Geophys. J. R. astr. Soc. (1987) **88**, 231–240

Surface-wave ray tracing equations and Fermat's principle in an anisotropic earth

Toshiro Tanimoto *Seismological Laboratory 252-21, California Institute of Technology, Pasadena, California 91125, USA*

Accepted 1986 July 11. Received 1986 July 11; in original form 1986 February 4

Summary. Ray tracing equations for surface waves in an anisotropic earth are derived in two ways: first, from the Hamilton's canonical equations, and secondly, from Fermat's principle. Phase velocity, including its azimuthal variation, is required to solve the equations, but group velocity is eliminated from the equations. The difference of direction between the wave vector and the ray path is one of the features of wave propagation in an anisotropic media and the equations explicitly show dependence upon such an angle. By putting that angle to be zero, ray tracing equations in a transversely isotropic (or simply isotropic) medium are obtained.

In an isotropic medium, phase traveltime, which is an integration of phase slowness along the ray path, is stationary. In an anisotropic medium, phase travel-times is not stationary. Instead, phase slowness projected onto the ray path and integrated along the ray path is stationary. Ray tracing by the bending method in an anisotropic media should utilize such a stationary quantity. In a weakly anisotropic medium, however, the angle (ψ_a) between the wave vector and the ray path is small and $\cos\psi_a \approx 1$ up to first order in ψ_a. Thus phase traveltime is approximately stationary in a weakly anisotropic medium.

Key words: Ray tracing, surface waves, Fermat's principle, anisotropy

1 Introduction

The Earth's lateral heterogeneity and anisotropy causes deflection of surface wave ray paths from the great circle path that contains the source and the receiver. In the long period range (longer than about 200–300 s) the Earth's structure looks almost laterally homogeneous and thus the assumption of the wave propagation along the great circle path has been quite successful. This assumption comes from Fermat's principle (for phase velocity studies), which states that a ray path is stationary and is not perturbed by lateral heterogeneity to the first order.

In the shorter period range, however, path deviation becomes quite large. Ray tracing is required in such a case and many studies have already been incorporating such effects to study lateral heterogeneity (e.g. Gjevik 1974; Woodhouse 1974; Sobel & von Seggern 1978; Lay & Kanamori 1985). Recent studies also include amplitude calculations with ray tracing algorithms (e.g. Wong & Woodhouse 1984; Yomogida & Aki 1985). However, most studies are concerned with an isotropic medium. Ray tracing equations in an anistropic medium, especially when azimuthal variation of phase velocity exists, have not been discussed in detail.

In view of the fact that some researchers have tried to recover azimuthal variations of surface wave phase velocities (e.g. Forsyth 1975; Tanimoto & Anderson 1984, 1985; Suetsugu & Nakanishi 1985; Nishimura & Forsyth 1985; Montagner & Nataf 1985), it seems to be timely to derive ray tracing equations in an anisotropic medium in a readily computable form. We derive the ray tracing equations, first from the Hamilton's canonical equations, which explicitly show the directional difference of the wave vector (direction of phase velocities) and ray path (direction of energy propagation or group velocity). This difference is one of the most important features of wave propagation in an anisotropic medium. In fact, by putting this angle difference and partial derivative of phase velocity with respect to azimuth to zero, we can obtain ray tracing equations in a transversely isotropic medium. By taking the distance as an integration variable, group velocity is completely eliminated from the equations. Phase velocity, including its azimuthal variation, must be supplied to solve the equations.

We also discuss Fermat's principle in an anisotropic medium and derive the same ray tracing equations from this principle. In an isotropic medium, phase slowness integrated along the ray path, which is phase traveltime, is stationary. In an anisotropic medium, since the direction of the wave vector is not the same as the ray path direction while projection of phase slowness on the ray path integrated along the ray trajectory becomes stationary.

In Section 2, we derive the ray tracing equations from the Hamilton's canonical equations and in Section 3 we derive the same equations from the Fermat's principle and discuss the differences between anisotropic and transversely isotropic cases.

2 Ray tracing equations

We assume a laterally, smoothly varying medium in which the dispersion relation varies gradually on a scale of wavelength and also the wavenumber changes gradually along the ray path. The waves are then locally approximately sinusoidal and displacement can be expressed by

$$u(x_1, x_2, t) = Q(x_1 x_2, t) \exp\,[i\alpha(x_1, x_2\ t)], \tag{1}$$

where $(x_1,\ x_2)$ are the coordinates in the surface, and Q and α are amplitude and phase functions, respectively. The local wave number, k_i, and frequency, ω, are given by

$$\frac{\partial\alpha}{\partial x_i} = -\,k_i \tag{2}$$

$$\frac{\partial\alpha}{\partial t} = \omega. \tag{3}$$

The dispersion relation takes the form

$$\omega = \bar{\omega}(k_1, k_2, x_1, x_2). \tag{4}$$

From (3) and (4), we have

$$\frac{\partial \alpha}{\partial t} = \bar{\omega}(k_1, k_2, x_1, x_2). \tag{5}$$

Differentiating this by x_j, we get

$$\frac{\partial^2 \alpha}{\partial x_j\, \partial t} = \sum_\sigma \left(\frac{\partial \bar{\omega}}{\partial k_\sigma}\right)_{x_1, x_2} \left(-\frac{\partial^2 \alpha}{\partial x_\sigma\, \partial x_j}\right) + \left(\frac{\partial \bar{\omega}}{\partial x_j}\right)_{k_1, k_2}, \tag{6}$$

or

$$\frac{\partial k_j}{\partial t} + \sum_\sigma \left(\frac{\partial \bar{\omega}}{\partial k_\sigma}\right)_{x_1, x_2} \frac{\partial k_j}{\partial x_\sigma} = -\left(\frac{\partial \bar{\omega}}{\partial x_j}\right)_{k_1, k_2}. \tag{7}$$

By the method of characteristics, (7) can be transformed to the following Hamilton's equations

$$\frac{dx_\sigma}{dt} = \left(\frac{\partial \bar{\omega}}{\partial k_\sigma}\right)_{x_1, x_2}, \tag{8}$$

$$\frac{dk_\sigma}{dt} = -\left(\frac{\partial \bar{\omega}}{\partial x_\sigma}\right)_{k_1, k_2}, \tag{9}$$

where $\sigma = 1, 2$.

Another important equation is

$$\frac{d\bar{\omega}}{dt} = \sum_\sigma \left[\left(\frac{\partial \bar{\omega}}{\partial k_\sigma}\right)_{x_1, x_2} \frac{dk_\sigma}{dt} + \left(\frac{\partial \bar{\omega}}{dx_\sigma}\right)_{k_1, k_2} \frac{dx_\sigma}{dt}\right] = 0,$$

which is obtained by using (8) and (9) and means that the frequency is constant along the ray path.

Equations (8) and (9) are the basic equations for ray tracing. Lighthill (1978) claims that the solutions are readily computable given the initial values of x_σ and k_σ (Lighthill 1978, p. 320). In principle, this is true, but in practice the dispersion relation function ω is not so completely known, including both spatial and wavenumber derivative (group velocity). Quite often, only phase velocity is given. We will thus manipulate (8) and (9) and obtain the ray tracing equations in a more computable form. The final form contains only spatial derivatives of phase velocity.

Hereafter, the spatial coordinate will be (θ, ϕ) which are colatitude and longitude in a sphere, and the wave vector is (k_θ, k_ϕ). Wavenumber k is defined by

$$k^2 = \frac{1}{a^2} k_\theta^2 + \frac{1}{a^2 \sin^2 \theta} k_\phi^2, \tag{10}$$

where a is the radius of the earth.

Instead of using k_θ and k_ϕ, we use the following expression for the dispersion relation:

$$\omega = W(k, \psi, \theta, \phi), \tag{11}$$

where ψ is the azimuth of the wave vector measured anti-clockwise from the south (Fig. 1), i.e.

$$\cos \psi = \frac{k_\theta}{ak} \tag{12}$$

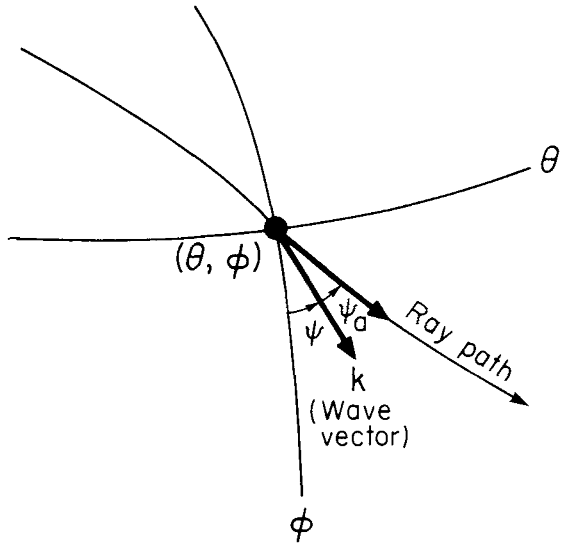

Figure 1. The azimuth of the wave vector at (θ, ϕ) is given by ψ and the angle between the wave vector and the ray path is shown by ψ_a. One of the most important features of wave propagation in an anisotropic media is this angle difference, ψ_a, between the wave vector (phase velocity) and the ray path (group velocity or the direction of energy propagation).

and

$$\sin\psi = \frac{1}{a\sin\theta}\frac{k_\phi}{k} \tag{13}$$

and the dispersion function changes from $\bar{\omega}$ to W because different arguments are used.

Equations (8) and (9) become

$$\frac{d\theta}{dt} = \left(\frac{\partial W}{\partial k}\right)_{\psi,\theta,\phi}\frac{\partial k}{\partial k_\theta} + \left(\frac{\partial W}{\partial \psi}\right)_{k,\theta,\phi}\frac{\partial \psi}{\partial k_\theta} \tag{14}$$

$$\frac{d\phi}{dt} = \left(\frac{\partial W}{\partial k}\right)_{\psi,\theta,\phi}\frac{\partial k}{\partial k_\phi} + \left(\frac{\partial W}{\partial \psi}\right)_{k,\theta,\phi}\frac{\partial \psi}{\partial k_\phi} \tag{15}$$

$$\frac{dk_\theta}{dt} = -\left(\frac{\partial W}{\partial k}\right)_{\psi,\theta,\phi}\frac{\partial k}{\partial \theta} - \left(\frac{\partial W}{\partial \psi}\right)_{k,\theta,\phi}\frac{\partial \psi}{\partial \theta} - \left(\frac{\partial W}{\partial \theta}\right)_{k,\psi,\phi} \tag{16}$$

$$\frac{dk_\phi}{dt} = -\left(\frac{\partial W}{\partial k}\right)_{\psi,\theta,\phi}\frac{\partial k}{\partial \phi} - \left(\frac{\partial W}{\partial \psi}\right)_{k,\theta,\phi}\frac{\partial \psi}{\partial \phi} - \left(\frac{\partial W}{\partial \phi}\right)_{k,\psi,\theta}. \tag{17}$$

By differentiating (10), (12) and (13), we obtain

$$\frac{\partial k}{\partial k_\theta} = \frac{1}{a}\cos\psi, \quad \frac{\partial k}{\partial k_\phi} = \frac{\sin\psi}{a\sin\theta}, \quad \frac{\partial k}{\partial \theta} = -\frac{k\cos\theta}{\sin\theta}\sin^2\psi, \quad \frac{\partial k}{\partial \phi} = 0 \tag{18}$$

and

$$\frac{\partial \psi}{\partial k_\theta} = -\frac{\sin\psi}{ak}, \quad \frac{\partial \psi}{\partial k_\phi} = \frac{\cos\psi}{ak\sin\theta}, \quad \frac{\partial \psi}{\partial \theta} = -\frac{\cos\theta}{\sin\theta}\cos\psi\sin\psi, \quad \frac{\partial \psi}{\partial \phi} = 0. \tag{19}$$

Substituting (18) and (19) in (14)–(17), we get

$$\frac{d\theta}{dt} = \left(\frac{\partial W}{\partial k}\right)_{\psi,\theta,\phi} \frac{\cos\psi}{a} - \left(\frac{\partial W}{\partial \psi}\right)_{k,\theta,\phi} \frac{\sin\psi}{ak} \tag{20}$$

$$\frac{d\phi}{dt} = \left(\frac{\partial W}{\partial k}\right)_{\psi,\theta,\phi} \frac{\sin\psi}{a\sin\theta} + \left(\frac{\partial W}{\partial \psi}\right)_{k,\theta,\phi} \frac{\cos\psi}{ak\sin\theta} \tag{21}$$

$$\frac{dk_\theta}{dt} = \left(\frac{\partial W}{\partial k}\right)_{\psi,\theta,\phi} \frac{k\cos\theta}{\sin\theta}\sin^2\psi + \left(\frac{\partial W}{\partial \psi}\right)_{k,\theta,\phi} \frac{\cos\theta}{\sin\theta}\cos\psi\sin\psi - \left(\frac{\partial W}{\partial \theta}\right)_{k,\psi,\phi} \tag{22}$$

$$\frac{dk_\phi}{dt} = -\left(\frac{\partial W}{\partial \phi}\right)_{k,\psi,\theta}. \tag{23}$$

The group velocity U can be expressed by

$$U = a\sqrt{\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta\left(\frac{d\phi}{dt}\right)^2}.$$

From (20) and (21), this can be written by

$$U = \sqrt{\left[\left(\frac{\partial W}{\partial k}\right)_{\psi,\theta,\phi}\right]^2 + \left[\frac{1}{k}\left(\frac{\partial W}{\partial \psi}\right)_{k,\theta,\phi}\right]^2}. \tag{24}$$

Note that in a transversely isotropic media, the group velocity is given by $U = (\partial W/\partial k)_{\theta,\phi}$. (24) shows that in an anisotropic media, an additional term appear because of azimuthal dependence of the dispersion relation.

Now let us introduce an infinitesimal angular distance along the ray $d\Delta$, which is

$$a\,d\Delta = U\,dt.$$

By introducing the angle ψ_a, which we define by

$$\cos\psi_a = \frac{(\partial W/\partial k)_{\psi,\theta,\phi}}{U} \tag{25}$$

and

$$\sin\psi_a = \frac{1/k\,(\partial W/\partial\psi)_{k,\theta,\phi}}{U}, \tag{26}$$

we can rewrite equations (20)–(23) as

$$\frac{d\theta}{d\Delta} = \cos(\psi + \psi_a)$$

$$\frac{d\phi}{d\Delta} = \frac{1}{\sin\theta}\sin(\psi + \psi_a)$$

$$\frac{1}{a}\frac{dk_\theta}{d\Delta} = \cos\psi_a\left[\frac{k\cos\theta}{\sin\theta}\sin^2\psi - \left(\frac{\partial k}{\partial\psi}\right)_{W,\theta,\phi}\frac{\cos\theta}{\sin\theta}\cos\psi\sin\psi + \left(\frac{\partial k}{\partial\theta}\right)_{W,\psi,\phi}\right] \tag{27}$$

$$\frac{1}{a}\frac{dk_\phi}{d\Delta} = \cos\psi_a\left(\frac{\partial k}{\partial\phi}\right)_{W,\psi,\theta}, \tag{28}$$

where we used

$$\left(\frac{\partial W}{\partial \psi}\right)_{k,\theta,\phi} = -\left(\frac{\partial W}{\partial k}\right)_{\psi,\theta,\phi}\left(\frac{\partial k}{\partial \psi}\right)_{W,\theta,\phi}, \tag{29}$$

$$\left(\frac{\partial W}{\partial \theta}\right)_{k,\psi,\phi} = -\left(\frac{\partial W}{\partial k}\right)_{\psi,\theta,\phi}\left(\frac{\partial k}{\partial \theta}\right)_{W,\psi,\phi}, \tag{30}$$

and

$$\left(\frac{\partial W}{\partial \phi}\right)_{k,\psi,\theta} = -\left(\frac{\partial W}{\partial k}\right)_{\psi,\theta,\phi}\left(\frac{\partial k}{\partial \phi}\right)_{W,\psi,\theta}. \tag{31}$$

From (12) and (13)

$$\frac{1}{a}\frac{dk_\theta}{d\Delta} = \frac{dk}{d\Delta}\cos\psi - k\sin\psi\frac{d\psi}{d\Delta} \tag{32}$$

and

$$\frac{1}{a}\frac{dk_\phi}{d\Delta} = \frac{dk}{d\Delta}\sin\theta\sin\psi + k\cos\theta\sin\psi\frac{d\theta}{d\Delta} + k\sin\theta\cos\psi\frac{d\psi}{d\Delta}. \tag{33}$$

Then (27) x $\sin\theta\cos\psi$ + (28) x $\sin\psi$ gives

$$\frac{dk}{d\Delta} = \cos\psi_a\left[-\left(\frac{\partial k}{\partial \psi}\right)_{W,\theta,\phi}\cos\theta\sin\psi + \left(\frac{\partial k}{\partial \theta}\right)_{W,\psi,\theta}\sin\theta\cos\psi + \left(\frac{\partial k}{\partial \phi}\right)_{W,\psi,\theta}\sin\psi\right], \tag{34}$$

where we used (25), (29), (31) and (32).

Also from (27) x ($-\sin\theta\sin\psi$) + (28) x $\cos\psi$, we obtain

$$\frac{d\psi}{d\Delta} = \cos\psi_a\left[-\frac{\cos\theta}{\sin\theta}\sin\psi - \frac{1}{k}\left(\frac{\partial k}{\partial \theta}\right)_{W,\psi,\phi}\sin\psi + \frac{1}{k}\left(\frac{\partial k}{\partial \phi}\right)_{W,\psi,\theta}\frac{\cos\psi}{\sin\theta}\right]. \tag{35}$$

Thus the ray tracing equations, using phase velocity c instead of wavenumber $k(=\omega/c)$, become

$$\frac{d\theta}{d\Delta} = \cos(\psi + \psi_a) \tag{36}$$

$$\frac{d\phi}{d\Delta} = \frac{1}{\sin\theta}\sin(\psi + \psi_a) \tag{37}$$

$$\frac{d\psi}{d\Delta} = \cos\psi_a\left[-\cos\theta\sin\psi + \frac{1}{c}\left(\frac{\partial c}{\partial \phi}\right)_{W,\psi,\phi}\sin\psi - \frac{1}{c}\left(\frac{\partial c}{\partial \phi}\right)_{W,\psi,\theta}\frac{\cos\psi}{\sin\theta}\right] \tag{38}$$

$$\frac{1}{k}\frac{dk}{d\Delta} = \cos\psi_a\left[\frac{1}{c}\left(\frac{\partial c}{\partial \psi}\right)_{W,\theta,\phi}\cos\theta\sin\psi - \frac{1}{c}\left(\frac{\partial c}{\partial \theta}\right)_{W,\psi,\phi}\sin\theta\cos\psi - \frac{1}{c}\left(\frac{\partial c}{\partial \phi}\right)_{W,\psi,\theta}\sin\psi\right]. \tag{39}$$

Note that the three equations (36)–(38) are sufficient to calculate the ray trajectories. The additional equation (39) provides the information on the change of wavenumber along the ray path.

If we can express ψ_a by phase velocity, group velocity is eliminated from the equations (36)–(39). This can be done as follows; equations (25) and (26) can be written

$$\cos\psi_a = \frac{1}{\sqrt{1+z^2}} \tag{40}$$

and

$$\sin\psi_a = \frac{z}{\sqrt{1+z^2}}, \tag{41}$$

where

$$z = \frac{1/k(\partial W/\partial\psi)_{k,\theta,\phi}}{(\partial W/\partial k)_{\psi,\theta,\phi}}. \tag{42}$$

From (29), this is simply

$$z = -\frac{1}{k}\left(\frac{\partial k}{\partial\psi}\right)_{W,\theta,\phi} = \frac{1}{c}\left(\frac{\partial c}{\partial\psi}\right)_{W,\theta,\phi}. \tag{43}$$

Therefore, if we know the azimuthal variation of the phase velocity c, we can obtain ψ_a from (40), (41) and (43). Thus if phase velocity c at a particular frequency ω is given as a function of θ, ϕ and ψ, all quantities that appear in equations (36)–(39) can be calculated and ray trajectories in an anisotropic medium can be traced.

Equations (36)–(39) also show the difference between the transversely isotropic (or simply isotropic) medium and the anisotropic medium. Equations for the transversely isotropic media are, of course, obtained by putting $\psi_a = 0$ in (36)–(39) and also dropping the term related to differentiation of phase velocity with respect to ψ in (39). This can be verified, for example, by comparing the equations in Aki & Richards (1980, p. 725). Equations (36) and (37) show that the ray direction (the direction of energy propagation) makes an angle of ψ_a with the direction of the wave vector (Fig. 1), which is one of the most important features of wave propagation in an anisotropic media. In addition, equations (38) and (39) tell us that infinitesimal increments of ψ and $\ln k$ (natural logarithm of k) are given by the projection ($\cos\psi_a$) of quantities in the brackets [ ], which are the increments for ψ and $\ln k$ in a transversely isotropic medium [since there is no azimuthal dependence of phase velocity in such a media, $\partial c/\partial\psi = 0$ in (39)]. It is interesting to note that the difference of the wave vector from the ray path direction in anisotropic media is corrected simply by a projection.

3 Fermat's principle

In the following, we show that the equations in the previous section can be derived from the Fermat's principle.

Fermat's principle is given, in general, by

$$\delta \int_{\text{ray}} k_\alpha\, dx^\alpha = 0, \tag{44}$$

where the summation over α is assumed and the integral is taken along the ray path (e.g. Landau & Lifshitz 1975, section 53). In the present problem, which is a surface wave problem in a sphere, this becomes

$$\delta \int_{\text{ray}} (k_\theta \, d\theta + k_\phi \, d\phi) = 0$$

Let us take the integration variable as Δ, the angular distance along the ray, and use (12) and (13):

$$\delta \ a\left[\int_{\text{ray}} \left(k \cos\psi \frac{d\theta}{d\Delta} + k \sin\theta \sin\psi \frac{d\phi}{d\Delta}\right) d\Delta\right] = 0, \tag{45}$$

where we have an additional constraint

$$\left(\frac{d\theta}{d\Delta}\right)^2 + \sin^2\theta \left(\frac{d\phi}{d\Delta}\right)^2 = 1. \tag{46}$$

This constraint allows us to assume the form

$$\frac{d\theta}{d\Delta} = \cos\psi' \tag{47}$$

and

$$\frac{d\phi}{d\Delta} = \frac{1}{\sin\theta} \sin\psi'. \tag{48}$$

Dropping the constant a, we can take the Lagrangian, L, to be

$$L\left(\theta, \frac{d\theta}{d\Delta}, \phi, \frac{d\phi}{d\Delta}\right) = k \cos\psi \frac{d\theta}{d\Delta} + k \sin\theta \sin\psi \frac{d\phi}{d\Delta}, \tag{49}$$

where k is a function of θ, ϕ and ψ, i.e. $k(\theta, \phi, \psi)$.
From the Euler's equation for θ

$$\frac{d}{d\Delta}\left(\frac{\partial L}{\partial (d\theta/d\Delta)}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{50}$$

and (47) and (48), we can obtain

$$\frac{d}{d\Delta}(k \cos\psi) = \cos(\psi' - \psi)\left[\frac{k \cos\theta}{\sin\theta} \sin^2\psi - \left(\frac{\partial k}{\partial \psi}\right)_{W,\theta,\phi} \frac{\cos\theta}{\sin\theta} \cos\psi \sin\psi + \left(\frac{\partial k}{\partial \theta}\right)_{W,\psi,\phi}\right], \tag{51}$$

where we used the relation for $\partial\psi/\partial\theta$ in (19).
From the other Euler's equation for ϕ

$$\frac{d}{d\Delta}\left(\frac{\partial L}{\partial (d\phi/d\Delta)}\right) - \frac{\partial L}{\partial \phi} = 0 \tag{52}$$

and (47) and (48), we get

$$\frac{d}{d\Delta}(k \sin\theta \sin\psi) = \cos(\psi' - \psi)\left(\frac{\partial k}{\partial \phi}\right)_{W,\psi,\theta}. \tag{53}$$

Noting that $k \cos \psi = k_\theta/a$ and $k \sin \theta \sin \psi = k_\phi/a$, it is obvious that the equations (47), (48), (51) and (53) are equivalent to (36), (37), (27) and (28) respectively, if we define ψ_a by $\psi' - \psi = \psi_a$. Thus the ray tracing equations in the previous section can be obtained solely from (45) with the constraint (46).

We can point out a subtle difference between the derivation in Section 2 and the one in this section from Fermat's principle. In Section 2, frequency was shown to be constant along the ray from the original equations in (2)–(4). Fermat's principle, however, assumes the constant frequency along the ray and (44) is given at a fixed frequency.

In transversely isotropic media, ψ' is equal to ψ, thus (45) becomes

$$\delta \int_{\text{ray}} k\, d\Delta = 0, \tag{54}$$

where we dropped a and still have the constraint (46). Dividing (54) by the frequency ω, we obtain

$$\delta \int_{\text{ray}} s\, d\Delta = 0, \tag{55}$$

where $s(=k/\omega)$ is the phase slowness or the inverse of phase velocity. This shows the well-known fact that phase traveltime along the ray path is stationary. This holds, however, only in transversely isotropic media.

In an anisotropic medium, we obtain a slightly different formula, because of the directional difference of the wave vector and the ray path: (45) becomes

$$\delta \left(\int_{\text{ray}} s \cos \psi_a\, d\Delta \right) = 0, \tag{56}$$

which states that phase slowness projected on the ray path and integrated along the ray, is the stationary quantity. Ray tracing by the bending method (Julian & Gubbins 1977) should be done with (56) instead of (55) in anisotropic media.

In a weakly anisotropic medium, the angle $\psi_a(= \psi' - \psi)$ is small. From (40), (41) and (43), this is closely related to the fact that azimuthal variation of phase velocity is a small quantity compared with phase velocity itself. In this case, $\cos \psi_a \approx 1$ up to first order in ψ_a and thus phase traveltime becomes approximately stationary.

4 Conclusion

A set of ray tracing equations for surface waves in an anisotropic earth has been derived. Phase velocity, including its azimuthal variation, is required to solve the equations, but group velocity is eliminated from the equations. The formulation explicitly shows the dependence upon ψ_a, which is the angle difference between the wave vector and the ray path direction and is not zero except in a transversely isotropic media. Ray tracing equations in a transversely isotropic (or simply isotropic) media are obtained from these equations by putting $\psi_a = 0$ and dropping the term with partial derivatives of phase velocity with respect to azimuth. The same equations can also be derived from the Fermat's principle.

In transversely isotropic media, Fermat's principle states that phase travel-time, which is phase slowness integrated along the ray, is stationary. In an anisotropic medium, phase slowness projected on to the ray path and integrated along the ray path is stationary.

Development of the bending method for ray tracing should be done with such a quantity instead of phase travel-time in an anisotropic medium. However, in a weakly anisotropic medium, phase travel-time is approximately stationary.

Acknowledgments

The author wishes to thank Professor Don L. Anderson and Dr Kiyoshi Yomogida for reviewing the manuscript. This research was supported by the National Science Foundation grant number EAR-8507653. Contribution No. 4306, Division of Geological and Planetary Sciences, California Institute of Technology, Pasadena, California 91125.

References

Aki, K. & Richards, P. G., 1980. *Quantitative Seismology, Theory and Methods*, Vol. 2, W. H. Freeman, San Francisco.

Forsyth, D. W., 1975. The early structural evolution and anisotropy of the oceanic upper mantle, *Geophys. J. R. astr. Soc.*, **43,** 103–162.

Gjevik, B., 1974. Ray tracing for seismic surface waves, *Geophys. J. R. astr. Soc.*, **39,** 29–39.

Julian, B. & Gubbins, D., 1977. Three-dimensional seismic ray tracing, *J. Geophys.*, **43,** 95–113.

Landau, L. D. & Lifshitz, E. M., 1975. *The Classical Theory of Fields,* 4th edn, Pergamon Press, Oxford.

Lay, T. & Kanamori, H., 1985. Geometric effects of global lateral heterogeneity on long-period surface wave propagation, *J. geophys. Res.*, **90,** 605–621.

Lighthill, J., 1978. *Waves in Fluids*, Cambridge University Press.

Montagner, J. P. & Nataf, H. C., 1985. Inversion of the azimuthal anisotropy of surface waves: application to the 3-dimensional structure of the Indian ocean, *EOS,* **66,** p. 966.

Nishimura, C. E. & Forsyth, D. W., 1985. Rayleigh wave phase velocities in the Pacific with implications for azimuthal anizotropy and lateral heterogeneity, *Geophys. J. R. astr. Soc.*, submitted.

Sobel, P. A. & von Seggern, D. H., 1978. Applications of surface wave ray tracing, *Bull. seism. Soc. Am.*, **68,** 1359–1380.

Suetsugu, D. & Nakanishi, I., 1985. Regional azimuthal dependence of phase velocities of mantle Rayleigh waves in the Pacific Ocean, *Phys. Earth planet. Int.*, submitted.

Tanimoto, T. & Anderson, D. L., 1984. Mapping convection in the mantle, *Geophys. Res. Lett.*, **11,** 287–290.

Tanimoto, T. & Anderson, D. L., 1985. Lateral heterogeneity and azimuthal anisotropy of the upper mantle: Love and Rayleigh waves 100–250 s, *J. geophys. Res.*, **90,** 1842–1858.

Wong, Y. K. & Woodhouse, J. H., 1984. Amplitude, phase and path anomalies for mantle waves, *EOS,* **244,**

Woodhouse, J. H., 1974. Surface waves in a laterally varying layered structure, *Geophys. J. R. astr. Soc.*, **37,** 461–490.

Yomogida, K. & Aki, K., 1985. Waveform synthesis of surface waves in a laterally heterogeneous earth by the Gaussian Beam method, *J. geophys. Res.*, **90,** 7665–7688.