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Surface modelling of non-radial pulsators: alternative formalisms within the linear approximation

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ABSTRACT

The problem of modelling the surface of stars undergoing non-radial pulsation is reviewed. Linear-approximation expressions for the surface radius, temperature, velocity and geometry of a pulsating star are derived and discussed using both the Lagrangian (fixed-element) and Eulerian (fixed-position) formalisms. In each case, small numerical discrepancies are found between the perturbed states predicted by these alternative approaches. These discrepancies are shown to scale quadratically with the pulsation amplitude, and are therefore attributed to a transgression of the linear-approximation limits.

Singled out for particular attention are the expressions for the surface geometry perturbations predicted by each formalism. Marked differences are apparent between these expressions: terms containing the horizontal fluid displacement appear explicitly in the Lagrangian result, but are absent from the corresponding Eulerian one. By examining the physical origin of these terms, it is demonstrated that the two formalisms are, in fact, perfectly consistent with regard to the geometry perturbations, and – as with all other perturbations – simply furnish alternative representations of the same physical processes. The conclusion is that either formalism is an appropriate choice when modelling the surface of a pulsating star.

Key words: hydrodynamics – methods: numerical – stars: atmospheres – stars: early-type – stars: oscillations.

1 INTRODUCTION

There exists strong evidence that non-radial pulsation is the underlying mechanism behind the periodic light fluctuations seen in many disparate classes of variable star (see, e.g., Unno et al. 1989, for a comprehensive review of the topic). In systems where a large number of separate components are detected in the variability signature (e.g. the Sun – Duvall et al. 1988), the non-radial pulsation hypothesis may be tested by comparisons between observed and theoretically inferred frequency spectra. However, in systems where only a few components are found (e.g. early-type stars – Fullerton, Gies & Bolton 1996), a different procedure is often adopted, that of direct modelling of the observed light and line-profile variations, based on theoretical predictions of the effects of pulsation on the geometry and physical conditions at the surface of the (putatively pulsating) star.

Typically, this surface modelling proceeds by sampling the stellar surface with a grid of points, which are subjected to the various perturbations arising from the excitation of one or more pulsation modes. By summing the observer-directed flux for all visible points, suitably weighted by the associated projected surface area, time-resolved spectroscopic and photometric data may be synthesized for the model star, suitable for comparison with observations.

Evidently, even if the non-radial pulsation hypothesis is correct for a given star, the degree of agreement between modelling and observations will be limited by the sophistication with which perturbations are applied to the surface grid. Osaki (1971), in simulating the line-profile variations of β Cephei stars, considered the effects on the stellar surface of pulsation-originated velocity fields alone. Subsequently, Dziembowski (1977) and Buta & Smith (1979) included in their models the additional influence of perturbations to the temperature, surface area and surface normal vector. Considered together, these three treatments constitute the foundations upon which more recent models, both photometric (e.g. Watson 1988; Cugier, Dziembowski & Pamyatnykh 1994; Heynderickx, Waelkens & Smeyers 1994; Townsend 2002) and spectroscopic (e.g. Gies 1991; Lee, Jeffery & Saio 1992; Schrijvers et al. 1997; Townsend 1997), have been developed.

Heynderickx et al. (1994) claim to differentiate their photometric model from others, by considering the influence of horizontal (i.e. tangential) fluid displacements on the geometry of the stellar surface. However, the present work demonstrates that the inclusion of these displacements is not synonymous with a more sophisticated model, but corresponds to adopting a Lagrangian framework for describing perturbations, rather than the Eulerian framework favoured by

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other authors. Furthermore, it will be argued that *either* formalism is appropriate when constructing models of the perturbed surface of a pulsating star.

This last point, despite its tautological flavour, is deemed important in light of the misleading acknowledgment made by Balona (1987) to Y. Osaki, for pointing out that Eulerian expressions for surface velocity fields, rather than corresponding Lagrangian ones, are necessary when undertaking spectroscopic modelling of pulsating stars. It is already well established (e.g. Pesnell 1990) that the two formalisms lead to equivalent results when solving the interior pulsation equations – as indeed they should – but Balona's remark demonstrates that there evidently remains some confusion over the issue at the stellar surface.

The present paper attempts to address any such confusion, by reviewing the way in which the two formalisms describe the perturbed surface of a star undergoing non-radial pulsation. An examination of the spectroscopic and photometric variations, which result from pulsation-originated perturbations, has already been made by a number of other authors (see, for instance, the references given above) and is not, therefore, included herein; rather, the focus is placed wholly on the procedure by which each formalism develops a mathematical representation of the perturbed stellar surface, and an examination of the ways in which these representations might differ.

2 SURFACE DESCRIPTION

In the following sections, expressions describing the perturbed surface properties of a pulsating star are derived, within the alternative Lagrangian and Eulerian formalisms. First, however, it is useful to review a few of the basic concepts and results of non-radial pulsation theory.

The visible surface of a star, in its equilibrium state, may be regarded as the ensemble of fluid elements, which together constitute the photosphere. Under the influence of a pulsation-originated perturbation, these elements will undergo displacements in both radial and horizontal directions. However, neglecting phenomena such as mass ejection or convection-like mixing (i.e. assuming that the Jacobian associated with the perturbation is everywhere well defined), all elements will remain part of the photosphere (this is a general property of surface wave propagation – see, e.g., Shivamoggi 1997).

The displacement undergone by a given photospheric fluid element, which in equilibrium is at spherical-polar co-latitude θ and azimuth ϕ, is described by the displacement vector $\tilde{\boldsymbol{\xi}}(\theta, \phi)$. If $\{\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi\}$ are the unit local basis vectors in the $\{r, \theta, \phi\}$-directions, respectively, then $\tilde{\boldsymbol{\xi}}$ may be expressed in terms of its components as

$$\tilde{\boldsymbol{\xi}}(\theta, \phi) = \tilde{\xi}_r(\theta, \phi)\boldsymbol{e}_r + \tilde{\xi}_\theta(\theta, \phi)\boldsymbol{e}_\theta + \tilde{\xi}_\phi(\theta, \phi)\boldsymbol{e}_\phi, \tag{1}$$

where, and subsequently, the tilde ($\tilde{}$) is used to denote the value of dependent variables at the stellar surface. Formally, both $\tilde{\boldsymbol{\xi}}$ and its components are also functions of time t; however, unless otherwise indicated, this dependence is taken as implicit.

Expressions for the components of $\tilde{\boldsymbol{\xi}}$, pertinent to pulsation in a non-rotating star, are obtained by subjecting the equations of hydrodynamics to perturbations about the equilibrium state of the star. With the assumption that these perturbations are small, the equations may be linearized by discarding terms of second- or higher-order in the perturbation amplitude. In combination with appropriate boundary conditions, the resulting system of equations constitutes an eigenvalue problem, which admits to an infinite set of discrete normal-mode solutions. For an individual normal mode with non-zero eigenfrequency, the radial component of $\tilde{\boldsymbol{\xi}}$ is found (e.g. Unno et al. 1989) to be given by

$$\tilde{\xi}_r(\theta, \phi) = \epsilon \tilde{r} Y_\ell^m(\theta, \phi), \tag{2}$$

where ϵ is the parameter that governs the amplitude of radial displacements, $\tilde{r}$ is the radius of the star within the equilibrium state and Y_ℓ^m is a spherical harmonic (e.g. Arfken 1970) of azimuthal order m and harmonic degree ℓ. Likewise, the corresponding horizontal components $\{\tilde{\xi}_\theta, \tilde{\xi}_\phi\}$ are found as

$$\tilde{\boldsymbol{\xi}}_\perp(\theta, \phi) = K\tilde{r}\nabla_\perp \tilde{\xi}_r(\theta, \phi), \tag{3}$$

where

$$\tilde{\boldsymbol{\xi}}_\perp(\theta, \phi) \equiv \tilde{\xi}_\theta(\theta, \phi)\boldsymbol{e}_\theta + \tilde{\xi}_\phi(\theta, \phi)\boldsymbol{e}_\phi \tag{4}$$

is the horizontal part of the displacement vector $\tilde{\boldsymbol{\xi}}$, K is a constant that governs the amplitude of these horizontal displacements relative to radial ones (e.g. Osaki 1971), and

$$\nabla_\perp \equiv \frac{\boldsymbol{e}_\theta}{\tilde{r}}\frac{\partial}{\partial\theta} + \frac{\boldsymbol{e}_\phi}{\tilde{r}\sin\theta}\frac{\partial}{\partial\phi} \tag{5}$$

is the tangential part of the spherical-polar gradient operator.

In a rotating star, expressions for the displacement vector of normal modes are significantly more difficult to formulate, due to the combined influence of the centrifugal and Coriolis forces (see, e.g., Lee & Baraffe 1995). However, if the former force is neglected, so that the star is assumed to retain its spherically symmetric equilibrium state, then relatively simple expressions for $\tilde{\boldsymbol{\xi}}$, which include the Coriolis-force effects, may be found in certain limiting situations (e.g. Lee & Saio 1990; Aerts & Waelkens 1993; Townsend 1997).

The perturbations to other variables at the stellar surface may be represented within two alternative formalisms; both ultimately describe the same unperturbed and perturbed states of the stellar surface, but each adopts a different framework in which to regard the perturbations. The Lagrangian formalism considers the change $\delta\tilde{f}$ in some variable $\tilde{f}$, experienced by a given photospheric fluid element throughout the perturbation process. Thus, the perturbed state is written as

$$\tilde{f}_\mathrm{p}(\tilde{\theta}_\mathrm{p}, \tilde{\phi}_\mathrm{p}) = \tilde{f}(\theta, \phi) + \delta\tilde{f}(\theta, \phi) \tag{6}$$

with the subscript p being used throughout to distinguish perturbed variables from equilibrium ones; here (θ, ϕ) are the angular coordinates of the fluid element at equilibrium and $(\tilde{\theta}_\mathrm{p}, \tilde{\phi}_\mathrm{p})$ are the corresponding coordinates of the same element in the perturbed state. These latter quantities may be expressed as

$$\tilde{\theta}_\mathrm{p}(\theta, \phi) = \theta + \delta\tilde{\theta}(\theta, \phi) \tag{7}$$

and

$$\tilde{\phi}_\mathrm{p}(\theta, \phi) = \phi + \delta\tilde{\phi}(\theta, \phi). \tag{8}$$

The Lagrangian coordinate perturbations $\{\delta\tilde{\theta}, \delta\tilde{\phi}\}$ are related to the horizontal components of the displacement vector via

$$\delta\tilde{\theta}(\theta, \phi) \equiv \frac{1}{\tilde{r}}\tilde{\xi}_\theta(\theta, \phi) \tag{9}$$

and

$$\delta\tilde{\phi}(\theta, \phi) \equiv \frac{1}{\tilde{r}\sin\theta}\tilde{\xi}_\phi(\theta, \phi). \tag{10}$$

Within the alternative Eulerian framework, perturbations are represented by the change $\tilde{f}'$ in the variable $\tilde{f}$, as experienced at fixed angular coordinates (θ, ϕ):

$$\tilde{f}_\mathrm{p}(\theta, \phi) = \tilde{f}(\theta, \phi) + \tilde{f}'(\theta, \phi). \tag{11}$$

Generally speaking, the unperturbed and perturbed states represented by this expression refer to differing fluid elements, in contrast

to the Lagrangian expression (6) in which the same fluid element is followed through the perturbation process. Note that in *neither* expression does the radial coordinate r appear; since the surface of the star is under consideration, this coordinate is not an independent variable, but rather a function of θ and ϕ (see Section 2.1).

Unno et al. (1989) give an expression (their equation 13.21), which relates Lagrangian ($\delta\tilde{f}$) and Eulerian ($\tilde{f}'$) perturbations, to first order in the pulsation amplitude ϵ (and see also Cox 1980). Their expression is applicable to variables $f(r,\theta,\phi)$ defined on $\mathbb{R}^3$; however, in the present context of surface variables $\tilde{f}$, which are defined on $\mathbb{R}^2$, it is easy to show that the appropriate relation will be

$$\delta\tilde{f}(\theta,\phi) = \tilde{f}'(\theta,\phi) + \left[\tilde{\boldsymbol{\xi}}_\perp(\theta,\phi)\cdot\nabla_\perp\right]\tilde{f}(\theta,\phi). \tag{12}$$

Unlike the equivalent $\mathbb{R}^3$-expression, the radial component $\tilde{\xi}_r$ of the displacement vector plays no part here, reflecting the previously mentioned fact that the radial coordinate r is *not* an independent variable when considering the stellar surface. It should be observed that, in those situations where $\tilde{f}$ is constant across the surface in equilibrium state, the Lagrangian and Eulerian perturbations will be numerically equal – although, of course, their semantics remain different. Also note that the above relationship may be applied to vector quantities, such as the fluid velocity (see Section 2.2).

In the following sections, this relationship (12) is used to obtain expressions for the perturbed radius, velocity, temperature and geometry of a pulsating star, within both the Lagrangian and Eulerian frameworks, in terms of the radial component of the displacement vector $\tilde{\boldsymbol{\xi}}$. In each case, a numerical comparison of the derived expressions is undertaken, to ascertain the nature of any differences that might arise between the alternative frameworks.

2.1 Radius

The spatial location of a stellar surface may be described by the function $\tilde{r}(\theta,\phi)$, which specifies the radial coordinate of the surface in terms of the corresponding angular ones (θ,ϕ). In the unperturbed state, this function is trivially given by $\tilde{r}(\theta,\phi) = \tilde{r}$, the equilibrium radius introduced previously. Under the influence of pulsation, the new surface is described, in the Lagrangian formalism (cf. equation 6), by

$$\tilde{r}_\mathrm{p}(\tilde{\theta}_\mathrm{p},\tilde{\phi}_\mathrm{p}) = \tilde{r} + \delta\tilde{r}(\theta,\phi). \tag{13}$$

The Lagrangian perturbation $\delta\tilde{r}$ may be identified with the radial component of the displacement vector $\tilde{\boldsymbol{\xi}}$, since the latter quantity is the amount by which the photospheric fluid element, initially located at angular coordinates (θ,ϕ), is displaced outward by the pulsation. Hence, the above expression becomes

$$\tilde{r}_\mathrm{p}(\tilde{\theta}_\mathrm{p},\tilde{\phi}_\mathrm{p}) = \tilde{r} + \tilde{\xi}_r(\theta,\phi), \tag{14}$$

which constitutes the Lagrangian perturbed radius description. Observe that the horizontal components $\{\tilde{\xi}_\theta,\tilde{\xi}_\phi\}$ of the displacement vector appear implicitly in this expression, through the definitions of the perturbed angular coordinates (cf. equations 7–10).

Since the equilibrium radius $\tilde{r}$ is independent of θ and ϕ, the Eulerian radius perturbation $\tilde{r}'$ will be numerically equal to the Lagrangian one,

$$\tilde{r}'(\theta,\phi) = \delta\tilde{r}(\theta,\phi). \tag{15}$$

Application of equation (11) then leads to the Eulerian expression for the perturbed radius,

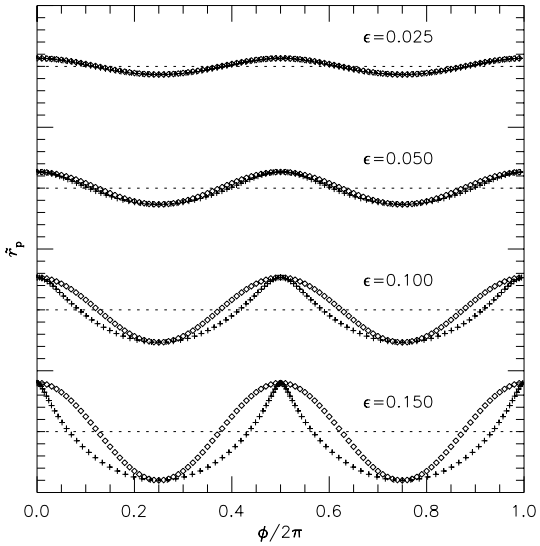

Figure 1. The perturbed radius around the equator for an $\{\ell, m\} = \{2, -2\}$ pulsation mode at four selected amplitudes ϵ. Crosses (diamonds) show profiles calculated using the Lagrangian (Eulerian) formalisms, and the dotted lines indicate the equilibrium state. The vertical (ordinate) scaling is arbitrary.

$$\tilde{r}_\mathrm{p}(\theta,\phi) = \tilde{r} + \tilde{\xi}_r(\theta,\phi). \tag{16}$$

In contrast to the Lagrangian result (34), the horizontal components of the displacement vector are entirely absent from this expression: only the radial component $\tilde{\xi}_r$ is required to locate the perturbed stellar surface within an Eulerian framework.

At this juncture, it is instructive to compare the surface radius distributions predicted by the two formalisms. Consider, therefore, an $\{\ell, m\} = \{2, -2\}$ prograde sectoral pulsation mode, with relative horizontal displacement amplitude $K = 1.25$ (cf. equation 3); these parameters are somewhat arbitrary, but the following discussion does not depend on their choice. Fig. 1 shows the perturbed radius $\tilde{r}_\mathrm{p}$ around the stellar equator for this mode, calculated using the appropriate expressions (14) and (16) at four selected values of the pulsation amplitude ϵ. In each case, the perturbed state was found by first representing the equilibrium equator with a grid of 100 equally spaced points, which were then subject to the appropriate variations in radius and, where necessary, azimuth. These points may directly be taken to represent physical fluid elements in the illustrated Lagrangian profiles; their interpretation in the Eulerian ones is, however, purely mathematical.

Although the perturbed radii predicted by each formalism agree at small amplitudes, both appearing sinusoidal, a discrepancy between the two becomes apparent as the amplitude is increased. The Eulerian $\tilde{r}_\mathrm{p}$ remain morphologically unchanged, but the Lagrangian ones exhibit a degree of cusping, with sharp crests and blunt troughs, which becomes progressively more pronounced toward larger values of ϵ. This cusping is due to the horizontal displacements $\{\tilde{\xi}_\theta,\tilde{\xi}_\phi\}$, which generate the coordinate perturbations $\{\delta\tilde{\theta},\delta\tilde{\phi}\}$ in the Lagrangian formalism (cf. equations 7–10). As mentioned previously, these displacements are not considered within the Eulerian framework, which explains why the surface profiles calculated using the latter remain undistorted at higher amplitudes.

The combined dependence of the Lagrangian $\tilde{r}_\mathrm{p}$ on radial and horizontal displacements suggests that the cusping will vary as the square of the pulsation amplitude. Fig. 2 shows $\Delta(\epsilon)$, the maximum absolute difference over the equatorial range $0 \leqslant \phi \leqslant 2\pi$ between

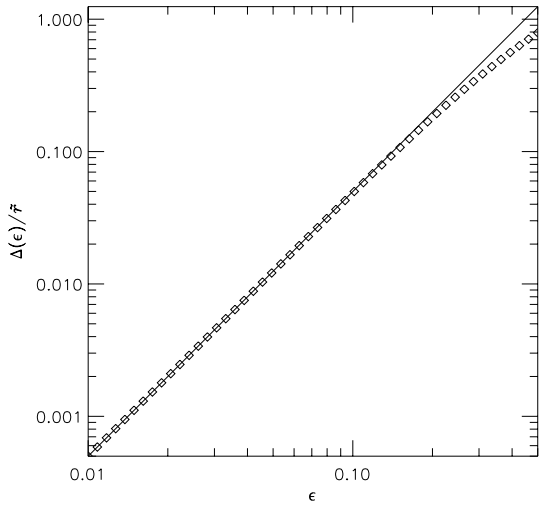

Figure 2. The maximum absolute difference $\Delta(\epsilon)$ (diamonds), normalized to the equilibrium radius $\tilde{r}$, between surface radius $\tilde{r}_\mathrm{p}$ calculated with the Lagrangian and Eulerian formalisms, as a function of the pulsation amplitude ϵ. The solid line has a slope of d log Δ/d log $\epsilon = 2$ and passes through the leftmost point.

the Lagrangian and Eulerian surface radii, as a function of the pulsation amplitude ϵ. For each value of ϵ considered, it was necessary to use cubic spline interpolation (Press et al. 1992) to calculate the Lagrangian radii on the same ϕ-abscissae as the corresponding Eulerian ones, since equation (14) represents the former as a function of $\tilde{\phi}_\mathrm{p}$ rather than ϕ. Also plotted in the figure is the line with slope d log Δ/dlog $\epsilon = 2$, which passes exactly through the point at the smallest value of ϵ. The close agreement between this line and the $\Delta(\epsilon)$ points indicates that the cusping – and the discrepancies between the two formalisms – does indeed increase quadratically with the pulsation amplitude.

The source of these discrepancies lies in the linearization procedure adopted in the derivation of the expressions (2) and (3) for the components of $\tilde{\boldsymbol{\xi}}$. Terms of second and higher order in the pulsation amplitude are discarded from the perturbed hydrodynamical equations for mass and momentum conservation (see Unno et al. 1989), and it is these missing non-linear terms that lead to the divergence between the two formalisms toward larger values of the amplitude ϵ. If the terms were retained, and in addition the missing higher-order terms in equation (12), then the discrepancies between the formalisms would vanish. Conversely, if ϵ is restricted to values small enough ($\ll 1$) that the linear approximation – and, by implication, expressions (2) and (3) – can be considered valid, then any discrepancies will be sufficiently small ($\sim\epsilon^2$) that they may be neglected.

2.2 Velocity

Over a pulsation cycle, a given photospheric element will follow a closed trajectory determined by the temporal evolution of the displacement vector of the element, $\tilde{\boldsymbol{\xi}}$. The instantaneous velocity of the element, as it moves around this trajectory, may be identified with the Lagrangian perturbation $\delta\tilde{\boldsymbol{v}}$ to the surface velocity – that is,

$$\delta\tilde{\boldsymbol{v}}(\theta,\phi) = \frac{\partial}{\partial t}\tilde{\boldsymbol{\xi}}(\theta,\phi,t), \tag{17}$$

where, for the moment, the time dependence of the displacement vector $\tilde{\boldsymbol{\xi}}$ is explicitly indicated. Since only static equilibrium configurations are considered herein, the unperturbed velocity $\tilde{\boldsymbol{v}}$ is zero; accordingly, the perturbed surface velocity within the Lagrangian framework will be given by

$$\tilde{\boldsymbol{v}}_\mathrm{p}(\tilde{\theta}_\mathrm{p},\tilde{\phi}_\mathrm{p}) = \frac{\partial}{\partial t}\tilde{\boldsymbol{\xi}}(\theta,\phi,t). \tag{18}$$

It should be remarked that, although the velocity of the fluid element is defined here at the perturbed angular coordinates $(\tilde{\theta}_\mathrm{p},\tilde{\phi}_\mathrm{p})$, the components of this velocity are relative to the unit basis vectors $\{\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\theta\}$ at the *unperturbed* coordinates (θ,ϕ): the derivative $\partial/\partial t$ does not operate on the unit vectors appearing in the definition (1) of $\tilde{\boldsymbol{\xi}}$. This fact is of importance when contrasting the components of a vector perturbed within each formalism (see below and Section 2.4).

Since $\tilde{\boldsymbol{v}}$ vanishes in the equilibrium configuration, the Eulerian and Lagrangian perturbations to the surface velocity will be equal,

$$\tilde{\boldsymbol{v}}'(\theta,\phi) = \delta\tilde{\boldsymbol{v}}(\theta,\phi), \tag{19}$$

as was the case when considering the radius, and the perturbed velocity within the Eulerian formalism is given by

$$\tilde{\boldsymbol{v}}_\mathrm{p}(\theta,\phi) = \frac{\partial}{\partial t}\tilde{\boldsymbol{\xi}}(\theta,\phi,t). \tag{20}$$

Evidently, although the horizontal displacement components $\{\tilde{\xi}_\theta,\tilde{\xi}_\phi\}$ play no part in the Eulerian expression (16) for the perturbed radius, they *are* important here in generating horizontal velocity fields.

As with the radius perturbations, it is useful to contrast the distribution of surface velocity predicted by the two formalisms. Accordingly, Fig. 3 shows $\tilde{\boldsymbol{v}}_{r,\mathrm{p}}$ and $\tilde{\boldsymbol{v}}_{\phi,\mathrm{p}}$ – the radial and azimuthal components of $\tilde{\boldsymbol{v}}_\mathrm{p}$ – around the equator of the star, evaluated for the previously adopted $\{\ell, m\} = \{2, -2\}$ pulsation parameters using equations (18) and (20). Since the former (Lagrangian) equation expresses $\tilde{\boldsymbol{v}}_\mathrm{p}$ in terms of the basis vectors at the unperturbed coordinates (θ,ϕ) (see above), it was necessary to use the orthogonal transformation

$$\begin{pmatrix}\boldsymbol{e}_r\\ \boldsymbol{e}_\theta\\ \boldsymbol{e}_\phi\end{pmatrix} = \begin{pmatrix}\cos\delta\tilde{\theta}\cos\delta\tilde{\phi} & -\sin\delta\tilde{\theta} & -\cos\delta\tilde{\theta}\sin\delta\tilde{\phi}\\ \sin\delta\tilde{\theta}\cos\delta\tilde{\phi} & \cos\delta\tilde{\theta} & -\sin\delta\tilde{\theta}\sin\delta\tilde{\phi}\\ \sin\delta\tilde{\phi} & 0 & \cos\delta\tilde{\phi}\end{pmatrix}\begin{pmatrix}\boldsymbol{e}_{r,\mathrm{p}}\\ \boldsymbol{e}_{\theta,\mathrm{p}}\\ \boldsymbol{e}_{\phi,\mathrm{p}}\end{pmatrix} \tag{21}$$

to find the components of $\tilde{\boldsymbol{v}}_\mathrm{p}$ relative to the unit basis vectors $\{\boldsymbol{e}_{r,\mathrm{p}}, \boldsymbol{e}_{\theta,\mathrm{p}}, \boldsymbol{e}_{\phi,\mathrm{p}}\}$ at the perturbed coordinates $(\tilde{\theta}_\mathrm{p},\tilde{\phi}_\mathrm{p})$. This transformation corresponds to rotating the axes used to define the components of $\tilde{\boldsymbol{v}}_\mathrm{p}$, while leaving the magnitude and direction of this vector unaltered.

Discrepancies are once again apparent between the two formalisms: with increasing amplitude ϵ, the Lagrangian velocity fields become progressively more distorted, relative to the Eulerian ones. For the azimuthal components of $\tilde{\boldsymbol{v}}_\mathrm{p}$, these distortions resemble the cusping already encountered in Fig. 1, while for the radial components they appear more as a slight asymmetry, where the maxima (minima) of $\tilde{\boldsymbol{v}}_{r,\mathrm{p}}$ are located at smaller (larger) values of ϕ in the Lagrangian formalism than in the Eulerian one. Using the same approach as adopted in Fig. 2, it was confirmed that these distortions, for both components of $\tilde{\boldsymbol{v}}_\mathrm{p}$, scale quadratically with ϵ. The reason why they are more pronounced in the azimuthal components, than in the radial ones, can be traced to the fact that the former are 2.5 ($=mK$) times stronger than the latter for the pulsation parameters adopted, and the onset of significant distortion therefore occurs at smaller ϵ.

The interpretation of these findings is essentially the same as that advanced previously in Section 2.1: the discrepancies between the velocity fields predicted within each formalism are purely a

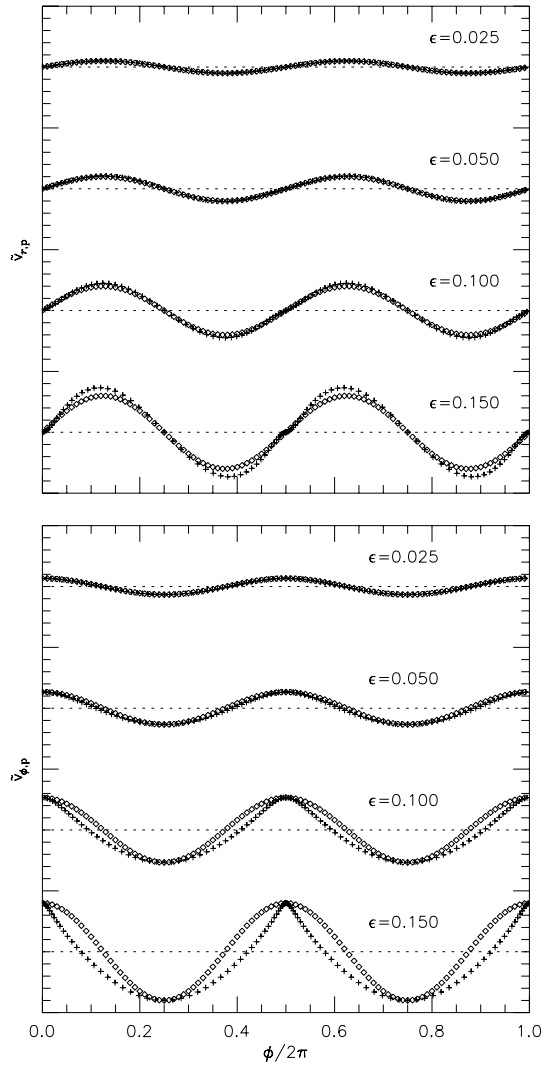

Figure 3. As in Fig. 1, except that the radial (upper panel) and azimuthal (lower panel) components of the perturbed surface velocity are shown. The arbitrary vertical (ordinate) scaling is different in the two panels.

symptom of the continued use of linear-theory results beyond the reasonable limits of the linear approximation. The two alternative expressions (18) and (20) for $\tilde{\boldsymbol{v}}_{\mathrm{p}}$ will lead to almost identical velocity distributions, if each component of $\tilde{\boldsymbol{v}}_{\mathrm{p}}$ remains small enough that the linear approximation can still be considered valid. Therefore, Balona's remark (cf. Section 1) is certainly misleading: if the Lagrangian and Eulerian formalisms predict equivalent velocity fields within the linear limit, why must the latter be used in preference to the former when calculating line profiles?

In fact, Osaki (personal communication) has revealed that his advice to Balona was in reference to rotating stars. When regarded from an inertial frame of reference, the velocity field $\tilde{\boldsymbol{v}}$ of a rotating star is both non-zero and non-uniform. Accordingly, equation (19) will not apply, and the Eulerian and Lagrangian velocity perturbations will differ. In the context of Balona's (1987) study, it was correct to use an Eulerian expressions for $\tilde{\boldsymbol{v}}_{\mathrm{p}}$, since his surface grid was itself Eulerian (i.e. he did not allow the angular coordinates of the grid points to vary).

However, he could have just as easily combined the Lagrangian expressions for $\tilde{\boldsymbol{v}}_{\mathrm{p}}$ (and other quantities) with a Lagrangian surface grid, where the grid points were free to move in all three coordinates. In fact, De Ridder (2001) has successfully applied such an approach in his models of pulsating B stars. Therefore, Osaki's advice clearly requires a qualifier, if it is not to mislead: when calculating line profiles for a pulsating star, one should use Eulerian expressions for the velocity perturbation, *if* one is working with an Eulerian grid.

2.3 Temperature

Within the adiabatic and Cowling (1941) approximations, where perturbations to the specific entropy and gravitational potential are neglected, Buta & Smith (1979) demonstrated that the Lagrangian perturbation to the surface temperature will be given by

$$\delta\tilde{T}(\theta,\phi) = \nabla_{\mathrm{ad}}\tilde{T}\left[K\ell(\ell+1) - 4 - \frac{1}{K}\right]\frac{\tilde{\xi}_r(\theta,\phi)}{\tilde{r}}, \tag{22}$$

where ∇_{ad} and $\tilde{T}$ are the adiabatic temperature gradient and temperature, respectively, of the equilibrium surface. Hence, the Lagrangian formalism gives the perturbed surface temperature as

$$\tilde{T}_{\mathrm{p}}(\tilde{\theta}_{\mathrm{p}},\tilde{\phi}_{\mathrm{p}}) = \tilde{T} + \left\{\nabla_{\mathrm{ad}}\left[K\ell(\ell+1) - 4 - \frac{1}{K}\right]\frac{\tilde{\xi}_r(\theta,\phi)}{\tilde{r}}\right\}\tilde{T}. \tag{23}$$

As was the case in preceding sections, the Eulerian and Lagrangian perturbations to $\tilde{T}$ will be numerically equal,

$$\tilde{T}'(\theta,\phi) = \delta\tilde{T}(\theta,\phi). \tag{24}$$

It should be stressed that this Eulerian perturbation $\tilde{T}'(\theta,\phi)$ is not to be confused with the similar-looking quantity $T'(\tilde{r},\theta,\phi)$. The former refers to the temperature change experienced by a point at fixed angular coordinates, which follows the perturbed stellar surface in and out in the radial direction; the latter, to the corresponding change experienced by a point at fixed angular coordinates *and* fixed radial coordinate $r = \tilde{r}$. Combining the above expression with equation (22) leads to the Eulerian result,

$$\tilde{T}_{\mathrm{p}}(\theta,\phi) = \tilde{T} + \left\{\nabla_{\mathrm{ad}}\left[K\ell(\ell+1) - 4 - \frac{1}{K}\right]\frac{\tilde{\xi}_r(\theta,\phi)}{\tilde{r}}\right\}\tilde{T}. \tag{25}$$

Both formulations (23) and (25) for $\tilde{T}_{\mathrm{p}}$ indicate that the temperature variations are proportional to the radial displacement $\tilde{\xi}_r$. Therefore, apart from differences in amplitude and sign, the perturbed surface temperature due to the $\{\ell, m\} = \{2, -2\}$ mode considered previously will follow the radius distributions shown in Fig. 1. As before, discrepancies between the two formalisms will be revealed toward larger ϵ, symptomatic of an abuse of the linear approximation.

Although the adiabatic approximation greatly simplifies the treatment of temperature perturbations, it often lacks physical realism in the outer layers of stars, where the thermal time-scale is short enough to permit significant thermal transfer between adjacent fluid elements (see, e.g., Cugier et al. 1994; Townsend 2002). One result of non-adiabaticity is to introduce a phase lag between radius and temperature variations, which will modify the nature of the discrepancies between the formalisms. Fig. 4 illustrates a typical case, showing the surface temperature distributions for the $\{\ell, m\} = \{2, -2\}$ mode, with the inclusion of a $\pi/3$-radian phase lag between $\tilde{\xi}_r$ and $\tilde{T}_{\mathrm{p}}$. Comparing this figure with Fig. 1, it is evident that the lag introduces an asymmetry in the Lagrangian distribution of $\tilde{T}_{\mathrm{p}}$, whereby the temperature maxima (minima) shift to smaller (larger) ϕ than found in the adiabatic case (this result explains the asymmetry seen in the Lagrangian $\tilde{v}_{r,\mathrm{p}}$ shown in Fig. 3, since the radial displacement lags the radial velocity by $\pi/2$ rad). In spite of this asymmetry, however, it was found that discrepancies between the two formalisms, in the presence of a phase lag, still scale quadratically with ϵ. Accordingly, the conclusions of Section 2.1 may be

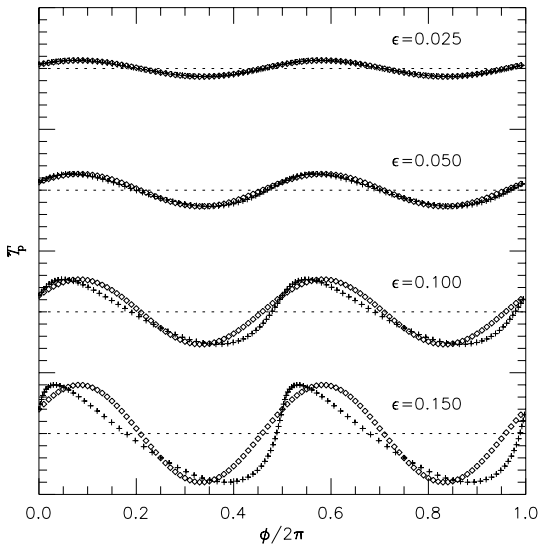

Figure 4. As in Fig. 1, except that the perturbed surface temperature is shown, assuming a $\pi/3$-radial phase lag between radius and temperature variations.

applied to temperature perturbations, notwithstanding the influence of non-adiabaticity: while ϵ remains small, the two formalisms will lead to equivalent perturbed temperature distributions at the stellar surface.

2.4 Geometry

In calculating the observer-directed flux for a given point of a perturbed stellar surface model grid (cf. Section 1), it is necessary to know the area element vector $\mathrm{d}\boldsymbol{A}_\mathrm{p}$ associated with the point. The magnitude of this vector gives the area of the surface 'patch' which the grid point represents, while its direction indicates the local surface normal. Both Buta & Smith (1979) and Heynderickx et al. (1994) give expressions for $\mathrm{d}\boldsymbol{A}_\mathrm{p}$ in a star undergoing non-radial pulsation. Unlike results presented in previous sections, however, these expressions appear markedly different; in particular, the coordinate perturbations $(\delta\tilde{\theta}, \delta\tilde{\phi})$ appear *explicitly* in the latter work, but are entirely absent in the former. As will be demonstrated, these differences can be traced, in their entirety, to the alternative formalisms adopted by the authors.

The Eulerian approach leads to the simpler expression for $\mathrm{d}\boldsymbol{A}_\mathrm{p}$, and is therefore a natural starting point for the discussion. This vector, for a surface patch spanning $\mathrm{d}\theta$ in co-latitude and $\mathrm{d}\phi$ in azimuth (both assumed small), may be written as (e.g. Korn & Korn 1968)

$$\mathrm{d}\boldsymbol{A}_\mathrm{p} = \left[\frac{\partial\tilde{\boldsymbol{r}}_\mathrm{p}(\theta,\phi)}{\partial\theta}\times\frac{\partial\tilde{\boldsymbol{r}}_\mathrm{p}(\theta,\phi)}{\partial\phi}\right]\mathrm{d}\theta\,\mathrm{d}\phi, \tag{26}$$

where $\tilde{\boldsymbol{r}}_\mathrm{p}$ is the position vector of the point on the perturbed surface at angular coordinates (θ, ϕ),

$$\tilde{\boldsymbol{r}}_\mathrm{p}(\theta,\phi) = \left[\tilde{r} + \tilde{\xi}_r(\theta,\phi)\right]\boldsymbol{e}_r. \tag{27}$$

Substituting this expression into equation (26), and discarding all quadratic terms in $\tilde{\xi}_r$ (in accordance with the linear approximation), leads to the result

$$\mathrm{d}\boldsymbol{A}_\mathrm{p}(\theta,\phi) = \mathrm{d}\boldsymbol{A}(\theta,\phi) + \left[(2\boldsymbol{e}_r - \tilde{r}\nabla_\perp)\frac{\tilde{\xi}_r(\theta,\phi)}{\tilde{r}}\right]\mathrm{d}A, \tag{28}$$

where

$$\mathrm{d}\boldsymbol{A}(\theta,\phi) = \boldsymbol{e}_r\,\mathrm{d}A \tag{29}$$

is the unperturbed area element vector, of magnitude $\mathrm{d}A \equiv \tilde{r}^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$. Taking into consideration the differing nomenclatures, the above two expression are in full agreement with those derived by Buta & Smith (1979), indicating that these authors chose an Eulerian framework in which to represent $\mathrm{d}\boldsymbol{A}_\mathrm{p}$.

The second term on the right-hand side of equation (28) may be identified with the Eulerian perturbation $\mathrm{d}\boldsymbol{A}'$ to the area element vector. It is instructive to discuss briefly the significance of the two operators appearing within the brackets of this term, which act on $\tilde{\xi}_r$. The first represents a translation of the surface patch in the radial direction. The extrinsic curvature of the equilibrium surface, which reflects its embedding in three-dimensional space (see, e.g. Schutz 1993), results in a change in the area $\mathrm{d}A$ of the patch, although the direction of its surface normal remains unaltered. This process is illustrated in panel (i) of Fig. 5. The second operator represent an in-place rotation of the patch, which tilts its surface normal away from the radial direction, but – to first order in $|\tilde{\xi}_r|$ – does not affect the area. This rotation is illustrated in panel (ii) of the figure.

It is possible to derive the Lagrangian perturbation $\delta(\mathrm{d}\boldsymbol{A})$ to the area element vector from the corresponding Eulerian one $\mathrm{d}\boldsymbol{A}'$. However, in this particular instance, the relationship (12) does not give the correct transformation between the two formalisms. This is because the area element vector scales with the dimensions, $\mathrm{d}\theta$ and $\mathrm{d}\phi$, of the patch under consideration, and therefore must be considered an extensive quantity – much like a volume or mass element. While equation (12) is appropriate for intensive quantities, such as the velocity or the temperature, it does not account for the influence of perturbations on the material boundaries of extensive quantities, and cannot be applied to the latter.

Instead, the Reynolds transport theorem must be used. Expressed in its canonical form (e.g. Shivamoggi 1997), this theorem relates the material (Lagrangian) and fixed-position (Eulerian) time derivatives of spatial integrals. By replacing the time derivatives with the corresponding one-off perturbations, and restricting the spatial

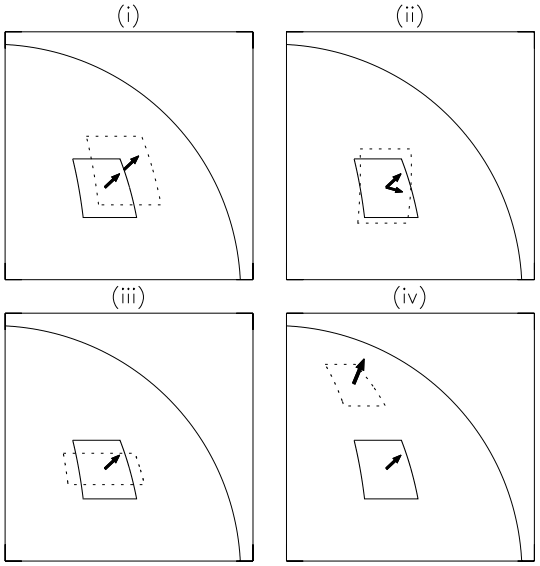

Figure 5. Illustrations of perturbations to a surface patch; solid (dotted) lines show the unperturbed (perturbed) patch boundaries, while the arrows show the corresponding local surface normal vectors. (i) radial translation, (ii) rotation (iii) horizontal expansion/contraction, (iv) horizontal translation.

integrals to a small-enough region, such that all integrands are constant, it is straightforward to demonstrate that the theorem leads to the relation

$$\delta(\mathrm{d}\boldsymbol{A})(\theta,\phi) = \mathrm{d}\boldsymbol{A}'(\theta,\phi) + \left\{\left[\nabla_\perp\cdot\tilde{\boldsymbol{\xi}}_\perp(\theta,\phi) + \tilde{\boldsymbol{\xi}}_\perp(\theta,\phi)\cdot\nabla_\perp\right]\boldsymbol{e}_r\right\}\mathrm{d}A, \quad (30)$$

where

$$\nabla_\perp\cdot \equiv \frac{\boldsymbol{e}_\theta\cdot}{\tilde{r}\sin\theta}\frac{\partial}{\partial\theta}\sin\theta + \frac{\boldsymbol{e}_\phi\cdot}{\tilde{r}\sin\theta}\frac{\partial}{\partial\phi} \quad (31)$$

is the tangential part of the spherical-polar divergence operator. The occurrence of this operator in the expression for $\delta(\mathrm{d}\boldsymbol{A})$ represents the horizontal expansion or contraction of a surface patch, arising from differential displacements of its material boundaries.

If, using equation (3), the horizontal displacement vector $\tilde{\boldsymbol{\xi}}_\perp$ is eliminated from the above relation (30), then the Lagrangian perturbed area element vector can be expressed in terms of $\tilde{\xi}_r$ as

$$\mathrm{d}\boldsymbol{A}_\mathrm{p}(\tilde{\theta}_\mathrm{p},\tilde{\phi}_\mathrm{p}) = \mathrm{d}\boldsymbol{A}(\theta,\phi) + \left\{[2\boldsymbol{e}_r - \tilde{r}\nabla_\perp - K\ell(\ell+1)\boldsymbol{e}_r + K\tilde{r}\nabla_\perp]\frac{\tilde{\xi}_r(\theta,\phi)}{\tilde{r}}\right\}\mathrm{d}A; \quad (32)$$

as with the velocity variations discussed in Section 2.2, it is understood that $\mathrm{d}\boldsymbol{A}_\mathrm{p}$ is defined here in terms of the unit basis vectors $\{\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi\}$ at the unperturbed coordinates. In deriving this expression, use has been made of the eigenvalue relation for the spherical harmonics (e.g. Arfken 1970),

$$\tilde{r}^2\nabla_\perp\cdot\nabla_\perp Y_\ell^m(\theta,\phi) = -\ell(\ell+1)Y_\ell^m(\theta,\phi), \quad (33)$$

and also of the algebraic identity

$$\left[\tilde{\boldsymbol{\xi}}_\perp(\theta,\phi)\cdot\nabla_\perp\right]\boldsymbol{e}_r = \frac{\tilde{\boldsymbol{\xi}}_\perp(\theta,\phi)}{\tilde{r}}. \quad (34)$$

The rightmost term of equation (32) is the Lagrangian perturbation $\delta(\mathrm{d}\boldsymbol{A})$ to the area element vector. As with the Eulerian perturbation $\mathrm{d}\boldsymbol{A}'$, it is instructive to discuss briefly the significance of the operators appearing within the brackets of this term. The first and second operators originate from the Eulerian perturbation, and have already been discussed (panels i and ii of Fig. 5, respectively). The third operator represents a change in the area $\mathrm{d}A$ of the patch, and originates from the divergence term in equation (30). This change can arise in one of two ways: either by perturbations to the dimensions ($\mathrm{d}\theta$ and $\mathrm{d}\phi$) of the patch, as shown in panel (iii) of Fig. 5, or by a horizontal translation of the patch in the polar direction. In the latter case, which is shown by the area variation in panel (iv) of the figure, the patch dimensions remain constant, but the intrinsic curvature of the equilibrium surface (see Schutz 1993) leads to a growth or shrinkage of $\mathrm{d}A$. Finally, the fourth operator in the $\delta(\mathrm{d}\boldsymbol{A})$ term arises from *any* horizontal translation of the patch across the stellar surface. The angular dependence of the basis vector $\boldsymbol{e}_r$, another manifestation of the previously mentioned extrinsic curvature, results in a change in the surface normal of the patch. This process is illustrated by the normal vector variation in panel (iv) of the figure.

Inspection of equation (32) reveals an interesting property of the Lagrangian $\mathrm{d}\boldsymbol{A}$ perturbation: when K is equal to unity, the tangential components of $\delta(\mathrm{d}\boldsymbol{A})$ vanish completely. In such situations, what is happening is that the rotation of a surface patch (panel ii) is being exactly offset by its translation around the star (panel iv), with the result that the patch normal remains parallel to the unperturbed radial unit vector $\boldsymbol{e}_r$. However, the normal will *not* be parallel to the radial unit vector $\boldsymbol{e}_{r,\mathrm{p}}$ at the perturbed angular coordinates $(\tilde{\theta}_\mathrm{p}, \tilde{\phi}_\mathrm{p})$ of the patch; accordingly, the stellar surface will not retain its spherical equilibrium geometry. Hence, the $K = 1$ situation, while mathematically interesting, is by no means anomalous.

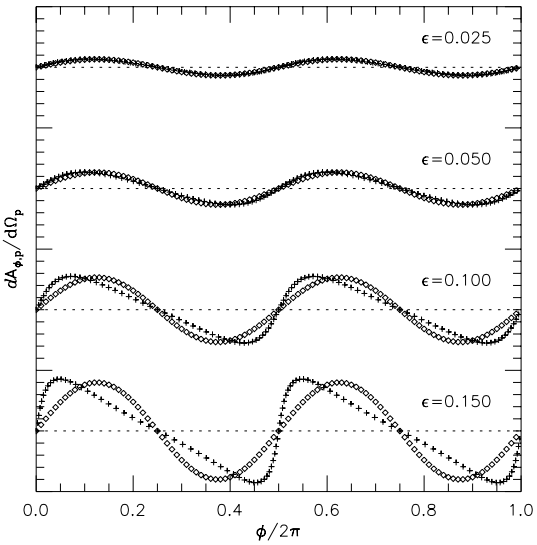

Figure 6. As in Fig. 1, except that the azimuthal component of the perturbed area element vector is shown, normalized by the perturbed solid angle $\mathrm{d}\Omega_\mathrm{p}$.

As was undertaken with the other surface variables, it is useful to contrast the perturbed area element vector predicted within each formalism. In the present case, a direct comparison is not appropriate, since the perturbed patches represented by $\mathrm{d}\boldsymbol{A}_\mathrm{p}$ subtend differing solid angles in the two formalisms, due to the above-mentioned growth and shrinkage of the patch within the Lagrangian framework. However, a like-for-like comparison *can* be made between the azimuthal components $\mathrm{d}\boldsymbol{A}_{\phi,\mathrm{p}}$ of $\mathrm{d}\boldsymbol{A}_\mathrm{p}$, if they are first normalized by the perturbed solid angle

$$\mathrm{d}\Omega_\mathrm{p} \equiv \frac{\mathrm{d}\boldsymbol{A}_{r,\mathrm{p}}}{\tilde{r}_\mathrm{p}^2} \quad (35)$$

associated with the patch that $\mathrm{d}\boldsymbol{A}_\mathrm{p}$ represents (there is little point in comparing the correspondingly normalized radial components $\mathrm{d}\boldsymbol{A}_{r,\mathrm{p}}$, since they will always be equal to $\tilde{r}_\mathrm{p}^2$).

Accordingly, Fig. 6 shows $\mathrm{d}\boldsymbol{A}_{\phi,\mathrm{p}}/\mathrm{d}\Omega_\mathrm{p}$ around the stellar equator, calculated within each formalism using the same pulsation parameters as in the preceding sections. As with the velocity variations (cf. Section 2.2), the transformation (21) was used to express the Lagrangian equation (28) in terms of the unit basis vectors $\{\boldsymbol{e}_{r,\mathrm{p}}, \boldsymbol{e}_{\theta,\mathrm{p}}, \boldsymbol{e}_{\phi,\mathrm{p}}\}$ at the perturbed angular coordinates $(\tilde{\theta}_\mathrm{p}, \tilde{\phi}_\mathrm{p})$. Once more, discrepancies are apparent between the formalisms, with the Lagrangian $\mathrm{d}\boldsymbol{A}_{\phi,\mathrm{p}}$ exhibiting the asymmetric cusping encountered previously. It comes as no surprise that these discrepancies scale quadratically with ϵ, and are therefore unimportant at amplitudes small enough that the linear approximation remains valid.

It is readily demonstrated that equation (32) is in full accordance with the expressions set out by Heynderickx et al. (1994) for the Cartesian components of $\mathrm{d}\boldsymbol{A}_\mathrm{p}$. Therefore, the treatment of these authors, which is evidently framed within the Lagrangian formalism, is seen to be perfectly consistent with the Eulerian approach of Buta & Smith (1979): both will result in essentially the same perturbed surface geometries for a pulsating star, so long as ϵ is kept small. The two extra K-dependent terms appearing in the Lagrangian expression (32) for $\mathrm{d}\boldsymbol{A}_\mathrm{p}$, which are absent from the corresponding Eulerian one (28), originate from the fact that the area element vector is (i)

an extensive variable and (ii) non-constant across the stellar surface in the equilibrium state.

3 DISCUSSION

In the preceding sections, Lagrangian and Eulerian expressions for non-radial perturbations to the surface radius, velocity, temperature and geometry have been derived within the linear approximation. In each case, the Eulerian result has been obtained from the corresponding Lagrangian one (cf. Sections 2.1–2.3), or vice versa (cf. Section 2.4), an approach chosen to emphasize that the two formalisms simply represent alternative frameworks within which to describe the same underlying physical processes.

It was found that, when linear-theory expressions are used for the displacement vector $\tilde{\boldsymbol{\xi}}$ and associated quantities at the stellar surface, discrepancies between the two formalisms become apparent toward larger pulsation amplitudes. By ascertaining that in all cases these discrepancies scale quadratically with the amplitude, it was established that they are in fact an artefact of the linearization procedure. Accordingly, as long as the amplitude is constrained to values consistent with adoption of the linear approximation (typically, $\epsilon \lesssim 2.5 \times 10^{-2}$, and similarly for the horizontal amplitude ϵK), any discrepancies between the formalisms will be small enough to be unimportant. It should be remarked that, in almost all stars observed to be undergoing non-radial pulsation, the inferred amplitudes of modes appear to fall within these 'recommended' limits.

In summary, although the alternative formalisms can lead to seemingly incompatible expressions for perturbed surface variables within the linear approximation (compare, in particular, equations (28) and (32) for the perturbed area element vector), both provide numerically equivalent descriptions of a stellar surface at small pulsation amplitudes. Accordingly, it is concluded that neither can be claimed to be more 'correct' than the other, and the choice between the two must be left to the discretion of the modeller.

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