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Surface waves in fibre-reinforced anisotropic elastic media

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Abstract. The aim of this paper is to investigate surface waves in anisotropic fibre-reinforced solid elastic media. First, the theory of general surface waves has been derived and applied to study the particular cases of surface waves – Rayleigh, Love and Stoneley types. The wave velocity equations are found to be in agreement with the corresponding classical result when the anisotropic elastic parameters tends to zero. It is important to note that the Rayleigh type of wave velocity in the fibre-reinforced elastic medium increases to a considerable amount in comparison with the Rayleigh wave velocity in isotropic materials.

Keywords. Fibre-reinforced medium; surface waves; Rayleigh waves; Love waves; Stoneley waves.

1. Introduction

Surface waves have been well recognized in the study of earthquake waves, seismology, geophysics and geodynamics. A good amount of literature is to be found in the standard books of Bullen (1965), Ewing *et al* (1957), Rayleigh (1885), Love (1911), Stoneley (1924) and Jeffreys (1959), regarding surface waves in classical elasticity. Sengupta and his research collaborators have also studied surface waves (Acharya & Sengupta 1978; Pal & Sengupta 1987; Mukherjee & Sengupta 1991; Das & Sengupta 1992; Das *et al* 1994).

In most previous investigations, the effect of reinforcement has been neglected. The idea of continuous self-reinforcement at every point of an elastic solid was introduced by Belfield *et al* (1983). The characteristic property of a reinforced concrete member is that its components, namely concrete and steel, act together as a single anisotropic unit as long as they remain in the elastic condition, i.e. the two components are bound together so that there can be no relative displacement between them.

In this paper the authors study the propagation of surface waves in fibre-reinforced anisotropic elastic solid media leading to particular cases such as Rayleigh waves, Love waves and Stoneley waves along with numerical results. The results reduce to corresponding classical results when the reinforced elastic parameters tend to zero and the medium becomes isotropic.

2. Formulation of the problem

The constitutive equations for a fibre-reinforced linearly elastic anisotropic medium with respect to a preferred direction $\bar{a}$ are (Belfield *et al* 1983)

$$\tau_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu_T e_{ij} + \alpha(a_k a_m e_{km}\delta_{ij} + e_{kk}a_i a_j) + 2(\mu_L - \mu_T)(a_i a_k e_{kj} + a_j a_k e_{ki}) + \beta(a_k a_m e_{km} a_i a_j), \tag{1}$$

where τ_{ij} are components of stress; $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ are components of strain; λ, μ_T are elastic parameters; $\alpha, \beta, (\mu_L - \mu_T)$ are reinforced anisotropic elastic parameters; u_i are the displacement vectors components and $\bar{a} = (a_1, a_2, a_3)$, where $a_1^2 + a_2^2 + a_3^2 = 1$. If $\bar{a}$ has components that are (1, 0, 0) so that the preferred direction is the x_1 axis, (1) simplifies, as given below

$$\begin{aligned}
\tau_{11} &= (\lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta)e_{11} + (\lambda + \alpha)e_{22} + (\lambda + \alpha)e_{33},\\
\tau_{22} &= (\lambda + \alpha)e_{11} + (\lambda + 2\mu_T)e_{22} + \lambda e_{33},\\
\tau_{33} &= (\lambda + \alpha)e_{11} + \lambda e_{22} + (\lambda + 2\mu_T)e_{33},\\
\tau_{23} &= 2\mu_T e_{23},\\
\tau_{13} &= 2\mu_T e_{13},\\
\tau_{12} &= 2\mu_T e_{12}.
\end{aligned} \tag{2}$$

The equations of motion in absence of body forces are

$$\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} = \rho \frac{\partial^2 u_1}{\partial t^2}, \tag{3}$$

$$\frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{23}}{\partial x_3} = \rho \frac{\partial^2 u_2}{\partial t^2}, \tag{4}$$

$$\frac{\partial \tau_{31}}{\partial x_1} + \frac{\partial \tau_{32}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3} = \rho \frac{\partial^2 u_3}{\partial t^2}, \tag{5}$$

where ρ is the density of the elastic medium. Using (2)–(5) and assuming all derivatives with respect to x_3 vanish, the equations of motion become

$$(\lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta)\frac{\partial^2 u_1}{\partial x_1^2} + (\alpha + \lambda + \mu_L)\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \mu_L \frac{\partial^2 u_1}{\partial x_2^2} = \rho \frac{\partial^2 u_1}{\partial t^2}, \tag{6}$$

$$\mu_L \frac{\partial^2 u_2}{\partial x_1^2} + (\alpha + \lambda + \mu_L)\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + (\lambda + 2\mu_T)\frac{\partial^2 u_2}{\partial x_2^2} = \rho \frac{\partial^2 u_3}{\partial t^2}, \tag{7}$$

$$(\mu_L - \mu_T)\frac{\partial^2 u_3}{\partial x_1^2} + \mu_T \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) u_3 = \rho \frac{\partial^2 u_3}{\partial t^2}. \tag{8}$$

To examine dilatational and rotational disturbances, we introduce two displacement potentials ϕ and ψ by the relations

$$u_1 = \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_2}, \qquad u_2 = \frac{\partial \phi}{\partial x_2} - \frac{\partial \psi}{\partial x_1}. \tag{9}$$

The component u_3 is associated with purely distortional movement. We note that ϕ, ψ and u_3 are respectively associated with P waves, SV waves and SH waves. The symbols have their usual significances.

3. General theory and boundary conditions

The propagation of general surface waves is examined here for a fibre-reinforced elastic solid semi-infinite medium M covered by another fibre-reinforced elastic medium M_1 (M_1 above M and mechanical properties different from M and which is welded in contact with M to prevent any relative motion or sliding during the disturbance). We consider an orthogonal Cartesian co-ordinate system $ox_1x_2x_3$ with the origin o at the common plane boundary surface and ox_2 directed normally into M.

We consider the possibility of a wave travelling in the direction ox_1 in such a manner that (a) the disturbance is largely confined to the neigbourhood of the boundary and (b) at any instant all particles in any line parallel to ox_3 have equal displacements. On account of (a) the wave is a surface wave and on account of (b) all the partial derivatives with respect to x_3 vanish.

Now using (9) in (6) we obtain the following wave equation in M satisfied by ϕ and ψ as

$$(\lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta)\frac{\partial^2\phi}{\partial x_1^2} + (\alpha + \lambda + 2\mu_L)\frac{\partial^2\phi}{\partial x_2^2} = \rho\frac{\partial^2\phi}{\partial t^2}, \tag{10}$$

$$(\alpha + 3\mu_L + \beta - 2\mu_T)\frac{\partial^2\psi}{\partial x_1^2} + \mu_L\frac{\partial^2\psi}{\partial x_2^2} = \rho\frac{\partial^2\psi}{\partial t^2}, \tag{11}$$

and similar relations in M_1 with $\rho, \lambda, \alpha, \mu_L, \beta$ replaced by $\rho_1, \lambda_1, \alpha_1, \mu_{L1}, \beta_1$. The general solutions for ϕ and ψ must satisfy (7).

3.1 *Boundary conditions*

The boundary conditions for the titled problem are:

(i) the component of displacement at the boundary surface between the media M and M_1 must be continuous at all times and places,

(ii) The stress components τ_{21}, τ_{22} and τ_{23} must be continuous across the interface of M and M_1 at all times and places,

where τ_{21}, τ_{22} and τ_{23} can be written in terms of ϕ and ψ in the medium M with the help of (2) and (9) as

$$\begin{aligned}
\tau_{21} &= \mu_L\left(2\frac{\partial^2\phi}{\partial x_1\partial x_2} + \frac{\partial^2\psi}{\partial x_1^2} - \frac{\partial^2\psi}{\partial x_2^2}\right),\\
\tau_{22} &= \lambda\nabla^2\phi + \alpha\left(\frac{\partial^2\phi}{\partial x_1^2} - \frac{\partial^2\psi}{\partial x_1\partial x_2}\right) + 2\mu_T\left(\frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_1\partial x_2}\right),\\
\tau_{23} &= \mu_T\frac{\partial u_3}{\partial x_2},
\end{aligned} \tag{12}$$

where ∇^2 is the two dimensional Laplacian operator given by

$$\nabla^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

Similar relations in M_1 with $\mu_L, \lambda, \alpha, \mu_T$ are replaced by $\mu_{L1}, \lambda_1, \alpha_1, \mu_{T1}$.

4. Solution of the problem

We seek harmonic solutions for (8), (10) and (11) in the form (Bullen 1965),

$$\phi, \psi, u_3 = \left\{\overline{\phi}(x_2), \overline{\psi}(x_2), \overline{u}_3(x_2)\right\} \exp\left\{i\omega(x_1 - ct)\right\}, \tag{13}$$

in M and similar relations in M_1 with the functions ϕ, ψ, u_3 being replaced by ϕ_1, ψ_1, u_3^1. This leads us to a particular solution corresponding to a group of simple harmonic waves of wavelength ${}^{2\pi}/_{\omega}$ travelling forward with speed c.

It is convenient to introduce h, r, s where

$$h = \left\{\frac{(\rho c^2 - \mu_L)}{\mu_T}\right\}^{1/2},$$
$$r = \left\{\frac{[\rho c^2 - (\lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta)]}{\alpha + \lambda + 2\mu_L}\right\}^{1/2},$$
$$s = \left\{\frac{[\rho c^2 - (\alpha + 3\mu_L + \beta - 2\mu_T)]}{\mu_L}\right\}^{1/2}, \tag{14}$$

and similar expressions h_1, r_1 and s_1 for the medium M_1. The positive value of the square root being taken in each case.

Now substituting from (13) into (8), (10) and (11) we obtain for the medium M

$$u_3 = C \exp[i\omega(-hx_2 + x_1 - ct)],$$
$$\phi = A \exp[i\omega(-rx_2 + x_1 - ct)],$$
$$\psi = B \exp[i\omega(-sx_2 + x_1 - ct)], \tag{15}$$

and for the medium M_1

$$u_3^1 = C_1 \exp[i\omega(h_1x_2 + x_1 - ct],$$
$$\phi_1 = A_1 \exp[i\omega(r_1x_2 + x_1 - ct],$$
$$\psi_1 = B_1 \exp[i\omega(s_1x_2 + x_1 - ct]. \tag{16}$$

In the above, for the effect to be essentially a surface one, each expression must diminish indefinitely with increasing distance from the boundary. This will be the case if each expression contains an exponential factor in which the exponent is real and negative. Hence h, r, s and similarly h_1, r_1, s_1 are taken to be imaginary.

Using (15) and (16) in the boundary conditions (i) and (ii) given in §(3.1) we obtain,

$$A + sB = A_1 - s_1B_1, \tag{17}$$

$$-Ar + B = A_1r_1 + B_1, \tag{18}$$

$$C = C_1, \tag{19}$$

$$\mu_L[2rA + (s^2 - 1)B] = \mu_{L1}[-2r_1A_1 + (s_1^2 - 1)B_1], \tag{20}$$

$$[(\lambda + \alpha) + r^2(\lambda + 2\mu_T)]A - (2\mu_T - \alpha)sB = [(\lambda_1 + \alpha_1) + r_1^2(\lambda_1 + 2\mu_{T1}]A_1 + (2\mu_{T1} - \alpha_1)s_1B_1, \tag{21}$$

$$-Ch\mu_T = C_1 h_1 \mu_{T1}. \tag{22}$$

It follows from (19) and (22) that both C and C_1 vanish, thus there is no propagation of displacement u_3. Finally we get the wave velocity equation in the common boundary of the media M and M_1 by eliminating the constants A, B, A_1 and B_1 from the equations (17), (18), (20) and (21) as

$$\begin{vmatrix} 1 & s & -1 & s_1 \\ -r & 1 & -r_1 & -1 \\ 2\mu_L r & \mu_L(s^2-1) & 2r_1\mu_{L1} & -(s_1^2-1)\mu_{L1} \\ \left[(\lambda+\alpha)+r^2\left(\lambda+2\mu_T\right)\right] & -(2\mu_T-\alpha)s & -\left[(\lambda_1+\alpha_1)+r_1^2(\lambda_1+2\mu_{T1}\right] & -(2\mu_{T1}-\alpha_1)s_1 \end{vmatrix} = 0. \tag{23}$$

5. Particular cases

5.1 *Rayleigh waves*

The particular case of M_1 replaced by vacuum was first examined by Rayleigh (1885). The absence of stress over the free surface enables us to replace the right-hand side of (20) and (21) by zero, giving

$$2rA + (s^2-1)B = 0, \tag{24}$$

$$[(\lambda+\alpha)+r^2(\lambda+2\mu_T)A - (2\mu_T-\alpha)s = 0. \tag{25}$$

Eliminating A and B from (24) and (25) we obtain the Rayleigh type of waves in the fibre-reinforced elastic medium as

$$(1-s^2)[(\lambda+\alpha)+r^2(\lambda+2\mu_T)] = 2rs(2\mu_T-\alpha), \tag{26}$$

where r and s have been defined in (14).

Now writing $\mu_L = \mu_L - \mu_T + \mu_T$ and making α, β and $|\alpha_L - \mu_T|$ all tend to zero, (26) reduces to the following form,

$$\left(2 - \frac{\rho c^2}{\mu_T}\right)^2 = 4\left(1 - \frac{\rho c^2}{\lambda+2\mu_T}\right)^{1/2}\left(1 - \frac{\rho c^2}{\mu_T}\right)^{1/2}, \tag{27}$$

which is the Rayleigh surface wave in isotropic materials.

5.1a *Numerical calculations for Rayleigh waves*: The following values of elastic constants and density are considered (Chattopadhyay 1998).

$\lambda = 5.65 \times 10^9\ Nm^{-2}$, $\mu_L = 5.66 \times 10^9\ Nm^{-2}$,
$\mu_T = 2.46 \times 10^9\ Nm^{-2}$, $\alpha = -1.28 \times 10^9\ Nm^{-2}$,
$\beta = 220.90 \times 10^9\ Nm^{-2}$, $\rho = 7800\ \ kg\,m^{-3}$.

Using (26) and the expressions given in (14) we obtain the following value of the wave velocity c as:

$\rho c^2/\mu_L$ (dimensionless) $= 40.817316$;

$\left(\rho c^2/\mu_L\right)^{1/2} = 6.38884$;
c^2 (m/s) $= 296 \times 10^5$;
c (km/s) $= 5.4406$.

The Rayleigh wave propagates very rapidly in fibre-reinforced elastic media according to this theory.

5.2 *Love waves*

For the existence of Love waves we consider a layered semi-infinite medium, in which M_1 is obtained by two horizontal plane surfaces, a finite distance H apart, and the medium M remains as before.

Now we investigate the displacement u_3 in the direction of x_2-axis. For the medium M the solution for the displacement component u_3 remains the same but for the medium M_1 we preserve the full solution, since the displacement component u_3 no longer diminishes with increasing distance from the boundary surface of two media. Hence,

$$u_3^1 = C_1 \exp[i\omega(h_1 x_2 + x_1 - ct)] + D_1 \exp[i\omega(-h_1 x_2 + x_1 - ct)], \tag{28}$$

where h_1 is now not necessarily imaginary. For M we still have h imaginary. In the present case the boundary conditions are:

(i) u_3 and τ_{23} are continuous at $x_2 = 0$

(ii) $\tau_{23} = 0$ at $x_2 = -H$.

Now using (12), (15) and (28) in the above boundary conditions we obtain

$$C - C_1 - D_1 = 0, \tag{29}$$

$$-\mu_T h C - \mu_{T1} h_1 C_1 + \mu_{T1} h_1 D_1 = 0, \tag{30}$$

$$\exp[-i\omega H h_1] C_1 - \exp[i\omega H h_1] D_1 = 0. \tag{31}$$

Eliminating C, C_1 and D_1 from the above equations we obtain

$$\mu_{T1} h_1 \tan(\omega H h_1) + i\mu_T h = 0. \tag{32}$$

Substituting for h and h_1 from (14) into (32) gives the equation for the velocity c of Love waves, namely

$$\mu_T \left(\frac{\mu_L - \rho c^2}{\mu_T}\right)^{1/2} - \mu_{T1} \left(\frac{\rho_1 c^2 - \mu_{L1}}{\mu_{T1}}\right)^{1/2} \tan\left[\omega H \left(\frac{\rho_1 c^2 - \mu_{L1}}{\mu_{T1}}\right)^{1/2}\right] = 0. \tag{33}$$

The requirement that h should be imaginary and hence that by (32) h_1 real is, by (14), satisfied if

$$\left(\mu_{L1}/\rho_1\right)^{1/2} < c < \left(\mu_L/\rho\right)^{1/2}. \tag{34}$$

Equation (33) shows that c is dependent on the particular value of ω and not a fixed constant so that in the present boundary conditions there is dispersion of the general wave form. We see from (33) that if ω is small, $c \to \left(\mu_L/\rho\right)^{1/2}$, while if ω is large, $c \to \left(\mu_{L1}/\rho_1\right)^{1/2}$.

5.2a *Numerical calculations for Love waves*: The upper limit for the wave velocity c given in the inequality (34) for the existence of propagation of Love waves in different elastic solid medium is given below.

Fibre reinforced medium $(\mu_L/\rho)^{1/2}$ (km/s) $= 0.851$;
Lead $(\mu/\rho)^{1/2}$ (km/s) $= 0.69836$;
Copper $(\mu/\rho)^{1/2}$ (km/s) $= 2.08$;
Iron $(\mu/\rho)^{1/2}$ (km/s) $= 3.10$;
Earth crust (km/s) $=$ 3.2–3.6.

5.3 *Stoneley waves*

Stoneley investigated a type of surface waves (Stoneley 1924) which are the generalised form of Rayleigh waves propagating at the common boundary of M and M_1. The Stoneley waves in fibre-reinforced elastic media along the common boundary of M and M_1 are, in fact, the same as the general discussion of surface waves presented at the starting and as such the wave velocity is determined by the root of the frequency equation (23). This equation of course reduces once more to the classical result when the parameters for the fibre-reinforced medium tend to zero.

6. Discussion

It is clear from the above investigation that the surface waves in the fibre-reinforced medium are affected by the reinforced parameters. In particular, the condition for the existence of propagation of Love waves, given by (34), depends upon the reinforced parameter μ_L and μ_{L1}. Also all the results reduce to the classical isotropic results when the anisotropic parameters for the fibre-reinforced medium tend to zero (if necessary writing $\mu_L = \mu_L - \mu_T$ and considering $|\mu_L - \mu_T| \to 0$).

From §5.1a, we conclude that the Rayleigh wave velocity in a fibre-reinforced elastic medium is considerably higher than the Rayleigh wave velocity in isotropic media. In this connection, terrestrial Rayleigh wave speed is about 3 km/s (Love 1911, p. 160). From §5.2a, the value of the upper limit of c given in (34) for the stable propagation of Love waves in the fibre-reinforced medium decreases in comparison with the upper limits in other elastic media except lead.

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