - f(\gamma(0))}{t} = \sum_{j=1}^{m} X_j f(x) \frac{\gamma_j(t) - \gamma_j(0)}{t} + o(1). \tag{3.12}$$

Let $\bar{\gamma} := (\gamma_1, ..., \gamma_m)$. Since

$$\dot{\gamma}(t) = \sum_{j=1}^{m} h_j(t) X_j(\gamma(t)), \qquad \text{for a.e. } t \in [0, T]$$

with $h = (h_1, ..., h_m)$ measurable coefficients such that $|h(t)| \leq 1$ a.e., from the special form of the vector fields (3.2) it follows that

$$\bar{\gamma}(t) = \bar{\gamma}(0) + \int_0^{\mathrm{t}} h(s)\, ds,$$

and thus $|\bar{\gamma}(t) - \bar{\gamma}(0)| \leq t$ for all $t \in [0, T]$. Thus if $t \in (0, T]$

$$\sum_{j=1}^{m} \Big| \frac{\gamma_j(t) - \gamma_j(0)}{t} \Big|^2 \leq 1,$$

and therefore, recalling (3.11) and (3.12)

$$1 = \Big| \sum_{j=1}^{m} X_j f(x) \frac{\gamma_j(t) - \gamma_j(0)}{t} + o(1) \Big| \leq |Xf(x)| + |o(1)|$$

for all $t \in (0, T)$. This proves that $|Xf(x)| \geq 1$. This inequality and the converse one (3.9) prove that $|Xf(x)| = 1$ for a.e. $x \in \mathbb{R}^n \setminus K$.

We now consider case B. Since d_K is $1-$Lipschitz, Theorem 2.5 implies, as above, that $|Xd_K(x)| \leq 1$ a.e., and moreover every $X_j d_K(x)$ exists as a $L^\infty_{loc}(\mathbb{R}^n \setminus A)$ function. But $|p_j|$ is locally strictly positive on $\mathbb{R}^n \setminus A$ and $p_j = p_j(x_1, ..., x_{j-1})$. Thus $\partial_j d_K \in L^\infty_{loc}(\mathbb{R}^n \setminus A)$ exist for $j = 1, ..., m$. By Rademacher Theorem it follows that d_K is differentiable almost everywhere on $\mathbb{R}^n$.

Now fix $x \in \mathbb{R}^n \setminus (K \cup A)$ such that $|Xd_K(x)| \leq 1$ and d_K is differentiable at x. Choose $\bar{x} \in K$ such that $d(x, \bar{x}) = d_K(x) := T$. There exists a geodesic $\gamma \in Lip([0, T]; \mathbb{R}^n) \cap C^1([T - \delta, T]; \mathbb{R}^n)$ such that

$\gamma(0) = (\bar{x})$ and $\gamma(T) = x$ for $\delta > 0$ small enough. Moreover there exist $h_1, ..., h_m \in C([T-\delta, T]; \mathbb{R}^n)$ for which

$$\dot{\gamma}(t) = \sum_{j=1}^{n} h_j(t) X_j(\gamma(t)), \quad \sum_{j=1}^{n} h_j^2 \leq 1$$

for all $t \in [T-\delta, T]$ (see for example [48]). By Proposition 2.4 we have $d_K(\gamma(t)) = t$, and we can differentiate this identity in $t = T$ to find

$$1 = \langle \nabla d_K(x), \dot{\gamma}(T) \rangle = \langle X d_K(x), h \rangle \leq |X d_K(x)|.$$

Finally we consider case C. If A is the matrix in (2.1) then $\operatorname{rank} A(x) = n$ for every $x \in \mathbb{R}^n$. Thus Chow-Hörmander's condition is satisfied and the C-C dinstance verifies (H1). Moreover it is well known that the metric space $(\mathbb{R}^n, d)$ is a Riemannian manifold and geodesics are (at least locally) regular curves (see, for instance, [39]). The function d_K is Lipschitz in the Euclidean sense, and thus differentiable almost everywhere. We can perform the same computations as in the previous case to show that $|X d_K(x)| = 1$ for a.e. $x \in \mathbb{R}^n \setminus K$ (see also [42]). □

4. Coarea formula in Carnot-Carathéodory spaces

In this section we slightly improve a coarea formula for vector fields. Let $\Omega \subset \mathbb{R}^n$ be an open set. If $f \in BV_X(\Omega)$ then $\mathbb{R} \ni t \to \|\partial E_t\|_X(\Omega)$ is $\mathcal{L}^1-$measurable and

$$\|Xf\|(\Omega) = \int_{-\infty}^{+\infty} \|\partial E_t\|_X(\Omega) dt, \tag{4.1}$$

where $E_t := \{x \in \Omega : f(x) > t\}$ are the level sets of the function f, and $\|Xf\|$ and $\|\partial E_t\|_X$ are the measures defined in (2.2) and (2.3). For the proof see [27, Theorem 2.3.5], [30, Theorem 5.2] and see also [29, Proposition 3.1] and [45].

Remark 4.1. Let $X_1, ..., X_m \in Lip(\mathbb{R}^n; \mathbb{R}^n)$, and let $E \subset \mathbb{R}^n$ be a X-Caccioppoli set. It can easily checked that $\|\partial E\|_X(\mathbb{R}^n \setminus \partial E) = 0$ and $\|\partial E\|_X = \|\partial(\mathbb{R}^n \setminus E)\|_X$.

Theorem 4.2. *Suppose that the vector fields $X_1, ..., X_m \in Lip(\mathbb{R}^n; \mathbb{R}^n)$ satisfy (H1). Let $f \in Lip(\mathbb{R}^n, d)$ and let $u : \mathbb{R}^n \to [0, +\infty]$ be $\mathcal{L}^n$-measurable. Then*

$$\int_{\mathbb{R}^n} u(x)|Xf(x)| dx = \int_{-\infty}^{+\infty} \Big(\int_{\{f=t\}} u \, d\mu_t \Big) dt, \tag{4.2}$$

where $\mu_t = \|\partial E_t\|_X$ is the perimeter measure of the level set E_t of f.

Proof. We shall divide the proof in two steps.

Step 1. Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $f \in BV_{X,loc}(\Omega)$ and let $B \subset \Omega$ be a Borel set, or alternatively let $f \in H^{1,1}_{X,loc}(\Omega)$ and let $B \subset \Omega$ be $\mathcal{L}^n-$measurable. In either cases the function $\mathbb{R} \ni t \to \|\partial E_t\|_X(B)$ is $\mathcal{L}^1-$measurable and

$$\|Xf\|(B) = \int_{-\infty}^{+\infty} \|\partial E_t\|_X(B)dt. \tag{4.3}$$

It is not restrictive to assume $f \in BV_X(\Omega)$ and, moreover, because of (4.1) we can assume that $\|\partial E_t\|_X$ is a Radon measure in Ω for $\mathcal{L}^1-$a.e. $t \in \mathbb{R}$, or, for the sake of semplicity, for every $t \in \mathbb{R}$. Let us denote by $\mathcal{M}_t(\Omega)$ the class of the $\|\partial E_t\|_X-$measurable sets $B \subset \Omega$ such that $\mathbb{R} \ni t \to \|\partial E_t\|_X(B)$ is $\mathcal{L}^1-$measurable, and let $\mathcal{M} = \bigcap_{t\in\mathbb{R}} \mathcal{M}_t(\Omega)$. We show that $\mathcal{M}$ contains the $\sigma-$algebra $\mathcal{B}(\Omega)$ of the Borel sets of Ω, thus proving the first statement. By a classical result of measure theory on monotone classes (see for instance [3, Remark 1.9]) it is sufficient to prove that

(i) $\mathcal{M}$ contains the open sets of Ω;
(ii) if $(B_h)_{h\in\mathbb{N}} \subset \mathcal{M}$ is an increasing sequence then $\bigcup_{h=1}^{\infty} B_h \in \mathcal{M}$;
(iii) if $B \in \mathcal{M}$ then $\Omega \setminus B \in \mathcal{M}$;
(iv) if B, C, $B \cup C \in \mathcal{M}$ then $B \cap C \in \mathcal{M}$.

It is easy to see that conditions (i)-(iv) follows from the properties of measurable functions and of the Radon measures $\|\partial E_t\|_X$.

The set function $\nu : \mathcal{B}(\Omega) \to [0 + \infty]$ defined by

$$\nu(B) = \int_{-\infty}^{+\infty} \|\partial E_t\|_X(B)\, dt$$

is a (Radon) measure on the $\sigma-$algebra $\mathcal{B}(\Omega)$ and $\nu(A) = \|Xf\|(A)$ for every open set $A \subset \Omega$. The coincidence criterion for measures in [3, Proposition 1.8] implies that $\nu(B) = \|Xf\|(B)$ for all $B \in \mathcal{B}(\Omega)$. This proves (4.3).

If $f \in H^{1,1}_{X,loc}$ and $B \subset \Omega$ is a $\mathcal{L}^n-$measurable set, then $\|Xf\| \ll \mathcal{L}^n$ and there exist two Borel sets $F, A \subset \Omega$ such that $F \subset B \subset A$ and $\|Xf\|(A \setminus F) = 0$. By (4.3) it follows that $\|\partial E_t\|_X(A \setminus F) = 0$ for $\mathcal{L}^1-$a.e. $t \in \mathbb{R}$, and thus $\mathbb{R} \ni t \to \|\partial E_t\|_X(B) = \|\partial E_t\|_X(A)$ is $\mathcal{L}^1-$measurable and (4.3) holds for any $\mathcal{L}^n-$measurable set $B \subset \Omega$.

Step 2. Since $f \in Lip(\mathbb{R}^n, d)$, by Theorem 2.5 the gradient $Xf(x)$ is defined almost everywhere, and $|Xf(x)| \le Lip(f)$. In particular $f \in H^{1,1}_{X,loc}(\mathbb{R}^n)$. Since u is measurable we can write $u(x) = \sum_{k=1}^{\infty} 1/k\chi_{A_k}(x)$ with $A_k \subset$

$\mathbb{R}^n$ measurable with finite measure (see [21, Theorem 1.1.7]). By the monotone convergence theorem and Step 1

$$\int_{\mathbb{R}^n} u(x)|Xf(x)|dx = \sum_{k=1}^{\infty} \frac{1}{k}\|Xf\|(A_k) = \sum_{k=1}^{\infty} \frac{1}{k} \int_{-\infty}^{+\infty} \|\partial E_t\|_X(A_k)dt$$
$$= \int_{-\infty}^{+\infty} \Big(\int_{\mathbb{R}^n} u(x)d\|\partial E_t\|_X \Big) dt.$$

Since $f \in Lip(\mathbb{R}^n, d)$ and since d is continuous with respect to the Euclidean topology, then f is continuous. It follows that $\partial\{x \in \mathbb{R}^n : f(x) > t\} \subset \{x \in \mathbb{R}^n : f(x) = t\}$. Thus Remark 4.1 implies that the support of the measure $\mu_t = \|\partial E_t\|_X$ is contained in $\{x \in \mathbb{R}^n : f(x) = t\}$. □

Remark 4.3. Let us observe that if the vector fields $X_1, ..., X_m$ satisfy condition (H1) then

(4.4)
$$\partial B(x_0, r) = \{x \in \mathbb{R}^n : d(x_0, r) = r\} = \partial\{x \in \mathbb{R}^n : d(x_0, r) > r\}$$

for every $x_0 \in \mathbb{R}^n$, $r > 0$. Indeed, (4.4) follows from the continuity of the metric d and the geodesic segment property of $(\mathbb{R}^n, d)$. Then, taking $u = \chi_{B(x_0,R)}$ and $f(x) = d(x, x_0)$ for fixed $x_0 \in \mathbb{R}^n$ and $R > 0$, we get from Theorem 4.2 that $B(x_0, r)$ has finite $X-$perimeter in $\mathbb{R}^n$ for a.e. $r > 0$. In particular (see Lemma 4.5) if $X_1, ..., X_m$ induce a structure of Carnot group on $\mathbb{R}^n$, then $B(x_0, r)$ has finite $X-$perimeter in $\mathbb{R}^n$ for every $r > 0$.

Corollary 4.4. *Let $X_1, ..., X_m$ be one of Cases A, B or C, and let d be the C-C metric induced by them. If $u \in L^1(\mathbb{R}^n)$ then*

(4.5)
$$\int_{\mathbb{R}^n} u(x)\, dx = \int_0^{+\infty} \Big(\int_{\partial B(0,r)} u(x) d\mu_r \Big) dr,$$

where $\partial B(0, r) = \{x \in \mathbb{R}^n : d(x, 0) = r\}$ and $\mu_r = \|\partial B(0, r)\|_X$.

Proof. By Theorem 3.1 we have $|Xd(x, 0)| = 1$ a.e. $x \in \mathbb{R}^n$, by Remark 4.1 and by (4.4) we can apply formula (4.2). □

Lemma 4.5. *Let $\mathbb{G} = (\mathbb{R}^n, \cdot, \delta_\lambda, d)$ be a Carnot group with canonical generating vector fields $X_1, ..., X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ and homogeneous dimension Q. Let $E \subset \mathbb{G}$ be measurable. Then*

$$\|\partial\delta_\lambda(E)\|_X(\delta_\lambda(A)) = \lambda^{Q-1}\|\partial E\|_X(A),$$

for $\lambda > 0$ and for every Borel set $A \subset \mathbb{R}^n$.

Proof. We prove first that, if $\psi \in C^1(\mathbb{R}^n)$, then

$$X_j(\psi \circ \delta_\lambda)(x) = \lambda(X_j\psi)(\delta_\lambda(x)), \tag{4.6}$$

for $j = 1, ..., m$ and $\lambda > 0$. Recall that $\delta_\lambda(x) = (\lambda^{\alpha_1} x_1, ..., \lambda^{\alpha_n} x_n)$ where $\alpha_1 = ... = \alpha_m = 1$ and $\alpha_{m+1}, ..., \alpha_n$ are integers greater or equal than 2. The vector fields are of the form $X_j(x) = \partial_j + \sum_{i=m+1}^n a_{ij}(x)\partial_i$, where $a_{ij}(\delta_\lambda(x)) = \lambda^{\alpha_i - 1} a_{ij}(x)$. Thus

$$\begin{aligned} X_j(\psi \circ \delta_\lambda)(x) &= \partial_j(\psi \circ \delta_\lambda)(x) + \sum_{i=m+1}^n a_{ij}(x)\partial_i(\psi \circ \delta_\lambda)(x) \\ &= \lambda\Big[\partial_j\psi(\delta_\lambda(x)) + \sum_{i=m+1}^n \lambda^{\alpha_i-1} a_{ij}(x)\partial_i\psi(\delta_\lambda(x))\Big] \\ &= \lambda(X_j\psi)(\delta_\lambda(x)) \end{aligned}$$

Without loss of generality we can assume A open. Take $\varphi \in F(\delta_\lambda(A); \mathbb{R}^m)$. Since the determinant of the jacobian of $\delta_\lambda(x)$ is λ^Q and $X_j^* = -X_j$, we can write

$$\int_{\delta_\lambda(E)\cap\delta_\lambda(A)} \mathrm{div}_X(\varphi)(x)\,dx = \lambda^Q \int_{E\cap A} \mathrm{div}_X(\varphi)(\delta_\lambda(x))\,dx =$$

$$= \lambda^Q \int_{E\cap A} \sum_{j=1}^m (X_j\varphi_j)(\delta_\lambda(x))\,dx = \lambda^{Q-1} \int_{E\cap A} \sum_{j=1}^m X_j(\varphi_j \circ \delta_\lambda)(x)\,dx.$$

Since $\varphi \circ \delta_\lambda \in F(A; \mathbb{R}^m)$ it immediately follows that

$$\|\partial\delta_\lambda(E)\|_X(\delta_\lambda(A)) \le \lambda^{Q-1}\|\partial E\|_X(A).$$

The converse inequality can be proved in the same way. □

Corollary 4.6. *Let* $\mathbb{G} = (\mathbb{R}^n, \cdot, \delta_\lambda, d)$ *be a Carnot group with canonical generating vector fileds* $X_1, ..., X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ *and homogeneous dimension* Q*. If* $u \in L^1(\mathbb{R}^n)$ *then*

$$\int_{\mathbb{R}^n} u(x)\,dx = \int_0^{+\infty} \Big(\int_{\partial B(0,1)} u(\delta_r(x)) r^{Q-1} d\mu\Big) dr, \tag{4.7}$$

where $\mu = \|\partial B(0,1)\|_X$.

Remark 4.7. Formula (4.7) gives an explicit representation of the (unique) surface measure whose existence for Carnot groups was proved in [23, Proposition 1.15].

Remark 4.8. It would be interesting to see whether in formula (4.2) the perimeter measure $\|\partial E_t\|_X$ could be replaced by the spherical Hausdorff measure $\mathcal{S}_d^{Q-1}$ (constructed according to Federer definition using the C-C dinstance d) in the case of a Carnot group of homogeneous dimension Q. However one can not always expect this replacement in a general C-C space. The reason is that in C-C spaces which are not Carnot groups the local Hausdorff dimension may change at different points of the space. Analogously, the boundary of a regular open set needs not have a uniform Hausdorff dimension.

For instance, consider in $\mathbb{R}^3$ the C-C metric d induced by the vector fields

$$X = \partial_x + 2y\partial_t, \quad Y = \partial_y - 2x\partial_t, \quad T = a(t)\partial_t,$$

where $a \in C^\infty(\mathbb{R})$ is $a(t) = \mathrm{e}^{-\frac{1}{t}}$ if $t > 0$ and $a(t) = 0$ if $t \le 0$. Thus, in the half space $\{t > 0\}$ X, Y and T are linearly independent and the Hausdorff dimension is 3, whereas in the half space $\{t < 0\}$ we have exactly the Heisenberg vector fields of Example A1 and thus the Hausdorff dimension here is 4. Now, let $\Omega \subset \mathbb{R}^3$ be an open set with C^∞ boundary. The set $\partial\Omega \cap \{t > 0\}$ has Hausdorff dimension 2, while the set $\partial\Omega \cap \{t < 0\}$ has Hausdorff dimension 3 (see [29, Corollary 7.7]). This shows that the right dimension for the Hausdorff measure to integrate over regular surfaces is different at different points of the space. The perimeter measure takes automatically account of such change of dimension.

5. Minkowski content and perimeter in Carnot-Carathéodory spaces

Let $E \subset \mathbb{R}^n$ be a bounded open set, and fix on $\mathbb{R}^n$ a C-C metric d induced by the vector fields $X_1, ..., X_m \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$. Let $d_{\partial E}(x) = \min_{y\in\partial E} d(x,y)$, and for $r > 0$ define the tubular neighborhood $I_r(\partial E) = \{x \in \mathbb{R}^n : d_{\partial E}(x) < r\}$. The *upper* and *lower Minkowski content* of ∂E in an open set $\Omega \subset \mathbb{R}^n$ are respectively

$$M^+(\partial E)(\Omega) := \limsup_{r\to 0^+} \frac{|I_r(\partial E) \cap \Omega|}{2r},$$
$$M^-(\partial E)(\Omega) := \liminf_{r\to 0^+} \frac{|I_r(\partial E) \cap \Omega|}{2r}.$$

In this section we prove that if E is regular and Ω has regular boundary then $M^+(\partial E)(\Omega) = M^-(\partial E)(\Omega)$, and this common value, which we shall call *Minkowski content* of ∂E in Ω and denote by $M(\partial E)(\Omega)$, coincides with the $X-$perimeter of E in Ω as defined in (2.3). The proof is based on a Riemmanian approximation of the C-C space $(\mathbb{R}^n, d)$. In this section and in the next one $\mathcal{H}^{n-1}$ stands for the $(n-1)-$dimensional Euclidean Hausdorff measure.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set with C^∞ boundary or $\Omega = \mathbb{R}^n$. Let $E \subset \mathbb{R}^n$ be a bounded open set with C^∞ boundary and suppose that $\mathcal{H}^{n-1}(\partial E \cap \partial \Omega) = 0$. Then $M^+(\partial E)(\Omega) = M^-(\partial E)(\Omega)$ and moreover*

$$M(\partial E)(\Omega) = \|\partial E\|_X(\Omega).$$

Proof. We prove separately that

$$M^-(\partial E)(\Omega) \geq \|\partial E\|_X(\Omega), \tag{5.1}$$

$$M^+(\partial E)(\Omega) \leq \|\partial E\|_X(\Omega). \tag{5.2}$$

The former statement follows from the lower semicontinuity of the perimeter. The latter one requires the Riemannian approximation.

Define

$$\varrho(x) = \begin{cases} d_{\partial E}(x) & \text{if } x \in E \\ -d_{\partial E}(x) & \text{if } x \in \mathbb{R}^n \setminus E. \end{cases}$$

Now, for $\varepsilon > 0$ define the function

$$\varphi_\varepsilon(x) = \begin{cases} \frac{1}{2\varepsilon}\varrho(x) + \frac{1}{2} & \text{if } |\varrho(x)| < \varepsilon \\ 1 & \text{if } \varrho(x) \geq \varepsilon \\ 0 & \text{if } \varrho(x) \leq -\varepsilon. \end{cases}$$

Since $|X\varrho| \leq 1$ a.e.

$$\begin{aligned} \|X\varphi_\varepsilon\|(\Omega) &= \frac{1}{2\varepsilon}\int_{\Omega\cap\{|\varrho(x)|<\varepsilon\}} |X\varrho(x)|dx \\ &\leq \frac{1}{2\varepsilon}|\{x \in \Omega : |\varrho(x)| < \varepsilon\}| = \frac{|I_\varepsilon(\partial E) \cap \Omega|}{2\varepsilon}. \end{aligned}$$

The total variation is lower semicontinuous (Proposition 2.1) and $\varphi_\varepsilon \to \chi_E$ in $L^1(\Omega)$, thus

$$\|\partial E\|_X(\Omega) \leq \liminf_{\varepsilon\to 0^+} \|X\varphi_\varepsilon\|(\Omega) \leq M^-(\partial E)(\Omega).$$

This proves (5.1). We shall now prove (5.2). Let A be the matrix $n \times m$ of the coefficients of the vector fields X_j defined in (2.1) and let $C = A^{\mathrm{T}}$. For $\varepsilon > 0$ consider the new family of vector fields $X_\varepsilon = (X_1, ..., X_m, \varepsilon\partial_1, ..., \varepsilon\partial_n)$, which generates a C-C metric d_ε. Define analogously the matrices A_ε and C_ε. One can prove the following statements:

(i) $d_\varepsilon(x, y) \leq d(x, y)$ for all x, y and in fact $d(x, y) = \sup_{\varepsilon>0} d_\varepsilon(x, y)$.

(ii) Let $d_{\varepsilon,\partial E}(x) = \min_{y\in\partial E} d_\varepsilon(x,y)$. The function

$$\varrho_\varepsilon(x) = \begin{cases} d_{\varepsilon,\partial E}(x) & \text{if } x \in E \\ -d_{\varepsilon,\partial E}(x) & \text{if } x \in \mathbb{R}^n \setminus E \end{cases}$$

is of class C^∞ in a neighborhood of ∂E, and $|X_\varepsilon \varrho_\varepsilon(x)| = 1$ in this neighborhood.

A proof of (i) can be found in [38], while statement (ii) relies on classical results in Riemannian geometry and on Theorem 3.1, case C.

Now define the upper and lower Minkowski content

$$M_\varepsilon^+(\partial E)(\Omega) := \limsup_{r\to 0^+} \frac{|\{x \in \Omega : |\varrho_\varepsilon(x)| < r\}}{2r},$$

$$M_\varepsilon^-(\partial E)(\Omega) := \liminf_{r\to 0^+} \frac{|\{x \in \Omega : |\varrho_\varepsilon(x)| < r\}}{2r}.$$

By (i) we have that $|\varrho_\varepsilon| \le |\varrho|$ and thus $\{x \in \Omega : |\varrho(x)| < r\} \subset \{x \in \Omega : |\varrho_\varepsilon(x)| < r\}$. It follows that

$$M^+(\partial E)(\Omega) \le M_\varepsilon^+(\partial E)(\Omega). \tag{5.3}$$

We shall soon prove that

$$M_\varepsilon^+(\partial E)(\Omega) = M_\varepsilon^-(\partial E)(\Omega) = \|\partial E\|_{X_\varepsilon}(\Omega). \tag{5.4}$$

Recalling the representation for the perimeter in Proposition 2.2 we find

$$\begin{aligned} \lim_{\varepsilon\to 0} \|\partial E\|_{X_\varepsilon}(\Omega) &= \lim_{\varepsilon\to 0} \int_{\Omega\cap\partial E} |C_\varepsilon(x)\nu(x)|\, d\mathcal{H}^{n-1} \\ &= \int_{\Omega\cap\partial E} |C(x)\nu(x)|\, d\mathcal{H}^{n-1} = \|\partial E\|_X(\Omega). \end{aligned} \tag{5.5}$$

In fact, $C_\varepsilon(x) \to C(x)$ pointwise. Thus, using (5.3) and (5.4) we get

$$M^+(\partial E)(\Omega) \le \lim_{\varepsilon\to 0^+} M_\varepsilon^+(\partial E)(\Omega) = \lim_{\varepsilon\to 0^+} \|\partial E\|_{X_\varepsilon}(\Omega) = \|\partial E\|_X(\Omega).$$

This completes the proof of the Theorem if we prove (5.4).

Let $E_s = \{x \in \mathbb{R}^n : \varrho_\varepsilon(x) > s\}$. Since $|X_\varepsilon \varrho_\varepsilon| = 1$ in a neighborhood of ∂E using the Coarea Formula (4.2) we can write

$$\frac{|\{x \in \Omega : |\varrho_\varepsilon(x)| < t\}|}{2t} = \frac{1}{2t} \int_{\{|\varrho_\varepsilon|<t\}\cap\Omega} dx =$$

$$= \frac{1}{2t} \int_{-t}^{+t} \int_{\{\varrho_\varepsilon = s\}\cap\Omega} \frac{1}{|X_\varepsilon \varrho_\varepsilon|} d\|\partial E_s\|_{X_\varepsilon}\, ds = \frac{1}{2t} \int_{-t}^{+t} \|\partial E_s\|_{X_\varepsilon}(\Omega) ds.$$

We consider first the case $\Omega = \mathbb{R}^n$ and $t > 0$. By Theorem 2.3

$$\int_{E_t\setminus E} \operatorname{div}_{X\varepsilon}(X_\varepsilon \varrho_\varepsilon)dx = \int_{\mathbb{R}^n} \langle X_\varepsilon \varrho_\varepsilon, \nu_{E_t}\rangle d\|\partial E_t\|_{X_\varepsilon} - \int_{\mathbb{R}^n} \langle X_\varepsilon \varrho_\varepsilon, \nu_E\rangle d\|\partial E\|_{X_\varepsilon},$$

and by (2.4)

$$\nu_E = \frac{C_\varepsilon n_E}{|C_\varepsilon n_E|} = \frac{X_\varepsilon \varrho_\varepsilon}{|X_\varepsilon \varrho_\varepsilon|},$$

where $n_E = \frac{\nabla \varrho_\varepsilon}{|\nabla \varrho_\varepsilon|}$ is the Euclidean normal to ∂E. We have an analogous representation formula for ν_{E_t} on ∂E_t. Thus, since $|X_\varepsilon \varrho_\varepsilon(x)| = 1$ in a neighborhood of ∂E

$$\begin{aligned}\int_{E_t\setminus E} &\operatorname{div}_{X\varepsilon}(X_\varepsilon \varrho_\varepsilon)dx \\ &= \int_{\mathbb{R}^n} \langle X_\varepsilon \varrho_\varepsilon, \frac{X_\varepsilon \varrho_\varepsilon}{|X_\varepsilon \varrho_\varepsilon|}\rangle d\|\partial E_t\| - \int_{\mathbb{R}^n} \langle X_\varepsilon \varrho, \frac{X_\varepsilon \varrho_\varepsilon}{|X_\varepsilon \varrho_\varepsilon|}\rangle d\|\partial E\|_{X_\varepsilon} \\ &= \|\partial E_t\|_{X_\varepsilon}(\mathbb{R}^n) - \|\partial E\|_{X_\varepsilon}(\mathbb{R}^n).\end{aligned}$$

Since $\operatorname{div}_{X\varepsilon}(X_\varepsilon \varrho_\varepsilon) \in L^1$ in a neighborhood of ∂E, the first term tends to zero when $t \to 0^+$, and we deduce that $\|\partial E_t\|_{X_\varepsilon}(\mathbb{R}^n) \to \|\partial E\|_{X_\varepsilon}(\mathbb{R}^n)$ as $t \to 0^+$. This concludes the proof if $\Omega = \mathbb{R}^n$.

Since $\chi_{E_t} \to \chi_E$ both in $L^1(\Omega)$ and in $L^1(\mathbb{R}^n \setminus \overline{\Omega})$ we have by the lower semicontinuity of perimeter

$$\begin{aligned}\|\partial E\|_{X_\varepsilon}(\Omega) &\le \liminf_{t\to 0^+} \|\partial E_t\|_{X_\varepsilon}(\Omega), \\ \|\partial E\|_{X_\varepsilon}(\mathbb{R}^n \setminus \overline{\Omega}) &\le \liminf_{t\to 0^+} \|\partial E_t\|_{X_\varepsilon}(\mathbb{R}^n \setminus \overline{\Omega}).\end{aligned} \tag{5.6}$$

From

$$\|\partial E_t\|_{X_\varepsilon}(\Omega) \le \|\partial E_t\|_{X_\varepsilon}(\overline{\Omega}) = \|\partial E_t\|_{X_\varepsilon}(\mathbb{R}^n) - \|\partial E_t\|_{X_\varepsilon}(\mathbb{R}^n \setminus \overline{\Omega}),$$

using the second inequality (5.6) and the convergence in $\mathbb{R}^n$ established above we find

$$\begin{aligned}\limsup_{t\to 0^+} \|\partial E_t\|_{X_\varepsilon}(\Omega) &\le \|\partial E\|_{X_\varepsilon}(\mathbb{R}^n) - \liminf_{t\to 0^+} \|\partial E_t\|_{X_\varepsilon}(\mathbb{R}^n \setminus \overline{\Omega}) \\ &\le \|\partial E\|_{X_\varepsilon}(\mathbb{R}^n) - \|\partial E\|_{X_\varepsilon}(\mathbb{R}^n \setminus \overline{\Omega}) \\ &\le \|\partial E\|_{X_\varepsilon}(\Omega) + \|\partial E\|_{X_\varepsilon}(\partial\Omega) \\ &\le \|\partial E\|_{X_\varepsilon}(\Omega) + \int_{\partial E\cap\partial\Omega} |C_\varepsilon \nu_E| d\mathcal{H}^{n-1} \\ &= \|\partial E\|_{X_\varepsilon}(\Omega).\end{aligned}$$

Together with the first inequality in (5.6) this proves that $\|\partial E_t\|_{X_\varepsilon}(\Omega) \to \|\partial E\|_{X_\varepsilon}(\Omega)$ as $t \to 0^+$. The case $t \to 0^-$ is quite similar. The theorem is now completely proved. □

Remark 5.2. The Riemannian type approximation in the proof of Theorem 5.1 is "Riemannian" only from the metric point of view. In fact we considered in $\mathbb{R}^n$ the Riemannian metric d_ε but in place of the Riemannian volume, which diverges as $\varepsilon \to 0$, we took the Lebesgue measure. The surface measure $\|\partial E\|_{X_\varepsilon}$ is not the classical Riemannian area, either, which still diverges when $\varepsilon \to 0$ (see [10] and see also Theorem 6.4). Namely, the Riemmanian area in the approximation is equal to the variational perimeter defined by the family of vector fields X_ε except for a term which is exactly the Riemmanian volume element, which makes it diverge.

Let A_ε be the $n \times (m+n)$ matrix defined in the proof of Theorem 5.1 and, according to our notation, write $C_\varepsilon = A_\varepsilon^{\mathrm{T}}$. The Riemannian tensor which induces on $\mathbb{R}^n$ the metric d_ε is given by the definite positive matrix $g_\varepsilon(x) = (C_\varepsilon(x)^{\mathrm{T}} C_\varepsilon(x))^{-1}$. Thus, if $E \subset \mathbb{R}^n$ is a bounded open set with regular boundary, the Riemannian volume of E and the Riemmanian area of ∂E are respectively

$$\begin{aligned}
\mathrm{Vol}_\varepsilon(E) &= \int_E \sqrt{\det g_\varepsilon(x)}\, dx = \int_E \frac{1}{\sqrt{\det(C_\varepsilon(x)^{\mathrm{T}} C_\varepsilon(x))}}\, dx, \\
\mathrm{Area}_\varepsilon(\partial E) &= \int_{\partial E} \langle g_\varepsilon^{-1} n(x), n(x)\rangle^{1/2} \sqrt{\det g_\varepsilon(x)}\, d\mathcal{H}^{n-1} \\
&= \int_{\partial E} \frac{|C_\varepsilon n(x)|}{\sqrt{\det(C_\varepsilon(x)^{\mathrm{T}} C_\varepsilon(x))}}\, d\mathcal{H}^{n-1},
\end{aligned}$$

where $n(x)$ is the *Euclidean* normal to ∂E at x.

Considering, for instance, in $\mathbb{R}^3$ the Heisenberg vector fields $X = \partial_x + 2y\partial_t$ and $Y = \partial_y - 2x\partial_t$ it can be easily checked that $\det(C_\varepsilon^{\mathrm{T}} C_\varepsilon) = \varepsilon^2(1+\varepsilon^2)[4(x^2+y^2)+1+\varepsilon^2]$ and thus it may happen

$$\lim_{\varepsilon\to 0} \mathrm{Vol}_\varepsilon(E) = \lim_{\varepsilon\to 0} \mathrm{Area}_\varepsilon(\partial E) = +\infty.$$

Remark 5.3. From the proof of Theorem 5.1 we get that $M_\varepsilon(\partial E)(\mathbb{R}^n) = \|\partial E\|_{X_\varepsilon}(\mathbb{R}^n)$ for every $\varepsilon > 0$ and that there exists the limit

$$\lim_{\varepsilon\to 0^+} M_\varepsilon(\partial E)(\mathbb{R}^n) = \|\partial E\|_X(\mathbb{R}^n)$$

for any family of vector fields $X = (X_1, ..., X_m)$ even not satisfying assumption (H1).

6. Variational approximations of the perimeter

In this section we prove that the $X-$ perimeter is the limit of "regular" functionals in the sense of $\Gamma-$ convergence. More precisely, we shall show that the C-C perimeter is both the limit of its elliptic-Riemannian approximation and of degenerate elliptic functionals. We recall first the definition of $\Gamma-$convergence (for a comprehensive introduction see [17]).

Definition 6.1. *Let (M, d) be a metric space, and let $F, F_h : M \to [-\infty, +\infty]$, $h \in \mathbb{N}$. F is said to be the $\Gamma-$limit of the sequence $(F_h)_{h\in\mathbb{N}}$, and we shall write $F = \Gamma(M) - \lim_{h\to\infty} F_h$, if the following conditions hold*

(6.1) $$\text{if } x \in M \text{ and } x_h \to x \quad \text{then} \quad F(x) \le \liminf_{h\to\infty} F_h(x_h),$$

(6.2) $$\forall x \in M\ \exists (x_h)_{h\in\mathbb{N}} \quad \text{such that } x_h \to x \quad \text{and } F(x) \ge \limsup_{h\to\infty} F_h(x_h).$$

We now state a "Reduction Lemma" whose proof can be found in [47].

Lemma 6.2. *Let (M, d) be a metric space, $F, F_h : M \to [-\infty, +\infty]$, $h \in \mathbb{N}$, $D \subset M$ and $x \in M$. Suppose that:*

(i) for every $y \in D$ there exists a sequence $(y_h)_{h\in\mathbb{N}} \subset M$ such that $y_h \to y$ in M and $\limsup_{h\to\infty} F_h(y_h) \le F(y)$;
(ii) there exists $(x_h)_{h\in\mathbb{N}} \subset D$ such that $x_h \to x$ and $\limsup_{h\to\infty} F(x_h) \le F(x)$.

Then there exists $(\bar{x}_h)_{h\in\mathbb{N}} \subset M$ such that $\limsup_{h\to\infty} F_h(\bar{x}_h) \le F(x)$.

We shall need a refinement of the approximation theorem for BV_X functions in order to bypass the following technical difficulty. In the Euclidean setting one of the main tool for the approximation of a set of finite perimeter in Ω by means of sets with regular boundary in $\mathbb{R}^n$ (not only in Ω) was the property of a function $u \in BV(\Omega) \cap L^\infty(\Omega)$ to be extended to a function $\widetilde{u} \in BV(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $\|D\widetilde{u}\|(\partial\Omega) = 0$, if Ω has Lipschitz boundary(see [46, Lemma 1]). It is not known whether such a property does hold for $BV_X(\Omega)$ functions. Nevertheless we can prove the following proposition.

Proposition 6.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^∞ boundary, and let $E \subset \Omega$ be a measurable set such that $\|\partial E\|_X(\Omega) < +\infty$. There exists a sequence $(E_h)_{h\in\mathbb{N}}$ of open sets in $\mathbb{R}^n$ such that*

(i) E_h is bounded and ∂E_h is of class C^∞ for all $h \in \mathbb{N}$;
(ii) $E_h \to E$ in $L^1(\Omega)$;
(iii) $\|\partial E_h\|_X(\Omega) \to \|\partial E\|_X(\Omega)$;
(iv) $\mathcal{H}^{n-1}(\partial E_h \cap \partial\Omega) = 0$ for all $h \in \mathbb{N}$.

Proof. Let $M = \sup_{x\in\Omega}\left(\sum_{i=1}^{n}\sum_{j=1}^{m}|a_{ij}(x)|^2\right)^{1/2}$ where the a_{ij} are the coefficients of the vector fields $X_1, ..., X_m$. The proof will be divided in three steps.

Step 1. Assume that ∂E is of class C^∞ in $\mathbb{R}^n$. For $\varepsilon > 0$ fixed we show that there exists a bounded open set $\widehat{E} \subset \mathbb{R}^n$ with C^∞ boundary such that:

(1) $|(E\Delta\widehat{E}) \cap \Omega| \le \varepsilon$;
(2) $|\|\partial E\|_X(\Omega) - \|\partial\widehat{E}\|_X(\Omega)| \le \varepsilon$;
(3) $\mathcal{H}^{n-1}(\partial\widehat{E} \cap \partial\Omega) = 0$.

Define the dinstance function

$$\delta(x) := \begin{cases} \min\limits_{y\in\partial\Omega}|x-y| & \text{if } x \in \Omega \\ -\min\limits_{y\in\partial\Omega}|x-y| & \text{if } x \in \mathbb{R}^n\setminus\Omega \end{cases}$$

There exists $\bar{\delta} > 0$ such that the function $x \to \delta(x)$ is of class C^∞ in the open set $\{x \in \mathbb{R}^n : |\delta(x)| < \bar{\delta}\}$ (see for example [46, Lemma 3] or [6]). If $t \in (0, \bar{\delta})$ define $\Omega_t = \{x \in \Omega : \delta(x) > t\}$ and notice that $\partial\Omega_t$ is of class C^∞. We can fix $t_0 \in (0, \bar{\delta})$ such that

$$|\Omega\setminus\Omega_{t_0}| \le \varepsilon, \quad \mathcal{H}^{n-1}(\partial E \cap \Omega\setminus\Omega_{t_0}) \le \varepsilon, \tag{6.3}$$

$$\begin{aligned} \|\partial E\|_X(\Omega\setminus\Omega_{t_0}) &= \int_{\partial E\cap(\Omega\setminus\Omega_{t_0})}|C(x)\nu(x)|\,d\mathcal{H}^{n-1} \\ &\le M\mathcal{H}^{n-1}(\partial E\cap(\Omega\setminus\Omega_{t_0})) \le M\varepsilon. \end{aligned} \tag{6.4}$$

Notice first that

$$\mathcal{H}^{n-1}(\partial\Omega_t \cap \partial E) = 0 \quad \text{for a.e. } t \in (0, \bar{\delta}). \tag{6.5}$$

By contradiction assume that $\mathcal{H}^{n-1}(\partial\Omega_t \cap \partial E) > 0$ for t belonging to a set of positive measure in $(0, \bar{\delta})$. Then

$$0 < \int_0^{\bar{\delta}}\mathcal{H}^{n-1}(\partial\Omega_t\cap\partial E)\,dt = \int_{\partial E\cap\Omega\setminus\Omega_{\bar{\delta}}}|\nabla\delta(x)|\,dx = |\partial E\cap\Omega\setminus\Omega_{\bar{\delta}}|$$

and this is not possible.

Now take a function $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi(s) = 1$ if $|s| \le \frac{1}{2}$, $0 < \varphi(s) \le 1$ if $\frac{1}{2} < |s| < 1$ and $\varphi(s) = 0$ if $|s| \ge 1$. For $x \in \mathbb{R}^n$ consider the vector field

$$N(x) := \begin{cases} -\varphi\left(\frac{\delta(x)}{t_0}\right)\nabla\delta(x) & \text{if } |\delta(x)| < t_0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\gamma_x : \mathbb{R} \to \mathbb{R}^n$ be the maximal solution to the following Cauchy problem

$$\begin{cases} \dot\gamma_x(s) = N(\gamma_x(s)) \\ \gamma_x(0) = x, \end{cases}$$

and define the flow $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by $\Phi(t,x) = \gamma_x(t)$. Write $\Phi_t(x) = \Phi(t,x)$ and notice that Φ_t restricted to Ω_{t_0} is the identity and $\Phi_t(\partial\Omega_t) = \partial\Omega$ if $t \le \frac{t_0}{2}$.

Choose $0 \le \bar t \le \frac{t_0}{2}$ such that

$$\mathcal{H}^{n-1}(\partial\Omega_{\bar t} \cap \partial E) = 0, \tag{6.6}$$

$$\sup_{x\in\Omega} |\nabla\Phi_{\bar t}(x)| \le 2. \tag{6.7}$$

This is possible because of (6.5), and by the fact that if $t = 0$ then Φ_t is the identity and the dependence on t is smooth.

Define $\widehat E = \Phi_{\bar t}(E)$. Since $\widehat E \cap \Omega_{t_0} = E \cap \Omega_{t_0}$ we have $|(\widehat E \Delta E) \cap \Omega| \le |\Omega \setminus \Omega_{t_0}| \le \varepsilon$. This is (1). Moreover

$$\mathcal{H}^{n-1}(\partial\Omega \cap \partial\widehat E) = \mathcal{H}^{n-1}(\Phi_{\bar t}(\partial\Omega_{\bar t} \cap \partial E)) = 0$$

since $\mathcal{H}^{n-1}(\partial\Omega_{\bar t} \cap \partial E) = 0$ and $\Phi_{\bar t}$ is a diffeomorphism. This proves (3).

Estimate the $X-$perimeter

$$\begin{aligned} &|\|\partial E\|_X(\Omega) - \|\partial\widehat E\|_X(\Omega)| \\ &\quad\le \int_{\partial E\cap\Omega\setminus\Omega_{t_0}} |C\nu(x)| d\mathcal{H}^{n-1} + \int_{\partial\widehat E\cap\Omega\setminus\Omega_{t_0}} |C\nu(x)| d\mathcal{H}^{n-1} \\ &\quad\le \varepsilon + M\mathcal{H}^{n-1}(\partial\widehat E \cap \Omega \setminus \Omega_{t_0}) \\ &\quad\le \varepsilon + 2^{n-1} M\mathcal{H}^{n-1}(\partial E \cap \Omega_{\bar t} \setminus \Omega_{t_0}) \\ &\quad\le (1 + 2^{n-1}M)\varepsilon, \end{aligned}$$

where we used - in order - Proposition 2.4, (6.4), (6.7) and (6.3).

Step 2. Assume that $\|\partial E\|_X(\partial\Omega) = 0$. First notice that

$$\begin{aligned} \|\partial E\|_X(\mathbb{R}^n) &= \|\partial E\|_X(\Omega) + \|\partial E\|_X(\partial\Omega) + \|\partial E\|_X(\mathbb{R}^n \setminus \overline{\Omega}) \\ &= \|\partial E\|_X(\Omega) < +\infty. \end{aligned}$$

By Theorem 7.1 there exists a sequence $(\widetilde E_h)_{h\in\mathbb{N}}$ of open subsets of $\mathbb{R}^n$ such that

(1) $\widetilde E_h$ is bounded and $\partial\widetilde E_h$ is of class C^∞ for all $h \in \mathbb{N}$;
(2) $\widetilde E_h \to E$ in $L^1(\Omega)$;
(3) $\|\partial\widetilde E_h\|_X(\mathbb{R}^n) \to \|\partial E\|_X(\mathbb{R}^n)$.

By the same argument as in [32, Proposition 1.13] we can also find

(4) $\|\partial\widetilde{E}_h\|_X(\Omega) \to \|\partial E\|_X(\Omega)$.

The claim follows from *Step 1* by a diagonal argument.

Step 3. Suppose that $\|\partial E\|_X(\partial\Omega) > 0$. We are going to reduce to *Step 2*. By Theorem 7.1 we can assume that $E \subset \Omega$ and $\partial E \cap \Omega$ is of class C^∞.

Let $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ be the diffeomorphism defined in *Step 1* . Fix $\varepsilon > 0$ and choose $0 < s < \bar{t} < t_0$ such that $|\Omega \setminus \Omega_{t_0}| \le \varepsilon$ and $\mathcal{H}^{n-1}(\partial E \cap \Omega_s \setminus \Omega_{t_0}) \le \varepsilon$. As before define $\Omega_s := \{x \in \Omega : \delta(x) > s\}$ and set

$$E_s := \Omega_s \cap E \quad \text{and} \quad \widehat{E} := \Phi_{\bar{t}}(E_s). \tag{6.8}$$

We can also assume that

$$\mathcal{H}^{n-1}(\partial\Omega_{t_0} \cap \partial E) = \mathcal{H}^{n-1}(\partial\Omega_{\bar{t}} \cap \partial E) = \mathcal{H}^{n-1}(\partial\Omega_s \cap \partial E) = 0.$$

Notice that E_s is a set with finite *Euclidean* perimeter in $\mathbb{R}^n$, i.e. $\|\partial E\|(\mathbb{R}^n) < +\infty$. Indeed

$$\begin{aligned}\|\partial E_s\|(\mathbb{R}^n) &= \|\partial E_s\|(\overline{\Omega}_s) + \|\partial E_s\|(\Omega \setminus \overline{\Omega}_s)\\ &= \|\partial E_s\|(\overline{\Omega}_s) = \mathcal{H}^{n-1}(\partial E \cap \Omega_s) + \mathcal{H}^{n-1}(E \cap \partial\Omega_s)\\ &< +\infty.\end{aligned}$$

On the other hand

$$\begin{aligned}\|\partial\widehat{E}\|_X(\Omega) &= \|\partial\widehat{E}\|_X(\Omega_{t_0}) + \|\partial\widehat{E}\|_X(\Omega \setminus \overline{\Omega}_{t_0})\\ &= \|\partial E\|_X(\Omega_{t_0}) + \|\partial\widehat{E}\|_X(\Omega \setminus \overline{\Omega}_{t_0}).\end{aligned} \tag{6.9}$$

By [32, Lemma 10.1] $\|\partial\widehat{E}\|(\mathbb{R}^n) < +\infty$ and by [27, Remark 2.1.9]

$$\begin{aligned}\|\partial\widehat{E}\|_X(\Omega \setminus \overline{\Omega}_{t_0}) &\le C\|\partial\widehat{E}\|(\Omega \setminus \overline{\Omega}_{t_0}) = C\|\partial\Phi_{\bar{t}}(E_s)\|(\Phi_{\bar{t}}(\Omega_{\bar{t}} \setminus \overline{\Omega}_{t_0}))\\ &\le C\|\partial E_s\|(\Omega_{\bar{t}} \setminus \overline{\Omega}_{t_0}) \le C\mathcal{H}^{n-1}(\partial E_s \cap \Omega_{\bar{t}} \setminus \overline{\Omega}_{t_0})\\ &\le C\varepsilon.\end{aligned}$$

Finally

$$\begin{aligned}\|\partial\widehat{E}\|_X(\partial\Omega) &\le C\|\partial\widehat{E}\|(\partial\Omega) = C\|\partial\Phi_{\bar{t}}(E_s)\|(\Phi_{\bar{t}}(\partial\Omega_{\bar{t}}))\\ &\le C\|\partial E\|(\partial\Omega_{\bar{t}}) = 0,\end{aligned} \tag{6.10}$$

and thus $\|\partial\widehat{E}\|_X(\partial\Omega) = 0$. □

Let $X = (X_1, ..., X_m)$ be a family of Lipschitz vector fields in $\mathbb{R}^n$ and for $\varepsilon > 0$ define the new family $X_\varepsilon = (X_1, ..., X_m, \varepsilon\partial_1, ..., \varepsilon\partial_n)$. Let $\Omega \subset \mathbb{R}^n$ be an open set and define the functionals $P, P_\varepsilon : L^1(\Omega) \to [0, +\infty]$

$$P(u) = \begin{cases} \|\partial E\|_X(\Omega) & \text{if } u = \chi_E \in BV_X(\Omega)\\ +\infty & \text{otherwise,}\end{cases}$$

and

$$P_\varepsilon(u) = \begin{cases} \|\partial E\|_{X_\varepsilon}(\Omega) & \text{if } u = \chi_E \in BV_{X_\varepsilon}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\varepsilon_h \to 0$ and write $P_h = P_{\varepsilon_h}$. In the following theorem we prove that the "elliptic-Riemannian" regularization of the perimeter $\Gamma-$converges to the perimeter.

Theorem 6.4. *If $\Omega \subset \mathbb{R}^n$ is a bounded open set with C^∞ boundary then*

$$P = \Gamma(L^1(\Omega)) - \lim_{h\to\infty} P_h$$

Proof. We will prove (6.1) and (6.2) with $M = L^1(\Omega)$. Let us begin with (6.1). Let $u \in L^1(\Omega)$ and $(u_h)_{h\in\mathbb{N}} \subset L^1(\Omega)$ be such that $u_h \to u$ in $L^1(\Omega)$. Possibly extracting a subsequence we can assume that $\liminf_{h\to\infty} P_h(u_h)$ $= \lim_{h\to\infty} P_h(u_h) < +\infty$ and $P_h(u_h) < +\infty$ for all $h \in \mathbb{N}$ and thus $u_h = \chi_{E_h}$ and $u = \chi_E$ for suitable measurable sets $E, E_h \subset \mathbb{R}^n$. Notice that by the definition of $X-$perimeter (see 2.3) it follows that

$$\|\partial F\|_X(\Omega) \le \|\partial F\|_{X_\varepsilon}(\Omega)$$

for all $\varepsilon > 0$ and for all measurable set $F \subset \mathbb{R}^n$. Thus

$$\|\partial E_h\|_X(\Omega) \le \|\partial E_h\|_{X_{\varepsilon_h}}(\Omega) = P_h(u_h),$$

and by the lower semincontinuity of perimeter we get

$$P(u) = \|\partial E\|_X(\Omega) \le \liminf_{h\to\infty} \|\partial E_h\|_X(\Omega) \le \liminf_{h\to\infty} P_h(u_h).$$

This proves (6.1).

Let us prove now (6.2). Take $u \in L^1(\Omega)$ and assume that $P(u) < +\infty$, that is to say $u = \chi_E \in BV_X(\Omega)$ (otherwise there is nothing to prove). By Proposition 6.3 there exists a sequence of bounded open sets $(E_h)_{h\in\mathbb{N}} \subset \mathbb{R}^n$ with C^∞ boundary such that $\chi_{E_h} \to \chi_E$ in $L^1(\Omega)$ and $\|\partial E_h\|_X(\Omega) \to \|\partial E\|_X(\Omega)$. By Lemma 6.2 with $F_h = P_h$, $F = P$ and $D = \{E_h\}$ it suffices to prove that if $u = \chi_E$ with ∂E of class C^∞ then there exists $(u_h)_{h\in\mathbb{N}} \in L^1(\Omega)$ such that

$$u_h \to \chi_E \text{ in } L^1(\Omega) \quad \text{and } P(u) \ge \limsup_{h\to\infty} P_h(u_h).$$

In fact, choosing $u_h = \chi_E$ for all $h \in \mathbb{N}$ and recalling the representation formula (2.4) we find

$$P_h(u_h) = \|\partial E\|_{X_{\varepsilon_h}}(\Omega) = \int_{\partial E\cap\Omega} |C_{\varepsilon_h}\nu(x)|\, d\mathcal{H}^{n-1}$$

and

$$\lim_{h\to\infty}\int_{\partial E\cap\Omega}|C_{\varepsilon_h}\nu(x)|\,d\mathcal{H}^{n-1}=\int_{\partial E\cap\Omega}|C\nu(x)|\,d\mathcal{H}^{n-1}=P(u).$$

□

We now prove the main $\Gamma-$convergence theorem. The classical result in Euclidean space is proved in [47] and [46]. The first example of regular approximation of the perimeter in the setting of Finsler manifolds is in [10]. Our proof is inspired by some ideas contained in [7], where the equality between perimeter and Minkowski content turns out to be the main tool for the approximation (see also [10]).

Fix a bounded open set $\Omega\subset\mathbb{R}^n$. For $\varepsilon>0$ define the functionals $F,F_\varepsilon:L^1(\Omega)\to[0,+\infty]$

$$F_\varepsilon(u)=\begin{cases}\displaystyle\int_\Omega(\varepsilon|Xu|^2+\frac{1}{\varepsilon}W(u))dx & \text{if } u\in H^1_X(\Omega)\\ +\infty & \text{otherwise}\end{cases}\tag{6.11}$$

where $W(u)=u^2(1-u)^2$, and

$$F(u)=\begin{cases}2\alpha\|\partial E\|_X(\Omega) & \text{if } u=\chi_E\in BV_X(\Omega)\\ +\infty & \text{otherwise,}\end{cases}\tag{6.12}$$

where $\alpha=\displaystyle\int_0^1\sqrt{W(s)}ds$. Let $\varepsilon_h\to 0$ and write $F_h:=F_{\varepsilon_h}$.

Theorem 6.5. *Suppose that $X_1,...,X_m\in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ satisfy hypotheses (H1) and (H2). If $\Omega\subset\mathbb{R}^n$ is a bounded open set with C^∞ boundary then*

$$F=\Gamma(L^1(\Omega))-\lim_{h\to\infty}F_h.\tag{6.13}$$

Proof. We will prove (6.1) and (6.2) with $M=L^1(\Omega)$. Let us begin with (6.1). Let $u_h\to u$ in $L^1(\Omega)$. It is not restrictive to assume that $\liminf_{h\to\infty}F_h(u_h)<\infty$, and - possibly extracting a subsequence - we can also assume that $u_h(x)\to u(x)$ for a.e. $x\in\Omega$. By Fatou Lemma

$$\int_\Omega W(u(x))\,dx\le\liminf_{h\to\infty}\int_\Omega W(u_h(x))\,dx\le\liminf_{h\to\infty}\varepsilon_hF_h(u_h)=0.$$

We deduce that $u(x)\in\{0,1\}$ for a.e. $x\in\Omega$. We shall write $u=\chi_E$ where $E:=\{x\in\Omega:u(x)=1\}$.

Define the increasing function $\varphi\in C^1(\mathbb{R})$ by $\varphi(t)=\displaystyle\int_0^t\sqrt{W(s)}\,ds$, and put

$$w(x)=\varphi(u(x)),\quad w_h(x)=\varphi(u_h(x)).$$

Observe that by [31, Lemma 3.16] $w_h \in H^1_X(\Omega)$. By the coarea formula (4.1)

$$\begin{aligned}\|Xw\|_X(\Omega) &= \int_{-\infty}^{+\infty} \|\partial\{x \in \Omega : \varphi(u(x)) > t\}\|_X(\Omega)\, dt \\ &= \int_0^1 \|\partial\{x \in \Omega : u(x) > s\}\|_X(\Omega)\varphi'(s)\, ds \\ &= \|\partial E\|_X(\Omega) \int_0^1 \sqrt{W(s)}\, ds = 1/2F(u).\end{aligned}$$

Replacing u_h with $\bar{u}_h(x) = \max\{0, \min\{u_h(x), 1\}\}$ and observing that $F_h(\bar{u}_h) \leq F_h(u_h)$, we can assume that $0 \leq u_h(x) \leq 1$. Thus, from

$$\int_\Omega |w_h(x) - w(x)|\, dx \leq \sup_{t\in[0,1]} |\varphi'(t)| \int_\Omega |u_h(x) - u(x)|\, dx$$

we deduce that $w_h \to w$ in $L^1(\Omega)$. Using the lower semicontinuty of the total variation we find

$$\begin{aligned}F(u) = 2\|Xw\|(\Omega) &\leq 2 \liminf_{h\to\infty} \int_\Omega |Xw_h(x)|\, dx \\ &\leq 2 \liminf_{h\to\infty} \int_\Omega |Xu_h(x)||\varphi'(u_h(x))|\, dx \\ &\leq \liminf_{h\to\infty} \int_\Omega (\varepsilon_h |Xu_h(x)|^2 + \frac{1}{\varepsilon_h} W(u_h(x)))\, dx \\ &\leq \liminf_{h\to\infty} F_h(u_h),\end{aligned}$$

and then (6.1) follows. We shall now prove (6.2). By Proposition 6.3 we can assume $u = \chi_E$, $E \subset \mathbb{R}^n$ bounded open set with C^∞ boundary and such that $\mathcal{H}^{n-1}(\partial\Omega \cap \partial E) = 0$. Let $d_{\partial E}(x) = \inf_{y\in\partial E} d(x, y)$, d being the C-C metric induced by the vector fields. Define $\varrho : \Omega \to [0, +\infty)$

$$\varrho(x) = \begin{cases} d_{\partial E}(x) & \text{if } x \in \Omega \cap E \\ -d_{\partial E}(x) & \text{if } x \in \Omega \setminus E. \end{cases}$$

Now define $\chi_0 : \mathbb{R} \to \mathbb{R}$ by

$$\chi_0(t) = \begin{cases} 1 \text{ if } t > 0 \\ 0 \text{ if } t \leq 0. \end{cases}$$

If $x \in \Omega$ we can write $u(x) = \chi_0(\varrho(x))$. Consider now the one dimensional functional (see [10], [1])

$$J_\varepsilon(\chi) := \int_{\mathbb{R}} (\varepsilon(\chi')^2 + \frac{1}{\varepsilon} W(\chi)) dt.$$

Fix $\varepsilon = 1$ and determine χ as a solution of the minimum problem

(6.14)
$$M = \min\{J_1(\chi) : \chi \in H^1_{loc}(\mathbb{R};[0,1]),\ \lim_{t\to\infty} \chi(t) = 1,\ \lim_{t\to-\infty} \chi(t) = 0\}.$$

One can prove that $\chi(t) = \dfrac{\mathrm{e}^{\mathrm{t}}}{1+\mathrm{e}^{\mathrm{t}}}$, unique solution of the Cauchy problem

$$\begin{cases} \chi' = \sqrt{W(\chi)} \\ \chi(0) = \frac{1}{2}, \end{cases} \tag{6.15}$$

actually is a solution to problem (6.14) with $M = 2\displaystyle\int_0^1 \sqrt{W(s)}ds$ (see [1, Proposition 2]). Let us follow now the proof contained in [10]. Fix $\varepsilon > 0$ and write $t_\varepsilon = 3\varepsilon \log \frac{1}{\varepsilon}$. Define the function $\Lambda_\varepsilon : \mathbb{R} \to \mathbb{R}$ in the following way

$$\Lambda_\varepsilon(t) = \begin{cases} \chi(t) & \text{if } 0 \le t < \frac{t_\varepsilon}{\varepsilon} \\ p_\varepsilon(t) & \text{if } \frac{t_\varepsilon}{\varepsilon} \le t < \frac{2t_\varepsilon}{\varepsilon} \\ 1 & \text{if } t \ge \frac{2t_\varepsilon}{\varepsilon} \\ 1 - \Lambda_\varepsilon(-t) & \text{if } t < 0. \end{cases}$$

where $p_\varepsilon : \mathbb{R} \to \mathbb{R}$ is the uniquely determined polinomial of degree 3 for which $\Lambda_\varepsilon \in C^{1,1}(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{\pm\frac{t_\varepsilon}{\varepsilon}, \pm\frac{2t_\varepsilon}{\varepsilon}\})$ (see [11] for the construction of p_ε). Now define $\chi_\varepsilon(t) = \Lambda_\varepsilon(\frac{t}{\varepsilon})$ for $t \in \mathbb{R}$. Then, χ_ε verifies the equations

$$\varepsilon \chi'_\varepsilon(t)^2 = \frac{1}{\varepsilon} W(\chi_\varepsilon(t)), \quad \chi_\varepsilon(-t) = 1 - \chi_\varepsilon(t) \tag{6.16}$$

for $t \in [-t_\varepsilon, t_\varepsilon]$.

Now put $u_\varepsilon(x) := \chi_\varepsilon(\varrho(x))$ for $x \in \Omega$. By [31, Lemma 3.16] we have $u_\varepsilon \in H^{1,\infty}_X(\Omega)$ and we can write write for a.e. $x \in \Omega$ the chain rule

$$Xu_\varepsilon(x) = \chi'_\varepsilon(\varrho(x)) X\varrho(x). \tag{6.17}$$

We first prove that

$$\lim_{\varepsilon\to 0} \int_\Omega |u_\varepsilon(x) - u(x)| dx = 0. \tag{6.18}$$

We have $|u_\varepsilon(x) - u(x)| = |\chi_\varepsilon(\varrho(x)) - \chi_0(\varrho(x))| \le 2$ for a.e. $x \in \Omega$ and

$$\lim_{\varepsilon\to 0} \chi_\varepsilon(t) = \begin{cases} 1 & \text{if } t > 0 \\ \frac{1}{2} & \text{if } t = 0 \\ 0 & \text{if } t < 0. \end{cases}$$

Thus, if $\varrho(x) \ne 0$ then $\lim_{\varepsilon\to 0} |\chi_\varepsilon(\varrho(x)) - \chi_0(\varrho(x))| = 0$. Since the set $\{x \in \Omega : \varrho(x) = 0\}$ has zero Lebesgue measure, the $L^1(\Omega)$ convergence (6.18) follows by the dominated convergence theorem.

We now prove that

$$F(u) = \lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon). \tag{6.19}$$

Consider the sets

$$A_\varepsilon := \{x \in \Omega : |\varrho(x)| < t_\varepsilon\} \quad \text{and}$$
$$B_\varepsilon := \{x \in \Omega : t_\varepsilon \le |\varrho(x)| \le 2t_\varepsilon\}.$$

Recalling the definition of u_ε, (6.17) and that $|X\varrho(x)| = 1$ a.e. in $\Omega \setminus \partial E$, we find

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &= \int_{A_\varepsilon} \Big(\varepsilon |Xu_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon)\Big) dx + \int_{B_\varepsilon} \Big(\varepsilon |Xu_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon)\Big) dx \\ &= \int_{A_\varepsilon} \Big(\varepsilon \chi_\varepsilon'(\varrho(x))^2 + \frac{1}{\varepsilon} W(\chi_\varepsilon(\varrho(x)))\Big) dx \\ &\quad + \int_{B_\varepsilon} \Big(\varepsilon p_\varepsilon' \left(\frac{|\varrho(x)|}{\varepsilon}\right)^2 + \frac{1}{\varepsilon} W(p_\varepsilon \left(\frac{|\varrho(x)|}{\varepsilon}\right) \Big) dx = \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon. \end{aligned}$$

In order to show that $\mathrm{II}_\varepsilon \to 0$ it suffices to notice that $\lim_{\varepsilon \to 0} |B_\varepsilon| = 0$ and that there exists a constant $C > 0$ not depending on ε such that, if $x \in B_\varepsilon$ then

$$\varepsilon p_\varepsilon' \left(\frac{|\varrho(x)|}{\varepsilon}\right)^2 + \frac{1}{\varepsilon} W(p_\varepsilon \left(\frac{|\varrho(x)|}{\varepsilon}\right) \le C.$$

In fact (see [11, Sect. 6]) it is not restrictive to assume $\|p_\varepsilon - 1\|_{L^\infty(\frac{t_\varepsilon}{\varepsilon}, \frac{2t_\varepsilon}{\varepsilon})} = O(\varepsilon^5)$ and $\|p_\varepsilon'\|_{L^\infty(\frac{t_\varepsilon}{\varepsilon} \frac{2t_\varepsilon}{\varepsilon})} = O(\varepsilon^6)$. If we show that

$$\lim_{\varepsilon \to 0} \mathrm{I}_\varepsilon = F(u), \tag{6.20}$$

the Theorem is proved. Using the coarea formula (4.2) we can write

$$\mathrm{I}_\varepsilon = \int_{-t_\varepsilon}^{t_\varepsilon} \Big(\varepsilon \chi_\varepsilon'(s)^2 + \frac{1}{\varepsilon} W(\chi_\varepsilon(s))\Big) \|\partial E_s\|_X(\Omega) ds,$$

where $E_s = \{x \in \Omega : \varrho(x) > s\}$. Notice that by (6.16) we have for all $s \in [-t_\varepsilon, t_\varepsilon]$

$$f_\varepsilon(s) := \varepsilon \chi_\varepsilon'(s)^2 + \frac{1}{\varepsilon} W(\chi_\varepsilon(s)) = \varepsilon \chi_\varepsilon'(-s)^2 + \frac{1}{\varepsilon} W(\chi_\varepsilon(-s)) = f_\varepsilon(-s)$$

and thus

(6.21)
$$\mathrm{I}_\varepsilon = \int_0^{t_\varepsilon} \Big(\varepsilon \chi_\varepsilon'(s)^2 + \frac{1}{\varepsilon} W(\chi_\varepsilon(s))\Big) (\|\partial E_s\|_X(\Omega) + \|\partial E_{-s}\|_X(\Omega)) ds,$$

Let $V(t) := |\{x \in \Omega : |\varrho(x)| \leq t\}|$ for $t \geq 0$. Then for $t > 0$ we have

(6.22)
$$V(t) = \int_{-t}^{t} \|\partial E_s\|_X(\Omega) ds \quad \text{and} \quad V'(t) = \|\partial E_t\|_X(\Omega) + \|\partial E_{-t}\|_X(\Omega)$$

for a.e. $t > 0$. From (6.21) and from (6.22) it follows

$$\begin{aligned} \mathrm{I}_\varepsilon &= \int_0^{t_\varepsilon} \Big(\varepsilon\chi_\varepsilon'(s)^2 + \frac{1}{\varepsilon}W(\chi_\varepsilon(s))\Big)V'(s)\, ds \\ &= \int_0^{t_\varepsilon} f_\varepsilon(s)V'(s)\, ds = V(t_\varepsilon)f_\varepsilon(t_\varepsilon) - \int_0^{t_\varepsilon} f_\varepsilon'(s)V(s)\, ds. \end{aligned} \tag{6.23}$$

By Theorem 5.1 we have

$$\lim_{t\to 0^+} \frac{V(t)}{2t} = L := \|\partial E\|_X(\Omega),$$

and thus there exists a function $\delta : [0, \infty) \to \mathbb{R}$ such that

$$V(t) = 2Lt + \delta(t)t \quad \text{and} \quad \lim_{\varepsilon\to 0^+} \sup_{t\in[0,t_\varepsilon]} |\delta(t)| = 0.$$

We can write (6.23) in the following way

$$\begin{aligned} \mathrm{I}_\varepsilon &= V(t_\varepsilon)f_\varepsilon(t_\varepsilon) - \int_0^{t_\varepsilon} s\delta(s)f_\varepsilon'(s)\, ds - 2L\int_0^{t_\varepsilon} sf_\varepsilon'(s)\, ds \\ &= V(t_\varepsilon)f_\varepsilon(t_\varepsilon) - \int_0^{t_\varepsilon} s\delta(s)f_\varepsilon'(s)\, ds - 2Lt_\varepsilon f_\varepsilon(t_\varepsilon) + 2L\int_0^{t_\varepsilon} f_\varepsilon(s)\, ds \\ &= \sigma_\varepsilon + L\int_{-t_\varepsilon}^{t_\varepsilon} f_\varepsilon(s)\, ds. \end{aligned}$$

In order to prove (6.20) it suffices to show that

$$\lim_{\varepsilon\to 0^+} \sigma_\varepsilon = 0, \tag{6.24}$$

$$\lim_{\varepsilon\to 0^+} \int_{-t_\varepsilon}^{t_\varepsilon} f_\varepsilon(s)\, ds = 2\int_0^1 \sqrt{W(s)}\, ds = 2\alpha. \tag{6.25}$$

We begin with (6.25). Using (6.16) we find

$$\begin{aligned} \int_{-t_\varepsilon}^{t_\varepsilon} f_\varepsilon(s)\, ds &= \int_{-t_\varepsilon}^{t_\varepsilon} \Big(\varepsilon\chi_\varepsilon'(s)^2 + \frac{1}{\varepsilon}W(\chi_\varepsilon(s))\Big)\, ds \\ &= 2\int_{-t_\varepsilon}^{t_\varepsilon} \chi_\varepsilon'(s)\sqrt{W(\chi_\varepsilon(s))}\, ds = 2\int_{\chi_\varepsilon(-t_\varepsilon)}^{\chi_\varepsilon(t_\varepsilon)} \sqrt{W(s)}\, ds. \end{aligned}$$

Since $\chi_\varepsilon(-t_\varepsilon) \to 0$ and $\chi_\varepsilon(t_\varepsilon) \to 1$, one gets (6.25). We now prove (6.24). Notice that

$$(V(t_\varepsilon) - 2Lt_\varepsilon)f(t_\varepsilon) = \delta(t_\varepsilon)t_\varepsilon\Big(\varepsilon\chi_\varepsilon'(t_\varepsilon)^2 + \frac{1}{\varepsilon}W(\chi_\varepsilon(t_\varepsilon))\Big) =$$

$$= 2\delta(t_\varepsilon)t_\varepsilon\frac{W(\chi_\varepsilon(t_\varepsilon))}{\varepsilon} \le 2t_\varepsilon\frac{\delta(t_\varepsilon)}{\varepsilon}(1-\chi(\frac{t_\varepsilon}{\varepsilon})) =$$

$$= \frac{2\delta(t_\varepsilon)t_\varepsilon}{\varepsilon}\frac{\varepsilon}{1+\varepsilon} = \frac{2\varepsilon\log\frac{1}{\varepsilon}\delta(t_\varepsilon)}{1+\varepsilon} \to 0.$$

Furthermore

$$|\int_0^{t_\varepsilon} s\delta(s)f_\varepsilon'(s)ds| \le \sup_{s\in[0,t_\varepsilon]}|\delta(s)|\int_0^{t_\varepsilon} s|f_\varepsilon'(s)|ds.$$

Our thesis will be proved if we show that the integral is bounded. Now

$$f_\varepsilon'(s) = \Big(\varepsilon\chi_\varepsilon'(s)^2 + \frac{1}{\varepsilon}W(\chi_\varepsilon(s))\Big)' = 2\varepsilon\Big(\chi_\varepsilon'(s)^2\Big)' = 4\varepsilon\chi_\varepsilon'(s)\chi_\varepsilon''(s),$$

and thus

$$\int_0^{t_\varepsilon} s|f_\varepsilon'(s)|ds = 4\varepsilon\int_0^{t_\varepsilon} s|\chi_\varepsilon'(s)\chi_\varepsilon''(s)|ds =$$

$$= 4\int_0^{3\log\frac{1}{\varepsilon}} s|\chi'(s)\chi''(s)|ds \le 4\int_0^{\infty} s|\chi'(s)\chi''(s)|ds < \infty.$$

This concludes the proof of Theorem 6.5. □

Remark 6.6. If $(\mathbb{R}^n, d)$ is a C-C space induced by the vector fields $X_1, \dots,$ X_m which satisfy one of the cases A, B or C of Sect. 3, and $\Omega \subset \mathbb{R}^n$ is a bounded open set with C^∞ boundary then (6.13) holds.

Consider, for instance, the Heisenberg group $\mathbb{H}^1 = (\mathbb{R}^3, \cdot, \delta_\lambda, d)$ (see Sect. 3, Example A1). Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with C^∞ boundary and let $F, F_\varepsilon : L^1(\Omega) \to [0+\infty]$ be the functionals given in (6.11) and (6.12) with $X = \nabla_{\mathbb{H}} = (X_1, Y_1)$. F_ε can be written as

$$F_\varepsilon(u) = \int_\Omega (\varepsilon\langle BDu, Du\rangle + \frac{1}{\varepsilon}W(u))\,dxdydt$$

if $u \in C^1(\Omega)$, with

$$B(x,y,t) = \begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \\ 2y & -2x & 4x^2+4y^2 \end{pmatrix}.$$

The matrix B is degenerate at every point of $\mathbb{R}^3$, namely $\det(B(x,y,t)) = 0$ for all $(x,y,t) \in \mathbb{R}^3$.

7. Appendix

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $E \subset \Omega$ be a measurable set such that $|E| + \|\partial E\|_X(\Omega) < +\infty$. There exists a sequence $(E_h)_{h\in\mathbb{N}}$ of open sets contained in Ω such that*

(i) $\partial E_h \cap \Omega$ *is of class* C^∞ *for all* $h \in \mathbb{N}$;
(ii) $\chi_{E_h} \to \chi_E$ *in* $L^1(\Omega)$;
(iii) $\|\partial E_h\|_X(\Omega) \to \|\partial E\|_X(\Omega)$.

Proof. The proof is the same as in the Euclidean setting (see for example [3]) and we give only a sketch of it. By [27, Theorem 2.2.2] the function χ_E can be approximated by $C^\infty(\Omega)$ functions. Using Sard Lemma and the coarea formula (4.1) one can choose suitable level sets of these functions with total variation converging to that of χ_E. □

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