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Surface Roughness Effects in Hydrodynamic Lubrication: the Flow Factor Method

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SURFACE ROUGHNESS EFFECTS IN HYDRODYNAMIC LUBRICATION:
THE FLOW FACTOR METHOD

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ABSTRACT

The average flow model of Patir and Cheng [1,2] for obtaining an average Reynolds equation in the presence of two-dimensional surface roughness is extended and generalized. Expectation values of the flow factors appearing in the formalism are calculated by means of a perturbation expansion of the pressure in a nominal parallel film. Terms in the series are evaluated using the unperturbed Green function which permits ensemble averaging to be performed directly on the solution. Calculations are carried to second order which involves only two-point correlation functions of the two rough surfaces. Perturbation results agree well with results of the earlier numerical simulation until surface contact becomes important when both approaches are inadequate. The theory displays the dependence of the flow factors on the roughness parameters in simple closed form, leading to improved understanding of the average flow method.

INTRODUCTION

In the theoretical study of the effects of surface roughness on the performance of lubricant films, the objective is to develop a form of the Reynolds equation from which the expected pressure at each point in the fluid may be determined. While limitations on the ability of the Reynolds equation to describe flow between rough boundaries continues to be a topic for much discussion [3], the wide variety of practical surface textures falling within the generally accepted domain of validity nonetheless justifies this endeavor. Among the basic equations of mathematical physics, the Reynolds equation is unusual in that boundary conditions on the flow determine the form of the equation itself, rather than serving only to define a particular solution of a single equation. These boundary conditions, giving the shape and velocities of the surfaces, enter as a result of the partial integration of the Navier-Stokes equation across the thin dimension of the film. In the roughness problem, where the shape of the bounding surfaces is a stochastic process, it is this feature which offers the possibility of constructing the required ensemble-averaged form. It is not the purpose of this paper to discuss the many previous attempts to develop such equations of which several critical reviews already exist [4]. However, as noted by Elrod [3], most treatments of the Reynolds equation have not encompassed the general situation of two-dimensional roughness, with consequent limitations

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on their applicability. Prior to Elrod's own work, one of the first attacks
was made by Patir and Cheng [1,2][1] who proposed an ensemble-averaged
Reynolds equation in which roughness effects were built into a number of
flow factors. Since their pioneering study, many authors [5-7] have made
use of the results either directly or for comparative purposes and it is
timely to establish the PC approach on a more analytic footing. The present
work constitutes both a critical analysis and an extension of their average
flow method, concluding with a discussion of some of the conceptual problems
encountered in the partial lubrication regime and an indication of how such
problems are being tackled.

In the PC formulation the actual flow between rough surfaces is equated
to an averaged flow between nominally smooth surfaces, while parameters de-
scribing the roughness are included in the Reynolds equation through the
flow factors. The problem is thus reduced to an evaluation of these fac-
tors, accomplished by considering flow between nominal plane parallel bound-
aries. At this stage, PC adopt the direct approach of numerical simulation
of the rough boundaries together with numerical solution of this model flow
problem. Apart from the inherent difficulty associated with extracting
reliable expectation values (ensemble averages) of the flow factors found
for each sample roughness, this approach is also unsatisfactory in that the
functional dependence of the factors on the roughness parameters can only be
inferred from numerical experiments and expressed in fitted form.

We overcome these difficulties here by developing formal solutions of
the model Reynolds equation which may be explicitly ensemble-averaged. This
automatically introduces relevant statistical properties of the film bound-
aries. The analysis not only agrees quantitatively with the simulations of
PC but also places on a firm theoretical footing several conjectures made in
that work on the basis of the numerical experimentation.

The formal solution is written in terms of the Green function for ideal
smooth flow, enabling the pressure to be expressed in a form containing the
random height of each surface. The flow factors themselves, involving pro-
ducts of film thickness and pressure derivatives, may thus be directly aver-
aged with no appeal to an ergodic assumption. A perturbation series is
developed for the pressure in which the expansion parameter effectively
'turns on' the roughness, with zeroth order corresponding to smooth flow.
One important variable, H, governing the flow is the ratio of the nominal
film thickness to the standard deviation of the combined roughness. For
$H \lesssim 3$ areas of contact near surface summits become progressively more sig-
nificant both in blocking flow and in altering the statistical properties of
the surfaces. In this partial lubrication regime, our results begin to
deviate from those of the PC model, which in contrast to the present work
makes some allowance for contact.

NOMENCLATURE

g_i	smooth pressure gradient
G	Green function, solution of Eq. (10)
h	film thickness $= z_2 - z_1$
H	non-dimensional nominal film thickness $= h^*/\sigma$

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1. These two papers will be referred to together as "PC".

i,j	Cartesian tensor indices
p	film pressure
P_{ij}	tensor determining pressure flow, Eq. (16)
q	reduced film pressure $= (1 - e^{-\alpha p})/\alpha$
Q_i	volume flow
R_n	right-hand side of Eq. (A1) for q_n
S_{ij}	tensor determining shear flow, Eq. (16)
t	time coordinate
$\vec{u}_{1,2}$	surface translational velocities
$\vec{U}$	mean surface velocity $= (\vec{u}_2 + \vec{u}_1)/2$
$\vec{V}$	slip velocity $= \vec{u}_2 - \vec{u}_1$
x,y,z	Cartesian space coordinates
$z_{1,2}$	heights of bounding surfaces 1 and 2
Z	mean surface height $= (z_1 + z_2)/2$
Greek	
α	viscosity pressure index
α, β	correlation function parameters, Eq. (19)
γ	roughness anisotropy index (Peklenik number) $= (\beta/\alpha)^{1/2}$
δ	Dirac delta function
δ_{ij}	Kronecker delta $=$ unit matrix
$\delta_{1,2}$	roughness heights of bounding surfaces
ϵ	perturbation expansion parameter
$\lambda^{\pm}$	non-dimensional sum and difference roughness heights $= (\delta_2 \pm \delta_1)/h^*$
μ	coefficient of viscosity
ρ	generalized two-point correlation function
ρ_c	two-point correlation function of $(\delta_2 - \delta_1)$

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σ	rms value of $(\delta_2 - \delta_1)$
$\sigma_{1,2}$	rms values of $\delta_{1,2}$
ϕ_x, ϕ_y, ϕ_s	x,y pressure and shear flow factors given by PC
ϕ_{ij}^p, ϕ_{ij}^s	pressure and shear flow tensors
Φ_s	single surface shear flow factor

Special Symbols

* $< >$	expectation value operator for stochastic quantity

FORMALISM

The class of flow problems considered by PC is illustrated in Fig. 1 which shows a fluid film between two rough boundaries, z_1 and z_2, moving in the (x,y) plane with arbitrary translational velocity, $\vec{u}_1$ and $\vec{u}_2$ measured relative to a point in the fluid remote from the film region. This specification of surface velocity eliminates rigid rotation, although in cases where the nominal surfaces[2] z_1^* and z_2^* possess rotation axes fixed relative to one another, a straightforward extension of the formalism applies. The actual surface height $z_{1,2}$ is considered to be simply the sum of the specified nominal height and a random roughness height $\delta_{1,2}$, so that difficulties in choosing a wavelength cutoff for roughness are avoided. The random variable $\delta_{1,2}$ is a stationary stochastic process distributed with standard deviation $\sigma_{1,2}$ about zero mean. We need also the standard deviation σ of the roughness combination $(\delta_2 - \delta_1)$, which in cases where cross-correlation is absent will be just $(\sigma_1^2 + \sigma_2^2)^{1/2}$. For convenience we replace the boundary variables with the following combinations:

film thickness $h \equiv z_2 - z_1$; mean surface height $Z \equiv (z_2 + z_1)/2$;

slip velocity $\vec{V} \equiv \vec{u}_2 - \vec{u}_1$; mean surface velocity $\vec{U} \equiv (\vec{u}_2 + \vec{u}_1)/2$.

Making the usual thin film assumptions, the volume flow per unit width through the film is

$$Q_i = - \frac{h^3}{12\mu} \partial_i p + h U_i \tag{1}$$

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2. We use the asterisk to denote nominal or expectation values (ensemble averages) except where several factors are included when the angle bracket <> notation is clearer.

where p is the film pressure and μ the shear viscosity coefficient of the lubricant.[3] For incompressible flow the continuity equation $\partial_i Q_i = -\partial h/\partial t$ is applied in the usual way to derive the Reynolds equation for this problem

$$\partial_i \left(\frac{h^3}{12\mu} \partial_i p \right) = -V_i \partial_i Z \tag{2}$$

where the Euler transform has been used to interchange time and space derivatives of h.

From Eq. (2) it is clear that p is a stochastic variable correlated with h so that care is required in calculating average flow from Eq. (1). The goal is to write Q_i^* in terms of h^* and $\partial_i p^*$ so that applying continuity will yield a Reynolds equation in which only averaged quantities appear. There is some latitude in the way this step is taken and we follow FC as closely as possible. The Couette term presents no problem and is handled by replacing h with h^* so that changes in the form of Q_i^* arise only from the pressure term. We note from Eq. (2) that even in a nominal parallel film there will be a term in V_i in the pressure gradient, while likewise, even if V_i vanishes the cofactor of the pressure gradient is modified. We separate these two effects on the pressure and write the flow as

$$Q_i = \frac{-h^{*3}}{12\mu} \phi_{ij}^p \partial_j p^* + hU_i - \frac{\sigma}{2} \phi_{ij}^s V_j \tag{3}$$

This equation serves to define the pressure and shear flow factor tensors ϕ_{ij}^p and ϕ_{ij}^s, respectively. In general these tensors are symmetrical but not diagonal so that, for example, an x pressure gradient can produce a y flow under certain conditions of roughness anisotropy. While Eq. (3) is exact, its utility rests on the hypothesis that the relative variances of the flow factors are small compared to those of h and p, in which case the Reynolds equation derived from Q_i^* instead of Q_i will represent roughness effects satisfactorily. The basis for this hypothesis lies in Eq. (3) which shows that Q_i and the ϕ factors have the same relative fluctuations. Since continuity determines flow globally rather than locally, its fluctuations are indeed smaller than those of h or p. It is incorrect, however, to assume flow fluctuation is exactly zero when only h^* is specified and a distinction remains between ϕ and ϕ^*. The PC factors correspond to the latter which by comparing ensemble-averaged Eqs. (1) and (3) must satisfy

$$\left\langle \frac{h^3}{12\mu} \partial_i p \right\rangle = \phi_{ij}^{p*} \left(\frac{h^{*3}}{12\mu} \right) \partial_j p^* + \frac{\sigma}{2} \phi_{ij}^{s*} V_j \, . \tag{4}$$

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3. Henceforth subscripts i and j refer exclusively to Cartesian components with (x_1, x_2) denoting (x,y). Summation on repeated (i,j) subscripts is implied.

Finally, using Eq. (4) in ensemble-averaged Eq. (2) yields

$$\partial_i\left(\phi_{ij}^{p*}\frac{h^{*3}}{12\mu}\partial_j p^*\right) = -V_j\partial_i(Z^*\delta_{ij} + \frac{1}{2}\sigma\phi_{ij}^{s*}) \tag{5}$$

where δ_{ij} is the unit matrix (Kronecker delta). This average Reynolds equation is essentially that proposed by PC apart from the appearance of the off-diagonal flow factor components, allowing Eq. (5) somewhat greater generality. In the following section where we calculate the complete flow factor tensors, we compare components with those of PC according to the the correspondence $\phi_{11}^{p*} \equiv \phi_x, \phi_{22}^{p*} \equiv \phi_y$ and $\phi_{11}^{s*} \equiv \phi_s$. Other components do not appear in PC. From Eq. (4) we see that $\phi_{ij}^{p*} \to \delta_{ij}$ and $\phi_{ij}^{s*} \to 0$ in the smooth limit so that $\phi_{x,y}$ represents fractional increase in (x,y) flow in absence of slip, while the other components represent various modes of entrained flow not present in the smooth case.

CALCULATION OF FLOW FACTORS

The flow factors are calculated from Eq. (4) by first solving the exact Reynolds equation (2) for p and then taking expectation values. We apply Eq. (2) to the flow between nominally plane parallel surfaces with separation h^* and develop the solution in perturbation series to obtain the flow factors as functions of h^* and certain roughness parameters. In contrast, PC solve Eq. (2) numerically using a population of carefully generated rough surfaces. Their ensemble-averaging is then executed in two stages, first invoking the ergodic hypothesis to take a spatial average of the flow across the model bearing area for each sample, followed by a population average. Elrod [3] also requires the ergodic hypothesis in his perturbation treatment.

Before proceeding, one further simplification is possible if the pressure dependence of the viscosity is restricted to the exponential form $\mu = \mu_0 e^{\alpha p}$. The reduced pressure $q \equiv (1 - e^{-\alpha p})/\alpha$ satisfies

$\mu_0^{-1}\partial_i q = \mu^{-1}\partial_i p$ so that μ and p may be replaced by μ_0 and q.

If the surfaces $z_{1,2}$ are represented by $z_1 = -h^*/2 + \delta_1$ and $z_2 = h^*/2 + \delta_2$, then the variables h and Z become:

$h = h^* + \delta_2 - \delta_1 = h^*(1+\lambda^-)$ and $Z = (\delta_2 + \delta_1)/2 = h^*\lambda^+/2$

which defines the non-dimensional sum and difference roughness coordinates $\lambda^\pm = (\delta_2 \pm \delta_1)/h^*$. In the spirit of perturbation theory, we introduce an attenuation factor ϵ, $0 < \epsilon \leq 1$, such that roughness effects are introduced by letting $\epsilon \to 1$. Using ϵ as the formal expansion parameter we write $q = \sum_{n=0}^{\infty} \epsilon^n q_n$ which, upon extracting the coefficient of ϵ^n, converts

Eq. (2) into a set of equations for q_n. Because of the factor h^3, the general form occurs for $n \geq 3$ but here we are concerned mainly with special forms for $n = 0$ and 1. These are

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$$n = 0 \ , \ \partial_i \partial_i q_0 = 0 \tag{6}$$

$$n = 1 \ , \ \partial_i \partial_i q_1 = -3\partial_i(\lambda^- \partial_i q_0) - \frac{6\mu_0}{h*^2} V_i \partial_i \lambda^+ \tag{7}$$

Up to terms in ϵ^2 the expression required in Eq. (4) becomes

$$\frac{h^3}{12\mu_0} \partial_i q = \frac{h^{*3}}{12\mu_0} \left\{ \partial_i q_0 + \epsilon[3\lambda^- \partial_i q_0 + \partial_i q_1] \right.$$

$$\left. + \epsilon^2 \left[3(\lambda^-)^2 \partial_i q_0 + 3\lambda^- \partial_i q_1 + \partial_i q_2 \right] + O(\epsilon^3) \right\} \tag{8}$$

Comparison of the expectation value of Eq. (8) with Eq. (4) will then yield the ϕ tensors correct to second order in roughness. In the Appendix it is shown that $\langle \partial_i q_n \rangle \equiv \partial_i q_n^* = 0$ for $n > 0$ so that roughness leaves the pressure gradient unchanged from its smooth value. Moreover from Eq. (6) this may be assigned the constant value, g_i, equal to the mean applied gradient. With this simplification, Eq. (8) may be averaged to give

$$\left\langle \frac{h^3}{12\mu_0} \partial_i q \right\rangle = \frac{h^{*3}}{12\mu_0} \left\{ g_i + 3\epsilon^2[H^{-2}g_i + \langle \lambda^- \partial_i q_1 \rangle] + O(\epsilon^3) \right\} \tag{9}$$

Since q_2 no longer appears, to this order the only term requiring attention is $\langle \lambda^- \partial_i q_1 \rangle$ which contains all the interesting effects of roughness.

The solution for q_1 is written in terms of the Green function G of the smooth problem, which from Eq. (6) satisfies

$$\partial_i \partial_i G(x_j ; x_j^1) = \delta(x_j - x_j^1) \tag{10}$$

where δ is the two-dimensional Dirac delta function. Ignoring the complementary function, the particular integral of Eq. (10) is well known to be

$$G(x_i ; x_i^1) = \frac{1}{4\pi} \ell n \ [(x_i - x_i^1)(x_i - x_i^1)] \tag{11}$$

which has the important property of depending only on the difference of its arguments. Using the notation ∂_i^1 to denote $\partial/\partial x_i^1$, the solution of Eq. (7) for q_1 becomes

$$q_i(x) = -3g_i \iint G(x-x^1)\partial_i^1 \lambda^-(x^1)dx^1 - \frac{6\mu_0}{h*^2} V_i \iint G(x-x^1)\partial_i^1 \lambda^+(x^1)dx^1 \tag{12}$$

where subscripts have been omitted from spatial variables. From this the expectation value appearing on the right of Eq. (9) is found to be

$$\langle \lambda^- \partial_i \eta_1 \rangle = -3g_j \iint \partial_i G(x-x^1) \partial_j \tfrac{1}{j} \langle \lambda^-(x)\lambda^-(x^1) \rangle dx^1$$

$$- \frac{6\mu_0}{h*^2} V_j \iint \partial_i G(x-x^1) \partial_j \tfrac{1}{j} \langle \lambda^-(x)\lambda^+(x^1) \rangle \, dx^1 \tag{13}$$

This shows that all the required statistical information lies in the two two-point correlation functions (CF) defined here. The first CF, ρ_c, is that for the combined roughness variable $(\delta_2 - \epsilon_1)$ which in conventional form becomes

$$\langle \lambda^-(x)\lambda^-(x^1) \rangle = H^{-2} \rho_c(x-x^1) \tag{14}$$

using the definition of $\sigma^2 = h*^2 \langle \lambda^-(x)\lambda^-(x) \rangle$. In many cases of interest, the cross-correlation between the two surfaces $\langle \delta_1(x)\delta_2(x^1) \rangle$ will be zero so that $\sigma^2 = \sigma_1^2 + \sigma_2^2$ and ρ_c reduces to

$\dfrac{\sigma_2^2}{\sigma^2} \rho_2 + \dfrac{\sigma_1^2}{\sigma^2} \rho_1$, where $\rho_{1,2}$ are the autocorrelation functions of the

two surfaces. The second CF, ρ_D, is related to a difference of the ACF's of the two surfaces and may be written

$$\langle \lambda^-(x)\lambda^+(x^1) \rangle = H^{-2} \rho_D(x-x^1) \; . \tag{15}$$

By symmetry any cross-correlation vanishes from ρ_D which thus becomes $\dfrac{\sigma_2^2}{\sigma^2} \rho_2 - \dfrac{\sigma_1^2}{\sigma^2} \rho_1$. In showing the dependence of these CF's on the argument

difference, the stationarity of the roughness process has been invoked.

It is useful to assign names P_{ij} and S_{ij} to the tensors defined by the integrals in Eq. (13) according to their association with pressure and shear flow, thus

$$\begin{bmatrix} P_{ij} \\ S_{ij} \end{bmatrix} = \iint \partial_i \, G(x) \partial_j \begin{bmatrix} \rho_c(x) \\ \rho_D(x) \end{bmatrix} dx \tag{16}$$

in which the difference argument no longer need appear. Combining Eqs. (9), (13) and (16) and comparing with Eq. (4) then leads to

$$\phi_{ij}^{p*} = \delta_{ij} + 3H^{-2}(\delta_{ij} + 3P_{ij}) \tag{17}$$

$$\phi_{ij}^{s*} = 3H^{-1} S_{ij} \tag{18}$$

This completes the formal part of the calculation. The ϕ tensors in Eqs. (17) and (18) lead via Eq. (3) to an averaged flow Q_i^* structurally

similar to that derived by Elrod. The essential difference lies in the direct appearance of the CF's in Eq. (16) by contrast to their Fourier transforms (power spectral densities) in that earlier study. In other respects the Green function technique complements the Fourier transform method as a means of solving the Reynolds equation.

RESULTS

For numerical comparisons it is now necessary to construct some model CF's to insert in Eq. (16). As we have seen in connection with Eqs. (14) and (15), ρ_C and ρ_D in the absence of cross-correlation are just a weighted sum and difference respectively of the individual ACF's of each surface and the ϕ factors are correspondingly expressed in sum or difference form. Our discussion of the general form for the CF applies to any one of these four functions. Even where cross-correlation is involved we assume the form of the CF still falls within the general description. Hence, appropriate combination of results for a single CF yields results for any given pair of surfaces. The flow factors in any case are not expected to be sensitive to the precise CF form provided they refer to physically similar surface textures.

Our calculations correspond to textures having no long-range order, for which the CF necessarily decays to zero as the correlation distance increases in any direction from the central summit, $\rho = 1$, at the origin. In general, the contours of ρ possess only the symmetry of inversion through the origin but if any preferred direction exists as a result of the finishing process, the existence of two perpendicular planes of reflection symmetry for the CF is assured. In such a situation which includes isotropy as the special case where all planes are reflection planes, it is possible to define Cartesian 'roughness axes' and since no restriction has been placed here on the boundary velocities or pressure gradients, no generality is sacrificed by choosing (x,y) to coincide with these axes. This corresponds formally to a coordinate rotation which diagonalizes the flow factor tensors, the simplest representation from a computational viewpoint. When roughness axes exist, the Peklenik γ factor [8] is well-defined, being the ratio of the x and y intercepts of the $\rho = 1/2$ contour. While any γ value still embraces an infinite variety of surface textures, it is a useful characterization adopted by a number of authors, including PC, to describe a surface whose asperity contours typically have an x dimension γ times their y dimension. In many cases these will be the maximum and minimum dimensions. The special values 1, 0 and ∞ correspond respectively to isotropic roughness and one-dimensional ridges parallel to y or x.

Conforming to these most general specification, PC chose a CF decaying linearly with distance and having rectangular contours. This form is convenient for their numerical simulation. We have performed the integration of Eq. (16) for this choice and find that the results differ unimportantly from a Gaussian form for ρ which has the advantage of a remarkably simple final expression for the flow factor components. Thus, for purposes of illustration we adopt

$$\rho = \exp - [\alpha(x-x^1)^2 + \beta(y-y^1)^2] \tag{19}$$

for which the contours are elliptical with $\gamma = \sqrt{\beta/\alpha}$. With this form for ρ, the range of integration in Eq. (16) may be extended to infinity with insignificant loss of accuracy, as discussed for example by Bush and Gibson [7]

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in a Green function analysis of the inclined slider bearing. Substituting Eq. (19) into Eq. (16) for ρ_c then leads to the result $P_{11}(\gamma) = -(\gamma+1)^{-1}$, while $P_{22}(\gamma) = P_{11}(1/\gamma)$. Thus, in PC notation

$$\phi_x(H,\gamma) = 1 + 3\frac{\gamma-2}{\gamma+1} H^{-2} \qquad (20)$$

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$$\phi_y(H,\gamma) = \phi_x(H,1/\gamma) . \qquad (21)$$

Similarly if ρ denotes ρ_1 or ρ_2, we obtain

$$\phi_s(H,\gamma_1,\gamma_2) = \frac{\sigma_1^2}{\sigma^2} \phi_s(H,\gamma_1) - \frac{\sigma_2^2}{\sigma^2} \phi_s(H,\gamma_2) \qquad (22)$$

where

$$\phi_s(H,\gamma) = \frac{3 H^{-1}}{\gamma + 1} \qquad (23)$$

is the shear flow factor for a single rough surface. These closed expressions may now be compared directly with the PC numerical results.

DISCUSSION

In Fig. 2 are shown some plots of Eqs. (20) and (23) together with the numerical results of PC. As we should expect, there is insignificant disagreement between the two as H grows large and effects of roughness on flow diminish. Differences being to appear for $H<3$ in the partial lubrication regime. Since contact effects are not included in the present work, results are plotted only for $H>2$.

This calculation has several interesting features. The asymptotic agreement of numerical simulation with the perturbation approach to solution of the Reynolds equation indicates that both techniques are accurate in the absence at least of significant degree of contact. In the two special one-dimensional cases, for which independent estimates of the flow factors are available [9], the perturbation method again agrees asymptotically. It thus supplies a useful analytic expression connecting these two extremes. The most striking feature of the pressure flow factors exhibited by PC is the crossover from monotonic increasing to decreasing behaviour occurring for a γ value between 1 and 3. Eq. (20) indeed suggests the transition occurs exactly at $\gamma = 2$ for which ϕ_x is constant. Although this exact constancy depends on the choice of model CF (with the linear CF used by PC, ϕ_x is found to increase very slightly for $\gamma = 2$), this qualitative agreement is nonetheless most encouraging. It sheds some light on the physical basis of this curious transition from reduced to enhanced flow as the longitudinal contour dimension (i.e., parallel to the pressure gradient) exceeds twice the transverse. Eq. (20) shows that transverse roughness is twice as effective in reducing flow as longitudinal roughness is in enhancing it. Thus if cross-correlation is zero, a combination of trans-

verse furrows on one surface with longitudinal furrows on the other has no resultant effect on flow ($\phi_x = 1$) , provided the variance of the longitudinal surface is just twice the transverse variance.

In general, the distinction between our flow factors and those of PC is that in each case ours predict a relatively larger roughness effect. Thus, our ϕ_s factor inevitably must be larger at small H since contact does not cut off the flow, and likewise the enhancement of pressure flow for longitudinal types of roughness will be overestimated. It is not so easy to interpret the present overestimation, at least relative to PC , of the reduction in flow produced by transverse texture. It is, however, clear that as γ decreases, fluctuations in ϕ_x calculated by simulation become more important and may make such techniques unreliable.

The analysis presented here also shows clearly how the flow factors are determined by the combined stochastic properties of the two surfaces. The appearance of two distinct CF's, ρ_C and ρ_D in pressure and shear flow respectively, is an important feature not discussed by PC. Whereas Eq. (22) here follows directly from Eq. (16), PC are only able to infer it from numerical experimentation. Similarly we can demonstrate another conjecture that so long as the γ factor is uniquely definable, the precise form of the CF does not strongly influence the numerical results. As already indicated, integration of Eq. (16) for the linear ACF used by PC alters our results by much less than the differences apparent in Fig. 2, while using yet another CF , Elrod's results agree closely with ours [3]. The idea receives further support in the results of Bush and Gibson [7], while the power of the form of the CF to predict tribological parameters in general has been emphasized in the work of Whitehouse and Phillips [10].

OUTLOOK

At this point the chief advantage of the Green function perturbation approach to the flow problem in general becomes apparent, namely the explicit way the statistical properties of the surfaces appear. Continuation of the perturbation series for pressure to arbitrary order n leads, as shown in the Appendix, to the involvement of all ℓ-point CF's up to $\ell = n$. Insofar as such CF's are experimentally measurable, their information could be included directly in the calculation of flow factors which goes some way to answering Tønder's objection [2,5] that the parameters σ and γ insufficiently characterize roughness. Even to second order there are no restrictions beyond the general ones discussed by the form given to the CF's. The formalism, however, is more difficult to apply to surfaces with long-range order of the type constructed by Tønder which have no unique γ value.

The chief cause of breakdown of the present approach, however, is not truncation of the perturbation series but the changes in statistical properties resulting from contact growth at small H. Contact not only alters the surface ACF's but also introduces cross-correlation. At higher degrees of contact the most important effect for flow is the shape and connectivity of the contact spots. Meeting the required condition of zero flow normal to any contact boundary, together with zero flow within pools of fluid trapped inside a closed boundary, involves the topology of the entire interface. The PC method which consists of setting local flow zero whenever the simulated heights indicate surface overlap makes some allowance for contact but grows progressively more inadequate as H decreases. It is likely that for $H \lesssim 2$ a percolation description of flow is more appropriate. Percola-

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tion theory, for example, demonstrates that when the fraction of contact area in a random isotropic ($\gamma = 1$) case reaches 44 percent, there no longer exist any open paths for flow [11]. On a Gaussian surface this occurs for H ~ 0.15. Rather surprisingly then the flow factors should vanish altogether at this point, even though the expectation of the film thickness (mean gap) is nonzero.

Such an approach to the partial lubrication regime emphasizes the random nature of the contact geometry. An approach described by Shukla [12] with somewhat the same viewpoint treats the interface as a random network, presenting an effective permeability to the lubricant, the apparent viscosity of which depends on the degree of contact. Such methods appear well suited to extending our understanding of this regime.

APPENDIX

We wish to establish that $\partial_i q_n^* = 0$ for $n > 0$ is consistent with the Reynolds equation for q^*. This result, required in arriving at Eq. (9), expresses the fact that the expectation of the pressure gradient may be written as the constant smooth value $\partial_i q_0 = g_i$. By substitution of the series for q into Eq. (2), the general equation for q_n is obtained:

$$\partial_i \partial_i q_n = -3\partial_i[\bar{\lambda}^- \partial_i q_{n-1}] - 3\partial_i[(\bar{\lambda}^-)^2 \partial_i q_{n-2}] - \partial_i[(\bar{\lambda}^-)^3 \partial_i q_{n-3}] \equiv R_n \qquad (A1)$$

In terms of the Green function defined by Eqs. (10) and (11), the solution for q_n is

$$q_n(x) = \int G(x-x^1) R_n(x^1) dx^1 \qquad (A2)$$

Recursive substitution of lower order q_k on the right of Eq. (A1) making use of Eq. (A2) leads to a series of terms in R_n of which the general form is an $(\ell - 1)$ − fold double integration with respect to $x^2, x^3, \ldots x^\ell$, $2 \leq \ell \leq n$, of a product of $(\ell - 1)$ Green functions and n factors of $\lambda^\pm$ distributed between the ℓ arguments $x^1, x^2, \ldots x^\ell$. For pressure flow this general term may be explicitly written

$$\int \cdots \int \left\{ \partial_i^1 [\bar{\lambda}^-(x^1)]^{P_1} \partial_i^1 G(x^1-x^2) \right\} \left\{ \partial_j^2 [\bar{\lambda}^-(x^2)]^{P_2} \partial_j^2 G(x^2-x^3) \right\}$$

$$\cdots \left\{ \partial_r^\ell [\bar{\lambda}^-(x^\ell)]^{P_\ell} g_r \right\} dx^2 dx^3 \ldots dx^\ell$$

times a numerical factor, depending on the set of integers P_m which must satisfy $\sum_{m=1}^{\ell} P_m = n$. For shear flow the final brace is replaced by

$$\left\{ \mu_o [\bar{\lambda}^-(x^\ell)]^{P_\ell - 1} \partial_r^\ell [\lambda^+(x^\ell)] V_r / h^{*2} \right\} ,$$ while the numerical factor is also changed.

Forming the expectation value of the integrand before performing the differentiation of each λ factor, a valid commutation of operations, we obtain a CF of order n between ℓ different sample points of the stochastic rough surface. By stationarity this CF depends on the $\ell(\ell - 1)/2$ argument differences. Carrying out next the required differentiations of the CF with respect to $x^2, x^3, \ldots x^\ell$ followed by the $(\ell - 1)$ − fold double integration then leads to a result independent of x^1, since the Green function derivatives also depend only on variable differences. It should be noted that this proof depends on the assumed rapid decay of the CF with distance, allowing the range of integration to be extended to infinity.

The ∂_1^1 operator now remaining to the left of the expression reduces it to zero. Thus $R^* = 0$ and hence by Eq. (A1) $\partial_i q^*$ may be chosen zero. This choice satisfies the boundary conditions and establishes the desired result.

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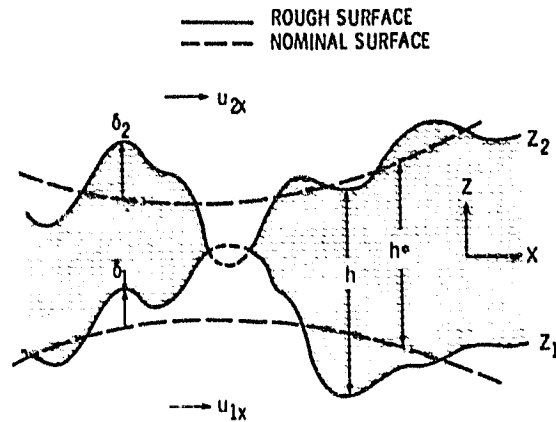12. Shukla, J. B., "A New Theory of Lubrication for Rough Surfaces," Wear, Vol. 49, 1978, pp. 33-42.

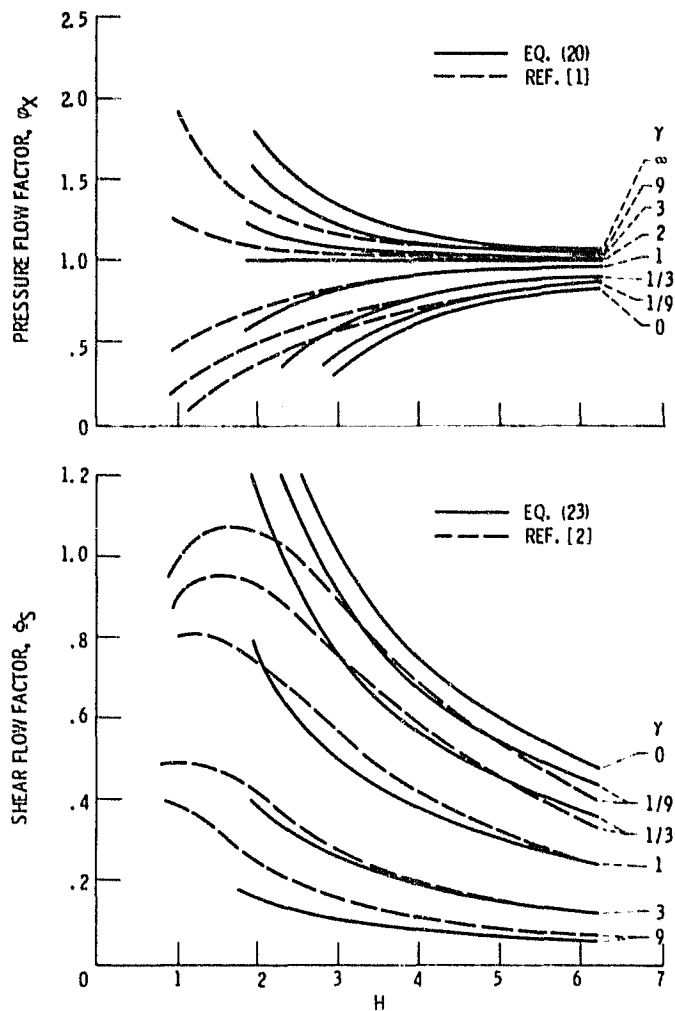Figure 1. - Contact of rough surfaces.

Figure 2. - Comparison of calculated flow factors with those of Patir
and Cheng[1, 2].