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Surface Waves in a Laterally Varying Layered Structure

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Summary

A kind of ray theory is developed for surface waves in a layered elastic medium in which there are gradual lateral variations in the thicknesses of the layers and in the elastic parameters characterizing each layer. An approximation equivalent to the JWKB approximation is used, and equations governing the slow variations in amplitude, frequency and wavenumber in a nearly uniform sinusoidal wavetrain are deduced. In the first approximation these equations are found to be those given by Whitham's average Lagrangian method. An equation governing the slow variation in phase is also deduced. The solution of these equations by the method of characteristics gives ray-tracing equations and an amplitude equation similar to those given by standard ray theory for body waves. The analysis also leads to a straightforward method for finding the total energy flux and energy density of the waves without performing an integration over the depth co-ordinate.

1. Introduction

Elastic surface waves in non-uniform waveguides have long been studied in seismology and attention has mainly been given to problems with varying layer thickness. De Noyer (1961) derived the period equation for harmonic Love waves propagating in a layer whose thickness varies sinusoidally in the direction of propagation and deduced, from energy conservation, the variation in amplitude across the structure. Ghosh (1963) used a Green's function technique to study the attenuation of Love waves at a continental boundary and the representation theorems of elastodynamics have been used by several authors (Herrera 1964; Knopoff & Hudson 1964; Herrera & Mal 1965; Mal & Knopoff 1965; Mal & Herrera 1965; Knopoff & Mal 1967) to investigate the scattering of surface waves by regions of varying layer thickness or lateral heterogeneity. Wolf (1967, 1970) has presented a method for determining the scattered field when a Love wave is incident upon a small irregularity in the free surface and this method has been extended (Slavin & Wolf 1970) to treat the case in which the irregularity cannot be considered small. The scattered field is represented as an integral in the complex plane over the modes of the unperturbed structure and the kernal of the integral is then chosen in such a way that the boundary condition is approximately satisfied on the irregular part of the boundary. Thapar (1970) has presented some theoretical and experimental results for the amplitude variations in Rayleigh waves as they traverse a small irregularity in the free surface of a uniform half-space or an irregularity in the interface of a two-layered half-space model. Boore (1970) has performed finite difference calculations for a Love wave pulse incident upon a thickening layer and Lysmer & Drake (1971) have applied the finite element method to determine the response to incident harmonic Love waves of quite complex structures.

Except for the simple treatment of De Noyer (1961), all the methods cited above require large amounts of computing time to obtain the response of a given structure and except for the finite difference calculations of Boore (1970) all methods deal with, waves of a single frequency. The aim of the present paper is to develop a theory analogous to standard ray theory for body waves which will give good results at high frequencies and which will be simpler to apply than the full wave solutions. The usefulness of the theory will be in determining travel times, shadow zones, and approximations to amplitudes for surface waves travelling over the two-dimensional surface of a layered structure in which the surface altitude and the underlying structure may vary gradually with the two horizontal co-ordinates. The theory will allow, however, for a wave to be propagated over many wavelengths, and to pass into a medium differing greatly in structure from the original medium. The theory suffers from the same inadequacies as ray theory, in that reflected waves are neglected together with mode conversion.

An approximate and general theory for nearly uniform harmonic wavetrains has been developed by Whitham (1965a, b) in terms of an average Lagrangian and this theory has been applied by Gjevik (1973) to Love waves in a layer whose thickness varies in the direction of propagation. The method gives equations governing the slow variations in frequency, wavenumber and amplitude of the waves. It has been shown (Bretherton 1968) that for linear waves in a slowly varying waveguide Whitham's equations may be obtained from a perturbation procedure. An advantage of this latter method is that it shows how higher approximations may be obtained; in fact the equations governing the first approximation are deduced from a necessary condition that the higher approximations should exist. In the present paper we apply this perturbation procedure to trapped elastic waves in a solid layered structure, taking advantage of the propagator matrix formalism for the unperturbed problem. Our results reduce to those of Gjevik, and are essentially equivalent to those of De Noyer in the restricted cases they consider.

2. Formulation of the problem and equations of motion

We shall consider a semi-infinite elastic medium (Fig. 1)

$$z \geqslant \eta_0(x, y)$$

where x, y, z are cartesian co-ordinates. We shall use Greek letters to denote suffices taking only the values 1, 2 and shall use the summation convention throughout. We set

$$x_1 = x, \quad x_2 = y, \quad x_3 = z$$

and also define

$$X_\sigma = \varepsilon x_\sigma, \quad T = \varepsilon t$$

where ε is a small parameter. The elastic parameters c_{ijkl} and the density ρ are allowed to have discontinuities at surfaces

$$z = \eta_r(x_\sigma), \qquad r = 0, 1, 2 \ldots N \tag{2.1}$$

where it is assumed that

$$\frac{\partial \eta_r}{\partial X_\sigma} = O(1), \quad r = 0, 1, 2 \ldots N.$$

This means that as $\varepsilon \to 0$ the slopes of the boundaries tend to zero.

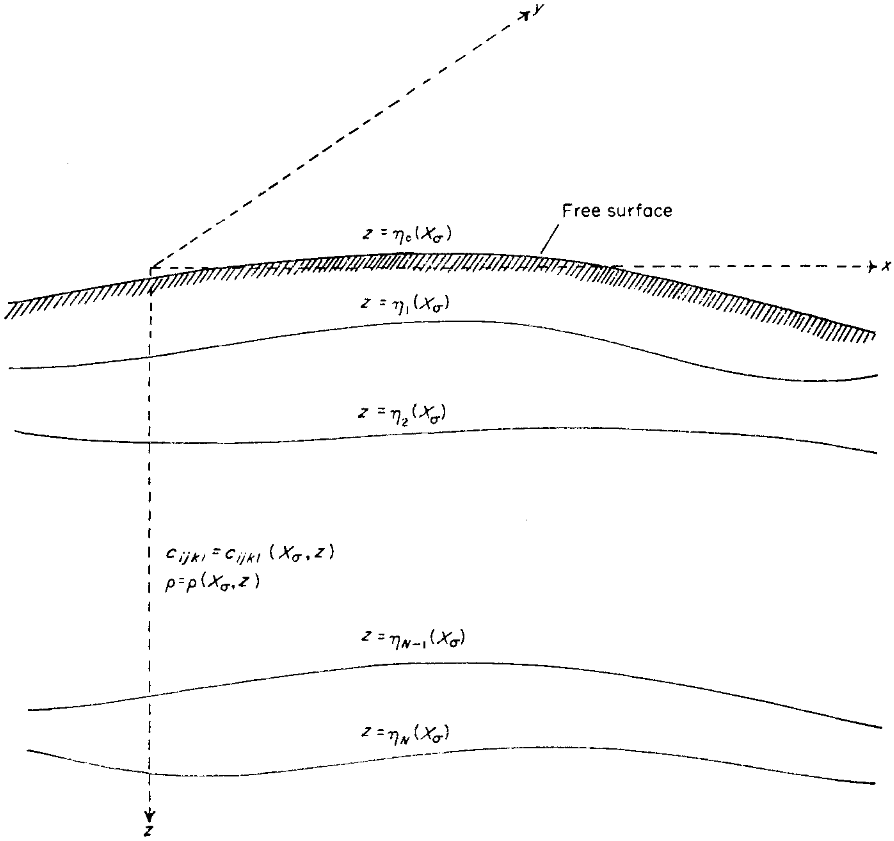

FIG. 1. A laterally heterogeneous layered structure.

Now if $\varepsilon = 0$ the medium is a laterally homogeneous layered structure for which the theory of harmonic waves travelling parallel to the surface and confined to the region of the layers is well known. We shall here perform a perturbation in the small parameter ε to find the behaviour of a nearly uniform wavetrain in a laterally varying structure, provided that this variation is small within a wavelength.

We shall assume that the disturbance in the solid is governed by the linear stress-strain relation:

$$\tau_{ij} = c_{ijkl}\, u_{k,l} \tag{2.2}$$

where $\mathbf{u}$ is the elastic displacement vector and c_{ijkl} has the usual symmetries:

$$c_{ijkl} = c_{ijlk} = c_{jikl} = c_{klij}.$$

Moreover it is assumed that the potential energy density

$$\tfrac{1}{2} c_{ijkl}\, u_{i,j}\, u_{k,l}$$

is a positive definite quadratic form in the quantities $u_{i,j}$ when they are real. The momentum equation (in the absence of body forces and sources) is

$$\frac{\partial}{\partial x_j}\tau_{ij} = \rho\frac{\partial^2 u_i}{\partial t^2} \tag{2.3}$$

and equations (2.2), (2.3) may be combined into the single equation

$$\frac{\partial}{\partial x_j}(c_{ijkl}\, u_{k,l}) = \rho \frac{\partial^2 u_i}{\partial t^2}. \tag{2.4}$$

We shall apply the boundary conditions that the surface tractions vanish at the free surface, and are continuous at the surfaces of discontinuity (2.1); i.e.

$$\left[\tau_{i3} - \tau_{i\sigma}\frac{\partial \eta_r}{\partial x_\sigma}\right]_{\eta_r-0}^{\eta_r+0} = 0, \quad r = 0, 1, 2 \dots N. \tag{2.5}$$

The free surface condition is included in equation (2.5) by defining

$$\tau_{ij} \equiv 0 \quad \text{for} \quad z < \eta_0(x, y)$$

and equation (2.5) will be referred to from now on as the continuity condition. We also require that the displacements $\mathbf{u}$ be continuous at the surfaces of discontinuity, and we shall assume this to be the case throughout without again explicitly stating the condition.

Let us define the matrices $\mathbf{C}_{ij}$ and vectors $\boldsymbol{\tau}$, $\mathbf{u}$, $\boldsymbol{\tau}_\sigma$ by the following relations:

$$\begin{aligned}
(\mathbf{C}_{ij})_{kl} &= c_{kilj} \\
(\boldsymbol{\tau})_i &= \tau_{i3} \\
(\boldsymbol{\tau}_\sigma)_i &= \tau_{i\sigma} \\
(\mathbf{u})_i &= u_i \\
\mathbf{Q}_{\sigma\nu} &= \mathbf{C}_{\sigma\nu} - \mathbf{C}_{\sigma 3}\,\mathbf{C}_{33}^{-1}\,\mathbf{C}_{3\nu}.
\end{aligned}$$

It is easily shown that $\mathbf{C}_{33}$ must be non-singular for the potential energy density to be positive definite. The lateral stress vectors $\boldsymbol{\tau}_\sigma$ may now be written

$$\boldsymbol{\tau}_\sigma = \mathbf{Q}_{\sigma\nu}\frac{\partial \mathbf{u}}{\partial x_\nu} + \mathbf{C}_{\sigma 3}\,\mathbf{C}_{33}^{-1}\,\boldsymbol{\tau} \tag{2.6}$$

and the equations of motion (2.2), (2.3) take the following form involving only $\mathbf{u}$ and $\boldsymbol{\tau}$:

$$\left.\begin{aligned}
\frac{\partial \mathbf{u}}{\partial z} &= \mathbf{C}_{33}^{-1}\,\boldsymbol{\tau} - \mathbf{C}_{33}^{-1}\,\mathbf{C}_{3\sigma}\frac{\partial \mathbf{u}}{\partial x_\sigma} \\
\frac{\partial \boldsymbol{\tau}}{\partial z} &= -\frac{\partial}{\partial x_\sigma}\left(\mathbf{Q}_{\sigma\nu}\frac{\partial \mathbf{u}}{\partial x_\nu}\right) + \rho\frac{\partial^2 \mathbf{u}}{\partial t^2} - \frac{\partial}{\partial x_\sigma}(\mathbf{C}_{\sigma 3}\,\mathbf{C}_{33}^{-1}\,\boldsymbol{\tau})
\end{aligned}\right\}. \tag{2.7}$$

Now if $\varepsilon = 0$ these equations have a solution in the form of a uniform harmonic wavetrain with

$$\mathbf{u} = \mathbf{u}(z)\exp\big(i(k_\sigma x_\sigma - \omega t)\big) \tag{2.8}$$

and the equation

$$\frac{\partial \mathbf{f}}{\partial z} = \mathbf{A}\mathbf{f} \tag{2.9}$$

is obtained where we have defined the 6-vector

$$\mathbf{f} = \left(\frac{\mathbf{u}}{\boldsymbol{\tau}}\right)$$

and the 6×6 partitioned matrix:

$$\mathbf{A} = \left(\begin{array}{c|c} \mathbf{T}^{\dagger} & \mathbf{C}_{33}^{-1} \\ \hline -\mathbf{S} & -\mathbf{T} \end{array}\right) \tag{2.10}$$

with

$$\mathbf{S} = \rho\omega^2\,\mathbf{1} - k_\sigma k_\nu \mathbf{Q}_{\sigma\nu}$$

$$\mathbf{T} = ik_\sigma \mathbf{C}_{\sigma 3}\mathbf{C}_{33}^{-1}$$

and where " † " denotes Hermitian conjugation. Equation (2.9) forms the basis of the propagator matrix formalism, introduced by Gilbert & Backus (1966), which is used in Section 6 of the present paper. For an isotropic material we have

$$c_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}$$

and if the vector $\mathbf{k}$ is directed along the x_1 axis we have:

$$\left.\begin{array}{l}
\mathbf{T} = \begin{pmatrix} 0 & 0 & \dfrac{ik_1\lambda}{\lambda+2\mu} \\ 0 & 0 & 0 \\ ik_1 & 0 & 0 \end{pmatrix} \qquad
\mathbf{C}_{33}^{-1} = \begin{pmatrix} \dfrac{1}{\mu} & 0 & 0 \\ 0 & \dfrac{1}{\mu} & 0 \\ 0 & 0 & \dfrac{1}{\lambda+2\mu} \end{pmatrix} \\[2ex]
\mathbf{S} = \begin{pmatrix} \rho\omega^2 - k_1{}^2\left[(\lambda+2\mu) - \dfrac{\lambda^2}{(\lambda+2\mu)}\right] & 0 & 0 \\ 0 & \rho\omega^2 - \mu k_1{}^2 & 0 \\ 0 & 0 & \rho\omega^2 \end{pmatrix}
\end{array}\right\}. \tag{2.11}$$

3. Variational principles

(a) *General considerations*

We shall assume throughout that the fields considered decrease sufficiently rapidly as $z \to \infty$ for the relevant integrals to exist. Let us consider the Lagrangian density:

$$L(\tilde{u}_i, u_i) \equiv \int_{\eta_0}^{\infty} \tfrac{1}{2}\left\{\rho\frac{\partial\tilde{u}_i}{\partial t}\frac{\partial u_i}{\partial t} - c_{ijkl}\,\tilde{u}_{i,j}\,u_{k,l}\right\} dz \tag{3.1}$$

where, for the moment, $\mathbf{u}$, $\tilde{\mathbf{u}}$ are independent vector fields subject to independent increments $\delta\mathbf{u}$, $\delta\tilde{\mathbf{u}}$. We find that

$$\delta L = \frac{\partial}{\partial t}\int_{\eta_0}^{\infty} \tfrac{1}{2}\left\{\delta\tilde{u}_i\,\rho\frac{\partial u_i}{\partial t} + \frac{\partial\tilde{u}_i}{\partial t}\rho\delta u_i\right\} dz$$

$$-\frac{\partial}{\partial x_\sigma}\int_{\eta_0}^{\infty}\tfrac{1}{2}\left\{c_{i\sigma kl}\,\delta\tilde{u}_i\,u_{k,l}+c_{ijk\sigma}\,\tilde{u}_{i,j}\,\delta u_k\right\}dz$$

$$+\int_{\eta_0}^{\infty}\tfrac{1}{2}\left\{\delta\tilde{u}_i\left[\frac{\partial}{\partial x_j}(c_{ijkl}\,u_{k,l})-\rho\frac{\partial^2 u_i}{\partial t^2}\right]\right.$$

$$\left.+\left[\frac{\partial}{\partial x_j}(c_{ijkl}\,\tilde{u}_{k,l})-\rho\frac{\partial^2\tilde{u}_i}{\partial t^2}\right]\delta u_i\right\}dz$$

$$+\tfrac{1}{2}\sum_{r=0}^{N}\left[\delta\tilde{u}_i\left(c_{i3kl}-\frac{\partial\eta_r}{\partial x_\sigma}c_{i\sigma kl}\,u_{k,l}\right)\right.$$

$$\left.+\left(c_{ijk3}\,\tilde{u}_{i,j}-\frac{\partial\eta_r}{\partial x_\sigma}c_{ijk\sigma}\,\tilde{u}_{i,j}\right)\delta u_k\right]_{\eta_r-0}^{\eta_r+0}.$$

We see that if δu_i, $\delta\tilde{u}_i$ are required to be continuous, and if these variations vanish on the boundary $\mathscr{B}$ of a region $\mathscr{R}$ in (x_σ, t) space, then the variational equation:

$$\delta\iiint_{\mathscr{R}} L\,dx_1\,dx_2\,dt = 0 \tag{3.3}$$

gives rise to the equations of motion (2.4) and the continuity condition (2.5) for the two fields u_i, $\tilde{u}_i$.

Alternatively we may consider the Lagrangian density:

$$L'(u_i) \equiv L(u_i, u_i) \tag{3.4}$$

which, for a real vector field **u** has the usual interpretation as the difference of the kinetic and potential energy densities and leads to the same equations and continuity conditions for the single vector field **u**.

In the case where the displacement is $Re(u_i)$ where u_i is complex a third Lagrangian density is useful:

$$\mathscr{L} = \tfrac{1}{2}L(u_i{}^*, u_i) \tag{3.5}$$

(' * ' indicates complex conjugation) which again leads to the equations (2.4), (2.5) for the single vector field **u**. In a laterally homogeneous solid in which the displacement takes the form (2.8) $\mathscr{L}$ may be interpreted as the average value of $L'(Re\,u_i)$, the average being taken over one period of the exponentials occuring in (2.8).

(b) *A laterally homogeneous solid*

If the solid is laterally homogeneous we may consider displacements

$$u_i = u_i(z)\exp(i(k_\sigma x_\sigma-\omega t)). \tag{3.6}$$

Let us consider the Lagrangian density $\mathscr{L}$ defined by equation (3.5). If $u_i(z)$ is subject to variations $\delta u_i(z)$ we see that in order to obtain a variational principle we may drop the demand that δu_i vanishes on a certain surface in (x_σ, t) space since the integrands in the first two terms on the right-hand side of equation (3.2) are independent of x_σ, t. The variational equation

$$\delta\iiint_{\mathscr{R}} \mathscr{L}\,dx_1\,dx_2\,dt = 0$$

takes the simple form

$$\delta\mathscr{L} = 0.$$

By considering certain specific variations $\delta\mathbf{u}$ we may obtain alternative forms for the integral $\mathscr{L}$ which will be useful later, Let us first take

$$\delta u_i = \gamma u_i$$

with γ real, and assume that $\mathbf{u}$ satisfies the equations of motion and continuity conditions, but not necessarily the boundary condition at the free surface. (The reason we consider such fields is that they are easily constructed using propagator matrices, as we shall show in Section 6.) From equation (3.2) we find

$$\delta\mathscr{L} = 2\gamma\mathscr{L} = \tfrac{1}{4}\gamma(\mathbf{u}^\dagger\,\boldsymbol{\tau}+\boldsymbol{\tau}^\dagger\,\mathbf{u})|_{z=\eta_0}$$

and so

$$\mathscr{L} = \tfrac{1}{8}(\mathbf{u}^\dagger\,\boldsymbol{\tau}+\boldsymbol{\tau}^\dagger\,\mathbf{u}) \tag{3.7}$$

where the notation is that used in Section 2.

Now take

$$\delta u_i = i\gamma u_i$$

so that

$$\delta\mathscr{L} = 0 = \tfrac{1}{4}i\gamma(\boldsymbol{\tau}^\dagger\,\mathbf{u}-\mathbf{u}^\dagger\,\boldsymbol{\tau}). \tag{3.8}$$

Hence

$$\mathscr{L} = \tfrac{1}{4}(\mathbf{u}^\dagger\,\boldsymbol{\tau})|_{z=\eta_0} = \tfrac{1}{4}(\boldsymbol{\tau}^\dagger\,\mathbf{u})|_{z=\eta_0} \tag{3.9}$$

and when the free surface condition

$$(\boldsymbol{\tau})|_{z=\eta_0} = 0$$

is satisfied $\mathscr{L} = 0$ giving the familiar result that average potential and kinetic energies are equal in a linear wave system. Also since $\mathscr{L}$ is stationary with respect to small variations $\delta u_i(z)$ from displacements $\mathbf{u}$ satisfying all equations of motion and continuity conditions, we obtain Rayleigh's principle for the calculation of ω when k_σ is known. It is easily shown that the average vertical energy flux

$$-Av\left[Re(\tau_{i3})\,Re\left(\frac{\partial u_i}{\partial t}\right)\right]$$

is given by:

$$-\frac{i\omega}{4}(\mathbf{u}^\dagger\,\boldsymbol{\tau}-\boldsymbol{\tau}^\dagger\,\mathbf{u})$$

so that equation (3.8) also expresses the fact that no energy is transmitted across the free surface.

It is of interest to find the form taken by the integral $\mathscr{L}$ in terms of the fields $\mathbf{u}$ and $\boldsymbol{\tau}$. Using equation (2.7) we may eliminate $\partial u_i/\partial z$ from $\mathscr{L}$ in favour of the stress components τ_{i3}. We obtain

$$\mathscr{L} = \int_{\eta_0}^{\infty}\tfrac{1}{4}(\mathbf{u}^\dagger\,\mathbf{S}\mathbf{u}-\boldsymbol{\tau}^\dagger\,\mathbf{C}_{33}^{-1}\,\boldsymbol{\tau})\,dz \tag{3.10}$$

which we shall write as $\mathscr{L}(\mathbf{u},\,\boldsymbol{\tau})$, where

$$\left.\begin{aligned}\boldsymbol{\tau} &= \mathbf{C}_{33}\frac{\partial\mathbf{u}}{\partial z}+ik_\sigma\,\mathbf{C}_{3\sigma}\,\mathbf{u}\\ \boldsymbol{\tau}^\dagger &= \frac{\partial\mathbf{u}^\dagger}{\partial z}\,\mathbf{C}_{33}-ik_\sigma\,\mathbf{u}^\dagger\,\mathbf{C}_{\sigma 3}.\end{aligned}\right\} \tag{3.11}$$

The variational problem is to find $\mathbf{u}$, $\boldsymbol{\tau}$ such that (3.10) is stationary, under the constraint (3.11). This constraint may be incorporated into the Lagrangian by adding terms

$$\tfrac{1}{4}\tilde{\boldsymbol{\phi}}\left(\boldsymbol{\tau}-\mathbf{C}_{33}^{-1}\frac{\partial \mathbf{u}}{\partial z}-ik_\sigma\mathbf{C}_{3\sigma}\mathbf{u}\right),\ \tfrac{1}{4}\left(\boldsymbol{\tau}^\dagger-\frac{\partial \mathbf{u}^\dagger}{\partial z}\mathbf{C}_{33}+ik_\sigma\mathbf{u}^\dagger\mathbf{C}_{\sigma 3}\right)\boldsymbol{\phi}$$

to the integrand in (3.10), where the row vector $\tilde{\boldsymbol{\phi}}$ and the column vector $\boldsymbol{\phi}$ are independent Lagrange multipliers (Seliger & Whitham 1968). We may now use the variational principle with $\boldsymbol{\tau}$, $\mathbf{u}$ subject to independent variations. Variations with respect to $\boldsymbol{\tau}$, $\boldsymbol{\tau}^\dagger$ yield

$$\tilde{\boldsymbol{\phi}} = \boldsymbol{\tau}^\dagger\mathbf{C}_{33}^{-1}$$

$$\boldsymbol{\phi} = \mathbf{C}_{33}^{-1}\boldsymbol{\tau}$$

and the Lagrangian takes the form:

$$\mathscr{L} = \int_{\eta_0}^{\infty}\tfrac{1}{4}\left\{\mathbf{u}^\dagger ik_\sigma\mathbf{C}_{\sigma 3}\mathbf{C}_{33}^{-1}\boldsymbol{\tau}-\boldsymbol{\tau}^\dagger ik_\sigma\mathbf{C}_{33}^{-1}\mathbf{C}_{3\sigma}\mathbf{u}-\mathbf{u}^\dagger k_\sigma k_\nu\mathbf{Q}_{\sigma\nu}\mathbf{u}\right.$$
$$\left.+\rho\omega^2\mathbf{u}^\dagger\mathbf{u}-\frac{\partial\mathbf{u}^\dagger}{\partial z}\boldsymbol{\tau}-\frac{\partial\boldsymbol{\tau}^\dagger}{\partial z}\mathbf{u}\right\}dz. \tag{3.12}$$

This may be written:

$$\mathscr{L} = \int_{\eta_0}^{\infty}\tfrac{1}{4}\left\{\mathbf{f}^\dagger\mathbf{B}\mathbf{f}-\frac{\partial\mathbf{f}^\dagger}{\partial z}\mathbf{M}\mathbf{f}-\mathbf{f}^\dagger\mathbf{M}^\dagger\frac{\partial\mathbf{f}}{\partial z}\right\}dz \tag{3.13}$$

with

$$\mathbf{B} \equiv \left(\begin{array}{c|c}\mathbf{S} & ik_\sigma\mathbf{C}_{\sigma 3}\mathbf{C}_{33}^{-1}\\ \hline -ik_\sigma\mathbf{C}_{33}^{-1}\mathbf{C}_{3\sigma} & \mathbf{C}_{33}^{-1}\end{array}\right),\qquad \mathbf{M} = \left(\begin{array}{c|c}0 & \mathbf{1}\\ \hline 0 & 0\end{array}\right) \tag{3.14}$$

where $\mathbf{1}$ is the unit 3×3 matrix. We do not justify the procedure of introducing the Lagrange multipliers $\tilde{\boldsymbol{\phi}}$, $\boldsymbol{\phi}$ but simply check that the form (3.13) leads to the correct equations of motion and continuity conditions. We have:

$$\delta\mathscr{L} = \int_{\eta_0}^{\infty}\tfrac{1}{4}\left\{\delta\mathbf{f}^\dagger\left[\mathbf{B}\mathbf{f}-(\mathbf{M}^\dagger-\mathbf{M})\frac{\partial\mathbf{f}}{\partial z}\right]+\left[\mathbf{f}^\dagger\mathbf{B}-\frac{\partial\mathbf{f}^\dagger}{\partial z}(\mathbf{M}-\mathbf{M}^\dagger)\right]\delta\mathbf{f}\right\}dz$$
$$+\sum_{r=0}^{N}\tfrac{1}{4}\Big[\delta\mathbf{f}^\dagger\mathbf{M}\mathbf{f}+\mathbf{f}^\dagger\mathbf{M}^\dagger\mathbf{f}\,\delta\mathbf{f}\Big]_{\eta_r-0}^{\eta_r+0} \tag{3.15}$$

which leads immediately to the equation

$$\frac{\partial\mathbf{f}}{\partial z} = (\mathbf{M}-\mathbf{M}^\dagger)\mathbf{B}\mathbf{f} \tag{3.16a}$$

and the condition that $\boldsymbol{\tau}$ is continuous and vanishes on the free surface. It is straightforward to check from (3.14) that

$$(\mathbf{M}-\mathbf{M}^\dagger)\mathbf{B} = \mathbf{A} \tag{3.16b}$$

and hence (3.16a) is identical with (2.9).

An alternative derivation of (2.9) from (3.10), (3.11) makes use of the classical Hamiltonian formalism. Defining a Lagrangian from the integrand of (3.10)

$$\Lambda = \boldsymbol{\tau}^\dagger \mathbf{C}_{33}^{-1} \boldsymbol{\tau} - \mathbf{u}^\dagger \mathbf{S}\mathbf{u}$$

we have conjugate momenta

$$\mathbf{p}^\dagger = \frac{\partial \Lambda}{\partial \mathbf{u}_{,z}} = \boldsymbol{\tau}^\dagger$$

$$\mathbf{p} = \frac{\partial \Lambda}{\partial \mathbf{u}^\dagger{}_{,z}} = \boldsymbol{\tau}.$$

Now define

$$\mathscr{H} \equiv \boldsymbol{\tau}^\dagger \mathbf{u}_{,z} + \mathbf{u}^\dagger{}_{,z} \boldsymbol{\tau} - \Lambda$$

$$= \mathbf{u}^\dagger \mathbf{S}\mathbf{u} + \boldsymbol{\tau}^\dagger \mathbf{C}_{33}^{-1} \boldsymbol{\tau} - \boldsymbol{\tau}^\dagger \mathbf{C}_{33}^{-1} ik_\sigma \mathbf{C}_{3\sigma} \mathbf{u} + \mathbf{u}^\dagger ik_\sigma \mathbf{C}_{\sigma 3} \mathbf{C}_{33}^{-1} \boldsymbol{\tau}.$$

Hamilton's equations expressing the stationary principle take the simple form:

$$\frac{\partial \mathbf{u}}{\partial z} = \frac{\partial \mathscr{H}}{\partial \boldsymbol{\tau}^\dagger} = \mathbf{C}_{33}^{-1} \boldsymbol{\tau} - \mathbf{C}_{33}^{-1} ik_\sigma \mathbf{C}_{3\sigma} \mathbf{u}$$

$$\frac{\partial \boldsymbol{\tau}}{\partial z} = -\frac{\partial \mathscr{H}}{\partial \mathbf{u}^\dagger} = -\mathbf{S}\mathbf{u} + ik_\sigma \mathbf{C}_{\sigma 3} \mathbf{C}_{33}^{-1} \boldsymbol{\tau}$$

and these equations are equivalent to (2.9). This formulation of the equations in terms of a Hamiltonian is also noted by Kennett (1974).

4. Perturbation theory

We shall now return to the original problem when there are gradual lateral variation in the properties of the medium. Following Bretherton (1968) we seek an expansion for $\mathbf{u}$, $\boldsymbol{\tau}$ in the form

$$\left.\begin{aligned} \mathbf{u} &= \sum_{s=0}^{\infty} \varepsilon^s e^{i\theta} \mathbf{u}^{(s)}(z, X_\sigma, T) \\ \boldsymbol{\tau} &= \sum_{s=0}^{\infty} \varepsilon^s e^{i\theta} \boldsymbol{\tau}^{(s)}(z, X_\sigma, T) \end{aligned}\right\} \qquad (4.1)$$

θ is a function of x_σ, t and we define

$$\left.\begin{aligned} k_\sigma &\equiv \frac{\partial \theta}{\partial x_\sigma} \\ \omega &\equiv -\frac{\partial \theta}{\partial t} \end{aligned}\right\}. \qquad (4.2)$$

We shall assume that k_σ, ω are slowly varying functions of x_σ, t, i.e.

$$k_\sigma = k_\sigma(X_\sigma, T)$$

$$\omega = \omega(X_\sigma, T)$$

and it follows from (4.2) that

$$\left.\begin{aligned} &\frac{\partial k_\sigma}{\partial T} + \frac{\partial \omega}{\partial X_\sigma} = 0 \\ &\frac{\partial k_\nu}{\partial X_\sigma} - \frac{\partial k_\sigma}{\partial X_\nu} = 0. \end{aligned}\right\} \quad (4.3)$$

Now substituting the expressions (4.1) into equations (2.7) we obtain

$$\left.\begin{aligned} &\sum_{s=0}^{\infty} \varepsilon^s e^{i\theta} \frac{\partial \mathbf{u}^{(s)}}{\partial z} = \sum_{s=0}^{\infty} \varepsilon^s e^{i\theta} \left\{ \mathbf{C}_{33}^{-1} \boldsymbol{\tau}^{(s)} - \mathbf{C}_{33}^{-1} \mathbf{C}_{3\sigma} \left(\varepsilon \frac{\partial}{\partial X_\sigma} + ik_\sigma \right) \mathbf{u}^{(s)} \right\} \\ &\sum_{s=0}^{\infty} \varepsilon^s e^{i\theta} \frac{\partial \boldsymbol{\tau}^{(s)}}{\partial t} = \sum_{s=0}^{\infty} \varepsilon^s e^{i\theta} \left\{ - \left(\varepsilon \frac{\partial}{\partial X_\sigma} + ik_\sigma \right) \mathbf{Q}_{\sigma\nu} \left(\varepsilon \frac{\partial}{\partial X_\nu} + ik_\nu \right) \mathbf{u}^{(s)} \right. \\ &\qquad + \rho \left(\varepsilon \frac{\partial}{\partial T} - i\omega \right) \left(\varepsilon \frac{\partial}{\partial T} - i\omega \right) \mathbf{u}^{(s)} \\ &\qquad \left. - \left(\varepsilon \frac{\partial}{\partial X_\sigma} + ik_\sigma \right) \mathbf{C}_{\sigma 3} \mathbf{C}_{33}^{-1} \boldsymbol{\tau}^{(s)} \right\} \end{aligned}\right\} \quad (4.4)$$

and using (2.6) with (2.5) the continuity conditions become:

$$\sum_{s=0}^{\infty} \varepsilon^s e^{i\theta} \left[\boldsymbol{\tau}^{(s)} - \varepsilon \frac{\partial \eta_r}{\partial X_\sigma} \left\{ \mathbf{Q}_{\sigma\nu} \left(\varepsilon \frac{\partial}{\partial X_\nu} + ik_\nu \right) \mathbf{u}^{(s)} + \mathbf{C}_{\sigma 3} \mathbf{C}_{33}^{-1} \boldsymbol{\tau}^{(s)} \right\} \right]_{\eta_r - 0}^{\eta_r + 0} = 0,$$

$$r = 0, 1, 2 \ldots N. \quad (4.5)$$

The zero order terms in ε in (4.4), (4.5) give the equations

$$\left.\begin{aligned} &\frac{\partial \mathbf{f}^{(0)}}{\partial z} = \mathbf{A}(X_\sigma, T) \mathbf{f}^{(0)} \\ &[\boldsymbol{\tau}^{(0)}]_{\eta_r - 0}^{\eta_r + 0} = 0, \qquad t = 0, 1, 2 \ldots N \end{aligned}\right\} \quad (4.6)$$

where

$$\mathbf{f}^{(0)} \equiv \begin{pmatrix} \mathbf{u}^{(0)} \\ \boldsymbol{\tau}^{(0)} \end{pmatrix}.$$

$\mathbf{A}$ is the matrix defined by equation (2.10) although k_σ, c_{ijkl}, ρ, ω will all now be functions of X_σ, T. These 'stretched' co-ordinates enter into (4.6) as parameters only, and for each X_σ, T the equation may be solved in the same way as in the case of a laterally homogeneous medium. Thus the first approximation gives displacements which vary with depth in the same way as those for a wave with frequency ω travelling in a laterally homogeneous structure with the same properties as those at X_σ, T. In addition the wave motion is required to be in a definite mode and ω, k_σ are related to one another by the local dispersion relation for this mode:

$$\omega = \omega(k_\sigma, X_\sigma), \tag{4.7}$$

the stretched co-ordinates entering through the dependence on X_σ of η_k, ρ, c_{ijkl}. For ω, k_σ on this dispersion surface the first approximation $\mathbf{f}^{(0)}$ takes the form

$$\mathbf{f}^{(0)} = a(X_\sigma, T)\,\hat{\mathbf{f}}(k_\sigma, \omega, X_\sigma, z) \tag{4.8}$$

where $\hat{\mathbf{f}}$ is a known solution of (4.6) with fixed phase and normalization, and $a(X_\sigma, T)$ is a phase and normalization factor which remains to be determined.

The first order terms in ε in (4.4) (4.5) give:

$$\left.\begin{aligned} &\frac{\partial \mathbf{u}^{(1)}}{\partial z} - \mathbf{C}_{33}^{-1}\,\boldsymbol{\tau}^{(1)} + \mathbf{C}_{33}^{-1}\,ik_\sigma\,\mathbf{C}_{\sigma 3}\,\mathbf{u}^{(1)} = -\mathbf{C}_{33}^{-1}\,\mathbf{C}_{3\sigma}\frac{\partial}{\partial X_\sigma}\,\mathbf{u}^{(0)} \\ &\frac{\partial \boldsymbol{\tau}^{(1)}}{\partial z} + (\rho\omega^2 - k_\sigma k_\nu \mathbf{Q}_{\sigma\nu})\,\mathbf{u}^{(1)} + ik_\sigma\,\mathbf{C}_{\sigma 3}\,\mathbf{C}_{33}^{-1}\,\boldsymbol{\tau}^{(1)} \\ &\qquad = -\left(\frac{\partial}{\partial X_\sigma}\mathbf{Q}_{\sigma\nu}\,ik_\nu + ik_\sigma\,\mathbf{Q}_{\sigma\nu}\frac{\partial}{\partial X_\nu}\right)\mathbf{u}^{(0)} \\ &\qquad\quad -\rho\left(\frac{\partial}{\partial T}\,i\omega + i\omega\frac{\partial}{\partial T}\right)\mathbf{u}^{(0)} - \frac{\partial}{\partial X_\sigma}\left(\mathbf{C}_{\sigma 3}\,\mathbf{C}_{33}^{-1}\,\boldsymbol{\tau}^{(0)}\right) \end{aligned}\right\} \tag{4.9}$$

$$\left[\boldsymbol{\tau}^{(1)} - (\mathbf{Q}_{\sigma\nu}\,ik_\nu\,\mathbf{u}^{(0)} + \mathbf{C}_{\sigma 3}\,\mathbf{C}_{33}^{-1}\,\boldsymbol{\tau}^{(0)})\frac{\partial \eta_r}{\partial X_\sigma}\right]_{\eta_r - 0}^{\eta_r + 0} = 0. \tag{4.10}$$

Now the equations (4.9) may be written

$$\frac{\partial \mathbf{f}^{(1)}}{\partial z} - \mathbf{A}\mathbf{f}^{(1)} = \boldsymbol{\chi} \tag{4.11}$$

where $\boldsymbol{\chi}$ represents the right-hand sides of (4.9), and given that $\mathbf{f}^{(0)}$ is a solution of the corresponding homogeneous equation (equation (4.6)) we may find a necessary condition that a solution $\mathbf{f}^{(1)}$ of equations (4.10), (4.11) exists. Let us consider the integral

$$\begin{aligned} I &\equiv \int_{\eta_0}^{\infty} \mathbf{f}^{(0)\dagger}\left(\mathbf{B} - (\mathbf{M}^\dagger - \mathbf{M})\frac{\partial}{\partial z}\right)\mathbf{f}^{(1)}\,dz \\ &= \int_{\eta_0}^{\infty} \mathbf{f}^{(0)\dagger}\,(\mathbf{M} - \mathbf{M}^\dagger)\,\boldsymbol{\chi}\,dz. \end{aligned} \tag{4.12}$$

Integrating the right-hand side of equation (4.12) by parts we find

$$I = \int_{\eta_0}^{\infty}\left(\mathbf{f}^{(0)\dagger}\,\mathbf{B} - \frac{\partial \mathbf{f}^{(0)\dagger}}{\partial z}\,(\mathbf{M} - \mathbf{M}^\dagger)\right)\mathbf{f}^{(1)}\,dz - \sum_{r=0}^{N}\,\left[\mathbf{f}^{(0)\dagger}\,(\mathbf{M} - \mathbf{M}^\dagger)\,\mathbf{f}^{(1)}\right]_{\eta_r - 0}^{\eta_r + 0}.$$

The integrand here vanishes by equation (4.6) so that

$$\int_{\eta_0}^{\infty} \mathbf{f}^{(0)\dagger} (\mathbf{M}-\mathbf{M}^{\dagger})\,\chi\, dz + \sum_{r=0}^{N} \left[\mathbf{f}^{(0)\dagger} (\mathbf{M}-\mathbf{M}^{\dagger})\mathbf{f}^{(1)}\right]_{\eta_r-0}^{\eta_r+0} = 0. \tag{4.13}$$

Substituting χ from (4.9) we get

$$\int_{\eta_0}^{\infty} \Bigg\{ \boldsymbol{\tau}^{(0)\dagger} \mathbf{C}_{33}^{-1} \mathbf{C}_{3\sigma} \frac{\partial \mathbf{u}^{(0)}}{\partial X_\sigma} - \mathbf{u}^{(0)\dagger} \Bigg[\left(\frac{\partial}{\partial X_\sigma} \mathbf{Q}_{\sigma\nu}\, ik_\nu + ik_\sigma \mathbf{Q}_{\sigma\nu} \frac{\partial}{\partial X_\nu} \right)$$

$$+ \rho \left(\frac{\partial}{\partial T}\, i\omega + i\omega \frac{\partial}{\partial T} \right) \Bigg] \mathbf{u}^{(0)}$$

$$- \mathbf{u}^{(0)\dagger} \frac{\partial}{\partial X_\sigma} \mathbf{C}_{\sigma 3} \mathbf{C}_{33}^{-1} \boldsymbol{\tau}^{(0)} \Bigg\} dz$$

$$+ \sum_{k=0}^{N} \left[\mathbf{u}^{(0)\dagger} \boldsymbol{\tau}^{(1)} - \boldsymbol{\tau}^{(0)\dagger} \mathbf{u}^{(1)} \right]_{\eta_k-0}^{\eta_k+0} = 0.$$

By use of the continuity conditions (4.10) on $\mathbf{u}^{(1)}$, $\boldsymbol{\tau}^{(1)}$ this becomes an equation in $\mathbf{u}^{(0)}$, $\boldsymbol{\tau}^{(0)}$ only and rearranging the derivatives with respect to X_σ, T the equation may be written:

$$\frac{\partial}{\partial X_\sigma} \int_{\eta_0}^{\infty} \{ \mathbf{u}^{(0)\dagger} \mathbf{Q}_{\sigma\nu}\, ik_\nu\, \mathbf{u}^{(0)} + \tfrac{1}{2} (\mathbf{u}^{(0)\dagger} \mathbf{C}_{\sigma 3} \mathbf{C}_{33}^{-1} \boldsymbol{\tau}^{(0)} - \boldsymbol{\tau}^{(0)\dagger} \mathbf{C}_{33}^{-1} \mathbf{C}_{3\sigma} \mathbf{u}^{(0)}) \} dz$$

$$+ \frac{\partial}{\partial T} \int_{\eta_0}^{\infty} (\mathbf{u}^{(0)\dagger} \rho i\omega \mathbf{u}^{(0)}) dz$$

$$= \int_{\eta_0}^{\infty} \Bigg\{ \left(\mathbf{u}^{(0)\dagger} \frac{\overleftarrow{\partial}}{\partial X_\sigma} \mathbf{C}_{\sigma 3} \mathbf{C}_{33}^{-1} \boldsymbol{\tau}^{(0)} + \boldsymbol{\tau}^{(0)\dagger} \mathbf{C}_{33}^{-1} \mathbf{C}_{3\sigma} \frac{\overrightarrow{\partial}}{\partial X_\sigma} \mathbf{u}^{(0)} \right)$$

$$+ \mathbf{u}^{(0)\dagger} \left(\frac{\overleftarrow{\partial}}{\partial X_\sigma} \mathbf{Q}_{\sigma\nu}\, ik_\nu - ik_\sigma \mathbf{Q}_{\sigma\nu} \frac{\overrightarrow{\partial}}{\partial X_\nu} \right) \mathbf{u}^{(0)}$$

$$+ \mathbf{u}^{(0)\dagger} \left(\rho \frac{\overleftarrow{\partial}}{\partial T}\, i\omega - i\omega \frac{\overrightarrow{\partial}}{\partial T} \rho \right) \mathbf{u}^{(0)} \Bigg\} dz \tag{4.14}$$

$$- \frac{\partial}{\partial X_\sigma} \int_{\eta_0}^{\infty} \tfrac{1}{2} \{ \mathbf{u}^{(0)\dagger} \mathbf{C}_{\sigma 3} \mathbf{C}_{33}^{-1} \boldsymbol{\tau}^{(0)} + \boldsymbol{\tau}^{(0)\dagger} \mathbf{C}_{33}^{-1} \mathbf{C}_{3\sigma} \mathbf{u}^{(0)} \} dz$$

where the arrows indicate the directions in which the differential operators act. Now it is clear by inspection that the left-hand side of this equation is pure imaginary and the right-hand side is real so both sides must vanish. We shall consider the implications of putting the right-hand side to zero in Section 9. The left-hand side gives:

$$\frac{\partial}{\partial X_\sigma}\int_{\eta_0}^{\infty}(\boldsymbol{\tau}^{(0)\dagger}\,i\,\mathbf{C}_{33}^{-1}\,\mathbf{C}_{3\sigma}\,\mathbf{u}^{(0)}-\mathbf{u}^{(0)\dagger}\,i\,\mathbf{C}_{\sigma 3}\,\mathbf{C}_{33}^{-1}\,\boldsymbol{\tau}^{(0)}+2\mathbf{u}^{(0)\dagger}\,\mathbf{Q}_{\sigma\nu}\,k_\nu\,\mathbf{u}^{(0)})dz$$

$$+\frac{\partial}{\partial T}\int_{\eta_0}^{\infty}2\mathbf{u}^{(0)\dagger}\,\omega\rho\,\mathbf{u}^{(0)}dz=0. \tag{4.15}$$

Comparing this equation with equation (3.12), we see that it may be written:

$$\frac{\partial}{\partial T}\left(\frac{\bar{\partial}\mathscr{L}}{\partial\omega}\right)-\frac{\partial}{\partial X_\sigma}\left(\frac{\bar{\partial}\mathscr{L}}{\partial k_\sigma}\right)=0 \tag{4.16}$$

where $\mathscr{L}=\mathscr{L}(\mathbf{u}^{(0)},\,\boldsymbol{\tau}^{(0)})$ and where, for the moment, the differential operators

$$\frac{\bar{\partial}}{\partial\omega},\frac{\bar{\partial}}{\partial k_\sigma}$$

are understood to operate only upon the explicit dependence of (3.12) on k_σ, ω, and not on the dependence contained in $\mathbf{u}^{(0)}$, $\boldsymbol{\tau}^{(0)}$ which are solutions to the unperturbed problem for the locally equivalent uniform waveguide.

We may show however that these derivatives may be replaced by ordinary derivatives. First we note that the functions $\hat{\mathbf{f}}$ of equation (4.8) are only uniquely defined for k_σ, ω lying on the dispersion surface (4.7). We shall assume however that these functions are extended in some arbitrary (but differentiable) way to points not on the dispersion surface. The derivatives

$$\frac{\partial\hat{\mathbf{f}}}{\partial k_\sigma},\frac{\partial\hat{\mathbf{f}}}{\partial\omega}$$

may then be defined in terms of this extended definition of $\hat{\mathbf{f}}$. In the examples of Section 7 this extension is accomplished by defining $\hat{\mathbf{f}}$ to be a solution of (4.6) except that one component of the surface traction (τ_{3r}, say) is allowed to be non-zero on the free surface. Setting this final stress to zero gives the dispersion relation, but $\hat{\mathbf{f}}$ is in general defined for arbitrary k_σ, ω. This particular way of extending the functions has the advantage that the Lagrangian $\mathscr{L}$ takes a particularly simple form:

$$\mathscr{L}=\frac{aa^*}{4}\,(\mathbf{u})_r{}^*\,(\boldsymbol{\tau})_r \qquad \text{(not summed)} \tag{4.17}$$

which we have obtained from equation (3.9). Having defined the derivatives

$$\frac{\partial\mathbf{f}^{(0)}}{\partial k_\sigma},\frac{\partial\mathbf{f}^{(0)}}{\partial\omega}$$

we may write:

$$\frac{\partial\mathscr{L}}{\partial k_\sigma}=\frac{\bar{\partial}\mathscr{L}}{\partial k_\sigma}+\int_{\eta_0}^{\infty}\tfrac{1}{4}\left\{\frac{\partial\mathbf{f}^{(0)\dagger}}{\partial k_\sigma}\,\mathbf{Bf}-\frac{\partial^2\mathbf{f}^{(0)\dagger}}{\partial k_\sigma\,\partial z}\,\mathbf{Mf}^{(0)}-\frac{\partial\mathbf{f}^{(0)\dagger}}{\partial k_\sigma}\,\mathbf{M}^\dagger\,\frac{\partial\mathbf{f}^{(0)}}{\partial z}+\mathbf{f}^{(0)\dagger}\,\mathbf{B}\frac{\partial\mathbf{f}^{(0)}}{\partial k_\sigma}\right.$$

$$\left.-\frac{\partial\mathbf{f}^{(0)\dagger}}{\partial z}\,\mathbf{M}\,\frac{\partial\mathbf{f}^{(0)}}{\partial k_\sigma}-\mathbf{f}^{(0)\dagger}\,\mathbf{M}^\dagger\,\frac{\partial^2\mathbf{f}^{(0)}}{\partial k_\sigma\,\partial z}\right\}dz \tag{4.18}$$

and using equation (4.6) we find

$$\frac{\bar{\partial}\mathscr{L}}{\partial k} = \frac{\partial \mathscr{L}}{\partial k_\sigma} - \tfrac{1}{4}\sum_{r=0}^{N}\left[\frac{\partial \mathbf{f}^{(0)\dagger}}{\partial k_\sigma}\mathbf{M}\mathbf{f}^{(0)} + \mathbf{f}^{(0)\dagger}\mathbf{M}^{\dagger}\frac{\partial \mathbf{f}^{(0)}}{\partial k_\sigma}\right]_{\eta_r-0}^{\eta_r+0}$$

and it is now clear that because of the continuity conditions satisfied by $\mathbf{f}^{(0)}$ the additional terms vanish when ω, k_σ lie on the dispersion surface giving

$$\frac{\partial \mathscr{L}}{\partial k_\sigma} = \frac{\bar{\partial}\mathscr{L}}{\partial k_\sigma}.$$

Similarly it may be shown that

$$\frac{\partial \mathscr{L}}{\partial \omega} = \frac{\bar{\partial}\mathscr{L}}{\partial \omega}$$

and equation (4.16) becomes:

$$\frac{\partial}{\partial T}\left(\frac{\partial \mathscr{L}}{\partial \omega}\right) - \frac{\partial}{\partial X_\sigma}\left(\frac{\partial \mathscr{L}}{\partial k_\sigma}\right) = 0. \tag{4.19}$$

It is shown in Section 8 that this equation leads to an equation governing the wave amplitude in the first approximation. For the quasi-static (ω = constant) two-dimensional problem the results of the next section show that equation (4.19) reduces to the condition used by De Noyer (1961) of constant energy transport across the structure.

When the form (4.8) is substituted into $\mathscr{L}$ given by (3.13) we obtain

$$\mathscr{L} = \mathscr{L}(a, \omega, k_\sigma, X_\sigma)$$

a known function of its arguments. Equation (4.19) may be derived from the variational problem:

$$\delta\int \mathscr{L}(a, \omega, k_\sigma, X_\sigma)\, dX_1\, dX_2\, dT = 0$$

with constraints (4.3) namely:

$$\frac{\partial \omega}{\partial X_\sigma} + \frac{\partial k_\sigma}{\partial T} = 0$$

$$\frac{\partial k_\sigma}{\partial X_\nu} - \frac{\partial k_\nu}{\partial X_\sigma} = 0$$

and this is the average Lagrangian principle as formulated by Whitham (1965b). The variational equation for a is simply

$$\frac{\partial \mathscr{L}}{\partial a} = 0$$

which is equivalent to the result

$$\mathscr{L} = 0$$

since a enters into $\mathscr{L}$ only as a multiplying factor.

5. Conservation equations

We shall here obtain two exact conservation equations for the laterally heterogeneous structure and examine the form these take for the approximate solution $\mathbf{f}^{(0)}$. The first equation is closely related to an energy equation and the second to conservation of a quantity which has been called the 'wave action' (e.g. Bretherton 1968). The Lagrangian (3.1) has two obvious invariances; firstly it is not an explicit function of t (we assume that the structure does not change with time) and secondly it is unchanged to first order if we simultaneously replace $\tilde{\mathbf{u}}$ by $\tilde{\mathbf{u}}(1+\gamma)$ and $\mathbf{u}$ by $\mathbf{u}(1-\gamma)$ where γ is a small parameter. Using equation (3.2) (or equivalently by Noether's theorem—see for example Seliger and Whitham 1968) these invariances give rise to two conservation equations:

$$\frac{\partial}{\partial t}\int_{\eta_0}^{\infty}\tfrac{1}{2}\left\{\frac{\partial \tilde{u}_i}{\partial t}\rho\frac{\partial u_i}{\partial t}+c_{ijkl}\,\tilde{u}_{i,j}\,u_{k,l}\right\}dz + \frac{\partial}{\partial x_\sigma}\int_{\eta_0}^{\infty}-\tfrac{1}{2}\left\{c_{i\sigma kl}\frac{\partial \tilde{u}_i}{\partial t}u_{k,l}+c_{ijk\sigma}\,\tilde{u}_{i,j}\frac{\partial u_k}{\partial t}\right\}dz=0 \qquad (5.1)$$

$$\frac{\partial}{\partial t}\int_{\eta_0}^{\infty}\tfrac{1}{2}\left\{\frac{\partial \tilde{u}_i}{\partial t}\rho u_i-\tilde{u}_i\,\rho\frac{\partial u_i}{\partial t}\right\}dz - \frac{\partial}{\partial x_\sigma}\int_{\eta_0}^{\infty}\tfrac{1}{2}\left\{c_{ijk\sigma}\,\tilde{u}_{i,j}\,u_k-c_{i\sigma kl}\,\tilde{u}_i\,u_{k,l}\right\}dz=0. \qquad (5.2)$$

These are exact equations satisfied by any two vector fields $\mathbf{u}$, $\tilde{\mathbf{u}}$ which satisfy the equations of motion and continuity conditions. If $\mathbf{u}$ is a real vector field and we set $\tilde{\mathbf{u}}=\mathbf{u}$ the first integrand of (5.1) becomes the energy density and the second integrand becomes the lateral energy flux vector, so that equation (5.1) is a conservation equation for the lateral flow of energy.

If the medium is laterally homogeneous and we set

$$\mathbf{u}=Re\{\mathbf{u}(z)\exp(i(k_\sigma x_\sigma-\omega t))\}$$

we can average over one period of the exponential to obtain the 'average' energy density:

$$+\int_{\eta_0}^{\infty}\tfrac{1}{4}\{\mathbf{u}^\dagger(z)\,(2\rho\omega^2-\mathbf{S})\,\mathbf{u}(z)+\boldsymbol{\tau}^\dagger(z)\,\mathbf{C}_{33}^{-1}\,\boldsymbol{\tau}(z)\}\,dz$$

and an 'average' energy flux vector:

$$-\int_{\eta_0}^{\infty}\frac{i\omega}{4}\{\mathbf{u}^\dagger(z)\,\boldsymbol{\tau}_\sigma(z)-\boldsymbol{\tau}_\sigma{}^\dagger(z)\,\mathbf{u}(z)\}\,dz.$$

Using the fact that $\mathscr{L}$ given by (3.10) vanishes, and the results of Section 4, these may be written*:

$$\omega\frac{\partial\mathscr{L}}{\partial\omega},\ -\omega\frac{\partial\mathscr{L}}{\partial k_\sigma}$$

* These expressions provide a straightforward way of finding the total energy flux and energy density of surface waves in a laterally homogeneous or heterogeneous layered solid.

respectively, where

$$\mathscr{L} = \mathscr{L}\left(\mathbf{u}(z)\exp\left(i(k_\sigma x_\sigma - \omega t)\right)\right).$$

Considering now the approximate solution $\mathbf{f}^{(0)}$ for the perturbed structure and arguing along the lines of Whitham (1965a) we may expect a conservation equation to hold for the flow of average energy; i.e. we may expect

$$\frac{\partial}{\partial T}\left(\omega\frac{\partial\mathscr{L}}{\partial\omega}\right) - \frac{\partial}{\partial X_\sigma}\left(\omega\frac{\partial\mathscr{L}}{\partial k_\sigma}\right) = 0 \tag{5.3}$$

where $\mathscr{L} = \mathscr{L}\ (\mathbf{u}^{(0)}, \boldsymbol{\tau}^{(0)})$. This equation may, indeed, be deduced from (4.19) and the fact that $\mathscr{L}$ does not depend explicitly upon t. Since $\mathscr{L}$ vanishes for k_σ, ω on the dispersion surface we have:

$$0 = \frac{\partial\mathscr{L}}{\partial T} = \frac{\partial\mathscr{L}}{\partial k_\sigma}\frac{\partial k_\sigma}{\partial T} + \frac{\partial\mathscr{L}}{\partial\omega}\frac{\partial\omega}{\partial T}. \tag{5.4}$$

(The derivatives with respect to T are with X_σ fixed, and those with respect to ω, k_σ are those obtained by differentiating the functional form $\mathscr{L} = \mathscr{L}\ (a, \omega, k_\sigma, X_\sigma)$). Using (4.3) and (4.19) in (5.4) equation (5.3) is immediately deduced.

The second conservation equation (5.2) leads to equation (4.19) if we substitute

$$\tilde{u}_i{}^* = u_i = \sum_{s=0}^{\infty} \varepsilon^s e^{i\theta} u_i^{(s)}(z, X_\sigma, T)$$

and retain only zero order terms in ε.

6. Propagator matrices

The equations (4.6) governing the z dependence of the first approximation in the perturbation expansions are identical with those governing wave propagation in a uniform waveguide, with the ' stretched ' co-ordinates X_σ entering only as parameters. They may be solved by standard propagator matrix methods (Haskell 1953; Gilbert & Backus 1966) and we shall simply present some results for these matrices without proof.

The propagator matrix $\mathbf{P}(z_1 z_0)$ is defined as the (unique) continuous solution of the matrix equations

$$\left.\begin{aligned} \frac{d}{dz}\mathbf{P}(z, z_0) &= \mathbf{A}(z)\,\mathbf{P}(z, z_0) \\ \mathbf{P}(z_0, z_0) &= \mathbb{1} \end{aligned}\right\} \tag{6.1}$$

where the notation is that used in Section 2 and $\mathbb{1}$ is the unit 6×6 matrix. $\mathbf{P}$ has the following properties:

(i) $$\mathbf{P}^{-1}(z_1, z_2) = \mathbf{P}(z_2, z_1)$$

(ii)* $$\mathbf{P}^\dagger(z_1, z_2)\,(\mathbf{M}-\mathbf{M}^\dagger) = (\mathbf{M}-\mathbf{M}^\dagger)\,\mathbf{P}^{-1}(z_1, z_2)$$

(iii) $$\mathbf{P}(z_1, z_2)\,\mathbf{P}(z_2, z_3) = \mathbf{P}(z_1, z_3)$$

(iv) If $\mathbf{A}(z)$ is constant in the region

$$z_1 \leqslant z \leqslant z_2$$

* Results (ii) and (vi) appear to be new and are proved in the Appendix.

then

$$\mathbf{P}(z_1, z_2) = \exp(\mathbf{A}(z_1 - z_2))$$

(v) The solution of the initial value problem

$$\frac{\partial \mathbf{f}}{\partial z} = \mathbf{A}\mathbf{f}$$

$$\mathbf{f}(z_0) = \mathbf{f}_0$$

is given by

$$\mathbf{f}(z) = \mathbf{P}(z_1 z_0)\,\mathbf{f}_0$$

(vi)* If the material is transversely isotropic throughout

$$\mathbf{NP}(z_1, z_2) = \mathbf{P}^*(z_1, z_2)\,\mathbf{N}$$

where

$$\mathbf{N} \equiv \begin{pmatrix} -1 & & & & & \\ & -1 & & & \bigcirc & \\ & & 1 & & & \\ & & & -1 & & \\ & \bigcirc & & & -1 & \\ & & & & & 1 \end{pmatrix}$$

Now let us suppose that the value of $\mathbf{f}^{(0)}$ on the lowest surface of discontinuity is given by

$$\mathbf{f}^{(0)}(\eta_N) = \mathbf{f}_N.$$

In order that the wave motion is confined to the region of the layers we require

$$\lim_{z \to \infty} \mathbf{P}(z_1\, \eta_N)\,\mathbf{f}_N = 0 \tag{6.2}$$

and using (3.7) $\mathscr{L}$ is given by

$$\mathscr{L} = \tfrac{1}{8}\,\mathbf{f}_N{}^\dagger\, \mathbf{P}^\dagger(\eta_0, \eta_N)\,(\mathbf{M} + \mathbf{M}^\dagger)\,\mathbf{P}(\eta_0, \eta_N)\,\mathbf{f}_N. \tag{6.3}$$

The condition that the stresses vanish at the free surface may be written

$$\mathbf{MP}(\eta_0, \eta_N)\,\mathbf{f}_N = 0. \tag{6.4}$$

It is shown in Section 10 how equations (6.2) to (6.4) may be applied to find $\mathscr{L}(a, \omega, k_\sigma, X_\sigma)$.

For an isotropic medium in which $\mathbf{k}$ is parallel to the x_1 axis we have:

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & -ik & 1/\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\mu & 0 \\ -\dfrac{ik\lambda}{\lambda+2\mu} & 0 & 0 & 0 & 0 & \dfrac{1}{\lambda+2\mu} \\ -\rho\omega^2 + k^2\left[(\lambda+2\mu) - \dfrac{\lambda^2}{\lambda+2\mu}\right] & 0 & 0 & 0 & 0 & -\dfrac{ik\lambda}{\lambda+2\mu} \\ 0 & -\rho\omega^2 + \mu k^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\rho\omega^2 & -ik & 0 & 0 \end{pmatrix} \tag{6.5}$$

* Results (ii) and (vi) appear to be new and are proved in the Appendix.

Let us define

$$\left.\begin{aligned} \nu &= \left(k^2 - \frac{\omega^2\rho}{\lambda+2\mu}\right)^{\frac{1}{2}} \\ \nu' &= \left(k^2 - \frac{\omega^2\rho}{\mu}\right)^{\frac{1}{2}} \end{aligned}\right\} \tag{6.6}$$

with ν, ν' being either positive real or positive imaginary. Then **A** has eigenvectors given by the columns of a matrix **R**, belonging to eigenvalues $-\nu'$, $-\nu$, ν', ν, $-\nu'$, ν', where

$$\mathbf{R} = \begin{pmatrix} -i\nu' & ik & i\nu' & ik & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ k & -\nu & k & \nu & 0 & 0 \\ i(2\mu k^2-\rho\omega^2) & -2ik\mu\nu & i(2\mu k^2-\rho\omega^2) & 2ik\mu\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu\nu' & \mu\nu' \\ -2\mu k\nu' & 2\mu k^2-\rho\omega^2 & 2\mu k\nu' & 2\mu k^2-\rho\omega^2 & 0 & 0 \end{pmatrix} \tag{6.7}$$

and $\mathbf{R}^{-1}$ is given by:

$$\mathbf{R}^{-1} = \begin{pmatrix} -\dfrac{i(2\mu k^2-\rho\omega^2)}{2\nu'\rho\omega^2} & 0 & \dfrac{k\mu}{\rho\omega^2} & \dfrac{i}{2\rho\omega^2} & 0 & -\dfrac{k}{2\nu'\rho\omega^2} \\ -\dfrac{ik\mu}{\rho\omega^2} & 0 & \dfrac{1}{2\nu\rho\omega^2}(2\mu k^2-\rho\omega^2) & \dfrac{ki}{2\nu\rho\omega^2} & 0 & -\dfrac{1}{2\rho\omega^2} \\ \dfrac{i(2\mu k^2-\rho\omega^2)}{2\nu'\rho\omega^2} & 0 & \dfrac{k\mu}{\rho\omega^2} & \dfrac{i}{2\rho\omega^2} & 0 & \dfrac{k}{2\nu'\rho\omega^2} \\ -\dfrac{ik\mu}{\rho\omega^2} & 0 & -\dfrac{1}{2\nu\rho\omega^2}(2\mu k^2-\rho\omega^2) & \dfrac{-ki}{2\nu\rho\omega^2} & 0 & -\dfrac{1}{2\rho\omega^2} \\ 0 & \frac{1}{2} & 0 & 0 & -\dfrac{1}{2\mu\nu'} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \dfrac{1}{2\mu\nu'} & 0 \end{pmatrix} \tag{6.8}$$

so that

$$\mathbf{R}^{-1}\mathbf{A}\mathbf{R} = \mathrm{Diag}\,(-\nu', -\nu, \nu', \nu, -\nu', \nu')$$

and

$$\exp(\mathbf{A}z) = \mathbf{R}\,\mathrm{Diag}(e^{-\nu' z}, e^{-\nu z}, e^{\nu' z}, e^{\nu z}, e^{-\nu' z}, e^{\nu' z})\,\mathbf{R}^{-1}. \tag{6.9}$$

7. Geometric properties

The properties derived in this section are true for any transversely isotropic, laterally homogeneous, layered structure with z axis as axis of symmetry, whether completely isotropic or not. Let us consider the 3×3 rotation matrix

$$\mathbf{V}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.1}$$

Because of transverse isotropy we have

$$c_{ijkl} = V_{im} V_{jn} V_{kp} V_{lq} c_{mnpq}$$

where the θ dependence is understood. In terms of the matrices of Section 2 this may be written

$$\mathbf{V}^\dagger \mathbf{C}_{mn} \mathbf{V} = V^\dagger_{mp} \mathbf{C}_{pq} V_{qn}$$

and for the matrices $\mathbf{S}$, $\mathbf{T}$, $\mathbf{C}_{33}$ occurring in $\mathbf{A}$ we find

$$\left.\begin{aligned} \mathbf{V}^\dagger \mathbf{T}\mathbf{V} &= i(\mathbf{V}\mathbf{k})_\sigma \mathbf{C}_{\sigma 3} \\ \mathbf{V}^\dagger \mathbf{S}\mathbf{V} &= \rho\omega^2 \mathbf{1} - (\mathbf{V}\mathbf{k})_\sigma (\mathbf{V}\mathbf{k})_\nu \mathbf{Q}_{\sigma\nu} \\ \mathbf{V}^\dagger \mathbf{C}_{33} \mathbf{V} &= \mathbf{C}_{33} \end{aligned}\right\}. \tag{7.2}$$

If the dependence of $\mathbf{A}$ upon $\mathbf{k}$ is made explicit the results (7.2) may be combined to give:

$$\mathbf{U}^\dagger \mathbf{A}(\mathbf{k}) \mathbf{U} = \mathbf{A}(\mathbf{V}\mathbf{k}) \tag{7.3a}$$

where $\mathbf{U}$ is the 6×6 partitioned matrix

$$\mathbf{U}(\theta) \equiv \begin{pmatrix} \mathbf{V}(\theta) & 0 \\ 0 & \mathbf{V}(\theta) \end{pmatrix}. \tag{7.3b}$$

If we now choose $\theta = \kappa$ such that

$$k_1 = k\cos\kappa$$

$$k_2 = k\sin\kappa$$

where

$$k = (k_\sigma k_\sigma)^{\frac{1}{2}}$$

we find

$$\mathbf{U}^\dagger(\kappa)\, \mathbf{A}\mathbf{U}(\kappa) = \mathbf{A}'$$

where

$$\mathbf{A}' = \mathbf{A}\big(\mathbf{V}(\kappa)\mathbf{k}\big)$$

is the matrix $\mathbf{A}$ calculated with $\mathbf{k}$ parallel to the x_1 axis, i.e. with $k_1 = k, k_2 = 0$. Thus equation (2.9) becomes:

$$\frac{\partial \mathbf{f}}{\partial z} = \mathbf{U}(\kappa)\, \mathbf{A}'\, \mathbf{U}^\dagger(\kappa)\, \mathbf{f}$$

which clearly has solution

$$\mathbf{f} = \mathbf{U}(\kappa)\, \mathbf{f}'$$

where

$$\frac{\partial \mathbf{f}'}{\partial z} = \mathbf{A}' \mathbf{f}'.$$

Let us now consider the calculation of $\mathscr{L}$ using (3.10) for instance; we have:

$$\mathscr{L} = \int_{\eta_0}^{\infty} \tfrac{1}{4}(\mathbf{u}'^{\dagger}\, \mathbf{U}^{\dagger}(\kappa)\, \mathbf{S}\mathbf{U}(\kappa)\, \mathbf{u}' - \boldsymbol{\tau}'^{\dagger}\, \mathbf{U}^{\dagger}(\kappa)\, \mathbf{C}_{33}^{-1}\, \mathbf{U}(\kappa)\, \boldsymbol{\tau}')\, dz$$

$$= \int_{\eta_0}^{\infty} \tfrac{1}{4}(\mathbf{u}'^{\dagger}\, \mathbf{S}'\, \mathbf{u}' - \boldsymbol{\tau}'^{\dagger}\, \mathbf{C}_{33}^{-1}\, \boldsymbol{\tau}')\, dz \tag{7.4}$$

where

$$\mathbf{S}' = \rho\omega^2\, \mathbf{1} - k^2(\mathbf{C}_{11} - \mathbf{C}_{13}\, \mathbf{C}_{33}^{-1}\, \mathbf{C}_{31})$$

and

$$\mathbf{f}' = \left(\frac{\mathbf{u}'}{\boldsymbol{\tau}'}\right).$$

The expression (7.4) is independent of the specific direction of the wave and hence to calculate $\mathscr{L}$ we need only consider waves parallel to the x_1 axis, that is with $k_1 = k$, $k_2 = 0$. This may be summarized by saying that $\mathscr{L}$ is a scalar with respect to rotations in the x_1, x_2 plane. This has an important consequence; the wavenumber components k_1, k_2 enter into $\mathscr{L}(a, \omega, k_\sigma, X_\sigma)$ only through the variable $k = (k_\sigma k_\sigma)^{\frac{1}{2}}$, and therefore the average energy flux vector,

$$-\omega \frac{\partial \mathscr{L}}{\partial k_\sigma} = -\omega \frac{k_\sigma}{k} \frac{\partial \mathscr{L}}{\partial k}.$$

is in the direction of $\mathbf{k}$. Hence for a transversely isotropic medium energy transport is in the direction of the wave vector.

8. Solution of the equation by the method of characteristics

The equations governing the gradual variations in amplitude, wave vector and frequency may be written:

$$\frac{\partial \omega}{\partial x_\sigma} + \frac{\partial k_\sigma}{\partial t} = 0 \tag{8.1}$$

$$\frac{\partial k_\sigma}{\partial x_\nu} - \frac{\partial k_\nu}{\partial x_\sigma} = 0 \tag{8.2}$$

$$\frac{\partial}{\partial x_\sigma}\left(\frac{\partial \mathscr{L}}{\partial k_\sigma}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \omega}\right) = 0. \tag{8.3}$$

We have replaced the stretched co-ordinates X_σ, T by the original co-ordinates of the problem, but still demand that the lateral variations in the medium be gradual. The derivatives

$$\frac{\partial}{\partial t}, \frac{\partial}{\partial x_\sigma}$$

are total in that they include the effects of varying wave vector, amplitude and frequency. We now define characteristic curves (Courant & Hilbert 1962) by the equations:

$$\frac{dt}{(\partial \mathscr{L}/\partial \omega)} = -\frac{dx_1}{(\partial \mathscr{L}/\partial k_1)} = -\frac{dx_2}{(\partial \mathscr{L}/\partial k_2)}, \tag{8.4}$$

i.e.

$$\dot{x}_\sigma = -\frac{(\partial \mathscr{L}/\partial k_\sigma)}{(\partial \mathscr{L}/\partial \omega)} \tag{8.5}$$

where '$\cdot$' denotes a derivative with respect to time along the characteristic curve. Now since $\mathscr{L}(a, \omega, k_\sigma, x_\sigma)$ vanishes identically when the dispersion relation

$$\omega = \omega(k_\sigma, x_\sigma) \tag{8.6}$$

is satisfied, (8.5) may be witten

$$\dot{x}_\sigma = \frac{\partial \omega}{\partial k_\sigma} \tag{8.7}$$

$\partial\omega/\partial k_\sigma$ is the local group velocity, and from (8.5) and (5.3) we see that it is also the velocity of energy transport. Substituting the form (8.6) into (8.1) and using (8.2) and (8.7) we find

$$\dot{k}_\sigma = -\frac{\partial \omega}{\partial x_\sigma} \tag{8.8}$$

where the derivative on the right-hand side is that obtained from (8.6) keeping k_σ constant; it is a known function of k_σ, x_σ. Recalling that k_σ, ω are defined in terms of the phase θ by equations (4.2) we obtain from (8.6) the following equation for θ:

$$\frac{\partial \theta}{\partial t} + \omega\left(\frac{\partial \theta}{\partial x_\sigma}, x_\sigma\right) = 0. \tag{8.9}$$

Equations (8.7), (8.8) are in canonical form with the dispersion relation $\omega = \omega(k_\sigma, x_\sigma)$ taking the place of the Hamiltonian $H = H(p_\sigma, q_\sigma)$ and equation (8.9) is the Hamilton–Jacobi equation corresponding to this Hamiltonian. This correspondence between wave propagation and the equations of classical mechanics has been known since the work of Hamilton (1828–1830) and has been used in a geophysical context by Backus (1962) to determine the perturbation induced by the rotation of the Earth on elastic wave propagation. Block & Gilbert (1972) have recently used such a correspondence to calculate perturbations in phase as a surface wave traverses an anomalous region. It will be shown below that equation (8.3) enables perturbations in amplitude also to be calculated, and furthermore it is shown in Section 9 that there is an additional phase perturbation which is not considered in the work of Block & Gilbert and is not predicted by the average Lagrangian method of Whitham.

Since ω is not an explicit function of t we immediately deduce from (8.6), (8.7), (8.8)

$$\dot{\omega} = 0,$$

i.e. ω is constant along the ray (or characteristic curve). Equation (8.3) may be written

$$\left(\frac{\partial \mathscr{L}}{\partial \omega}\right)^{\cdot} + \frac{\partial \mathscr{L}}{\partial \omega}\frac{\partial \dot{x}_\sigma}{\partial x_\sigma} = 0. \tag{8.10}$$

Now let us suppose that initial conditions are given as

$$k_\sigma = \beta_\sigma(s_\nu)$$

on the surface having representation in terms of parameters s_1, s_2:

$$x_\sigma = \alpha_\sigma(s_\nu)$$
$$t = t_0(s_\nu).$$

The solutions of equations (8.7), (8.8) may be written:

$$x_\sigma = x_\sigma\big(t, \alpha_\sigma(s_\nu), \beta_\sigma(s_\nu), t_0(s_\nu)\big)$$
$$k_\sigma = k_\sigma\big(t, \alpha_\sigma(s_\nu), \beta_\sigma(s_\nu), t_0(s_\nu)\big).$$

It is easily checked that

$$\frac{\partial \dot{x}_\sigma}{\partial x_\sigma} = \frac{\partial \dot{x}_\sigma}{\partial s_\nu}\,\frac{\partial s_\nu}{\partial x_\sigma} = \frac{1}{J}\,\dot{J}$$

where

$$J = \frac{\partial(x_1, x_2)}{\partial(s_1, s_2)}$$

and equation (8.10) requires

$$\left(J\,\frac{\partial \mathscr{L}}{\partial \omega}\right) = 0. \tag{8.11}$$

This equation together with the 'ray' equations (8.7), (8.8) enables the amplitude factor aa^* to be calculated. J is simply a measure of the geometrical spreading of the rays and equation (8.11) means that for an observer moving with the local group velocity the product of the local energy density with the area of his ray tube remains constant. The correspondence of the present theory with ray theory for body waves in now apparent.

The equations governing the propagation of the derivatives needed in calculating J may be derived by differentiating the ray equations (8.7), (8.8) with respect to s_ν:

$$\left(\frac{\partial x_\sigma}{\partial s_\nu}\right) = \frac{\partial^2 \omega}{\partial k_\sigma\,\partial x_\mu}\left(\frac{\partial x_\mu}{\partial s_\nu}\right) + \frac{\partial^2 \omega}{\partial k_\sigma\,\partial k_\mu}\left(\frac{\partial k_\mu}{\partial s_\nu}\right) \tag{8.12}$$

$$\left(\frac{\partial k_\sigma}{\partial s_\nu}\right) = -\frac{\partial^2 \omega}{\partial x_\sigma\,\partial x_\mu}\left(\frac{\partial x_\mu}{\partial s_\nu}\right) - \frac{\partial^2 \omega}{\partial x_\sigma\,\partial k_\mu}\left(\frac{\partial k_\mu}{\partial s_\nu}\right). \tag{8.13}$$

The equations (8.7), (8.8), (8.12), (8.13) now form a system of differential equations for the twelve unknowns x_σ, k_σ, $\partial x_\sigma/\partial s_\nu$, $\partial k_\sigma/\partial s_\nu$ and may be integrated numerically in order to calculate J. This procedure is analogous to the time integration method of Wesson (1970) in the theory of body waves. The initial conditions from which the integration may be started are as follows:

$$\left.\begin{aligned} x_\sigma\big(t_0(s_\nu)\big) &= \alpha_\sigma(s_\nu) \\ k_\sigma\big(t_0(s_\nu)\big) &= \beta_\sigma(s_\nu) \\ \frac{\partial x_\sigma}{\partial s_\nu}\big(t_0(s_\nu)\big) &= \frac{\partial \alpha_\sigma}{\partial s_\nu}(s_\nu) \\ \frac{\partial k_\sigma}{\partial s_\nu}\big(t_0(s_\nu)\big) &= \frac{\partial \beta_\sigma}{\partial s_\nu}(s_\nu). \end{aligned}\right\} \tag{8.14}$$

We know from the previous analysis, that for the propagating ray

$$\mathscr{L}(a, \omega, k_\sigma, X_\sigma) = 0 \tag{8.15}$$

and it is easily shown that

$$\dot{\mathscr{L}}(a, \omega, k_\sigma, X_\sigma) = 0$$

is a direct consequence of equations (8.7), (8.8). Thus if (8.15) is satisfied at the initial point of the ray and the ray is found by numerical integration of (8.7), (8.8) then (8.15) is automatically satisfied everywhere on the ray. This has the useful consequence that if the dispersion relation is satisfied at the initial point it is satisfied everywhere along the ray so we do not need to solve for ω as a function of **k** at each step of the integration. This should greatly speed the numerical integration of the ray equations.

The equation governing the phase θ for an observer moving with the local group velocity along the characteristic curve is:

$$\dot{\theta} = k_\sigma \dot{x}_\sigma - \omega(k_\sigma, x_\sigma)$$

and hence

$$[\theta]_{t_0}^{t_1} = \int_{t_0}^{t_1} \{k_\sigma \dot{x}_\sigma - \omega(k_\sigma, x_\sigma)\}\, dt. \tag{8.16}$$

The integrand here is the Lagrangian corresponding to the Hamiltonian $\omega = \omega(k_\sigma, x_\sigma)$ and it is easily checked that the integral (8.16) is stationary with respect to small perturbations of x_σ, k_σ from their true values, provided that the end points $x_\sigma(t_0)$, $x_\sigma(t_1)$ are fixed.

9. An additional variation in phase

Equation (4.8) gives the first approximation to the stress-displacement field in terms of an amplitude factor a and equation (4.19) gives an equation for aa^* since $\mathscr{L}$ depends on a only through this quantity. As yet we have no equation governing the phase of a so that our solution is determined only up to a factor $\exp(ip(X_\sigma, T))$. To find an equation governing p we must return to the relation (4.14). We have used the fact that the left-hand side vanishes and we now consider the right-hand side.

Let us write

$$\mathbf{f}^{(0)} = \exp(ip(X_\sigma, T))\,\bar{\mathbf{f}}$$

$$\tag{9.1}$$

where

$$\bar{\mathbf{f}} = |a(X_\sigma, T)|\,\hat{\mathbf{f}}(k_\sigma, \omega, X_\sigma, z)$$

so that p is the phase of a occurring in equation (4.8). Substituting (9.1) into the right-hand side of equation (4.14) and setting this to zero we find

$$\frac{\partial p}{\partial T}\frac{\bar{\partial}\mathscr{L}}{\partial \omega} - \frac{\partial p}{\partial X_\sigma}\frac{\bar{\partial}\mathscr{L}}{\partial k_\sigma} = \frac{\partial}{\partial X_\sigma}\int_{\eta_0}^{\infty} \tfrac{1}{8}(\bar{\mathbf{u}}^\dagger\, \mathbf{C}_{\sigma 3}\, \mathbf{C}_{33}^{-1}\, \bar{\boldsymbol{\tau}} + \bar{\boldsymbol{\tau}}^\dagger\, \mathbf{C}_{33}^{-1}\, \mathbf{C}_{3\sigma}\, \bar{\mathbf{u}})\,dz$$

$$-\int_{\eta_0}^{\infty} \tfrac{1}{4}\left\{\left(\bar{\mathbf{u}}^\dagger \frac{\bar{\partial}}{\partial X_\sigma}\, \mathbf{C}_{\sigma 3}\, \mathbf{C}_{33}^{-1}\, \bar{\boldsymbol{\tau}} + \bar{\boldsymbol{\tau}}^\dagger\, \mathbf{C}_{33}^{-1}\, \mathbf{C}_{3\sigma} \frac{\bar{\partial}}{\partial X_\sigma}\, \bar{\mathbf{u}}\right)\right.$$

$$+\bar{\mathbf{u}}^\dagger\left(\frac{\bar{\partial}}{\partial X_\sigma}\, \mathbf{Q}_{\sigma\nu}\, ik_\nu - ik_\sigma\, \mathbf{Q}_{\sigma\nu}\frac{\bar{\partial}}{\partial X_\nu}\right)\bar{\mathbf{u}}$$

$$+\bar{\mathbf{u}}^\dagger\left(\rho\frac{\overleftrightarrow{\partial}}{\partial T}i\omega - i\omega\frac{\overleftrightarrow{\partial}}{\partial T}\rho\right)\bar{\mathbf{u}}\Big\}\,dz. \tag{9.2}$$

When the dispersion relation is satisfied the left-hand side of (9.2) becomes

$$\frac{\partial\mathscr{L}}{\partial\omega}\dot{p}$$

in the notation of Section 8. The right-hand side may, in theory, be evaluated since all quantities there are known from the preceding analysis. Hence by integrating along the rays we may find the additional phase correction p.

10. Examples

In order to apply the above theory we require $\mathscr{L}$ as a function of k, ω, X_σ, T. This function may be found explicitly only if the propagator matrices are known as functions, k_σ, ω and the elastic parameters. Such expressions for the propagators are known if the elastic parameters and density in each layer are independent of z and so in the examples considered we shall assume this to be the case. We therefore associate with each layer a set of elastic parameters and a density which depend only upon the

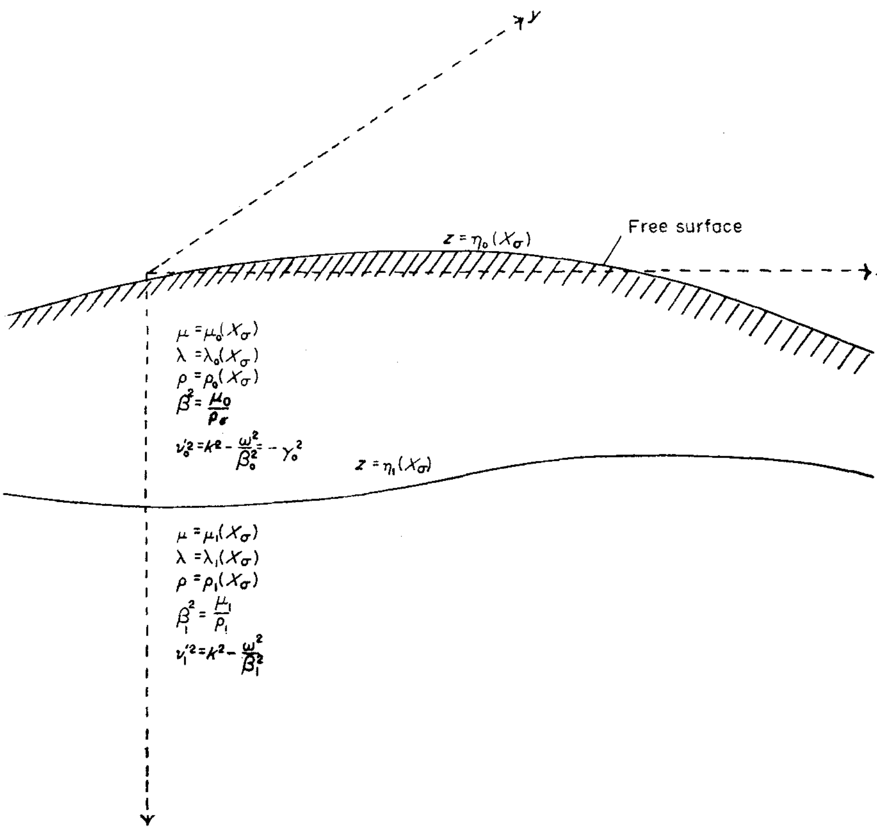

FIG. 2. A two-layeied structure with lateral heterogeneity.

horizontal co-ordinates x_1, x_2. As stated above the functions defining the layering are also functions of x_1, x_2.

(a) Love waves

We shall consider the three-dimensional layered solid shown in Fig. 2 and a disturbance with $u_3^{(0)} = 0$. The material of each layer is assumed to be completely isotropic and, as shown in Section 7, we may calculate $\mathscr{L}$ by considering a Love wave travelling parallel to the x_1 axis in a laterally homogeneous medium. Since $\hat{\mathbf{f}}$ has only two nonvanishing components $\hat{u}_2, \hat{\tau}_{32}$ we shall redefine

$$\mathbf{f} = \begin{pmatrix} u_2 \\ \tau_{32} \end{pmatrix}$$

and similarly redefine the matrix $\mathbf{A}$ and the propagators $\mathbf{P}$ by deleting all but their second and fifth rows and columns. The propagator $\mathbf{P}(\eta_0, \eta_1)$ takes the simple form

$$\mathbf{P}(\eta_0, \eta_1) = \begin{pmatrix} \cos(\gamma_0(\eta_1 - \eta_0)) & -\dfrac{1}{\mu_0 \gamma_0} \sin(\gamma_0(\eta_1 - \eta_0)) \\ \mu_0 \gamma_0 \sin(\gamma_0(\eta_1 - \eta_0)) & \cos(\gamma_0(\eta_1 - \eta_0)) \end{pmatrix}$$

where

$$\gamma_0 = -i\nu_0' = \left(\frac{\rho_0 \omega^2}{\mu_0} - k^2\right)^{\frac{1}{2}}.$$

$$\mathbf{f}_1^{(0)} \equiv \begin{pmatrix} u_2^{(0)} \\ \tau_{32}^{(0)} \end{pmatrix}\Bigg|_{z=\eta_1}$$

must satisfy equation (6.2) and it is easily seen that this implies that $\mathbf{f}_1^{(0)}$ is a linear combination of the eigenvectors of $\mathbf{A}$ (for the lower medium) with negative eigenvalues. Since there is only one such eigenvector in this case we have

$$\mathbf{f}_1^{(0)} = a\begin{pmatrix} 1 \\ -\mu_1 \nu_1' \end{pmatrix}$$

where a is a constant and hence

$$\mathbf{f}_0^{(0)} \equiv \mathbf{f}^{(0)}|_{z=\eta_0} = a\begin{pmatrix} \cos\gamma_0(\eta_1 - \eta_0) + \dfrac{\mu_1 \nu_1'}{\mu_0 \gamma_0} \sin\gamma_0(\eta_1 - \eta_0) \\ \gamma_0 \mu_0 \sin\gamma_0(\eta_1 - \eta_0) - \mu_1 \nu_1' \cos\gamma_0(\eta_1 - \eta_0) \end{pmatrix}.$$

Substituting into (4.17) we obtain

$$\mathscr{L} = \tfrac{1}{4} a a^* \left\{\cos\gamma_0(\eta_1 - \eta_0) + \frac{\mu_1 \nu_1'}{\mu_0 \gamma_0} \sin\gamma_0(\eta_1 - \eta_0)\right\} \times \{\gamma_0 \mu_0 \sin\gamma_0(\eta_1 - \eta_0) - \mu_1 \nu_1' \cos\gamma_0(\eta_1 - \eta_0)\}.$$

Although this average Lagrangian differs from that derived for Love waves by Gjevik (1973) it gives the same expressions for

$$\frac{\partial \mathscr{L}}{\partial \omega}, \frac{\partial \mathscr{L}}{\partial k_\alpha}$$

when the dispersion relation is satisfied. The results for $\mathscr{L}$ are different because Gjevik has (in effect) extended the functions $\hat{\mathbf{f}}$ to points not on the dispersion surface in a manner different from that used here (see Section 4).

The vanishing of the surface stresses gives the usual dispersion relation:

$$\tan \gamma_0(\eta_1-\eta_0) = \frac{\mu_1 \nu_1'}{\mu_0 \gamma_0} \tag{10.1}$$

and when this relation is satisfied $\mathscr{L}$ vanishes. We have shown in Section 7 that the same function $\mathscr{L}$ is obtained for Love waves moving in any direction over the surface. When the dispersion relation is satisfied we find

$$\frac{\partial \mathscr{L}}{\partial k_\sigma} = -\tfrac{1}{4}aa^* k_\sigma \left\{\mu_0 h(1+\xi^2)+\frac{\mu_0^2}{\mu_1 \nu_1'}\left(\xi^2+\frac{\mu_1^2}{\mu_0^2}\right)\right\}$$

$$\frac{\partial \mathscr{L}}{\partial \omega} = \tfrac{1}{4}aa^* \frac{\omega}{\beta_0^2}\left\{\mu_0 h(1+\xi^2)+\frac{\mu_0^2}{\mu_1 \nu_1'}\left(\xi^2+\frac{\mu_1 \rho_1}{\mu_0 \rho_0}\right)\right\}$$

where

$$h = h(X_\sigma) = \eta_1-\eta_0$$

$$\xi = \tan \gamma_0 h = \frac{\mu_1 \nu_1'}{\mu_0 \gamma_0}$$

$$\beta_0^2 = \frac{\mu_0}{\rho_0}.$$

The surface amplitude is found to be

$$s \equiv |\mathbf{u}^{(0)}|_{z=\eta_0} = |a|(1+\xi^2)^{\frac{1}{2}}$$

so

$$\left.\begin{aligned} \frac{\partial \mathscr{L}}{\partial k_\sigma} &= -\tfrac{1}{4}s^2 k_\sigma \mu_0\left\{h+\frac{\mu_1 \mu_0}{\nu'_1}\frac{\Delta(\nu'^2)}{\Delta(\mu^2 \nu'^2)}\right\} \\ \frac{\partial \mathscr{L}}{\partial \omega} &= \tfrac{1}{4}s^2 \omega \rho_0\left\{h+\frac{\rho_1 \mu_0}{\nu_1'}\frac{\Delta(\beta^2 \nu'^2)}{\Delta(\mu^2 \nu^2)}\right\} \end{aligned}\right\} \tag{10.2}$$

where Δ indicates that the difference of a quantity is to be taken between the surface layer and the underlying half space with the understanding that

$$\nu_0'^2 = -\gamma_0^2 = k^2-\frac{\omega^2}{\beta_0^2}.$$

Thus the following conservation equation is obtained:

$$\frac{\partial}{\partial X_\sigma}\left(s^2 k_\sigma \mu_0\left\{h+\frac{\mu_1 \mu_0}{\nu_1'}\frac{\Delta(\nu'^2)}{\Delta(\mu^2 \nu'^2)}\right\}\right)+\frac{\partial}{\partial T}\left(s^2 \omega \rho_0\left\{h+\frac{\mu_1 \mu_0}{\nu_1'}\frac{\Delta(\nu'^2)}{\Delta(\mu^2 \nu'^2)}\right\}\right) \tag{10.3}$$

The dispersion relation (10.1) implies a specific dependence

$$\omega = \omega(k_\sigma, x_\sigma)$$

of ω on k_σ, x_σ so that all quantities except a in equation (10.3) are known functions of X_σ, k_σ. This equation together with equations (4.3) are the propagation equations for s^2, k_σ as functions of X_σ, T.

The ray tracing equations may be written

$$\dot{x}_\sigma = \frac{k_\sigma}{\omega}\beta_0^2\left[\frac{h\nu_1'\,\Delta(\mu^2\nu'^2)+\mu_1\mu_0\,\Delta(\nu'^2)}{h\nu_1'\,\Delta(\mu^2\nu'^2)+\rho_1\mu_0\,\Delta(\beta^2\nu'^2)}\right]$$

$$\dot{k}_\sigma = -\frac{\nu\gamma^2\beta_0^2}{\omega}\left[\frac{\left(\dfrac{\omega^2 h}{\gamma_0^2\beta_0^3}\dfrac{\partial\beta_0}{\partial x_\sigma}-\dfrac{\partial h}{\partial x_\sigma}\right)\Delta(\mu^2\nu'^2)+\mu_1\nu\mu_0\,\Delta\left(\dfrac{\omega^2}{\nu'^2\beta^3}\dfrac{\partial\beta}{\partial x_\sigma}+\dfrac{1}{\mu}\dfrac{\partial\mu}{\partial x_\sigma}\right)}{\nu_1'\,h\Delta(\mu^2\nu'^2)+\rho_1\mu_0\,\Delta(\beta^2\nu'^2)}\right]$$

and these may be integrated by standard methods.

(*b*) *Rayleigh waves*

We shall consider an elastic half space with elastic parameters and density depending on the horizontal co-ordinates alone and with an undulating surface. We shall consider displacements in a Rayleigh mode, and therefore we may calculate the average Lagrangian $\mathscr{L}$ by considering a Rayleigh wave propagating in the direction of the x_1 axis in a homogeneous half space. $\mathbf{f}$ may be redefined as

$$\mathbf{f} = \begin{pmatrix} u_1 \\ u_3 \\ \tau_{13} \\ \tau_{33} \end{pmatrix}$$

and the matrices $\mathbf{A}$ and $\mathbf{P}$ are similarly redefined by deleting their second and fifth rows and columns. There are two eigenvectors of $\mathbf{A}$ with negative eigenvalues given by the first two columns of $\mathbf{R}$ in (6.7). A linear combination of these eigenvectors will give the surface stresses and displacements and if we demand that τ_{13} vanish at the surface we obtain

$$\mathbf{f}_0^{(0)} = a\begin{pmatrix} -i\nu'+\dfrac{i}{2\mu\nu}(2\mu k^2-\rho\omega^2) \\ \dfrac{\rho\omega^2}{2\mu k} \\ 0 \\ -2\mu k\nu'+\dfrac{(2\mu k^2-\rho\omega^2)^2}{2\mu k\nu} \end{pmatrix}$$

By (4.17)

$$\mathscr{L} = \tfrac{1}{4}|a|^2\frac{\rho\omega^2}{4\mu^2k^2\nu}\left((2\mu k^2-\rho\omega^2)^2-4\mu^2k^2\nu\nu'\right).$$

The requirement that $\tau_{33} = 0$ gives the usual dispersion relation:

$$(2\mu k^2 - \rho\omega)^2 - 4\mu^2 k^2 \nu\nu' = 0$$

and $\mathscr{L}$ vanishes when this is satisfied. The surface displacements are:

$$u_1^{(0)}|_{z=\eta_0} = \frac{ai}{2\mu\nu}(-2\mu\nu\nu' + 2\mu k^2 - \rho\omega^2)$$

$$u_3^{(0)}|_{z=\eta_0} = \frac{a\rho\omega^2}{2\mu k}$$

and we have

$$\frac{\partial \mathscr{L}}{\partial k_\sigma} = \frac{k_\sigma \rho\omega^2 |a|^2}{4k^2 \nu}\left\{4k^2 - \frac{2\rho\omega^2}{\mu} - 2\nu\nu' - k^2\left(\frac{\nu'}{\nu} + \frac{\nu}{\nu'}\right)\right\}$$

$$\frac{\partial \mathscr{L}}{\partial \omega} = \frac{\rho\omega^3|a|^2}{4\mu^2 k^2 \nu}\left\{\mu^2 k^2\left(\frac{\nu'}{\alpha^2 \nu} + \frac{\nu}{\beta^2 \nu'}\right) - \rho(2\mu k^2 - \rho\omega^2)\right\}.$$

The ray equations take the form:

$$\dot{x}_\sigma = \frac{k_\sigma}{\omega}\left[\frac{4k^2 - \dfrac{2\omega^2}{\beta^2} - 2\nu\nu' - k^2\left(\dfrac{\nu'}{\nu} + \dfrac{\nu}{\nu'}\right)}{\dfrac{1}{\beta^2}\left(2k^2 - \dfrac{\omega^2}{\beta^2}\right) - k^2\left(\dfrac{\nu'}{\alpha^2\nu} + \dfrac{\nu}{\beta^2\nu'}\right)}\right]$$

$$\dot{k}_\sigma = -\frac{\left\{(2\mu k^2 - \rho\omega^2)\left(\dfrac{\partial\mu}{\partial x_\sigma}k^2 - \tfrac{1}{2}\dfrac{\partial\rho}{\partial x_\sigma}\omega^2\right) - 2\mu\dfrac{\partial\mu}{\partial x_\sigma}k^2\nu\nu' - \mu^2 k^2\omega^2\left(\dfrac{\nu'}{\alpha^3\nu}\dfrac{\partial\alpha}{\partial x_\sigma} + \dfrac{\nu}{\beta^3\nu'}\dfrac{\partial\beta}{\partial x_\sigma}\right)\right\}}{\omega\left\{\rho(2\mu k^2 - \rho\omega^2) - \mu^2 k^2\left(\dfrac{\nu'}{\alpha^2\nu} + \dfrac{\nu}{\beta^2\nu'}\right)\right\}}$$

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References

Backus, G. E., 1962. The propagation of short elastic surface waves on a slowly rotating Earth, *Bull. seism. Soc. Am.*, **52,** 823–846.

Block, B. & Gilbert, F., 1972. Wave propagation and a secular theory of calculation based on Hamilton–Jacobi theory, *Geophys. J. R. astr. Soc.*, **30,** 343–352.

Boore, D. M., 1970. Love waves in nonuniform waveguides: Finite difference calculations, *J. geophys. Res.*, **75,** 1512–1527.

Bretherton, F. P., 1968. Propagation in slowly varying waveguides, *Proc. R. Soc.*, **A302**, 555–576.

Courant, R. & Hilbert D., 1962. *Methods of mathematical physics II*, Interscience, New York.

De Noyer, J., 1961. The effect of variations in layer thickness on Love waves, *Bull. seism. Soc. Am.*, **51,** 227–235.

Ghosh, M. L., 1963. On Love waves across the ocean, *Geophys. J. R. astr. Soc.*, **7,** 350–360.

Gilbert, F. & Backus, G. E., 1966. Propagator matrices in elastic wave and vibration problems. *Geophysics*, **31,** 326–332.

Gjevik, B., 1973. A variational method for Love waves in nonhorizontally layered structures, *Bull. seism. Soc. Am.*, **63,** 1013–1023.

Hamilton, W. R., 1828. Theory of systems of rays. *Trans. R. Irish Acad.* **15,** 69–174.

Hamilton, W. R., 1830. Supplement to an essay on the theory of systems of rays, *Trans. R. Irish.*, **16,** 4–71.

Hamilton, W. R., 1830. Second supplement to an essay on the theory of systems of rays, *Trans. R. Irish Acad.*, **16,** 91–126.

Hamilton, W. R., 1837. Third supplement to an essay on the theory of systems of rays, *Trans. R. Irish Acad.* **17,** 1–144.

Haskell, N. A., 1953. The dispersion of surface waves on multilayered media, *Bull. seism. Soc. Am.*, **43,** 17–34.

Herrera, I., 1964. A perturbation method for elastic wave propagation, 1. Non parallel boundaries, *J. geophys. Res.*, **69,** 3845–3851.

Herrera, I. & Mal, A. K., 1965. A perturbation method for elastic wave propagation, 2. Small inhomogeneities, *J. geophys. Res.*, **70,** 871–883.

Kennett. B. L. N., 1974. On variational principles and matrix methods in elastodynamics, *Geophys. J. R. astr. Soc.*, **37,** 391–405

Knopoff, L. & Hudson, J. A., 1964. Transmission of Love waves past a continental margin, *J. geophys. Res.*, **69.** 1649–1653.

Knopoff, L. & Mal, A. K., 1967. Phase velocity of surface waves in the transition zone of continental margins, *J. geophys. Res.*, **72,** 1769–1776.

Lysmer, J. & Drake, L. A., 1971. The propagation of Love waves across nonhorizontally layered structures, *Bull. seism. Soc. Am.*, **61,** 1233–1251.

Mal, A. K. & Herrera, I., 1965. Scattering of Love waves by a constriction in the crust, *J. geophys. Res.*, **70,** 5125–5133.

Mal, A. K. & Knopoff, L., 1965. Transmission of Rayleigh waves past a step change in elevation, *Bull seism. Soc. Am.*, **55,** 319–334.

Seliger, R. L. & Whitham, G. B., 1968. Variational principles in continuum mechanics, *Proc. R. Soc.*, **A305,** 1–25.

Slavin, L. M. & Wolf, B., 1970. Scattering of Love waves in a surface layer with an irregular boundary for the case of an underlying rigid half-space, *Bull. seism. Soc. Am.*, **60,** 859–877.

Thapar, M. R., 1970. Rayleigh wave propagation and perturbed boundaries, *Can. J. earth Sci.*, **7,** 1449–1461.

Wesson, R. L., 1970. A time integration method for computation of the intensities of seismic rays, *Bull. seism. Soc. Am.*, **60,** 307–316.

Whitham, G. B., 1965a. Nonlinear dispersive waves, *Proc. R. Soc.*, **A283**, 238–261.
Whitham, G. B., 1965b. A general approach to linear and nonlinear dispersive waves using a Lagrangian. *J. fluid. Mech.*, **22**, 273–283.
Wolf, B., 1967. Propagation of Love waves in surface layers of varying thickness, *Pure appl. geophys.*, **67**, 76–82.
Wolf, B., 1970. Propagation of Love waves in layers with irregular boundaries, *Pure appl. geophys.*, **78**, 48–57.

Appendix

In this appendix we shall prove the two symmetries of the propagator matrix given by properties (ii) and (vi) of Section 6.

From (3.14) and (3.16b) we see that

$$\mathbf{A} = (\mathbf{M}-\mathbf{M}^\dagger)\,\mathbf{B}.$$

Where $\mathbf{B}$ is Hermitian
Hence

$$\begin{aligned}\mathbf{A}^\dagger &= -\mathbf{B}(\mathbf{M}-\mathbf{M}^\dagger)\\ &= (\mathbf{M}-\mathbf{M}^\dagger)\,\mathbf{A}(\mathbf{M}-\mathbf{M}^\dagger).\end{aligned}$$

Now using (6.1) and its Hermitian conjugate we find

$$\frac{d}{dz}\left(\mathbf{P}^\dagger(\mathbf{M}-\mathbf{M}^\dagger)\,\mathbf{P}\right) = 0$$

and using the initial condition and continuity condition we have

$$\mathbf{P}^\dagger\,(\mathbf{M}-\mathbf{M}^\dagger)\,\mathbf{P} = \mathbf{M}-\mathbf{M}^\dagger$$

and therefore property (ii) is proved for a general layered solid.

Property (vi) is true for a transversely isotropic layered solid and may be proved as follows.

Substituting $\theta = \pi$ in equations (7.3a, b) we find

$$\mathbf{U}^\dagger(\pi)\,\mathbf{A}(\mathbf{k})\,\mathbf{U}(\pi) = \mathbf{A}(-\mathbf{k})$$

and from (2.10) we see that

$$\mathbf{A}(-\mathbf{k}) = \mathbf{A}^*(\mathbf{k})$$

and therefore

$$\mathbf{N}^\dagger\,\mathbf{A}\mathbf{N} = \mathbf{A}^*$$

where $\mathbf{N}$ the matrix defined in the statement of property (vi).

If we multiply equation (6.1) on the left by $\mathbf{N}^\dagger$ and on the right by $\mathbf{N}$ and then take the complex conjugate we find

$$\frac{d}{dz}(\mathbf{N}^\dagger\,\mathbf{P}^*\,\mathbf{N}) = \mathbf{A}(\mathbf{N}^\dagger\,\mathbf{P}^*\,\mathbf{N})$$

so that by uniqueness we have

$$\mathbf{N}^\dagger\,\mathbf{P}^*\,\mathbf{N} = \mathbf{P}$$

and property (vi) is proved.

It is interesting to note that properties (ii) and (vi) proved in this appendix are group properties which restrict $\mathbf{P}$ to a certain subgroup of the complete group of 6×6 complex matrices. Symmetries such as these will be explored further in a later paper.