QUARTERLY OF APPLIED MATHEMATICS

Vol. V APRIL, 1947 No. 1

SURFACE WAVES IN WATER OF VARIABLE DEPTH*

BY

J. J. STOKER

New York University

Introduction. The problem of irrotational gravity waves in water is, from the mathematical point of view, a problem in potential theory which involves a nonlinear boundary condition at the free surface. In addition, the shape of the free surface is itself not given a priori but is to be determined as part of the solution along with other quantities, such as the distribution of the velocity and the pressure. Very few successful attacks on the problem in this formulation have been made; among these are the proofs of the existence of steady periodic waves of special type by Levi-Civita [5],** and Struik [11], and a partial treatment of the problem of the so-called solitary wave by A. Weinstein [13].

Most of the literature on the subject of surface gravity waves is concerned with theories which result from the general nonlinear theory when simplifying assumptions of one kind or another are made. The present paper is concerned with two such theories:

1) Perhaps the best known and most extensively studied theory is that which results from the general theory when it is assumed that the amplitude of the waves at the surface and the velocity of the particles there are small enough that the free surface condition can be simplified by dropping the nonlinear terms; in addition, this condition may be prescribed at the original undisturbed surface of the water. The result is a problem in potential theory with a linear boundary condition of the mixed type. The greater part of our work here makes use of this theory. We shall refer to it as the *exact linear theory*, or simply as the *exact theory*.

2) The second theory furnishes an approximation to the exact linear theory which is based on the assumption that the depth of the water is small. The resulting theory yields a differential equation for the surface elevation of the water which turns out to be the wave equation. This approximate theory, which is often called the shallow water theory is used, for example, in discussing the tides in the ocean. In Sec. 7 we give a brief derivation of the shallow water theory which brings out the role of the depth of the water as the determining factor in the accuracy of the approximation. The usual derivation of the theory based on assuming that the pressure in the water is given by the same law as in hydrostatics does not bring this point out clearly. The

* Received Aug. 24, 1946. This paper presents results which were obtained at the Institute for Mathematics and Mechanics of New York University while pursuing investigations of various kinds under a contract with the Navy Department.

** Numbers in square brackets refer to the bibliography at the end of the paper.

approach to the theory through the hydrostatic pressure relation also does not lend itself easily to generalization to other cases, such as the derivation of the shallow water theory when floating bodies are present.

Our principal object in this paper is to solve the problem of determining progressing waves over a uniformly sloping bottom by making use of the exact linear theory, to discuss the solutions numerically, and to compare them with the approximate solutions furnished by the shallow water theory.

Solutions for waves on sloping beaches in terms of the exact theory have been obtained by Hanson [3], Bondi [2], Miche [9] and H. Lewy [6]. The first author obtained one type of standing wave solution. The second and third authors obtained two types of standing wave solutions for the case of motion in two dimensions from which progressing wave solutions can be constructed. The first three writers, as well as the writer of the present paper, are concerned only with cases in which the bottom slopes at the special angles $\pi/2n$, with n an integer. The method employed in the present paper is different from those of the first three authors; in particular, the progressing wave solutions are obtained here in a closed form which lends itself well to detailed discussion. Also, the method employed in the present paper yields three-dimensional progressing wave solutions, that is progressing waves which approach an arbitrary plane wave at infinity (cf. Sec. 9).

The investigations which led to the present paper were begun in collaboration with H. Lewy, who then later extended the method to two-dimensional motions (cf. [6]) for a bottom sloping at the angles $p\pi/2n$ in which p is any odd integer and n is any integer such that $2n>p$. (The cases $p\neq 1$ are very much more complicated than those for $p=1$, by the way.)

The basic idea of the method devised by H. Lewy and used by him and the author is to obtain a differential equation for the desired velocity potential, through use of the boundary conditions, which is *not* the potential equation and which, as it turns out, permits an explicit integration. In the case of two-dimensional motions the problems can be treated, of course, by making use of analytic functions of a complex variable; in this case Lewy's differential equation becomes an ordinary non-homogeneous differential equation with constant coefficients for the complex potential, and this equation can be integrated to yield the desired solutions. In the three dimensional cases the solutions can also be obtained, as explained in Sec. 9, but the results are more complicated and more difficult to handle numerically. It may be of interest to observe that the method developed here for the mixed boundary value problem in wedge-shaped regions of angle $\pi/2n$ is not confined in its usefulness to solutions of the potential equation—it could also be extended to other linear partial differential equations.

In Sec. 1 the exact linear theory (which apparently goes back to Poisson) is formulated briefly. In Sec. 2 the theory is applied to yield the well known solutions for steady progressing waves in water of infinite depth. In addition, we show in Sec. 2 that these solutions are uniquely determined if the amplitude and velocity of the waves is bounded at ∞.[1] The method used to obtain this result is essentially the same as that employed later to obtain progressing wave solutions over a sloping bottom.

[1] The author was led to these rather general conditions guaranteeing the uniqueness of the solutions as the result of a conversation with A. Weinstein, who had previously obtained the same result by quite different methods for water of finite and constant depth (cf. [12]).

In Sec. 3 convenient dimensionless independent variables are introduced, and these variables are used in the remainder of the paper except in Sections 7 and 8.

In Sec. 4 the case of progressing waves coming from infinity in an ocean of infinite depth but bounded on one side by a vertical cliff is treated in considerable detail. Only the two dimensional case is considered—that is, the wave crests and all other curves of constant phase are assumed to be horizontal straight lines parallel to the cliff. Since the problem is then a potential problem in two dimensions, it is convenient to solve it in terms of analytic functions of a complex variable. In Sec. 5 the method used for the case of a vertical cliff is generalized to yield solutions in water over a plane bottom sloping at any angle $\pi/2n$, with n an integer. Again only the two dimensional case is treated. The essential step in the generalization requires the derivation of Lewy's differential equation, which turns out to be a differential equation of order n for an angle $\pi/2n$. The solutions of the differential equation which satisfy the boundary and regularity conditions are then given in this section. The solutions obtained are shown to be uniquely determined if they have at most a logarithmic singularity at the origin (that is, at the shore line) and satisfy certain boundedness conditions at ∞.[2] The method of determining the arbitrary constants in such a way that the boundary conditions are satisfied is discussed in Appendix I. In Appendix II, the behavior of the solutions at ∞ is investigated; we find that the solutions behave as one would expect, i.e. that they tend to the simple standing wave solutions for water of infinite depth. It is then readily seen that standing waves of arbitrary phase and amplitude at ∞ can be constructed from our solutions. In this way the existence of a unique set of standing wave solutions is established. For the angles $\pi/2n$, then, the problem of progressing waves has been solved completely for the two dimensional case.

In Sec. 6 the theory of Sections 4 and 5 is applied to the cases $n=1$, $n=2$, and $n=15$, i.e. to the cases of bottom slopes of 90°, 45°, and 6° respectively.[3] The standing wave solutions are given numerically in the form of graphs for a distance of a few wave lengths from shore. The numerical evaluation requires the calculation of the values of complex integrals of the form

$$E(z) = e^{z}\int_{z}^{+\infty} t^{-1}e^{-t}\,dt$$

taken over appropritate paths in the t-plane. A table of values of $E(z)$ for the range of values of interest to us was computed and is included here as an Appendix. This table was based on a previous table calculated by the Mathematical Tables Project [8]. It might be noted that the calculations for the case of the 6° slope were very laborious.

In all cases (except that of water of infinite depth everywhere) there are two types of standing waves: the one type has a finite amplitude all the way to shore, the other type has amplitudes which become logarithmically infinite as the shore is approached. At infinity the amplitude and wave length may be arbitrarily prescribed for both types, but the wave length at ∞ and the frequency are connected by the same relation as in water which is infinite in depth everywhere. These two types of standing

[2] This result (which is not obtained in the papers by Miche, Bondi, and Lewy cited above) means that no non-trivial solution exists which dies out at ∞.

[3] The paper of Miche [9], which gives the solutions in the general case, contains graphs for the 45° and 30° cases as well as approximate solutions for small angles of slope.

waves are always out of phase at ∞, but the ones remaining finite (for any fixed bottom slope) at the shore all have the same phase at infinity. (The solutions obtained by Hanson [3] are those which remain finite at the shore.) Thus all progressing waves furnished by the exact theory necessarily have large amplitudes near shore.

In all three cases treated numerically, i.e. those with 90°, 45°, and 6° slopes, the most striking general result is the following: The wave lengths and amplitudes change very little from the values at ∞ until points about a wave length from shore (wave length at ∞ is meant) are reached. Closer in shore the amplitude of any progressing wave becomes large. It is curious, however, that the amplitude (that is, the relative maximum or minimum) of a progressing wave at a point a wave length or two from shore can actually be about 10 per cent *less* than the value at ∞. This was found in all three cases, including the 6° case.

In Sec. 7 the shallow water theory is derived. In this theory, the wave amplitudes for a uniform bottom slope die out at ∞ like $x^{-1/4}$ (where x denotes distance from shore) while the wave lengths (or, rather, the distances between successive nodes) increase without limit. Thus if the wave lengths and amplitudes were to be taken with the same value at a given point some distance off shore in the two theories (i.e. the exact theory and the shallow water theory) the amplitude near shore as given by the shallow water theory would be much too high while at great distances from shore it would be too low.

In Sec. 8 the results of the exact theory are compared with those of the shallow water theory for the case of water of finite and constant depth in order to bring out the known fact that the accuracy of the shallow water theory for a simple progressing wave depends upon having the ratio of depth to wave length sufficiently small.

Progressing waves moving toward shore as given by both theories for the case of the 6° beach are compared graphically in Sec. 8 assuming the same frequency in both cases and also the same phase and amplitude at a point two or three wave lengths from shore.The results from there on toward shore do not differ greatly except at points very close to shore. This was to be expected, since the depth of the water at the point where the phase and amplitude are the same is about one-eighth of the wave length and hence one might expect the shallow water theory to be quite accurate. However, if the amplitudes had been made equal at a point 15 or 20 wave lengths from shore, the amplitudes given by the shallow water theory three wave lengths from shore would be 50 per cent to 60 per cent higher than those furnished by the exact theory. In other words, the amplitude variation with decrease in depth cannot be correctly estimated over distances of many wave lengths from a given point using the inverse fourth root law unless the wave length at the point is eight or ten times the depth of the water. On the other hand, as the results of the exact theory indicate, if the depth remains an appreciable fraction of the wave length the amplitude changes very little with changes in depth. We draw this conclusion from the fact that the wave amplitudes as given by the exact theory approach their values at ∞ very quickly when the depth approaches say half the wave length.

Another feature of our numerical results which is of interest concerns the variation in the phase or propagation velocity c of a progressing wave when the depth h varies. In Sec. 8 graphs are given which show the results of calculations of the phase velocity for the 6° case, using both the exact theory and the shallow water theory. In the case of the shallow water theory, it is found that the phase velocity c is practically identical with that given by the formula $c=(gh)^{1/2}$. In the case of the exact theory, the phase velocity is given very accurately by the formula $c=[(\lambda g/2\pi)\tanh(2\pi h/\lambda)]^{1/2}$, that is, in both cases c is given by the formula which is exact (for that theory) only when the depth h is constant. (In the second formula λ is taken as the wave length at ∞.) In other words, the phase velocity at any point is given very accurately by the formula which is exact only for water of uniform depth equal to that at the point in question and for a steady progressing wave traveling in one direction only. At the same time, the results for the phase velocity as furnished by the two theories agree very well with each other for a distance of about three wave lengths from shore but from then on out the shallow water theory furnishes a value which is too high by an amount which increases without limit with increasing distance from shore. These remarks, it should be recalled, result from our calculations for a slope of 6°. With decreasing slope, it seems certain that the shallow water theory would be accurate up to distances comprising more and more wave lengths from shore.

All progressing wave solutions discussed above were obtained on the assumption that the wave comes from ∞ toward shore with no component which goes outward at ∞. Once the frequency and amplitude at ∞ are prescribed, the additional condition that the wave at ∞ is a progressing wave moving toward shore leads to a unique solution, which, as we have already mentioned, has a logarithmic singularity at the shore line. The solution is also uniquely determined if the singularity at the shore line is prescribed—the behavior at ∞ is then determined. Our theory thus furnishes us with two types of standing wave solutions from which solutions behaving like arbitrary simple harmonic progressing waves at ∞ can be constructed, but it furnishes no criterion by which one can decide what type of wave would actually occur in practice. Our assumption, in the numerical cases treated, that the waves move from ∞ toward shore with no reflection from the shore back to ∞ was the result of information on the phase velocities as measured on beaches with small slopes; these measurements agree rather well with the theoretical results discussed in the preceding paragraph for a progressing wave moving toward shore. The physical mechanism which prevents the reflection of waves from the shore can be understood as the result of the partial loss of energy from turbulence and the conversion of the remainder into an undertow through the occurrence of breakers. If, however, the slope of the beach is large it may well be that a standing wave, denoting perfect reflection, could occur.

In Sec. 9 we solve the problem of progressing waves in an ocean of infinite depth bounded on one side by a vertical cliff when the wave crests are not assumed to be parallel to the shore line (as in Sec. 4); that is, we solve a three-dimensional problem using the exact theory. Solutions are obtained which tend at ∞ to an arbitrary plane wave. In all of the solutions obtained in the preceding sections by means of the exact theory the discussion was greatly facilitated by the use of analytic functions of a complex variable. In the present three-dimensional case this approach is no longer possible. Nevertheless, the process of obtaining the solutions remains analogous to that using complex functions. The solution for the case of a vertical cliff only is obtained, but it is readily seen how three-dimensional progressing wave solutions for slopes of angles $\pi/2n$ could also be found. The solution for the case of the vertical cliff is also evaluated numerically in Sec. 9 for the case of a progressing wave with wave crests tending to a straight line at ∞ which makes an angle of 30° with the shore line. One of the figures given in Sec. 9 shows the contours of the wave surface. In this case, as well as in the previous two-dimensional cases, it turns out that there is a point near the cliff where the wave crests are lower than they are at ∞, although the elevation of the wave crests becomes infinite upon approaching the cliff.

Finally, the author takes pleasure in acknowledging the help and advice he received from a number of his colleagues and co-workers. The actual solution of Lewy's differential equation and the determination of the constants to satisfy the boundary conditions—no small task in itself—was carried out by E. Bromberg and E. Isaacson. The extensive numerical computations were completed by E. Isaacson, B. Grossmann, and J. Butler.

1. Resumé of general theory of surface waves of small amplitude. In this section we state briefly the well-known mathematical formulation of the problem of surface waves of small amplitude in water. (See, for example, Lamb: Hydrodynamics, Chap IX; or Milne-Thompson: Theoretical Hydrodynamics, Chap. XIV.) The water is assumed to fill the region $-h(x, z) \leqq y \leqq 0$ when at rest. The non-negative quantity h is the (variable) depth of the water. The motion is assumed to be irrotational, so that a velocity potential $\Phi(x, y, z; t)$ exists, in which Φ depends not only on x, y, z but also on the time t. Hence Φ satisfies the Laplace equation

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0. \tag{1.1}$$

A solution of this differential equation in the region $-h \leqq y \leqq 0$ is to be found which satisfies appropriate boundary conditions. The condition to be satisfied at the free surface $y=0$ is

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial y} = 0, \tag{1.2}$$

which results from the Bernoulli law and the assumption that nonlinear terms in the displacement and velocity of the free surface can be neglected. At the bottom $y = -h(x, z)$ we require of course that the derivative of Φ normal to the bottom surface should vanish:

$$\left.\frac{\partial\Phi}{\partial n}\right|_{y=-h} = 0. \tag{1.3}$$

We shall be interested solely in phenomena which are periodic in the time t. It is therefore convenient to replace Φ in the above equation by $e^{i\sigma t}\varphi$, in which φ depends only on x, y, z and not on t.

The conditions on φ are the same as those on Φ except that (1.2) becomes

$$g\frac{\partial\varphi}{\partial y} - \sigma^2\varphi = 0. \tag{1.2'}$$

In addition to the differential equation (1.1) and the boundary conditions (1.2′) and (1.3), the functions $\varphi(x, y, z)$ should be required to satisfy certain conditions at infinity which lead to unique solutions of the type desired on physical grounds. These conditions will be formulated in the following sections for the special cases of interest to us.

Once φ has been determined, the vertical displacement $\eta(x, z; t)$ of the free surface is determined (see the books of Lamb and Milne-Thompson cited above) by the formula

$$\eta = \frac{1}{g} \frac{\partial}{\partial t} (e^{i\sigma t}\varphi) \bigg|_{y=0}. \tag{1.4}$$

2. Plane traveling waves in water of infinite depth. We seek solutions of the boundary value problem partially formulated in equations (1.1), (1.2′) and (1.3) which have the form of plane traveling waves for the case of infinite depth of water. The potential function φ may be assumed to depend only upon x and y and not on z: $\varphi=\varphi(x, y)$, and it is to be determined in the entire half plane $y \leqq 0$.

The functions

$$\varphi = e^{my} \cos (mx + \alpha), \text{ with } m \text{ and } \alpha \text{ arbitrary}, \tag{2.1}$$

are familiar potential functions which obviously yield "standing waves" upon reintroduction of the time factor.[4] The amplitude of these waves decreases exponentially with the depth. The condition (1.2′) at the free surface is satisfied if the following relation holds:

$$\sigma^2 = gm = g \cdot \frac{2\pi}{\lambda}, \tag{2.2}$$

in which λ is the wave length. This yields, of course, a relation between the frequency and wave length which characterizes the type of dispersion encountered with surface gravity waves in water of infinite depth.

Since our problem is linear and homogeneous, we may take linear combinations of the standing wave solutions to obtain "traveling wave" solutions given by the velocity potential $\Phi(x, y, z, t)$ as follows:

$$\Phi = e^{my} \cos (mx + \sigma t + \alpha), \tag{2.3}$$

in which α is an arbitrary phase shift. This would represent a wave traveling in the direction of the negative x-axis. Of course, the relation (2.2) between wave length and frequency must always be satisfied. The above theory is the well-known classical theory, which is due to Poisson.

A point which seems not to have been raised in the standard treatises is that concerning the uniqueness of the solutions given by (2.1). It is of interest to deal with this question here since the reasoning used is later on generalized in such a way as to yield the solutions for the problem of waves in water over a uniformly sloping bottom.

We wish to show the following: If φ is a regular potential function in the half-plane $y \leqq 0$ which satisfies the free surface condition $\sigma^2\varphi = g\partial\varphi/\partial y$ on $y=0$ and if the function $\varphi(x, y)$ and its first derivatives φ_x and φ_y are uniformly bounded in (x, y) as $x^2+y^2 \to \infty$, then $\varphi \equiv Ae^{my} \cos (mx+\alpha)$, that is, if the velocity and the vertical displacement of the water are bounded at ∞, then $\varphi(x, y)$ is either identically zero or it is of the type (2.1) with $A \neq 0$. The mixed boundary condition at the free surface in conjunction with the relatively weak condition at ∞ thus leads to this rather narrow class of solutions.[5]

Our proof of this theorm requires the introduction of the analytic function

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[4] It is a curious fact that this theory, which deals with what is perhaps the most familiar of all wave motions in nature, is governed by the potential equation rather than the wave equation. Nevertheless, we shall use the terms *wave-length, amplitude, phase,* etc. with a meaning which is obvious from the context.

[5] The author's attention was called to this possibility by A. Weinstein, who had already proved the same theorem for the case in which the fluid has a finite and constant depth. See [12].

$f(z)=\varphi+i\psi$ of the complex variable[6] $z=x+iy$, the real part of which is the potential function φ. It is convenient at this point to translate our conditions at ∞ on $\varphi(x, y)$ into conditions on $f(z)$ at ∞. In both cases, of course, these functions are defined in the half-plane $y \leqq 0$. On account of the Cauchy-Riemann equations it follows at once that $|df/dz|$ is not greater than $|\varphi_x|+|\varphi_y|$ and thus is uniformly bounded at ∞ since this is the case for $|\varphi_x|$ and $|\varphi_y|$. Since $f(z)=\int^z f'(\xi)d\xi$ it is clear that we have

$$|f(z)| < M|z|,$$

where M a positive constant, for all sufficiently large $|z|$.

In addition, it is readily seen that $|df/dz|$ tends to zero when $|z| \to \infty$ along any ray in $y \leqq 0$ which is not parallel to the real axis. Since we wish to make use of a slightly more general result later on it is convenient to formulate here the following *Lemma: If $f(z)=\varphi+i\psi$ is analytic and regular in the interior of a sector of the complex plane and φ is uniformly bounded at ∞, the absolute value of any derivative of $f(z)$ tends to zero when $|z| \to \infty$ along any ray which is not parallel to either of the boundary rays of the sector.* The lemma is an almost immediate consequence of the assumption that φ is uniformly bounded at ∞. We consider $\varphi(x, y)$ to be expressed in terms of its values on the circumference of a circle with center at (x, y) and radius R through use of the Poisson integral formula. By differentiating both sides of this relation one easily obtains bounds for φ_x and φ_y at the point (x, y) of the form $|\varphi_x| < 2M_1/R$, $|\varphi_y| < 2M_1/R$ with M_1 a constant which may be taken as the maximum of $|\varphi|$ on the circle of radius R. It therefore follows that $|df/dz|_{z=x+iy} < 4M_1/R$ in view of the Cauchy-Riemann equations. We now assume that M_1 is a fixed upper bound for φ for all sufficiently large z—that such a bound exists was assumed. As $|z| \to \infty$ along any ray not parallel to the sides of our sector, it is clear that $|df/dz| \to 0$ since we may allow R to tend to infinity (i.e. our domain will accommodate circles of arbitrarily large radius with centers on such a ray) while M_1 is fixed. Since φ_x and φ_y are potential functions which are well known to be bounded at ∞ in any closed sub-domain of the sector[7] it follows that the second derivatives of φ tend to zero and therefore also $|d^2f/dz^2|$ tends to zero along the rays in question. That the higher derivatives of $f(z)$ behave in the same manner is now obvious.

The condition on φ at the free surface leads to the following condition on $f(z)=\varphi+i\psi$ on the real axis. We may write successively

$$\left(g\frac{\partial}{\partial y}-\sigma^2\right)\varphi = \Re\left(g\frac{\partial}{\partial y}-\sigma^2\right)(\varphi+i\psi)$$
$$= \Re\left(gi\frac{d}{dz}-\sigma^2\right)f(z).$$

($\Re$ means "the real part of" the expression which follows) the last step being a consequence of the fact that $f(z)$ is analytic. Thus the free surface condition may be written

$$\Re\left(gi\frac{d}{dz}-\sigma^2\right)f = 0 \quad \text{for } z \text{ real.} \tag{2.4}$$

[6] In all but the final section of this paper we deal with two-dimensional problems only so that no confusion between the complex variable z and the third space variable should arise, particularly since the complex variable is not used in the final section.

[7] The bounds $2M_1/R$ for $|\varphi_x|$ and $|\varphi_y|$ obtained above could be used to yield this result.

We next introduce the analytic function $F(z)$ defined by

$$F(z) = \left(gi\frac{d}{dz} - \sigma^2\right)f(z). \tag{2.5}$$

Clearly $F(z)$ is regular in the lower half plane. Since $f(z)$ satisfies (2.4) it follows that the real part of $F(z)$ is zero on the real axis, and hence $F(z)$ can be continued analytically by reflection into the entire upper half-plane; thus the resulting function is regular in the entire plane, and any bounds for $|F(z)|$ in the lower half plane also hold in the upper half plane. From (2.5) and our conditions on $f(z)$ at ∞ it is clear that $|F(z)| < M_2|z|$ for all sufficiently large $|z|$ in the lower half plane, and hence also in the entire plane. It follows that $F(z)$ is linear, by application of Liouville's theorem to the function $[F(z) - F(0)]/z$; that is, it can be written as $F(z) = c_1z + c_2$. If we now introduce $c_1z + c_2$ for $F(z)$ in (2.5) and integrate to obtain $f(z)$ the result is

$$f(z) = Ae^{-imz} + Bz + C, \tag{2.6}$$

with $m = \sigma^2/g$. The constant A is an arbitrary complex constant. The constant B must, however, be zero since $|df/dz|$ tends to zero along all rays not parallel to the real axis by the Lemma proved above,[8] and the same is true of $|de^{-imz}/dz|$ since m is real and $\Re e(-imz) \to -\infty$ along such rays. The constant C must be pure imaginary because of the boundary condition (2.4), as one readily sees. Thus the only non-vanishing potential functions φ for which the velocity and surface elevation are bounded at ∞ are of the form (2.1).

3. Introduction of dimensionless quantities. In dealing with surface waves in the remainder of this paper it is convenient to work with dimensionless space variables x_1 and y_1 defined by $x_1 = mx$, $y_1 = my$, in which m is given by

$$\sigma^2 = gm = g\cdot\frac{2\pi}{\lambda}, \tag{3.1}$$

with σ the circular frequency and λ as wave length. We also replace the time variable t by a new variable t_1 given by $t_1 = \sigma t$. In these variables the surface condition (1.2′) is readily seen to take the form

$$\frac{\partial\varphi}{\partial y_1} - \varphi = 0 \quad \text{for} \quad y_1 = 0, \tag{3.2}$$

where φ of course satisfies $\nabla^2\varphi = 0$ in the new variables. However, since nearly all of our work from now on is carried out in the new variables, *we shall drop the subscripts but retain the surface condition in the form* (3.2). The original variables can always be reintroduced by replacing x and y in all of our results by mx and my and t by σt. This means that the reciprocal of the quantity m defined by (3.1) is our unit of length. In the course of our discussion on waves over a sloping bottom it will be shown that the relation (cf. (2.2)) $\sigma^2 = gm = 2\pi g/\lambda$ between frequency and wave length for water of infinite depth holds asymptotically as the depth of the water becomes infinite, when λ is the "wave length at ∞." Thus our unit of length in these cases is proportional to the wave length at ∞.

[8] At this point we use the assumption that φ is bounded.

The standing wave solutions Φ corresponding to (2.1) are, in the new variables (after dropping subscripts):

$$\Phi = e^{it}e^{y} \cos (x + \alpha). \tag{3.3}$$

4. Plane traveling waves in an ocean of infinite depth bounded on one side by a vertical cliff. As stated in the introduction, the main purpose of this paper is to study plane traveling waves in an ocean with a uniformly sloping bottom. In this section we deal in detail with the special case in which the "bottom" is vertical. Most of the essentials of the method to be employed in the more general cases are well illustrated in this case, while the formal apparatus necessary is much simpler than that needed for the general case. In our treatment of the more general cases we shall then feel free to condense the presentation in many particulars.

We assume that all quantities depend upon x and y only so that all curves of constant phase (the loci of the wave crests, for example) are parallel to the line of intersection of the free surface and the cliff forming the vertical boundary of the water.

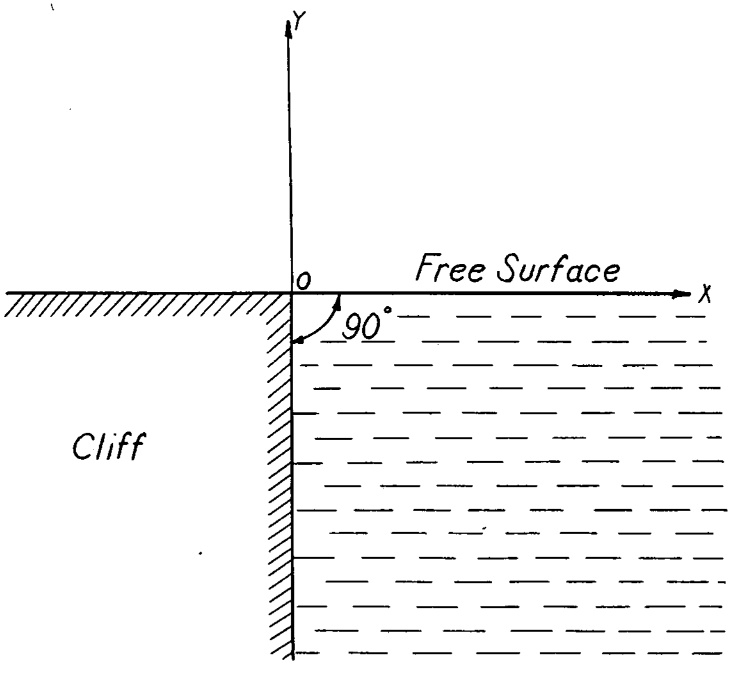

FIG. 1.

Thus we seek a potential function $\varphi(x, y)$ in the shaded area of Fig. 1 which satisfies the surface condition (see Sec. 3):

$$\frac{\partial \varphi}{\partial y} - \varphi = 0 \quad \text{when} \quad y = 0, \; x > 0 \tag{4.1}$$

and the condition at the vertical wall

$$\frac{\partial \varphi}{\partial x} = 0 \quad \text{when} \quad x = 0, \; y < 0. \tag{4.2}$$

As we have already stated, our purpose is to obtain potential functions φ which satisfy (4.1) and (4.2) and which behave at ∞ like traveling waves moving toward shore. It seems reasonable to expect that a velocity potential $\Phi(x, y, t)$ which behaves at ∞ like the solution $e^{it}e^{y}\cos(x+\alpha)$ for water of infinite depth everywhere (cf. (3.3)) will exist in the present case. Or, as we could also put it, we may expect that two potential functions φ_1, φ_2 can be found which behave on the surface at ∞ like $\sin x$ and $\cos x$ respectively, since a reintroduction of the time factor e^{it} would yield two "standing wave solutions" which could be combined linearly (since our problem is linear and homogeneous) to yield a solution behaving like a traveling wave at ∞.[9] In what follows we shall obtain two such potential functions which are 90° "out of phase" at ∞. One such solution which is bounded and regular can be obtained immediately: The boundary condition (4.2) permits an analytic continuation of φ by reflection in the negative y axis into the entire half plane $y \leqq 0$, and we have already obtained solutions for this case in Sec. 2. Since only an even function of x is in question it follows that $\varphi_1 = Ae^{y}\cos x$ is the only solution regular in the entire fourth quadrant which, together with its first derivatives, is uniformly bounded at ∞ in this quadrant, because of the fact that the solutions obtained in Sec. 2 were shown to be unique under these circumstances. (It is clear that bounds for φ at ∞ in the fourth quadrant hold, after reflection, in the half plane $y \leqq 0$.) In other words, all non-singular solutions which are bounded at ∞ have the same phase at ∞, i.e. they behave like $\cos x$ there.[10] To obtain solutions "out of phase" with $e^{y}\cos x$ at ∞ it is therefore essential to admit a singularity. On the other hand, it is rather natural on physical grounds to expect a singularity at the origin (i.e. at the water line on the vertical cliff) of the type of a source or sink if a progressing wave coming toward shore from ∞ occurs.[11] This point has already been discussed in the introduction.

We are now in a position to complete the formulation of our boundary value problem by prescribing conditions on φ at the origin and at ∞. We require, in accordance with the remarks above, that φ should possess a representation of the form

$$\varphi = \bar{\varphi} \log r + \bar{\bar{\varphi}} \tag{4.3$_0$}$$

valid near the origin, with $r = x^2 + y^2$ and $\bar{\varphi}$ and $\bar{\bar{\varphi}}$ functions which together with their first two derivatives are bounded in a neighborhood of the origin. At ∞ we require that φ and its partial derivatives of the first two orders be uniformly bounded, i.e. that a constant M exists such that

$$|\varphi| + \sum |\varphi^{(n)}| < M, \tag{4.3$_\infty$}$$

for all sufficiently large x^2+y^2, in which the sum is taken over all first and second partial derivatives of φ.[12] The conditions (4.1), (4.2), (4.3$_0$) and (4.3$_\infty$) constitute the

[9] Of course the motion will not be a steady wave motion in general, but one which "approaches" a steady motion at ∞.

[10] Upon reintroduction of the original space variables it is seen that this type of solution includes waves of all possible wave lengths.

[11] The paper of Hanson [3] mentioned in the introduction contains only the regular solution. It might be of interest to note that the starting point of the present investigation was the conjecture that solutions out of phase with those of Hanson at ∞ could be obtained by admitting a singularity corresponding to a source or sink at the origin.

[12] These requirements are more stringent than would be necessary to ensure the existence and uniqueness of the type of solutions desired. However, we are not interested in this paper in formulating conditions at ∞ and at the origin in the most general way possible, but only in formulating conditions which seem reasonable on physical grounds and which will lead to unique solutions of a type which interest us.

complete set of conditions on φ. We shall obtain all non-vanishing solutions of this problem by constructing them explicitly.

Our method of solving this boundary value problem requires the introduction of the analytic function $f(z)$ of the complex variable $z=x+iy$, the real part of which is the potential function φ:

$$f(z) = \varphi + i\psi.$$

We must reformulate our conditions on φ in terms of $f(z)$. The boundary conditions (4.1) and (4.2) can be written as

$$\begin{aligned}\left(\frac{\partial}{\partial y} - 1\right)\varphi &= \Re e\left(\frac{\partial}{\partial y} - 1\right)(\varphi + i\psi)\\ &= \Re e\left(i\frac{d}{dz} - 1\right)f(z) = \Re e L_1(D)f = 0 \text{ on the positive real axis} \qquad (4.4)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial x}(\varphi) &= \Re e\frac{\partial}{\partial x}(\varphi + i\psi)\\ &= \Re e\frac{df(z)}{dz} = \Re e L_2(D)f = 0 \text{ on the negative imaginary axis.} \qquad (4.5)\end{aligned}$$

The last step in each case is justified by the fact that $f(z)$ is analytic. The symbol $\Re e$ means, of course, that the real part of what follows is to be taken, and the symbols L_1 and L_2 refer to the linear operators defined by (4.4) and (4.5). ($D=d/dz$.)

The conditions (4.3_∞) at ∞ on φ lead to the following conditions on $f(z)$ at ∞: 1) Because of the fact that φ is uniformly bounded at ∞, the Lemma of Sec. 2 shows that $|df/dz|$ and $|d^2f/dz^2|$ tend to zero along all straight lines in the quarter-plane which are not parallel to its boundaries. 2) Because of the Cauchy-Riemann equations $|df/dz|$ and $|d^2f/dz^2|$ are uniformly bounded in z as $|z|\to\infty$.

We shall make use of the condition (4.3_0) at the origin in the following form: The analytic function $f(z)$ is such that $|df/dz| < M_1/|z|$ and $|d^2f/dz^2| < M_2/|z|^2$ with M_1 and M_2 constants, in a neighborhood of the origin. This statement follows immediately from the conditions (4.3_0) since $r\varphi_x$, $r\varphi_y$, $r^2\varphi_{xx}$, $r^2\varphi_{xy}$ are as a result bounded near the origin and this leads to finite bounds for $|zf'|$ and $|z^2f''|$ through use of the Cauchy-Riemann equations.

Our method of solution depends essentially upon the observation that the linear operators L_1 and L_2 defined by (4.4) and (4.5) have the following property:

$$\Re e L_1L_2(f) = \Re e L_2L_1(f) = 0 \qquad (4.6)$$

on both boundaries of our domain, i.e. on both the positive x-axis and the negative y-axis. This property is an immediate consequence of the linearity and special form of L_1 and L_2 and the boundary conditions (4.4) and (4.5). We are thus led to consider the analytic function $F(z)$ defined by (see (4.4) and (4.5))

$$F(z) = L_2L_1f(z) = \frac{d}{dz}\left(i\frac{d}{dz} - 1\right)f(z). \qquad (4.7)$$

We shall prove the following: *If the analytic function $f(z)$ having the properties we have postulated exists, then $F(z) \equiv Ai/z^2$*, with A an arbitrary real constant; *hence $f(z)$ would satisfy an ordinary differential equation with constant coefficients.*

What are the properties of $F(z)$, assuming that our $f(z)$ exists? From (4.6) we observe that the real part of $F(z)$ vanishes on both boundaries of the quarter-plane (i.e. on the positive real axis and the negative imaginary axis). $F(z)$ can therefore be continued analytically by the reflection process into the entire plane; the result will obviously be a single-valued function whose real part vanishes on the entire real as well as the entire imaginary axis. (Here we make essential use of the fact that the original domain is a sector of angle $\pi/2$.) At the origin $F(z)$ has at most a pole of order two since $|df/dz| < M_1/|z|$ and $|d^2f/dz^2| < M_2/|z|^2$ hold near $z=0$, and these bounds for the derivatives of f in the quarter-plane lead to the bound $|F(z)| < M_3/|z|^2$ in a full neighborhood of the origin as one sees from (4.7) and the fact that $F(z)$ is continued by reflection into a single-valued function in the entire plane. Hence $F(z)$ has at most a pole of order two, and not an essential singularity, at $z=0$. In the same way the conditions at ∞ on $f(z)$ yield for $F(z)$ the condition that $|F(z)|$ is uniformly bounded at ∞. Also $|F(z)|$ tends to zero when $|z| \to \infty$ along any ray which is not parallel to the real or the imaginary axis, since $|df/dz|$ and $|d^2f/dz^2|$ have this property. The only analytic function $F(z)$ with all of these properties is $F(z) = Ai/z^2$, with A an arbitrary real constant (zero included): It is well known that a single-valued analytic function defined in the entire plane is determined by its singularities, which in this case consist of a pole at the origin. This fact, together with the additional conditions on $F(z)$, leads easily to our result.

We can now be certain that the solutions $\varphi(x, y)$ of our potential problem must be the real part of an analytic function $f(z)$ which satisfies the ordinary differential equation

$$\frac{d}{dz}\left(i\frac{d}{dz} - 1\right)f(z) = A\frac{i}{z^2}, \quad A \text{ real}. \tag{4.8}$$

Our problem is therefore reduced to that of determining integration constants in the solution of (4.8) in such a way as to satisfy the boundary conditions (4.4) and (4.5) and the conditions at the origin and at ∞. We shall see that such solutions of (4.8) can be determined, which means that potential functions $\varphi(x, y) = \Re e f(z)$ satisfying our conditions will be shown to exist. It will also be shown that the solutions φ of our problem behave on the water surface at ∞ like $C\cos(x+\alpha)$ in which C and α, the "amplitude" and "phase" of φ, may have any values. Once the phase and amplitude at ∞ are prescribed, however, the solution is uniquely determined.

We proceed to solve the differential equation (4.8) and to fix the constants appropriately. One integration can evidently be carried out at once to yield

$$\left(i\frac{d}{dz} - 1\right)f(z) = -A\frac{i}{z}, \quad A \text{ real}. \tag{4.9}$$

The additive constant which arises through the integration would be imaginary because of the boundary condition (4.4); we have taken it to be zero since it would upon integrating (4.9) give rise only to an additive imaginary constant in the general solution for $f(z)$ and this in turn would contribute nothing to the real part φ of $f(z)$.

A solution $f_1(z)$ of (4.9) for $A=0$ (i.e. a solution of the homogeneous equation) can be found which satisfies all of our conditions. The solution is

$$f_1(z) = Be^{-iz}, \tag{4.10}$$

with B a real but otherwise arbitrary constant, as one can readily verify. The homogeneous differential equation thus furnishes solutions of the problem which are bounded. The corresponding real potential function $\varphi_1 = \Re e f_1(z)$ is, evidently

$$\varphi_1(x, y) = Be^y \cos x. \tag{4.10'}$$

We observe that these solutions differ in amplitude but not in phase. They are, in fact, the non-singular solutions of our problem mentioned earlier in this section.

Other types of solutions result from the non-homogeneous equation, i.e. for the case $A \neq 0$ and these will be singular at the origin. One solution of the non-homogeneous equation is given by

$$f(z) = -Ae^{-iz} \int_{+\infty}^{-iz} \frac{e^{-t}}{t}\, dt, \tag{4.11}$$

in which the path of integration is taken along the positive real axis from $+\infty$, then along a circular arc about the origin, and then along a ray to the point $z\beta_k$ with $\beta_k = -i$, as indicated in Fig. 2. The integral evidently converges. However, it is

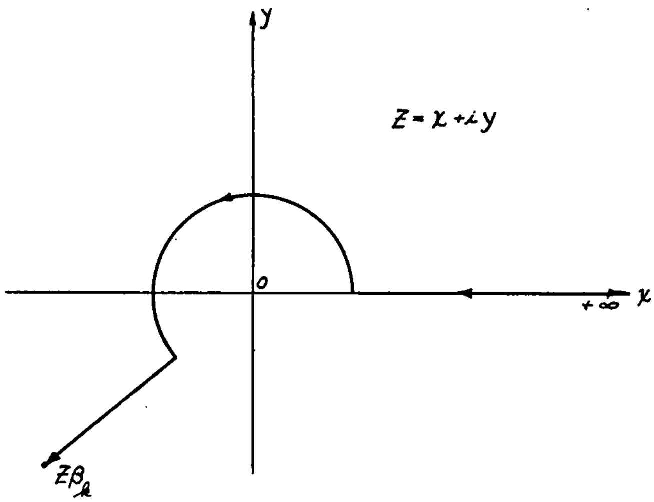

FIG. 2. Path of integration.

necessary to add an appropriately chosen solution of the homogeneous equation in order to satisfy the boundary condition on the negative imaginary axis. (The boundary condition on the real axis is automatically satisfied since $f(z)$ satisfies (4.9) with A real.) From (4.11) we find

$$\frac{df}{dz} = -A\left[-ie^{-iz}\int_{+\infty}^{-iz} \frac{e^{-t}}{t}\, dt + \frac{1}{z}\right], \tag{4.12}$$

and we are interested in the value of the expression on the right when z is a point on the negative imaginary axis. One sees readily that the straight line portions of the path of integration (both of which lie on the real axis in this case) contribute purely real quantities to $\int_\infty^{-iz}(e^{-t}/t)dt$, but that the semicircular part yields an imaginary contribution of amount[13] plus πi. Thus the real part of the right hand side of (4.12) for z on the negative imaginary axis reduces to the real part of $A\pi e^{-iz}$ there, since $1/z$ and ie^{-iz} are both pure imaginary on this axis and A is real. If, therefore, we add the function $+A\pi ie^{-iz}$ to the right hand side of (4.11) the result

$$f_2(z) = -A\left[e^{-it}\int_{+\infty}^{-iz}\frac{e^{-t}}{t}dt - \pi ie^{-it}\right] \tag{4.13}$$

will be a solution of the non-homogeneous equation for which $\Re e\, df_2/dz = 0$ on the negative imaginary axis, i.e. it will be a function satisfying the boundary condition on the imaginary axis.

Thus $f_1(z)$ and $f_2(z)$ as defined in (4.10) and (4.13) are two linearly independent analytic functions which satisfy the boundary conditions. We observe also that $f_1(z)$ and $f_2(z)$ behave in the prescribed manner at the origin: The function $f_1(z)$ is regular there while $f_2(z)$ obviously has a logarithmic singularity.

We have still to check the conditions at ∞. For $f_1(z)$ the conditions are obviously satisfied. To investigate the behavior of $f_2(z)$ as $z\to\infty$ we must consider the asymptotic behavior of the integral $\int_\infty^{iz}(e^{-t}/t)dt$. *We shall show in Appendix I that the* integral possesses the following asymptotic representation:[14]

$$\int_{+\infty}^{-iz}\frac{e^{-t}}{t}dt \cong 2\pi i - ie^{iz}\left[\frac{1}{z} + \cdots\right]$$

the dots representing higher order terms in $1/z$. Assuming that such a development holds, then $f_2(z)$ clearly possesses the following asymptotic development:

$$f_2(z) \cong -A\pi ie^{-iz} = -A\pi e^{-i(z-\pi/2)} \text{ (with } A \text{ real).} \tag{4.14}$$

The derivatives of $f_2(z)$ have essentially the same asymptotic behavior, as one can readily see. Thus all conditions at ∞ are satisfied. The function $\varphi_2(x, y) = \Re e f_2(z)$ behaves as follows at ∞:

$$\varphi_2(x, y) \cong -A\pi e^y \sin x. \tag{4.14'}$$

The solutions $\varphi_1(x, y)$ (cf. (4.10')) and $\varphi_2(x, y)$ are thus out of phase at ∞ and we have therefore achieved one of our main objects. Our conjecture that solutions behaving like φ_2 at ∞ would result by imposing a logarithmic singularity at the origin proves to be correct.

It is important to show that *the set of all analytic functions $f(z)$ which satisfy our boundary and regularity conditions is given by*

$$f(z) = f_1(z) + f_2(z) \tag{4.15}$$

with $f_1(z)$ and $f_2(z)$ the above defined solutions of (4.9) $-f_1$ of the homogeneous equation and f_2 of the non-homogeneous equation. This one proves as follows: Suppose that $g(z)$ is any solution of our problem and set $G(z) = f(z) - g(z)$, with $f(z)$ given by

[13] This is readily seen by expanding e^{-t} in powers of t and observing that the term $1/t$ furnishes the entire value of the integral.

[14] The present case corresponds to the case $\beta_k = -i$ of Appendix II, which in turn arises for $k=2$, $n=2$.

(4.15). It is clear that $G(z)$ satisfies the same boundary conditions as f and g. Since f and g both satisfy (4.9) with the same value of the constant A it follows that $(id/dz-1)G=0$ and the only solution of this equation satisfying our conditions is $G=Ce^{-iz}$, with C real. Thus $g(z)$ could differ from $f(z)$ only by an additive real multiple of $f_1(z)$—that is, g and f are really the same manifold of solutions. We observe that the set of solutions (4.15) contains two real constants A and B which are at our disposal.

From (4.10′) and (4.14′) we conclude that real potential functions solving our problem and having any prescribed amplitude and phase at ∞ can be obtained by superposition of φ_1 and φ_2, since A and B may be chosen arbitrarily, and since our boundary value problem for the function φ is linear and homogeneous. Conversely, it is evident that the constants A and B are uniquely determined when the phase and amplitude of any solution φ at ∞ are prescribed. Once this has been done the complex potential function $f(z)$ is uniquely determined (cf. (4.15) and the remarks concerning it) and hence also the real potential function $\varphi(x, y)$. In other words, our solutions φ exist and are uniquely determined when we prescribe the phase and amplitude at ∞.

We now reintroduce the original space variables by replacing x and y in all relations by mx and my, in which m satisfies the conditions $\sigma^2=gm=2\pi g/\lambda$ (cf. Sec. 3) and σ is the circular frequency. At ∞ our solutions φ_1 and φ_2 have been shown to behave as follows:

$$\varphi_1 = C_1e^{my}\cos mx,$$

$$\varphi_2 \cong C_2e^{my}\sin mx,$$

and consequently the quantity $\lambda=2\pi/m$ is the "wave length" at ∞. This substantiates the remark made in Sec. 3 that the asymptotic relation between the wave length at ∞ and the frequency is the same as that for water which is everywhere (i.e. for all values of x) infinite in depth.

The standing wave solutions φ_1 and φ_2 will be discussed in detail in Sec. 6.

5. Traveling waves over a sloping beach. In this section we shall generalize the method of the preceding section to yield solutions for waves on a beach which slopes at an angle $\pi/2n$ with the horizontal, with n any integer.[15] The method we use is in principle exactly the same as for our previous case of a vertical cliff. The only differences arise from the fact that the differential equation corresponding to (4.11) cannot be obtained quite so easily: it will be, in fact, a differential equation with constant coefficients of order $2n$ instead of one of order two. Naturally, the actual determination of the desired solution will therefore become more and more complicated as n increases, i.e. as the inclination angle of the beach decreases.

We formulate our problem at the outset in terms of the analytic function $f(z)=\varphi+i\psi$. We seek such a function in the sector of angle $\pi/2n$ indicated in Fig. 3. The function $f(z)$ should be regular in the interior of this domain, and have at most a

[15] The problem can be solved for other angles by similar methods, and probably also for any angle by extensions of the theory along the lines of the method of Sommerfeld used in diffraction problems. However, it seems certain that such solutions would be very complicated and would involve functions not easily handled numerically with the tables of functions now available. The cases we discuss in this section are, it happens, amenable to numerical treatment. In any case, for angles less than 90°, it seems certain that the main features of the wave motion will be completely revealed through study of our special cases. For angles greater than 90°—that is, for overhanging cliffs or docks—new features could be expected to arise, and these cases deserve study. As we mentioned in the introduction, H. Lewy [6] has solved the problem for angles $p\pi/2n$ with p any odd integer and n any integer such that $2n>p$.

logarithmic singularity at the origin, which we interpret to mean (cf. the remarks on this point in the preceding section) that $|d^kf(z)/dz^k| < M_k/|z|^k$ for $k=1, 2, \cdots, 2n$, with M_k certain constants. At ∞ we require that $|\Re e f(z)|$ and $|d^kf(z)/dz^k|$ for $k=1, 2, \cdots, 2n$ should remain uniformly bounded when $z\to\infty$ in the sector.[16] As a consequence, all derivatives of $f(z)$ tend to zero along certain rays. On the boundary the conditions on $\varphi(x, y)=\Re e f(z)$ lead, as before, to the following conditions on $f(z)$ at the boundary (cf. (4.4) and (4.5)):

$$\Re e L_1(D)\cdot f(z) = \Re e\left(-\, ie^{-i(\pi/2n)}\frac{d}{dz}\right)f = 0 \quad \text{for} \quad z = re^{-i(\pi/2n)},\ r > 0 \tag{5.1}$$

$$\Re e L_{2n}(D)\cdot f(z) = \Re e\left(i\frac{d}{dz} - 1\right)f = 0 \quad \text{for} \quad z = x > 0. \tag{5.2}$$

By D we mean, of course, differentiation with respect to z. The boundary condition (5.1) should state that the derivative of $\varphi(x, y)$ normal to the bottom vanishes; that it does can be checked easily, for example by inserting $\varphi+i\psi$ for f, replacing d/dz by $\partial/\partial x$, and using the Cauchy-Riemann equations. The condition (5.2) at the free surface is the same as (4.4).

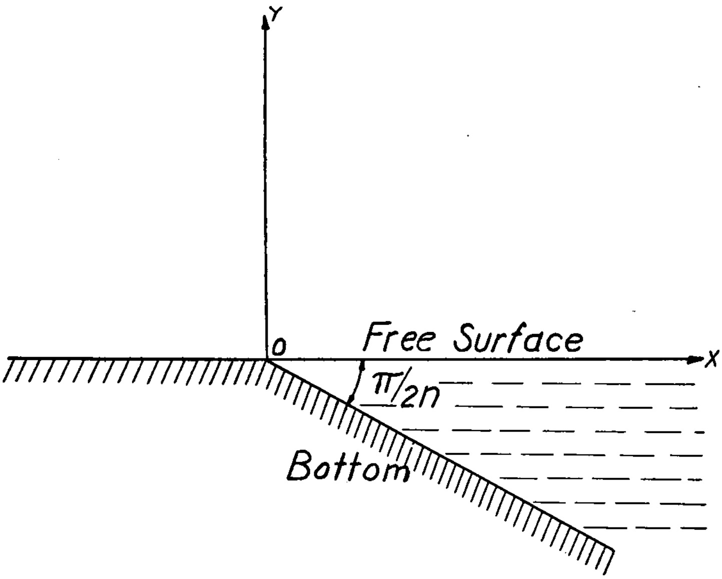

FIG. 3.

In the case treated in the preceding section, the operators L_1 and L_2 had the property $\Re e L_1\cdot L_2 f(z)=\Re e L_2\cdot L_1 f(z) = 0$ on both boundaries (cf. (4.6)). This is, however, not the case for the corresponding operators L_1 and L_{2n} defined in (5.1) and (5.2). It is necessary for our purposes, in fact, to make use of a set of $2n$ linear operators $L_1, L_2, \cdots, L_{2n-1}, L_{2n}$, of which L_1 and L_{2n} are the first and last members of the sequence, and which are so defined that $\Re e(L_1\cdot L_2\cdot L_3\cdot \cdots \cdot L_{2n-1}\cdot L_{2n}f)=0$ on both boundaries. It would be possible to derive these operators through geometrical constructions and arguments (involving essentially a succession of reflections in the lines $re^{-i(k\pi/2n)}$, $k=1, 2, \cdots$), but we prefer to give them at once and then examine them to see that they have the properties we wish. They are defined as follows:

$$L_k = \begin{cases}(\alpha_k D) & \text{if } k \text{ is odd,}\\ (\alpha_k D - 1) & \text{if } k \text{ is even,}\end{cases} \tag{5.3}$$

[16] It would be possible to weaken this requirement considerably.

in which the α_k are the following complex numbers:

$$\alpha_k = e^{-i\pi[k/n+1)/2}, \qquad k = 1, 2, \cdots, 2n. \tag{5.4}$$

It is useful to bear in mind the location of the points α_k in the complex plane, as given in Fig. 4. These numbers lie on the unit circle spaced at equal angles $\pi/2n$, all of them except $\alpha_{2n}=i$ having negative real parts.

We show that these operators have the required properties. To begin with, we verify at once that the operators L_1 and L_{2n} as given by (5.3) are the same as those given in (5.1) and (5.2). We write:

$$L(D)f = L_1 \cdot L_2 \cdots L_{2n} \cdot f(z) = (\alpha_1 D)(\alpha_2 D - 1) \cdots (\alpha_{2n-1}D)(\alpha_{2n}D - 1)f(z). \tag{5.5}$$

Our object is to show that $\Re e L(D)f=0$ on both boundaries of the sector. We know that $\Re e(\alpha_{2n}D-1)f(z)=0$ on the real axis (condition (5.2)). We proceed to show that the operator $P_1(D)$ defined through (5.5) by $L(D)=P_1(D)\cdot(\alpha_{2n}D-1)$ has all of its coefficients real. It is clear that we may write the polynomial $P_1(D)$ as the product of two factors: $P_1(D)=P_1'(D)P_1''(D)$, with $P_1'(D)$ and $P_1''(D)$ defined as follows:

$$P_1'(D) = [\alpha_1\alpha_{2n-1}D^2][\alpha_3\alpha_{2n-3}D^2]\cdots,$$
$$P_1''(D) = [(\alpha_2 D - 1)(\alpha_{2n-2}D - 1)][(\alpha_4 D - 1)(\alpha_{2n-4}D - 1)]\cdots.$$

That is, we separate the linear factors of P_1 into two groups, one containing all factors for which k is even, the other all those for which k is odd; these two groups are then arranged in the manner indicated. From the definition (5.4) of the α_k (cf. also Fig. 4) it is clear that $\alpha_k = \bar{\alpha}_{2n-k}$, in which the bar over a quantity means that the complex conjugate of the quantity is taken, and also $|\alpha_k|=1$ for all k. Hence both $\alpha_k\cdot\alpha_{2n-k}$ and $\alpha_k+\alpha_{2n-k}$ are real numbers for all $k=1, 2, \cdots, 2n-1$, and hence all of the quadratic factors in P_1' and P_1'' obviously have real coefficients. Since $P_1(D)$ contains an odd number of linear factors it follows that either P_1' or P_2' will contain one unpaired linear factor, i.e. the factor containing $\alpha_n D$. But since $\alpha_n=-1$ (cf. (5.4)) it follows that the polynomials P_1' and P_1'' have all their coefficients real and hence also the

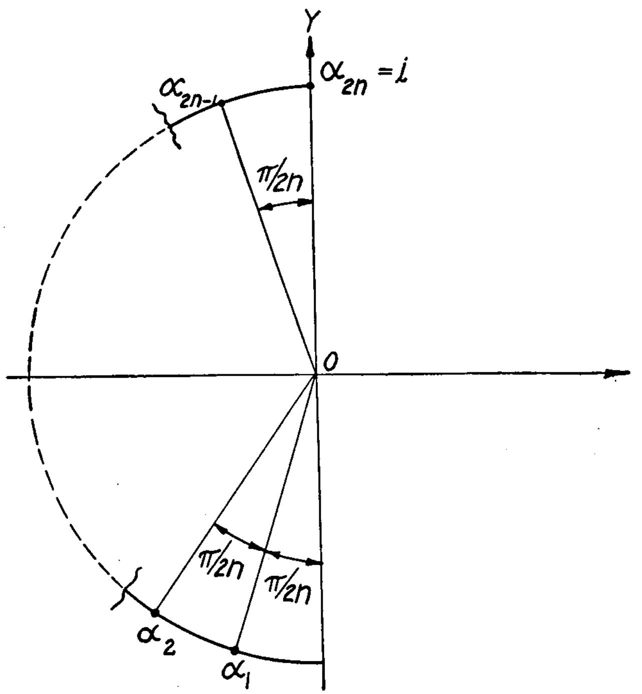

FIG. 4.

polynomial operator $P_1(D)$. Consequently when z varies along the real axis we may write

$$\Re e L(D)f = P_1\left(\frac{\partial}{\partial x}\right)\cdot \Re e(\alpha_{2n}D - 1)f,$$

since D may be replaced by $\partial/\partial x$ in this case and the coefficients of the operator $P_1(\partial/\partial x)$ are all real. Therefore, in view of (5.2) we may write

$$\Re e L(D)f = 0 \quad \text{on the real axis.} \tag{5.6}$$

To prove that the same relation holds along the bottom, i.e. for $z = re^{-i\pi/2n}$, we consider the operator $P_2(D)$ defined by $L(D) = (\alpha_1 D)P_2(D)$. Along $z = re^{-i\pi/2n}$ the operator $D = d/dz$ may evidently be replaced by the operator $e^{i\pi/2n}\partial/\partial r$. By $\partial/\partial r$ is meant, of course, the directional derivative along the bottom line. If, then, D is replaced by $e^{i\pi/2n}\partial/\partial r$ in $P_2(D)$ we observe that $\alpha_2 D, \alpha_3 D, \cdots, \alpha_{2n}D$ become $\alpha_1\partial/\partial r, \alpha_2\partial/\partial r, \cdots, \alpha_{2n-1}\partial/\partial r$ since $\alpha_{k+1}\cdot e^{i\pi/2n} = \alpha_k$ from the definition (5.4) of the α_k. It follows that the operator $P_2(D)$ along $z = re^{i\pi/2n}$ may be replaced by an operator $P_3(\partial/\partial r)$ such that all coefficients of P_3 will be real. This can be seen in exactly the same manner as in the preceding case. We may write, therefore, along this line:

$$\Re e L(D)f(z) = \Re e P_3\left(\frac{\partial}{\partial r}\right)(\alpha_1 D)f(z) = P_3\left(\frac{\partial}{\partial r}\right)\Re e(\alpha_1 D)f(z),$$

and, in view of the boundary condition (5.1), we see that

$$\Re e L(D)f(z) = 0 \quad \text{on} \quad z = re^{-i\pi/2n}. \tag{5.7}$$

Consequently, $\Re e L(D)f(z) = 0$ *on both boundaries.*

We now continue in the same fashion as in the preceding section by introducing the analytic function $F(z)$ defined in our sector by

$$F(z) = L(D)f(z), \tag{5.8}$$

and obtain a differential equation for $f(z)$ by determining $F(z)$ uniquely through use of the properties postulated for $f(z)$. The properties of $F(z)$ are the following: 1) Its real part vanishes on both boundaries of the sector, 2) It is either regular at the origin or it has a pole of order at most $2n$, 3) $|F(z)|$ remains uniformly bounded as $z \to \infty$ in the sector. We know therefore that the derivatives of $F(z)$ will tend to zero along any rays not parallel to the sides of the sector when $|z| \to \infty$, by the Lemma introduced in Sec. 2. The first property has just been established, and the second and third follow immediately from our conditions on $f(z)$ at the origin and at ∞, through the definition (5.8) of $F(z)$.

Because of property 1) and the fact that we are working in a sector of angle $\pi/2n$, with n an integer, it is clear that $F(z)$ can be continued over the boundaries of the sector by successive reflections to yield a single-valued function regular in the entire plane except possibly at the origin where it may have a pole of order $2n$ at most. In addition $|F(z)|$ is bounded at ∞ and the real part of $F(z)$ vanishes on all rays through the origin for which $z = re^{ik\pi/2n}$, $k = 1, 2, \cdots, 4n$. The only analytic function with these properties is $F(z) \equiv A_{2n}i/z^{2n}$, with A_{2n} an arbitrary real constant (zero not excluded). (We rejected a possible additive constant in $F(z)$ through use of the fact that $F(z)$ tends to zero along certain lines.) It follows therefore that the solutions $f(z)$ of our problem satisfy the differential equation

$$L(D)f = (\alpha_1 D)(\alpha_2 D - 1)(\alpha_3 D)(\alpha_4 D - 1) \cdots (\alpha_{2n-1} D)(\alpha_{2n} D - 1)f = A_{2n} \frac{i}{z^{2n}}, \quad (5.9)$$

with A_{2n} an arbitrary real constant. This differential equation may also be written in the form

$$\alpha_1\alpha_3\alpha_5 \cdots \alpha_{2n-1} D^n(\alpha_2 D - 1)(\alpha_4 D - 1)(\alpha_6 D - 1) \cdots (\alpha_{2n} D - 1)f(z) = A_{2n} \frac{i}{z^{2n}} \cdot (5.10)$$

If we integrate (5.10) once with respect to z on both sides the only change on the left side is that D^n becomes D^{n-1} while the right hand side becomes $-A_{2n}i/(2n-1)z^{2n-1}+C_1$ in which C_1 is an integration constant. But since $|d^k f/dz^k|$ for $k=1, 2, \cdots, 2n$ tends to zero as $|z| \to \infty$ along any ray not parallel to the sides of the sector it follows that $C_1=0$. The same process can be repeated—in particular, the successive integration constants will be zero for the same reason—until we obtain[17]

$$(\alpha_2 D - 1)(\alpha_4 D - 1) \cdots (\alpha_{2n} D - 1)f = A \frac{i}{z^n}, \quad (5.11)$$

with A real (but otherwise arbitrary) since the product $\alpha_1\alpha_3\alpha_5 \cdots \alpha_{2n-1}$ is real. The nth integration would also yield an additional constant on the right hand side of (5.11) which would not necessarily be zero, but which would be pure imaginary. This follows from the boundary condition $\Re e(\alpha_{2n}D-1)f=0$ on the real axis, the fact that $(\alpha_2 D-1)(\alpha_4 D-1) \cdots (\alpha_{2n-2}D-1)=P_1''(D)$ has only real coefficients (as we have seen a little earlier), and the obvious fact that $\Re e Ai/z^n$ (A real) is zero for real z. However, such an imaginary constant on the right hand side of (5.11) would clearly contribute to the general solution $f(z)$ of (5.11) only an additive pure-imaginary constant which would contribute nothing to the real potential function φ. Consequently we take this constant to be zero.

Once we have the differential equation (5.11) the general procedure as well as the results are exactly analogous to those of the preceding Section 4: the arbitrary constants in the general solution of (5.11) can be fixed so that two types of solutions $f_1(z)$ and $f_2(z)$ are obtained which satisfy the boundary conditions[18] as well as the conditions at the origin and at ∞. The solutions $f_1(z)$ are obtained from the homogeneous equation, while the $f_2(z)$ are solutions of the non-homogeneous equation. The solutions $f_1(z)$ are regular at the origin, while the solutions $f_2(z)$ have a logarithmic singularity there. All solutions $f(z)$ of our problem are then given by $f(z)=f_1(z)+f_2(z)$ and this set of solutions depends only on two real constants which occur as factors in f_1 and f_2. At ∞ the behavior of f_1 and f_2 is such that the real potential functions $\varphi_1=\Re e f_1$ and $\varphi_2=\Re e f_2$ behave like $C_1 e^y \cos(x+\alpha_1)$ and $C_2 e^y \cos(x+\alpha_2)$ in which C_1 and

[17] The differential equation (5.11) was obtained by H. Lewy in a different way through reflecting in the bottom and then working in a sector of angle π/n instead of one of angle $\pi/2n$. At ∞ Lewy assumes at the outset that the solutions behave like those in water of infinite depth, in contrast to the above treatment in which only the boundedness of certain derivatives is assumed.

[18] That this can be done is far from trivial, since our boundary conditions must be satisfied at all points on the straight lines composing the boundary of the sector.

C_2 are arbitrary, but α_1 and α_2 are fixed and differ by $\pi/2$. It then follows that any solution φ of our problem is uniquely determined once the phase and amplitude of φ at ∞ are prescribed.

The general solution of (5.11) containing n arbitrary constants is of course obtained by straightforward and elementary methods. However, the determination of these constants in order to satisfy the boundary conditions is not entirely trivial, particularly in the case of the solution $f_2(z)$ of the non-homogeneous equation. In Appendix I we discuss the method of determining the integration constants in such a way as to satisfy the boundary conditions; in the present section we simply give the results. The solutions are also seen to satisfy the conditions at the origin and at ∞.

In the homogeneous case, the solution is

$$f_1(z) = \frac{\pi}{(n-1)!\sqrt{n}} \cdot \sum_{k=1}^{n} c_k e^{z\beta_k}, \qquad k = 1, 2, \cdots, n. \tag{5.12}$$

The numerical factor before the summation sign is chosen for convenience in later computations. The constants c_k and β_k are the following complex numbers:

$$\beta_k = e^{i\pi(k/n+1/2)} \tag{5.13}$$

$$\begin{aligned} c_k &= e^{i\pi[(n+1)/4-k/2]} \cot\frac{\pi}{2n} \cot\frac{2\pi}{2n} \cdots \cot\frac{(k-1)\pi}{2n} \quad \text{for} \quad k = 2, 3, \cdots, n, \\ c_1 &= \bar{c}_n. \end{aligned} \tag{5.14}$$

By comparison with (5.4) we note that $\beta_k = \bar{\alpha}_{2k}$. (The bar on $\bar{c}_n$ and $\bar{\alpha}_{2k}$ means that the complex conjugate of these quantities is taken.) The constants c_k are uniquely determined (see Appendix I) within a real multiplying factor.

As $|z| \to \infty$ in the sector, all terms clearly die out exponentially except the term for $k=n$, which is $c_n e^{-iz}$, since all β_k's except β_n have negative real parts. Even the term for $k=n$ dies out exponentially except along lines parallel to the real axis. (The value of c_n, by the way, is $e^{-i\pi(n-1)/4}$ since the cotangents in (5.14) cancel each other for $k=n$.) This term thus yields the asymptotic behavior of $f_1(z)$:

$$f_1(z) \sim \frac{\pi}{(n-1)!\sqrt{n}} \cdot c_n e^{-iz}. \tag{5.15}$$

The general solution $f_2(z)$ of the non-homogeneous equation (5.11) when the real constant A is set equal to one is as follows:

$$f_2(z) = \sum_{k=1}^{n} a_k \left[e^{z\beta_k} \int_{+\infty}^{z\beta_k} \frac{e^{-t}}{t}\, dt - \pi i e^{z\beta_k} \right]. \tag{5.16}$$

The β_k's are defined by (5.13); and the a_k's are defined by

$$a_k = c_k/(n-1)!\sqrt{n}, \tag{5.17}$$

that is, they are a fixed multiple (for given n) of the c_k's defined by (5.14). The a_k's, like the c_k's, are uniquely determined within a real multiplying factor. The path of integration for all integrals in (5.16) is indicated in Fig. 2. That the points $z\beta_k$ lie in the left half of the complex plane (as indicated) can be seen from our definition (5.13) of the β_k and the fact that z is restricted to the sector $0 \leqq \arg z \leqq \pi/2n$.

The behavior of $f_2(z)$ at ∞ of course depends on the behavior of the integrals in

(5.16). In Appendix II it is shown that these integrals behave asymptotically as follows:

$$\int_{+\infty}^{z\beta_k} \frac{e^{-t}dt}{t} \sim \begin{cases} -\dfrac{e^{-z\beta_k}}{\beta_k}\left(\dfrac{1}{z} + \cdots\right), & \dfrac{\pi}{2} < \arg z\beta_k \leqq \pi, \\ 2\pi i - \dfrac{e^{-z\beta_k}}{\beta_k}\left(\dfrac{1}{z} + \cdots\right), & \pi < \arg z\beta_k \leqq \dfrac{3\pi}{2}\cdot \end{cases} \tag{5.18}$$

Once this fact is established it is clear from (5.17) and (5.16) that $f_2(z)$ behaves asymptotically as follows:

$$f_2(z) \sim \frac{\pi}{(n-1)!\sqrt{n}} \cdot c_n i e^{-iz}, \tag{5.19}$$

since the term for $k=n$ dominates all others (cf. (5.18)) and $\arg z\beta_k > \pi$ in this case. Comparison of (5.19) with (5.15) shows that the real parts of $f_1(z)$ and $f_2(z)$ would be $90°$ out of phase at ∞.

That the derivatives of $f_2(z)$ behave asymptotically in the same fashion as $f_2(z)$ itself is easily seen, since the only terms in the derivatives of a type different from those in (5.16) are of the form b_k/z^k, k an integer $\geqq 1$. Finally, it is clear that $f_2(z)$ has a logarithmic singularity at the origin. Hence $f_1(z)$ and $f_2(z)$ satisfy all requirements. Just as in the $90°$ case (cf. the preceding section) it can now be readily seen that $f(z) = b_1 f_1(z) + b_2 f_2(z)$, with b_1 and b_2 any real constants, contains *all* standing wave solutions of our problem. The proof that the real potential function $\varphi = \Re e f(z)$ is uniquely determined once the phase and amplitude at ∞ are fixed can also be carried out exactly as in the previous section for the $90°$ case.

The relations (5.15) and (5.19) yield for the asymptotic behavior of the real potential functions φ_1 and φ_2 the relations:

$$\varphi_1(x, y) = \Re e f_1 \sim \frac{\pi}{(n-1)!\sqrt{n}} e^y \cos\left(x + \frac{n-1}{4}\pi\right) \tag{5.20}$$

$$\varphi_2(x, y) = \Re e f_2 \sim \frac{\pi}{(n-1)!\sqrt{n}} e^y \sin\left(x + \frac{n-1}{4}\pi\right) \tag{5.21}$$

when it is observed that $c_n = e^{-i\pi(n-1)/4}$. It is now possible to construct either standing wave or progressing wave solutions which behave at ∞ like the known solutions for steady progressing waves in water which is everywhere infinite in depth. In particular we observe that it makes sense to speak of the wave length at ∞ in our cases and that the relation between wave length and frequency satisfies asymptotically the relation which holds everywhere in water of infinite depth. For this, it is only necessary to reintroduce the original space variables by replacing x and y by mx and my, with $m = \sigma^2/g$ (cf. Sec. 3), and to take note of (5.20) and (5.21).

Finally, we write down a solution $\Phi(x, y; t)$ which behaves at ∞ like $e^y \cos(x+t+\alpha)$, i.e. like a steady progressing wave moving toward shore:

$$\Phi(x, y; t) = A\left[\varphi_1(x, y)\cos(t+\alpha) - \varphi_2(x, y)\sin(t+\alpha)\right]. \tag{5.22}$$

The solutions (5.22) will be discussed numerically in Sec. 8 for the case of a beach sloping at an angle of $6°$ (i.e. for the case $n = 15$).

6. Numerical discussion of standing wave solutions for 90°, 45°, and 6° slopes. In this section we give graphs of the two types of standing wave solutions for the case of a vertical cliff (cf. Sec. 4) and for bottom slopes of 45° and 6°. We continue to make use of the dimensionless variables of Sec. 3. In particular, it should be recalled that the variable x means the distance from shore divided by $\lambda/2\pi$, in which λ is the wave length at infinity. In other words the quantity x is proportional to the wave length at ∞.

In the case of the vertical cliff, or 90° case, two standing wave solutions are given by

$$\Phi_1(x, y; t) = \pi e^{it} e^{y} \cos x \tag{6.1}$$

and

$$\Phi_2(x, y; t) = e^{it} e^{y}\left[\cos x \int_{\infty}^{x} \frac{\cos \xi}{\xi} d\xi + \sin x \int_{\infty}^{x} \frac{\sin \xi}{\xi} d\xi + \pi \sin x\right], \tag{6.2}$$

As one can readily verify, either from (4.10) in Sec. 4 or from (5.12) in Sec. 5 with $n=1$.

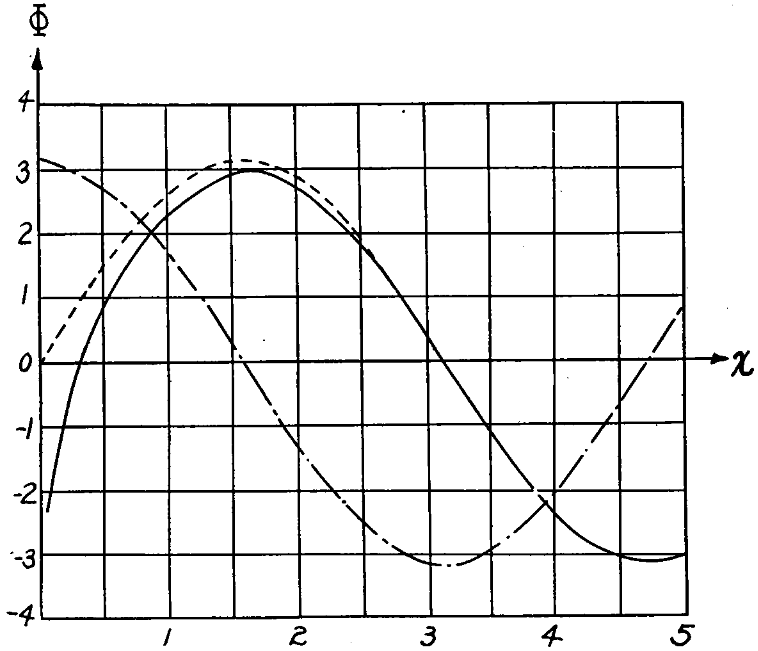

FIG. 5. Standing waves for a vertical cliff.

$\Phi_1(x, 0, 0)$ — - — -
$\Phi_2(x, 0, 0)$ ———
$\pi \sin x$ - - - - -

x = distance from shore/$(\lambda/2\pi)$
λ = wavelength at ∞

The functions Φ_1 and Φ_2 for $y=0$ and $t=0$ are plotted[19] in Fig. 5 together with the function $\pi \sin x$, which yields the asymptotic behavior of Φ_2 on the surface. The most

[19] The curves for Φ_1 and Φ_2 differ from the corresponding surface elevations η_1 and η_2 (cf. (1.4)) only by a phase shift and a constant multiplier.

notable feature of the curves of Fig. 5 is that the standing wave solution Φ_2, although it tends to become negative infinite as $x \to 0$, nevertheless differs little from the function $\pi \sin x$ until a point within a distance from the cliff of a fraction of the wave length at ∞ is reached. In other words, the asymptotic development of Φ_2 for x large holds with good accuracy until x is rather small. In addition, the last maximum of Φ_2 (i.e. the crest nearest to the cliff) has an amplitude slightly *less* by a little more than 10 per cent than the amplitude at ∞. The distance between the two zeros of Φ_2 nearest to shore is also 10 per cent less than half the wave length at ∞.

An interesting additional fact which is not difficult to prove is that the velocity of the water does not decrease exponentially with the depth y (as it does when no cliff is present), but only like $1/y$. The presence of the cliff thus reinforces the velocity. (The velocity of the water along the cliff is given by $\Re e(idf(z)/dz)$ with $f(z)$ defined by (4.15); the result indicated follows by calculating idf/dz and using the known asymptotic behavior of the complex integral which occurs.)

The standing wave solution Φ_1, Φ_2 for the case of a beach sloping at 45° are obtained from (5.12) and (5.16) with $n=2$. For Φ_1 we write

$$\Phi_1(x, y; t) = \frac{\pi}{\sqrt{2}} e^{it} \Re e\left[e^{i\pi/4} e^{-z} + e^{-i\pi/4} e^{-iz}\right]. \tag{6.3}$$

The unbounded standing wave solution is given by

$$\Phi_2(x, y; t) = \frac{e^{it}}{\sqrt{2}} \Re e\left\{e^{i\pi/4}\left[e^{-t}\int_\infty^z \frac{e^{-t}}{t}\,dt - \pi i e^{-z}\right] + e^{i\pi/4}\left[e^{-iz}\int_\infty^{-iz} \frac{e^{-t}}{t}\,dt - \pi i e^{-iz}\right]\right\}. \tag{6.4}$$

The surface values of Φ_1 and Φ_2 for $t=0$ are plotted in Fig. 6. These curves are obtained by using tables of the functions C_i, S_i, and E_i[20] computed by the Mathematical Tables Project [7]. In fact, Φ_2 can be written in the form

$$\Phi_2(x, y; t) = \frac{e^{it}e^{y}}{\sqrt{2}}\left\{C_i(x)[\sin x - \cos x] - \left[\frac{\pi}{2} + S_i(x)\right][\cos x + \sin x] - e^{-x}E_i(x)\right\}. \tag{6.4'}$$

At ∞, $\Phi_1(x, 0; 0)$ behaves like $(\pi/\sqrt{2})(\cos x - \sin x)$ while $\Phi_2(x, 0; 0)$ behaves like $-(\pi/\sqrt{2})(\cos x + \sin x)$.

The same general comments can be made about the curves of Fig. 6 as were made for those of Fig. 5. In particular, the minimum of Φ_2 at $x=1$ is about 10 per cent less in numerical value than the amplitude of Φ_2 at ∞, but Φ_2 differs very little from its asymptotic representation until x is quite small. The regular solution Φ_1 attains a maximum on shore which is $\sqrt{2}$ times the amplitude at ∞. The distance between successive zeros shortens on approaching shore, as in the preceding case, but the shortening is now more pronounced.

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[20] We use i unconventionally as a subscript to avoid confusion with $i=\sqrt{-1}$; in M.T.P. [7] the notation Ci, Si, and Ei is used.

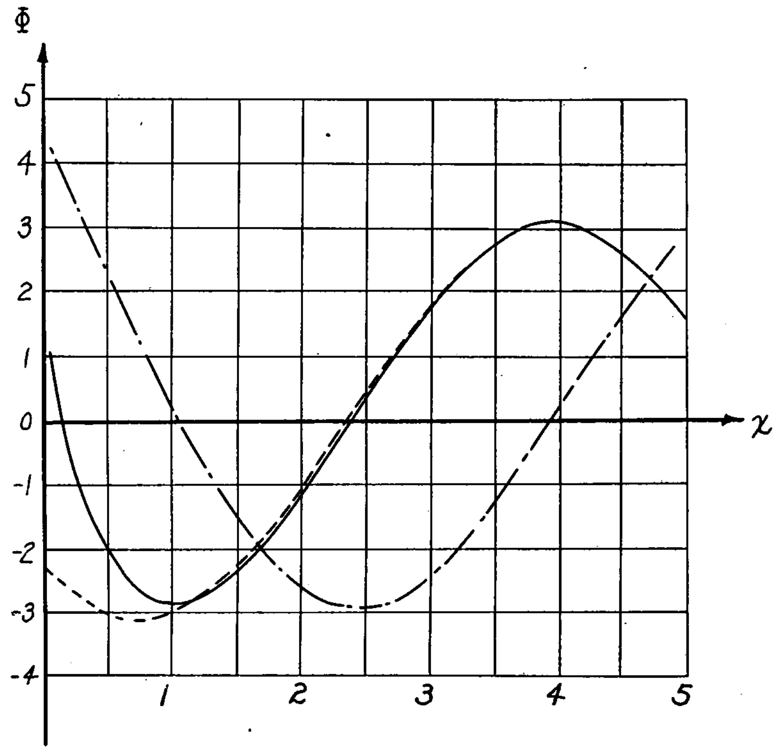

FIG. 6. Standing waves for a 45° bottom slope.

$\Phi_1(x, 0, 0)$ — - — - x = distance from shore/$(\lambda/2\pi)$
$\Phi_2(x, 0, 0)$ ——— λ = wavelength at ∞
$-\pi/\sqrt{2}(\cos x+\sin x)$ - - - -

Finally, we describe the two types of standing wave solutions for the case of a bottom slope of 6°. These solutions are obtained from (5.12) and (5.16) by taking $n=15$. The regular standing wave Φ_1 is taken in the form (cf. (5.12)):

$$\Phi_1(x, y; t) = e^{it}\Re e f_1(z) \tag{6.5}$$

with $f_1(z)$ defined by

$$\begin{aligned} f_1(z) &= \sum_{k=1}^{15} c_k e^{z\beta_k}, \\ \beta_k &= e^{i\pi(k/15+1/2)}, \\ c_k &= e^{-i\pi k/2} \cot\frac{\pi}{30} \cdots \cot\frac{(k-1)\pi}{30}, \quad k \neq 1, \\ c_1 &= -i, \end{aligned} \tag{6.6}$$

The fifteen quantities c_k are alternately real and pure imaginary and they vary in absolute value from 1 to approximately 819. For large values of x the function $\Phi_1(x, 0; 0)$ behaves as follows (cf. (5.20)):

$$\Phi_1(x, 0; 0) \sim \pi \sin x. \tag{6.7}$$

The standing wave solution Φ_2, which is infinite at the shore line, is defined by

$$\Phi_2(x, y; t) = -14!\sqrt{15}\, e^{it}\Re e f_2(z) \tag{6.8}$$

with $f_2(z)$ defined by

$$f_2(z) = \frac{1}{14!\sqrt{15}} \sum_{k=1}^{15} c_k \left\{ e^{z\beta_k} \int_{+\infty}^{z\beta_k} \frac{e^{-t}}{t}\, dt - \pi i e^{z\beta_k} \right\} \tag{6.9}$$

in which the c_k and β_k are defined as in (6.6). The path of integration is shown in Fig. 2 of a preceding section. This solution behaves for large x and for $y=0$, $t=0$ as follows (cf. (5.21)):

$$\Phi_2(x, 0; 0) \sim \pi \cos x. \tag{6.10}$$

In order to determine Φ_2 numerically the function

$$E(z) = e^z \int_z^\infty \frac{e^{-t}}{t}\, dt$$

was computed for values of z on the rays $z = re^{i\pi(k/15+1/2)}$, $k = 1, 2, \cdots, 7$.

The function $E(z)$ defined by (6.11) has been tabulated for values of z in the second quadrant by the Mathematical Tables Project [8]. However, the entries in this table

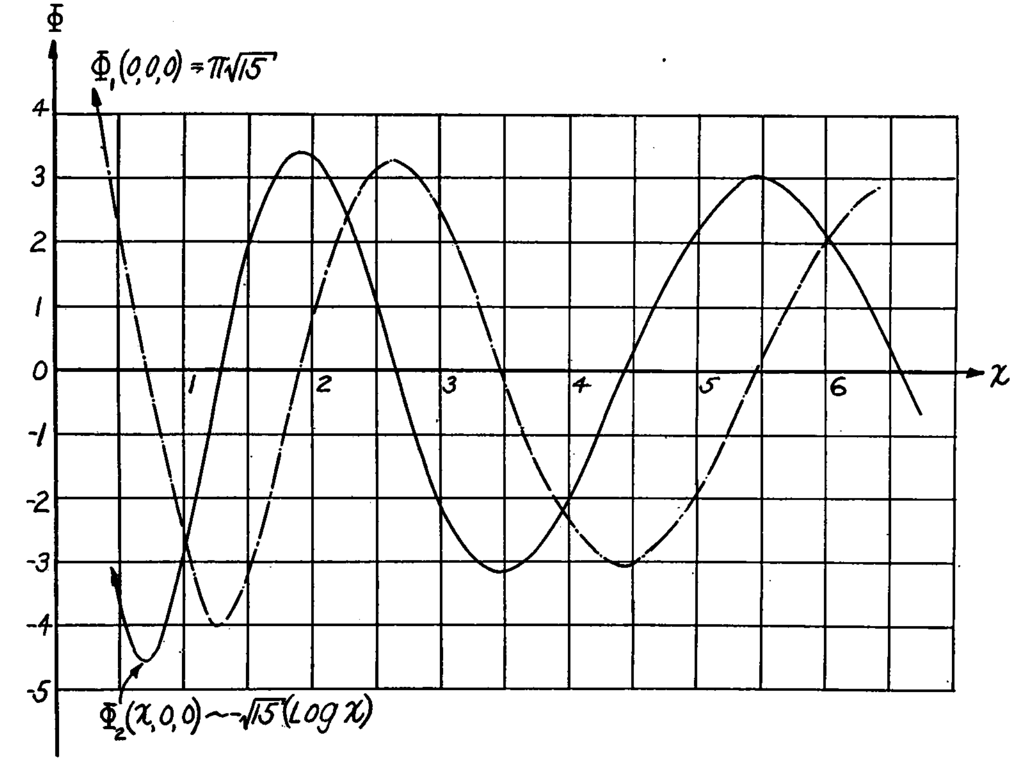

FIG. 7. Standing waves for a 6° bottom slope.

$\Phi_1(x, 0, 0)$ — - — - $\qquad$ x = distance from shore/$(\lambda/2\pi)$

$\Phi_2(x, 0, 0)$ ——— $\qquad$ λ = wavelength at ∞

are for a rather wide rectangular net of points in the z-plane. We therefore found it necessary to compute $E(z)$ along the above described rays as follows: For $r=|z|=3.00$ and $r=3.50$, we interpolated in the M.T.P. tables of $E(z)$ by means of a Taylor series expansion about the nearest tabulated points. (The derivatives of $E(z)$ are easily calculated.) The values at $r=3.00$ and 3.50 were checked by using them to compute the values at $r=3.25$. Power series developments were thus obtained for points along each of the seven rays for the range $1.00 \leqq r \leqq 7.00$ at an interval of 0.25, which means that more than 100 power series were derived. A table was then constructed by using these series to obtain values at intermediate points. Since integrals of this form would always occur in solving differential equations having constant coefficients and rational right hand sides, it seems worth while to include this table as an Appendix. The table is believed to be accurate within one unit in the fourth decimal place. The table would, of course, also be useful if computations were to be made for beach slopes at angles other than 6°.

Fig. 7 shows the surface values of Φ_1 and Φ_2 near shore, while Figs. 8 and 9 are graphs of Φ_1 and Φ_2 together with their asymptotic representation for large x. Again the same general comments are in order as in the preceding cases, except that it is now necessary to go further out from shore to obtain close agreement with the asymptotic solution in deep water. This is not very surprising since the depth of the

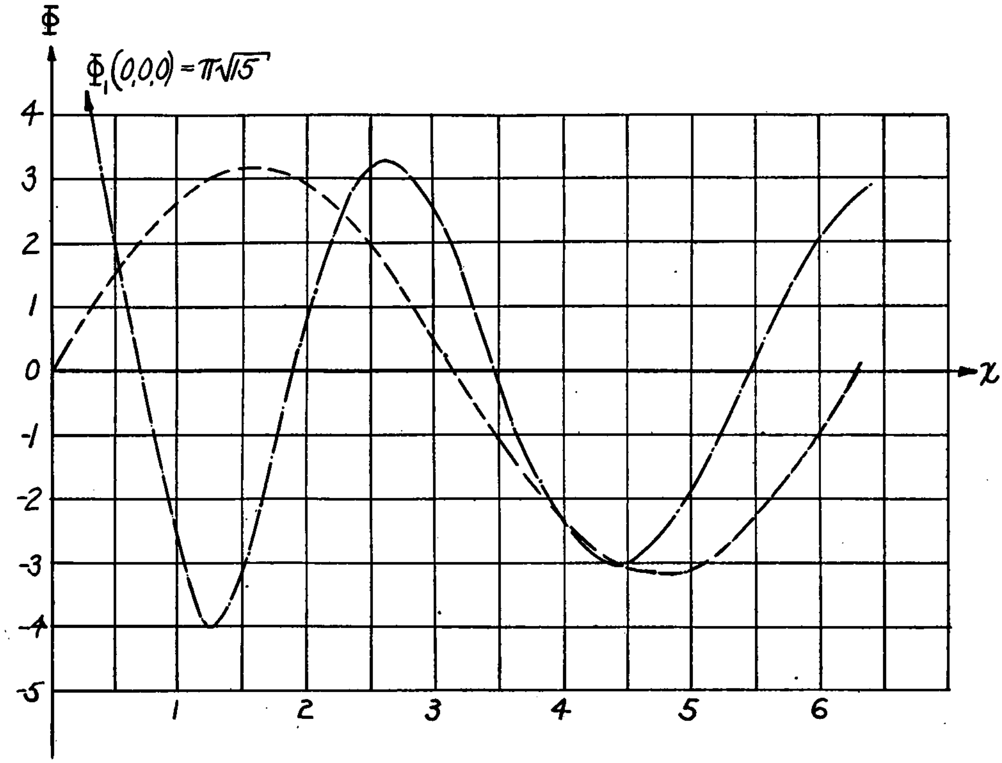

FIG. 8. Comparison of standing wave solution for a 6° slope, Φ_1, with its asymptotic limit $\pi \sin x$ (see Fig. 7).

$\Phi_1(x, 0, 0)$ —·—· $\quad x$ = distance from shore/$(\lambda/2\pi)$

$\pi \sin(x)$ - - - - $\quad \lambda$ = wavelength at ∞

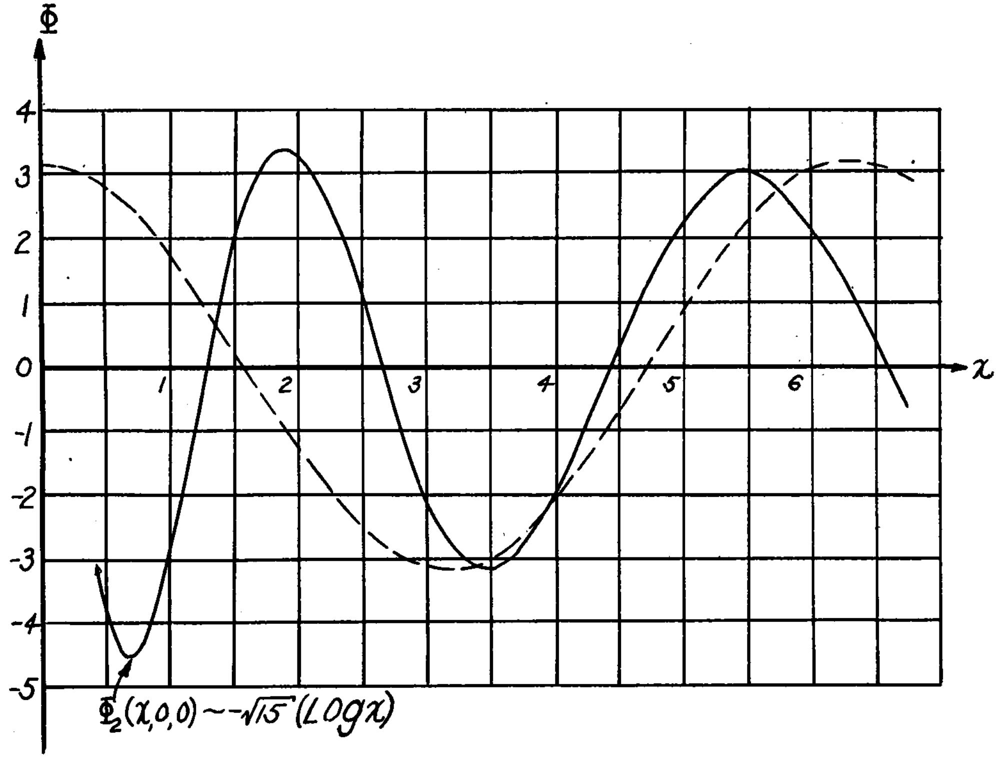

FIG. 9. Comparison of standing wave solution for a 6° slope, Φ_2, with its asymptotic limit $\pi \cos x$ (see Fig. 7).

$\Phi_2(x, 0, 0)$ ——— x = distance from shore/$(\lambda/2\pi)$

$\pi \cos (x)$ - - - - λ = wavelength at ∞

water at $x=7$ (the largest x for which we have numerical values) because of the small slope is less than 1/8 of the wave length at ∞. Comparison of Fig. 7 with Figs. 5 and 6 shows that the distance between successive zeros near shore has now shortened very materially as compared with the preceding cases. In fact, this effect would become more and more pronounced with decrease in the slope.[21] We observe also that the relative maximum of Φ_2 at $x=5.4$ and the relative maximum of Φ_1 at $x=6.5$ are less than the amplitude at ∞. Progressing waves for the case of a 6° slope will be discussed in Sec. 8.

7. Shallow water theory. In some gravity wave problems it is possible to obtain an accurate approximation to the exact linear theory by relatively simple means. Such an approximate theory, which is commonly referred to as the linear shallow water theory, has long been the basis for the theoretical study of the tides in the oceans and large estuaries. In this section we give a derivation of the shallow water theory for water of variable depth; in the next section we shall compare the results from both theories numerically in the case of progressing waves in water of constant depth and over a beach with a 6° slope. In what follows we consider only the two-dimensional case. We revert also to the original space and time variables.

[21] See the next section, where an approximate theory valid for water in which the depth is small compared with the wave length is discussed.

The exact linear theory requires the determination of a potential function $\Phi(x, y; t)$ in the region indicated in Fig. 10). As before, $y=0$ is the original undisturbed surface of the water and $y=-h(x)$ is the bottom profile. The elevation of the

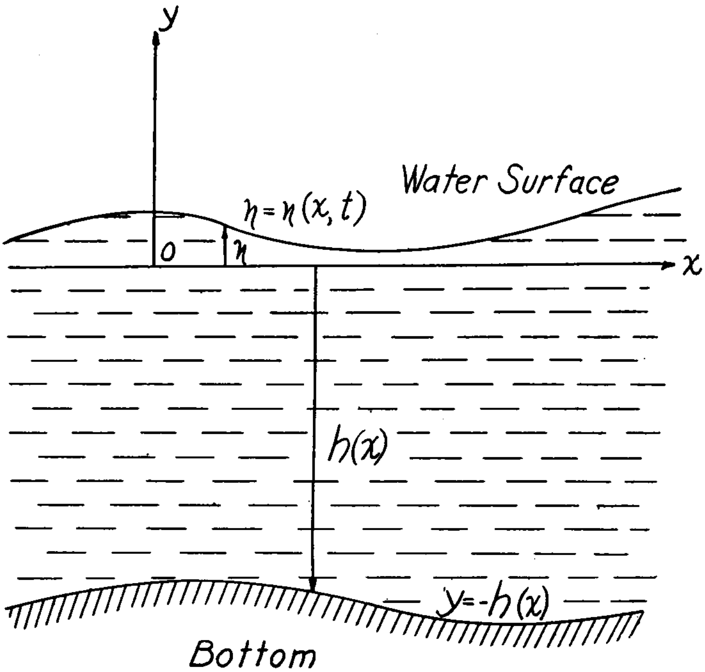

FIG. 10.

free surface in the course of the motion is denoted by $\eta(x, t)$. The conditions to be satisfied by Φ are (cf. Sec. 1).

$$\Phi_{xx} + \Phi_{yy} = 0 \qquad \text{for} \quad 0 \geqq y \geqq -h(x) \tag{7.1}$$

$$\Phi_{tt} = -g\Phi_y \qquad \text{for} \quad y = 0 \tag{7.2}$$

$$\Phi_y = -h_x\Phi_x \qquad \text{for} \quad y = -h(x). \tag{7.3}$$

For the purposes of the present section it is not necessary to formulate conditions at ∞. The surface elevation η is given by

$$\eta(x, t) = \frac{1}{g}\cdot\Phi_t\Big|_{y=0}. \tag{7.4}$$

In the following derivation it is convenient to denote quantities evaluated for $y=0$ by a bar over the quantity, and for $y=-h(x)$ (i.e. at the bottom) by a bar under the quantity. Thus conditions (7.2) and (7.3) could be written in the form $\overline{\Phi}_{tt}=-g\overline{\Phi}_y$ and $\underline{\Phi}_y=-h_x\underline{\Phi}_x$.

We may write

$$\int_{-h(x)}^{0} \Phi_{yy}dy = \overline{\Phi}_y - \underline{\Phi}_y = \overline{\Phi}_y + h_x\underline{\Phi}_x \tag{7.5}$$

from (7.3). On the other hand we may also write

$$\int_{-h(x)}^{0} \Phi_{yy} dy = -\int_{-h(x)}^{0} \Phi_{xx} dy = -\frac{\partial}{\partial x}\int_{-h(x)}^{0} \Phi_x dy + h_x \underline{\Phi}_x \tag{7.6}$$

by using (7.1) and observing that the lower limit $h(x)$ depends on x. By equating the right hand sides of (7.5) and (7.6) we find

$$\bar{\Phi}_y = -\frac{\partial}{\partial x}\int_{-h(x)}^{0} \Phi_x dy. \tag{7.7}$$

We consider next the relation

$$\int_{-h(x)}^{0} \Phi_x dy = h\underline{\Phi}_x - \int_{-h}^{0} y\Phi_{xy} dy \tag{7.8}$$

obtained through integration by parts, and also the relation

$$\int_{-h(x)}^{0} h(x)\Phi_{xy} dy = h(x)\bar{\Phi}_x - h(x)\underline{\Phi}_x. \tag{7.9}$$

The quantity $\underline{\Phi}_x$ in (7.8) can be eliminated by using (7.9), and this in turn lead-through use of (7.7) to

$$\begin{aligned}\bar{\Phi}_y &= -\frac{\partial}{\partial x}\left[h(x)\bar{\Phi}_x - \int_{-h(x)}^{0} (y+h)\Phi_{xy} dy\right] \\ &= -(h\bar{\Phi}_x)_x + \int_{-h}^{0} [(y+h)\Phi_{xxy} + h_x \Phi_{xy}] dy.\end{aligned}$$

From the boundary condition (7.2) we may replace $\bar{\Phi}_y$ by $-1/g\bar{\Phi}_{tt}$ to obtain

$$\bar{\Phi}_{tt} = (gh\bar{\Phi}_x)_x - g\int_{-h}^{0} [(y+h)\Phi_{xxy} + h_x \Phi_{xy}] dy. \tag{7.10}$$

Up to now we have made no assumptions in addition to those made in deriving the exact linear theory—we have simply carried out integrations with respect to y in such a way as to yield (7.10). We now assume that *Φ_{xxy} and Φ_{xy} are bounded for all x and t, and for $-h \leqq y \leqq 0$, and that h_x is small of the same order of magnitude as h.* It follows at once that (7.10) may be written in the form

$$\bar{\Phi}_{tt} = (gh\bar{\Phi}_x)_x + O(h^2) \tag{7.11}$$

in which $O(h^2)$ is a quantity of order h^2. Thus, if the depth h and slope h_x are sufficiently small, the surface value $\bar{\Phi}$ of the potential function $\Phi(x, y; t)$ should be given with good approximation by the differential equation

$$\bar{\Phi}_{tt} = (gh\bar{\Phi}_x)_x. \tag{7.12}$$

In the case $h=$const. (7.12) is the wave equation in one space variable with $c=(gh)^{1/2}$ as propagation speed. Equation (7.12) is the differential equation of the linear shallow water theory.[22] The above derivation of the theory is due to F. John.

Instead of $\bar{\Phi}$ we may introduce the surface elevation $\eta(x, t) = \bar{\Phi}_t/g$ (cf. (7.4)) as dependent variable; this leads easily to the equation

$$\eta_{tt} = (gh\eta_x)_x. \tag{7.13}$$

It is possible to derive the shallow water theory in such a way as to obtain in place of (7.10) a relation in which the integral on the right hand side is replaced by a power series in h which converges if the initial conditions (i.e. conditions for $t=0$) are assumed to be sufficiently smooth. That is, it would not be necessary in this version of the theory to assume at the outset that Φ_{xxy} and Φ_{xy} are bounded for all time.

We proceed to derive standing wave solutions of (7.12) for the case in which

$$h(x) = qx, \qquad q = \text{const.}, \tag{7.14}$$

i.e. for the case of a uniformly sloping bottom profile.[23] For this purpose it is convenient to set

$$\bar{\Phi}(x, t) = e^{i(\sigma t + \epsilon)} Z(x). \tag{7.15}$$

The function $Z(x)$ satisfies

$$(xZ_x)_x + \frac{m}{g} Z = 0, \qquad m = \frac{\sigma^2}{g}, \tag{7.16}$$

in which the quantity m has the same definition in terms of σ as in the exact theory of the preceding sections (cf. Sec. 3).

The differential equation (7.16) is a Bessel equation with the general solution

$$Z(x) = AJ_0\left(2\sqrt{\frac{mx}{q}}\right) + BY_0\left(2\sqrt{\frac{mx}{q}}\right). \tag{7.17}$$

J_0 and Y_0 are the regular and the singular Bessel functions of order zero respectively; thus Y_0 has a logarithmic singularity at $x=0$. For large values of x these functions are well known to behave as follows:

$$J_0\left(2\sqrt{\frac{mx}{q}}\right) \sim \frac{\cos\left(2\sqrt{\frac{mx}{q}} - \frac{\pi}{4}\right)}{\sqrt{\pi\sqrt{\frac{mx}{q}}}}$$

[22] The derivation often given for this theory (cf. for example [4] p. 254) starts with the assumption that the pressure p is determined by the same relation as in hydrostatics, i.e. $p = g(\eta - y)$. One would be inclined to think that such a relation would be on the whole more accurate the deeper the water since all motions die out in the depth, but it is easily seen on the basis of the simplest examples that the approximate theory cannot be accurate in sufficiently deep water: The exact solutions for steady periodic waves in water of constant depth yield waves in which the wave length depends essentially on the frequency, but the steady waves given by (7.12) in the case $h=$const. are without dispersion. In other words, the derivation of the approximate theory by means of the hydrostatic assumption is open to criticism since it does not indicate clearly the essential role played by the depth in determining the accuracy of the approximation. Cisotti has given another derivation of the shallow water theory (cf. [10] pp. 357, 379) which does not start with the hydrostatic assumption.

[23] This theory is, of course, well known. See, for example [4], p. 291.

$$Y_0\left(2\sqrt{\frac{mx}{q}}\right) \sim \frac{\sin\left(2\sqrt{\frac{mx}{q}} - \frac{\pi}{4}\right)}{\sqrt{\pi\sqrt{\frac{mx}{q}}}}. \tag{7.18}$$

With this theory it is therefore not possible to obtain either standing waves or progressing waves with non-vanishing amplitudes at ∞, in sharp contrast to solutions given by the exact theory. Also, for large values of x the wave length (defined as the distance between successive nodes, say) as given by the approximate theory would be roughly $2\pi(gx/m)^{1/2}$ and would therefore increase indefinitely with x. This is also in marked contrast with our exact solutions, in which the wave length at ∞ tends to a constant.

Nevertheless, it is possible to reintroduce the time factor and obtain for $\eta(x, t)$ solutions which have the form of progressing waves traveling toward shore, as follows:

$$\bar{\Phi}(x, t) = A\left[\cos(\sigma t - \epsilon)Y_0(2\sqrt{mx/q}) + \sin(\sigma t - \epsilon)J_0(2\sqrt{mx/q})\right]. \tag{7.19}$$

One could expect that such a solution might furnish in some cases at least a fair approximation to the exact solution for a not too large range of values of x. In particular, we note that the singularity at the origin (i.e. on the shore line) is of the same type as in the exact theory. In the next section a numerical comparison with the exact theory will be made.

8. Comparison of exact and shallow water theories. In the present section we compare the results obtained using the shallow water theory derived in the preceding section with those of the exact theory. We consider progressing wave solutions first for the case of water of uniform but finite depth and then for the case of a bottom slope of 6°. In this section it is preferable to use the original independent variables rather than the dimensionless variables of Sec. 3.

In the case of water of uniform depth h the velocity potential $\Phi(x, y; t)$ is given by (cf. [4], pp. 363–368)

$$\Phi(x, y; t) = A\frac{\cosh m(y + h)}{\cosh mh}\sin(mx + \sigma t), \tag{8.1}$$

in which h represents the constant depth of the water. The undisturbed surface of the water is the line $y=0$. This potential function evidently satisfies the condition $\partial\Phi/\partial y = 0$ at the bottom $y = -h$. The free surface condition (1.2) is also satisfied if

$$\sigma^2 = mg \tanh mh, \tag{8.2}$$

or as we may also write

$$\sigma^2/gh = m^2\left(1 - \frac{m^2h^2}{3} + \cdots\right). \tag{8.3}$$

If we introduce the phase velocity $c = \sigma/m$ we may write

$$c = \sqrt{gh\left(1 - \frac{m^2h^2}{3} + \cdots\right)} \tag{8.4}$$

or

$$c = \sqrt{gh\left[1 - \frac{1}{3}\left(\frac{2\pi h}{\lambda}\right)^2 + \cdots\right]} \tag{8.5}$$

if the wave length $\lambda = 2\pi/m$ is used. The relations (8.2) to (8.5) characterize the type of dispersion encountered in this case.[24] If the ratio of wave length λ to depth h is large, the relation (8.5) for the phase velocity c becomes

$$c \cong \sqrt{gh}. \tag{8.6}$$

In fact, if $\lambda/h = 5$, $c = .82(gh)^{1/2}$; while if $\lambda/h = 10$, $c = .94(gh)^{1/2}$.

In the same case of constant depth the shallow water theory gives the following approximation $\bar{\Phi}(x, t)$ to the surface value $\Phi(x, 0; t)$:

$$\bar{\Phi} = A \sin(mx + \sigma t). \tag{8.7}$$

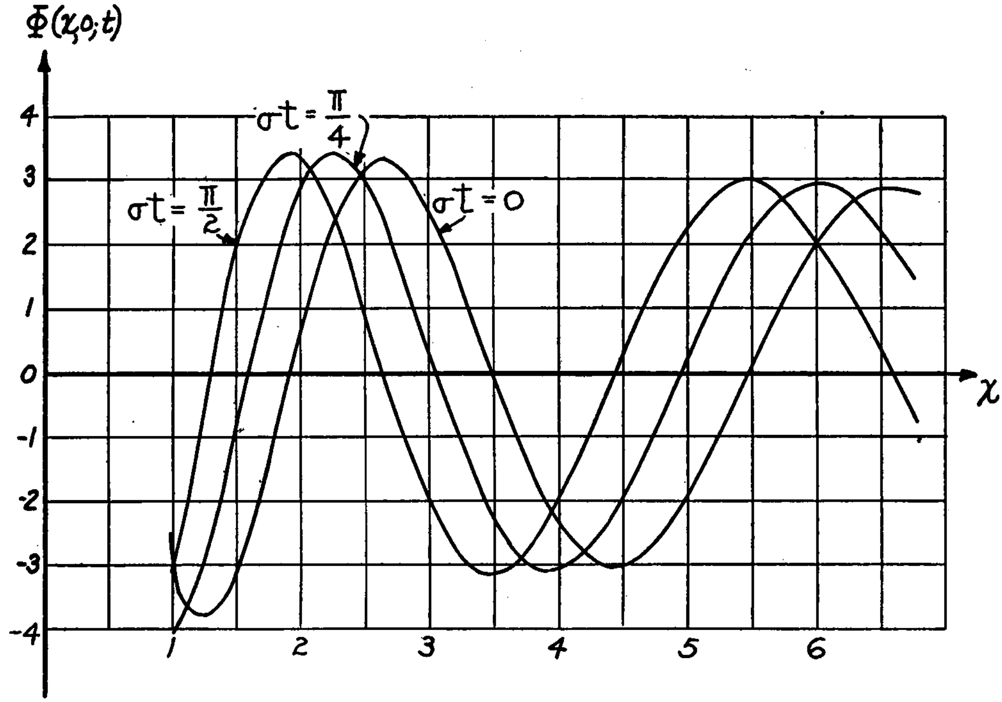

FIG. 11. Progressing waves for a 6° bottom slope. (Exact theory.)
λ = wavelength at ∞
x = distance from shore/$(\lambda/2\pi)$

This is a solution of (7.12) only if

$$\sigma^2/gh = m^2, \tag{8.8}$$

i.e. if the phase velocity $c = \sigma/m$ is given by

$$c = \sqrt{gh}. \tag{8.9}$$

[24] This theory appears to be due to Airy [1], who published it in 1845.

In other words, the approximate solution (8.7) furnished by the shallow water theory yields the phase velocity with good accuracy only if the ratio of the wave length to the depth is sufficiently great, as we see by comparing (8.9) with (8.5). In fact, (8.9) yields c correct, within about 5 per cent, only if the wave length is more than ten times the depth. For steady waves, therefore, the approximate theory is accurate only if the water is shallow compared with the wave length. For this reason the approximate theory is sometimes referred to as the long wave-shallow water theory.

We turn next to a comparison of the results of the two theories for progressing waves over a beach sloping at 6°. The progressing wave solution of the exact theory is given by (5.22), which behaves at ∞ like $\pi \sin (mx+\sigma t)$. Graphs of the numerical solutions for times $\sigma t=0$, $\pi/4$, and $\pi/2$ are shown in Fig. 11. (Again we note that the dimensionless variables of Section 3 are *not* used.) We observe that the "amplitude" of the progressing wave increases as the wave moves from the point $2\pi x/\lambda=6$

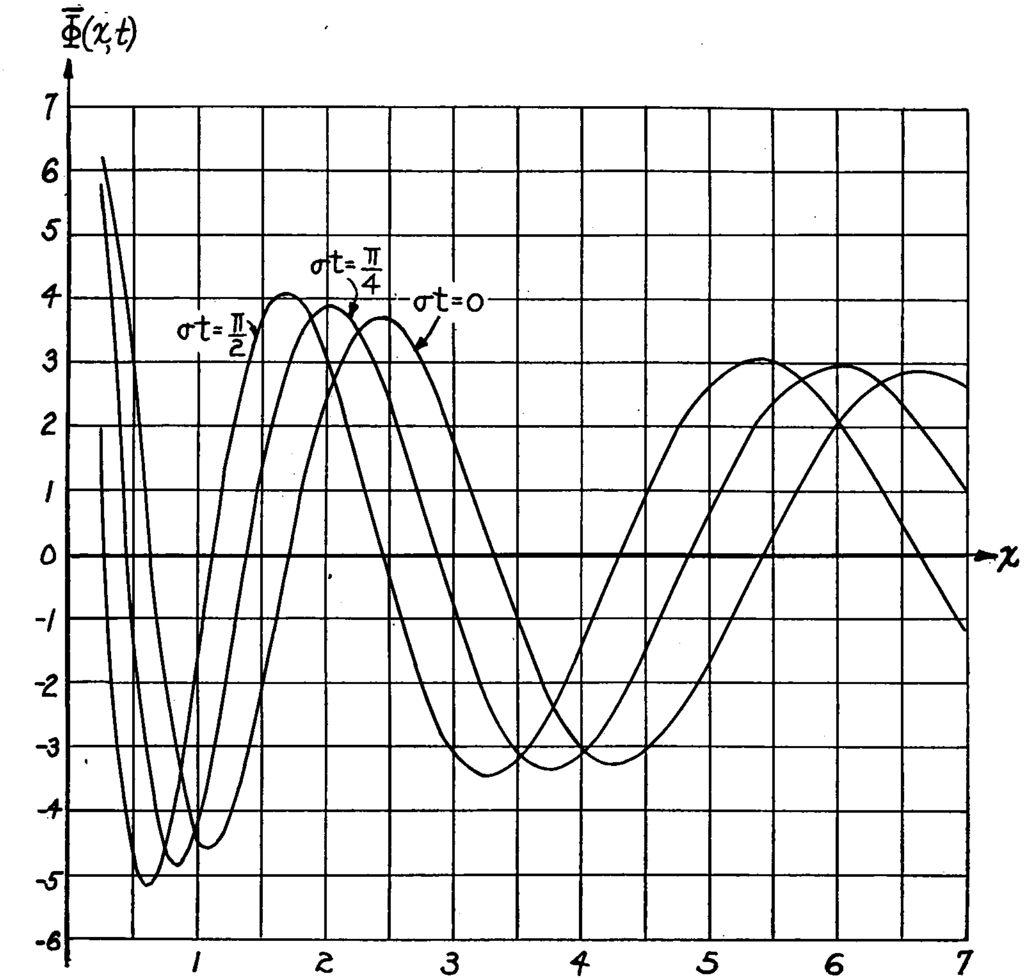

FIG. 12. Progressive waves for a 6° bottom slope (shallow water theory).
λ = wavelength at ∞ in exact linear case
x = distance from shore/($\lambda/2\pi$)

toward shore. However, the maximum at this point is 6 per cent less[25] than the amplitude at ∞.

In the preceding section it was already explained that the shallow water theory could not, in principle, furnish a good approximation to the progressing waves in the present case over the whole range of values of x since the amplitude tends to zero at ∞ in this theory. We have therefore chosen to make a comparison by assuming that the circular frequency σ is the same for both theories and that both solutions have the same relative maximum for $\sigma t=\pi/4$ and $2\pi x/\lambda=6$. This yields for the approximate solution $\bar{\Phi}(x, t)$ the following (cf. (7.19)):

$$\Phi(x, t) = -14.5\left[\cos(\sigma t - .98)Y_0\left(2\sqrt{\frac{mx}{q}}\right) + \sin(\sigma t - .98)J_0\left(2\sqrt{\frac{mx}{q}}\right)\right]. \tag{8.10}$$

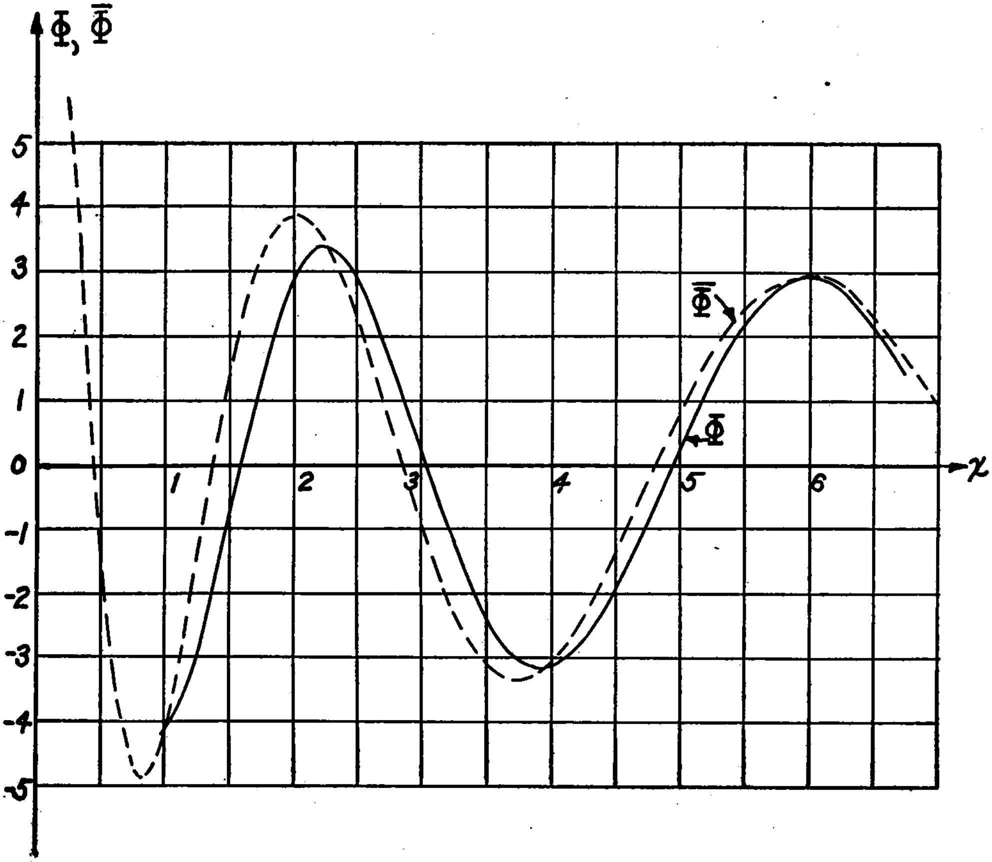

FIG. 13. Comparison of surface values from both theories.

Φ = Exact linear theory ———— λ = wavelength at ∞ (exact theory)

$\bar{\Phi}$ = Shallow water theory — — — — x = distance from shore/$(\lambda/2\pi)$

Phase and amplitude are the same at ($x=6$, $\sigma t=\pi/4$)

Frequencies are equal

[25] The author is told that this effect has been observed experimentally.

Fig. 12 is a graph of $\bar{\Phi}$ for $\sigma t=0$, $\pi/4$, $\pi/2$. The curves have the same general appearance as those of Fig. 11, but the amplitude increases somewhat more rapidly in the present case. This is brought out more directly by Fig. 13, which gives graphs of Φ and $\bar{\Phi}$ for $\sigma t=\pi/4$. The two theories agree fairly well as the shore is approached, although the amplitude given by the approximate theory at $2\pi x/\lambda \sim 2$ is about 15 per cent greater than that furnished by the exact theory. We have at our disposal the information necessary to describe how the continuation of the curves of Fig. 13 would appear for values of $2\pi x/\lambda$ greater than 7, since both curves are given with good accuracy in this range by their respective asymptotic representations. Thus Φ will be very closely the same as $\pi \sin(2\pi x/\lambda+\pi/4)$—that is, the exact solution would be one having an almost constant amplitude equal to π—while the amplitude of $\bar{\Phi}$ decreases like $1/(2\pi x/\lambda)^{-1/4}$ (cf. (7.19)). If, therefore, we were to compare the amplitudes at $2\pi x/\lambda=24$, the amplitude of $\bar{\Phi}$ would be about 40 per cent less than that of Φ, while at $2\pi x/\lambda=12$ the error would be about 20 per cent. On the other hand, it should be stated that at $2\pi x/\lambda=7$ the depth of the water for the 6° slope is somewhat less than 1/8 of the wave length. Thus the shallow water theory might be expected to yield fairly accurate results from this point in toward shore, but it could not be expected to do so for $2\pi x/\lambda=12$, much less for $2\pi x/\lambda=24$, on the basis of our discussion above for the case of water of uniform depth.

It is of some interest to compare the phase velocity c furnished by the two theories for $2\pi x/\lambda<7$. This was done by calculating the position of the zeros and the maxima and minima for a series of closely-spaced values of t; the velocities were then obtained by taking difference quotients. The results of such calculations for $2\pi x/\lambda \leqq 7$ are shown in Fig. 14. The asymptotic value for c as furnished by the exact theory is indicated. Up to a distance of about a wave length from shore the two theories yield the same phase velocity, but from this point on the phase velocity predicted by the shallow water theory is too high and becomes more and more inaccurate as the distance from shore (and therefore also the depth of the water) increases. At a distance of about 3 wave lengths from shore the phase velocity as given by the shallow water theory is in error by 10 per cent.

We remark, finally, that the curves of Fig. 14 can not be distinguished (to the scale used there) from the curves

$$c=\sqrt{gh}, \quad \text{and} \quad c=\sqrt{\frac{g\lambda}{2\pi}\tanh\frac{2\pi h}{\lambda}} \tag{8.11}$$

in which h is the depth corresponding to any given x and λ is the wave length at ∞. In other words, the phase velocities actually computed by using the two theories for $2\pi x/\lambda \leqq 7$ are practically identical in each of the two theories with those which would be obtained by calculating c for each value of x through use of the corresponding theory applied to water of the correct *uniform* depth.

9. A problem in three-dimensional wave motion. Except for the present section, we consider in this paper only motions which are the same in all planes parallel to an $x-y$-plane, and which therefore can be treated by using functions of a single complex variable. (The exact linear theory is, of course, in question here.) For motions which depend essentially upon three space variables it is not possible to make use of complex functions, but it is possible to extend the basic idea of the method used in the two-dimensional cases to surface wave problems in three dimensions. In this section we illustrate the method by treating the problem of progressing waves in an infinite ocean bounded on one side by a vertical cliff—in other words, the same problem as that of Sec. 4 except that we no longer require the waves to move with their crests parallel to the shore line.

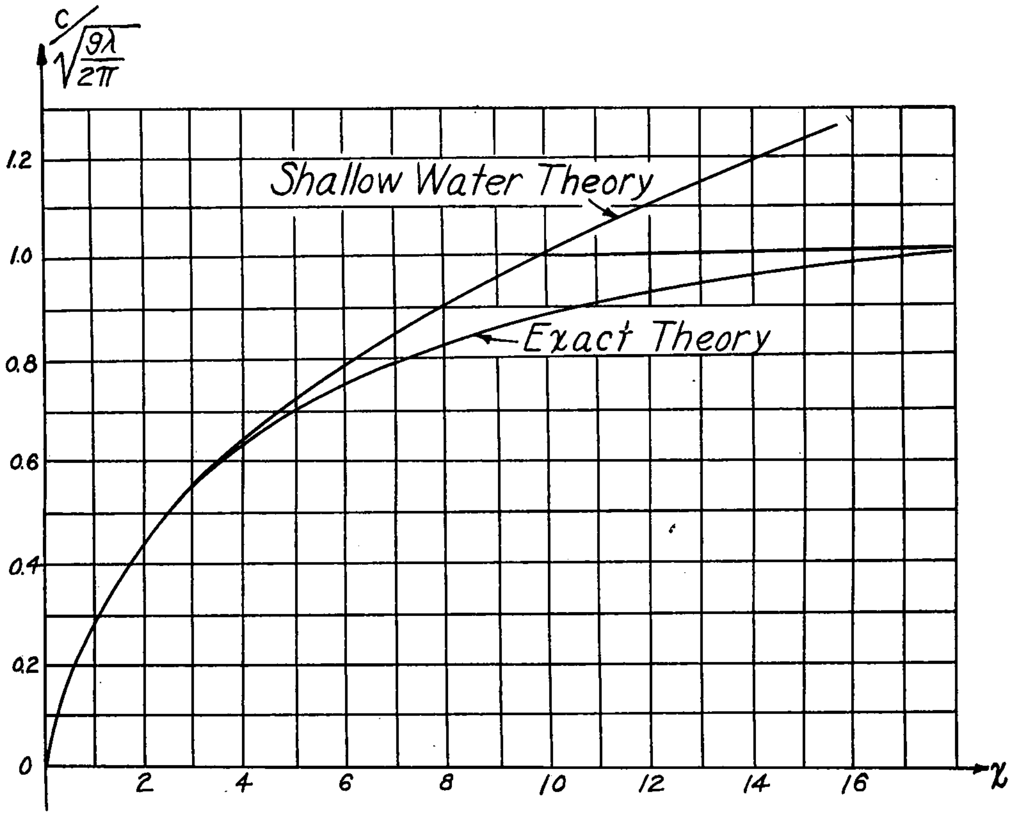

FIG. 14. Phase velocity for progressing waves over a 6° sloping beach
x = distance from shore/$(\lambda/2\pi)$
λ = wavelength at ∞ (exact theory)
c = phase velocity.

We seek solutions $\Phi(x, y, z; t)$ of $\nabla^2_{(x,y,z)}\Phi = 0$ in the region $x \geqq 0$, $y \leqq 0$, $-\infty < z < \infty$ with the y-axis taken normal to the undisturbed free surface of the water and the z-axis[26] taken along the "shore," i.e. at the water line on the vertical cliff $x = 0$. Progressing waves moving toward shore are to be found such that the wave crests (or other curves of constant phase) at large distances from shore tend to a straight line which makes an arbitrary angle with the shore line. For this purpose we seek solutions of the form

$$\Phi(x, y, z; t) = e^{i(t + kz + \beta_1)}\varphi(x, y) \tag{9.1}$$

[26] It has already been pointed out that functions of a complex variable are not used in this section, so that the reintroduction of the letter z to represent a space coordinate should cause no confusion with the use of z as a complex variable in earlier sections.

that is, solutions in which periodic factors in both z and t are split off. Of course, the function $\varphi(x, y)$ is not, as in our previous cases, a potential function; instead, it satisfies the differential equation

$$\nabla^2_{(x,y)}\varphi - k^2\varphi = 0, \tag{9.2}$$

as one readily sees. The free surface condition is taken in the form

$$\frac{\partial\varphi}{\partial y} - \varphi = 0 \quad \text{for} \quad y = 0. \tag{9.3}$$

which implies that the dimensionless space and time variables of Sec. 3 (including now z as well as x and y) are assumed at the outset. The condition at the cliff, is, of course,

$$\frac{\partial\varphi}{\partial x} = 0 \quad \text{for} \quad x = 0. \tag{9.4}$$

At the origin $x=0$, $y=0$ (i.e. at the shore line on the cliff) we require, as in former cases, that φ should be of the form

$$\varphi = \bar{\varphi}\log r + \bar{\bar{\varphi}}, \qquad r \ll 1, \tag{9.5}$$

for sufficiently small values of $r=(x^2+y^2)^{1/2}$, with $\bar{\varphi}$ and $\bar{\bar{\varphi}}$ certain bounded functions with bounded first and second derivatives in a neighborhood of the origin. The functions $\bar{\varphi}$ and $\bar{\bar{\varphi}}$ should be considered at present as certain given functions; later on, they will be chosen specifically.

For large values of r we wish to have $\Phi(x, y, z, t)$ behave like $e^{y}e^{i(t+kz+\alpha x+\beta)}$ with $k^2+\alpha^2=1$ but k and α otherwise arbitrary constants, so that progressing waves tending to an arbitrary plane wave at ∞ can be obtained. This requires that $\varphi(x, y)$ should behave at ∞ like $e^{y}e^{i(\alpha x+\beta_2)}$ because of (9.1). However, it is no more necessary here than it was in our former cases to require that φ should behave in this specific way at ∞; it suffices in fact to require that

$$|\varphi| + |\varphi_x| + |\varphi_{xy}| < M \quad \text{for} \quad r > R_0, \tag{9.6}$$

i.e. that φ and the two derivatives of φ occurring in (9.6) should be uniformly bounded at ∞. As we shall see, this requirement leads to solutions of the desired type.

We proceed to solve the boundary value problem formulated in equations (9.2) to (9.6). The procedure we follow is analogous to that used in the former cases in every respect. To begin with, we observe that

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} - 1\right)\varphi = 0 \quad \text{for both } x = 0 \quad \text{and} \quad y = 0, \tag{9.7}$$

because of the special form of the linear operator on the left hand side together with the fact that (9.3) and (9.4) are to be satisfied. A function $\psi(x, y)$ is introduced by the relation

$$\psi = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} - 1\right)\varphi. \tag{9.8}$$

The essential point of our method is that the function ψ is determined uniquely within an arbitrary factor if our function φ, having the properties postulated, exists. Furthermore, ψ can then be given explicitly without difficulty. The properties of ψ are as follows.

1) ψ satisfies the same differential equation as φ, i.e. equation (9.2), as one sees from the definition (9.8) of ψ.
2) ψ is regular in the quadrant $x>0$, $y<0$ and vanishes, in view of (9.7), on $x=0$, $y<0$ and $y=0$, $x>0$. Hence ψ can be continued over the boundaries by the reflection to yield a continuous and single valued function having continuous second derivatives ψ_{xx} and ψ_{yy} (as one can readily see since $\nabla^2\psi-k^2\psi=0$, and $\psi=0$ on the boundaries) in the entire x, y-plane with the exception of the origin. (Here we use the fact that our domain is a sector of angle $\pi/2$.)
3) At the origin, ψ has a possible singularity which is of the form $\bar{\psi}(x, y)/r^2$, with $\bar{\psi}$ regular, as one can see from (9.5) and (9.8). This statement clearly holds for the function ψ when it has been extended by reflection to a full neighborhood of the origin.
4) The condition (9.6) on φ clearly yields for ψ the condition that ψ is uniformly bounded at ∞ after ψ has been extended to the whole plane.

Thus ψ is a solution of $\nabla^2\psi-k^2\psi=0$ in the entire plane which is uniformly bounded at ∞. At the origin $\psi=\bar{\psi}/r^2$ with $\bar{\psi}$ a certain given regular function ($\bar{\psi}=0$ not excluded). In addition, $\psi=0$ on the entire x and y axes. It is not difficult to prove that the solution of this problem is unique.[27]

On the other hand, a solution ψ of the problem for a special function[28] $\bar{\psi}$ characterizing the singularity at the origin is readily given in polar coordinates (r, θ); it is

$$\psi(x, y) = AiH_2^{(1)}(ikr)\sin 2\theta, \qquad r=\sqrt{x^2+y^2}, \qquad 0 \leqq k \leqq 1, \tag{9.9}$$

in which $H_2^{(1)}$ is the Hankel function of order two which tends to zero as $r\to\infty$, and A is any real constant. The function ψ has real values when r is real. (The notation given in Jahnke-Emde, Tables of Functions, is used.) One can readily verify that this function really does satisfy all conditions imposed on ψ. For our purposes it is of advantage to write the solution ψ in the following form:

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[27] One way to do so is the following: The difference, Ψ, of two solutions would have all of the properties of ψ except that it would be regular at the origin. If Ψ_0 is the value of Ψ at any point (x_0, y_0) in the plane, then it is known (see, for example, Courant, Hilbert, Methoden d. Math. Phys., Bd. II, S.261) that the mean value formula

$$\Psi_0\cdot J_0(ikR) = M$$

holds. Here M is the mean value of Ψ over any circle of radius R and center at (x_0, y_0). The function J_0 is the regular Bessel function of order zero. If R is chosen large enough M remains less than a certain constant since Ψ is uniformly bounded at ∞. On the other hand, $J_0(ikR)$ behaves for large R like $e^{kR}(2\pi kR)^{-1/2}$ (see Jahnke-Emde, Tables of Functions, p. 138) and hence as $R\to\infty$, Ψ_0 would tend to zero. But since Ψ_0 is independent of R it follows that Ψ_0 is zero at any arbitrary point (x_0, y_0). Thus $\Psi\equiv 0$, and the uniqueness of the function ψ is proved.

[28] Our uniqueness theorem is less general in the present case than in the earlier cases since we prescribe the singularity at the origin so specifically in the present case.

$$\psi = A i \frac{\partial^2}{\partial x \partial y} H_0^{(1)}(ikr), \qquad r = \sqrt{x^2 + y^2}, \tag{9.10}$$

in which A is any real constant and $H_0^{(1)}$ is the Hankel function of order zero which is bounded as $r \to \infty$. It is readily verified that this solution differs from that given by (9.9) only by a constant multiplier; one can do so, for example, by using the well-known identities involving the derivatives of Bessel functions of different orders.

Once ψ is determined we may write (9.8) in the form

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} - 1\right)\varphi = A i \frac{\partial^2}{\partial x \partial y} H_0^{(1)}(ikr), \qquad A \text{ arbitrary.} \tag{9.11}$$

This means that our function φ, if it exists, must satisfy (9.11) as well as (9.2). By integration of (9.11) it turns out that we are able to determine φ explicitly without great difficulty on account of the simple form of the left hand side of (9.11).[29] This we proceed to do.

Integration of both sides of (9.11) with respect to x leads to

$$\left(\frac{\partial}{\partial y} - 1\right)\varphi = A i \frac{\partial}{\partial y} H_0^{(1)}(ikr) + g(y), \tag{9.12}$$

in which $g(y)$ is an arbitrary function. But $g(y)$ must satisfy (9.2), since all other terms in (9.12) satisfy it. Hence $d^2g/dy^2 - k^2 g = 0$. In addition $g(0) = 0$, since the other terms in (9.12) vanish for $y = 0$ because of (9.3) and the fact that $\partial/\partial y H_0^{(1)} = ik(y/r)dH_0^{(1)}/dr$. Finally, $g(y)$ is bounded as $y \to -\infty$ because of condition (9.6) and the fact that $\partial H_0^{(1)}/\partial y$ tends to zero as $r \to \infty$. The function $g(y)$ is therefore readily seen to be identically zero. By integration of (9.12) we obtain (after setting $g(y) = 0$):

$$\varphi = A i e^y \int_{+\infty}^{y} e^{-t} \frac{\partial}{\partial t} \left[H_0^{(1)}(ik\sqrt{x^2 + t^2})\right] dt + B(x) e^y. \tag{9.13}$$

The function $B(x)$ and the real constant A are arbitrary. The integral converges, since $\partial/\partial t(H_0^{(1)})$ dies out exponentially as $t \to \infty$.

We shall see that two solutions $\varphi_1(x, y)$ and $\varphi_2(x, y)$ satisfying all conditions of our problem can be obtained from (9.13) by taking $A = 0$ in one case and $A \neq 0$ in the other case, and that these solutions will be 90° "out of phase" at ∞. (This is exactly analogous to the behavior of the solutions in our previous cases.)

Consider first the case $A = 0$. The function φ given by (9.13) satisfies (9.2) only if

$$\frac{d^2 B(x)}{dx^2} + (1 - k^2) B(x) = 0. \tag{9.14}$$

[29] It can now be seen how the differential equation corresponding to (9.11) could be obtained for the case of waves coming in toward shore over a uniformly sloping beach at inclination $\pi/2n$ to the horizontal: The left hand side would be a differential operator on ϕ of order $2n$, which could be obtained from (5.9) by going over to real operators. The right hand side would be essentially the Hankel function $H_{2n}^{(1)}(ikr)$ of order $2n$ multiplied by $\sin 2n\theta$. In fact, the entire discussion of sec. 5) for the angles $\pi/2n$ could be generalized to yield three-dimensional progressing wave solutions by proceeding in the manner of the present section.

It is important to recall that $k^2<1$. The boundary condition $\varphi_x=0$ for $x=0$ requires that $B_x(0)=0$. The condition $\varphi_y-\varphi=0$ for $y=0$ is automatically satisfied because of (9.12) and $g(y)\equiv 0$. Hence $B(x)=A_1\cos\sqrt{1-k^2}x$, with A_1 arbitrary, and the solution $\varphi_1(x, y)$ is

$$\varphi_1(x, y) = A_1 e^y \cos\sqrt{1-k^2}\, x. \tag{9.15}$$

This leads to solutions Φ_1 in the form of standing waves,[30] as follows:

$$\Phi_1(x, y, z, t) = A_1 e^{it} e^y \cos\sqrt{1-k^2}\, x. \begin{Bmatrix}\cos kz\\ \sin kz\end{Bmatrix} \tag{9.15'}$$

for $k^2<1$. If $k=1$, the solution Φ_1 given by (9.15′) is valid, although it could not be obtained by our process since $B(x)$ would be identically zero then.

As we have already stated, we obtain solutions $\varphi_2(x, y)$ from (9.13) for $A\neq 0$ which behave for large x like $\sin\sqrt{1-k^2}x$ rather than like $\cos\sqrt{1-k^2}x$, and with these two types of solutions progressing waves approaching an arbitrary plane wave at ∞ can be constructed by superposition.

We begin by showing that (9.2) is satisfied for all $x>0$, $y<0$ by φ as given in (9.13) with $A\neq 0$, provided only that $B(x)$ satisfies (9.14). Since $x>0$, it is permissible to differentiate under the integral sign in (9.13), even though t takes on the value zero (since the upper limit y is negative). By differentiating we obtain

$$\begin{aligned}\nabla^2\varphi - k^2\varphi = Ai\Big\{e^y\int_\infty^y e^{-t}\frac{\partial}{\partial t}\Big[\frac{\partial^2}{\partial x^2}+(1-k^2)\Big]H_0^{(1)}dt \\ +\frac{\partial H_0^{(1)}}{\partial y}+\frac{\partial^2 H_0^{(1)}}{\partial y^2}\Big\} + B''(x)+(1-k^2)B.\end{aligned} \tag{9.16}$$

Since $H_0^{(1)}$ is a solution of (9.2) the operator $(\partial^2/\partial x^2-k^2)$ occurring under the integral sign can be replaced by $-\partial^2/\partial y^2$ and hence the integral can be written in the form

$$\int_\infty^y e^{-t}\Big[-\frac{\partial^3}{\partial t^3}+\frac{\partial}{\partial t}\Big]H_0^{(1)}(ikr)dt.$$

We introduce the following notation

$$I_m(x, y) = e^y\int_\infty^y e^{-t}\frac{\partial^m}{\partial t^m}H_0^{(1)}(ikr)dt,$$

and obtain through two integrations by parts the relation

$$I_m(x, y) = \Big[\frac{\partial^{m-1}}{\partial y^{m-1}}+\frac{\partial^{m-2}}{\partial y^{m-2}}\Big]\cdot H_0^{(1)} + e^y\int_{+\infty}^y e^{-t}\frac{\partial^{m-2}H_0^{(1)}}{\partial t^{m-2}}dt,$$

in which we have made use of the fact that the boundary terms are zero at the lower limit $+\infty$, since all derivatives of $H_0^{(1)}(ikr)$ tend to zero as $r\to+\infty$. The integral of interest to us is given obviously by I_1-I_3 and this in turn is given by

[30] The standing wave solutions of this type (but not of the type with a singularity) for beaches sloping at angles $\pi/2n$ were obtained by Hanson [3] by a quite different method.

$$-I_3 + I_1 = -\frac{\partial^2 H_0^{(1)}}{\partial y^2} - \frac{\partial H_0^{(1)}}{\partial y} - e^y \int_\infty^y e^{-t} \frac{\partial H_0^{(1)}}{\partial t} dt$$
$$+ e^y \int_\infty^y e^{-t} \frac{\partial H_0^{(1)}}{\partial t} dt = -\frac{\partial^2 H_0^{(1)}}{\partial y^2} - \frac{\partial H_0^{(1)}}{\partial y}$$

by use of the above relations for I_m. Hence the quantity in brackets in (9.16) is identically zero—in other words the term containing the integral on the right hand side of (9.13) is a solution of (9.2). Hence φ is a solution of (9.2) in the case $A \neq 0$ if $B(x)$ satisfies (9.14). Since (9.12) holds and $g(y) \equiv 0$ it follows that the free surface condition (9.3) is satisfied by φ in view of the fact that $\partial H_0^{(1)}(ikr)/\partial y = 0$ for $y = 0$.

We have still to show that a solution $B(x)$ of (9.14) can be chosen so that $\varphi_x = 0$ for $x = 0$, and that φ has the desired behavior for large values of r. Actually, these two things go hand in hand, just as in former cases. An integration by parts in (9.13) yields the following for φ:

$$\varphi = Aie^y \int_\infty^y e^{-t} H_0^{(1)}(ik\sqrt{x^2 + t^2})dt + AiH_0^{(1)}(ik\sqrt{x^2 + y^2}) + B(x)e^y, \tag{9.17}$$

provided that $x > 0$. It should be recalled that the upper limit y of the integral is negative; thus the integrand has a singularity for $x = 0$ since $t = 0$ is included in the interval of integration and $H_0^{(1)}(ikr)$ is singular for $r = 0$. We shall show that $\lim_{x \to 1} \partial\psi/\partial x = 0$ provided that $B_x(0) = -2A \neq 0$. We have, for $x > 0$ and $y < 0$:

$$\frac{\partial \varphi}{\partial x} = Aie^y \int_\infty^y e^{-t} \frac{\partial}{\partial x}[H_0^{(1)}(ik\sqrt{x^2 + t^2})]dt + Ai\frac{\partial}{\partial x}[H_0^{(1)}(ik\sqrt{x^2 + y^2})] + B_x(x)e^y.$$

The second term on the right hand side is readily seen to approach zero as $x \to 0$ since this term can be written as the product of x and a factor which is bounded for $y < 0$. For the same reason it is clear that the only contribution furnished by the integral in the limit as $x \to 0$ arises from a neighborhood of $t = 0$ since the factor x may be taken outside of the integral sign. We therefore consider the limit

$$\lim_{x \to 0} \int_\epsilon^{-\epsilon} e^{-t} \frac{\partial}{\partial x}[iH_0^{(1)}(ik\sqrt{x^2 + t^2})dt, \qquad \epsilon > 0.$$

The function $iH_0^{(1)}(ikr)$ has the following development valid near $r = 0$:

$$iH_0^{(1)}(ikr) = -\frac{2}{\pi}[J_0(ikr) \log r + p(r)]$$

in which $p(r)$ represents a convergent power series containing only even powers of r (including a zero power), and J_0 is the regular Bessel function with the following development

$$J_0(ikr) = 1 + \frac{(kr)^2}{2^2} + \cdots.$$

It follows that

$$\frac{\partial}{\partial x}\left[iH_0^{(1)}(ikr)\right] = -\frac{2}{\pi}\left[\frac{x}{r^2}J_0(ikr) + J_0'(ikr)\frac{x}{r}\log r + xg(r)\right]$$
$$= -\frac{2}{\pi}\left[\frac{x}{r^2}J_0(ikr) + \tfrac{1}{2}kx\log r + xg(r)\right]$$

in which $g(r)=(1/r)dp/dr$ is bounded as $x\to 0$ since $y<0$. The contribution of our integral in the limit is therefore easily seen to be given by

$$\lim_{x\to 0} -\frac{2}{\pi}\int_{\epsilon}^{-\epsilon} e^{-t}\cdot\frac{x}{x^2+t^2}dt = \lim_{x\to 0} -\frac{2}{\pi}\int_{\epsilon}^{-\epsilon}\frac{x}{x^2+t^2}dt.$$

By introducing $u=t/x$ as new integration variable and passing to the limit we may write

$$\lim_{x\to 0} -\frac{2}{\pi}\int_{\epsilon}^{-\epsilon}\frac{x}{x^2+t^2}dt = -\frac{2}{\pi}\int_{\infty}^{-\infty}\frac{du}{1+u^2} = 2.$$

It therefore follows that $\lim_{x\to 0}\partial\varphi/\partial x=0$ provided that

$$B_x(0) = -2A. \tag{9.18}$$

The function $B(x)$ which satisfies this condition and the differential equation (9.14) is

$$B(x) = -\frac{2A}{\sqrt{1-k^2}}\sin\sqrt{1-k^2}\,x. \tag{9.19}$$

Since $H_0^{(1)}(ikr)$ dies out exponentially as $r\to\infty$ it follows that the solution φ given by (9.17) with $B(x)$ defined by (9.19) behaves at ∞ like $e^y\sin\left[(1-k^2)^{1/2}x\right]$.

A solution φ_2 of our problem which is out of phase with φ_1 (cf. (9.15)) is therefore given by

$$\varphi_2(x,y) = A_2\left[ie^y\int_{\infty}^{y}e^{-t}H_0^{(1)}(ik\sqrt{x^2+t^2})dt + iH_0^{(1)}(ik\sqrt{x^2+y^2}) - \frac{2e^y}{\sqrt{1-k^2}}\sin\sqrt{1-k^2}\,x\right], \tag{9.20}$$

with A_2 an arbitrary real constant. A standing wave solution Φ_2 is then given by

$$\Phi_2 = A_2e^{it}\varphi_2(x,y)\cdot\begin{Bmatrix}\cos kz\\ \sin kz\end{Bmatrix}. \tag{9.20'}$$

By taking appropriate values of k progressing waves tending at ∞ to any arbitrary plane wave solution for water of infinite depth can be obtained by forming proper linear combinations of solutions of the type (9.15′) and (9.20′). For a progressing wave traveling toward shore, for example, we might write

$$\begin{aligned}\Phi(x,y,z;t) &= A\left[\varphi_1(x,y)\cos kz + \frac{\sqrt{1-k^2}}{2}\varphi_2(x,y)\sin kz\right]\cos t\\ &\quad - A\left[\varphi_1(x,y)\sin kz - \frac{\sqrt{1-k^2}}{2}\varphi_2(x,y)\cos kz\right]\sin t\end{aligned} \tag{9.21}$$

in which A_1 and A_2 in (9.15) and (9.20) are both taken equal to A. The solution (9.21) behaves at ∞ like $Ae^y \cos(\sqrt{1-k^2}x+kz+t)$ as one can readily verify by making use of the asymptotic behavior of $\varphi_1(x, y)$ and $\varphi_2(x, y)$.[31]

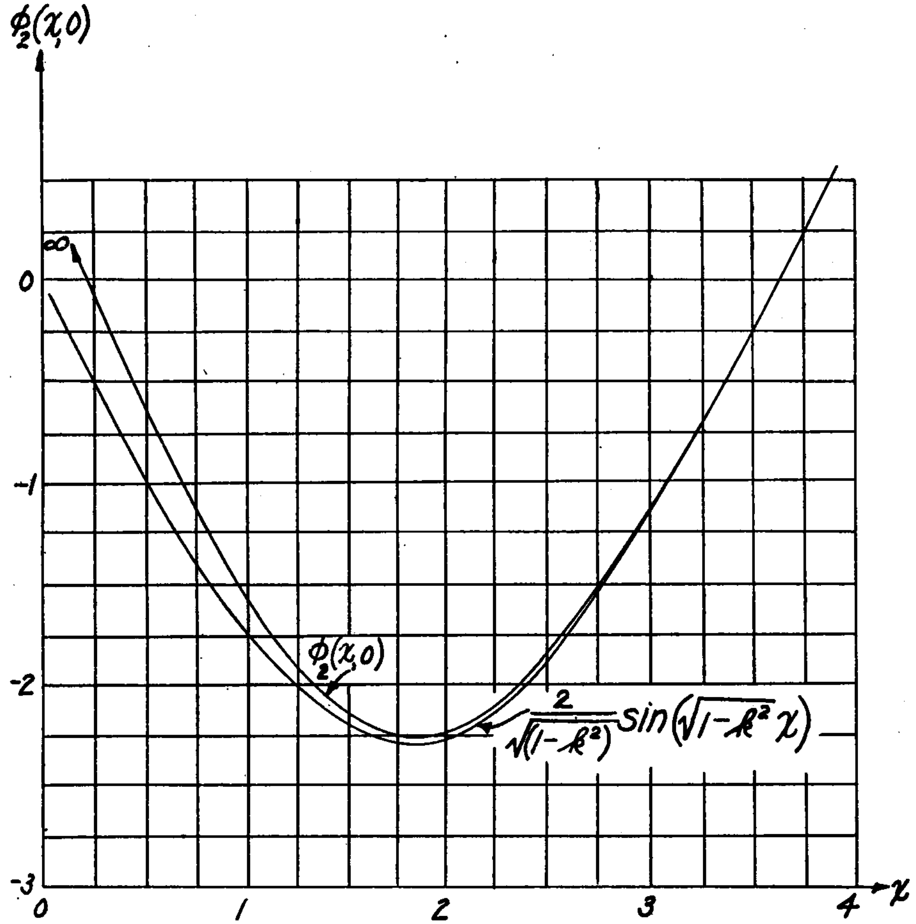

FIG. 15. Standing wave solution for a vertical cliff (with crests at an angle of 30° to shore).

The special case $k=1$ has a certain interest. It corresponds to waves which at ∞ have their crests at right angles to shore. One readily sees from (9.15) and (9.20) that as $k\to 1$ the progressing wave solution (9.21) tends to

$$\Phi(y, z; t) = Ae^y \cos(z + t) \tag{9.22}$$

that is, the progressing wave solution for this case is independent of x, is free of a singularity at the origin, and the curves of constant phase are straight lines at right angles to the shore line.

The progressing wave solution (9.21) has been discussed numerically for $k=1/2$, i.e. for the case in which the wave crests tend at ∞ to a straight line inclined at 30° to the shore line. The function $\varphi_2(x, 0)$ is plotted in Fig. 15. With the aid of these val-

[31] We remark once more that the original space and time variables can be reintroduced simply by replacing x, y, z by mx, my, mz and t by σt (cf. Sec. 3).

ues the contours for $\Phi(x, 0, z; 0)$, i.e. for the surface of the water at time $t=0$, were calculated and are given in Fig. 16. The water surface is shown between a pair of successive "nodes" of Φ, that is, curves for which $\Phi=0$. These curves go into the z-axis (the shore line) under zero angle, as do all other contour lines. This is seen at once from their equation (cf. (9.21) with $t=0$).

$$\varphi_1(x, 0) \cos kz + \frac{\sqrt{1-k^2}}{2} \varphi_2(x, 0) \sin kz = \eta = \text{const.} \tag{9.23}$$

Since $\varphi_2 \to \infty$ as $x \to 0$ while φ_1 remains bounded, it is clear that $\sin kz$ must approach

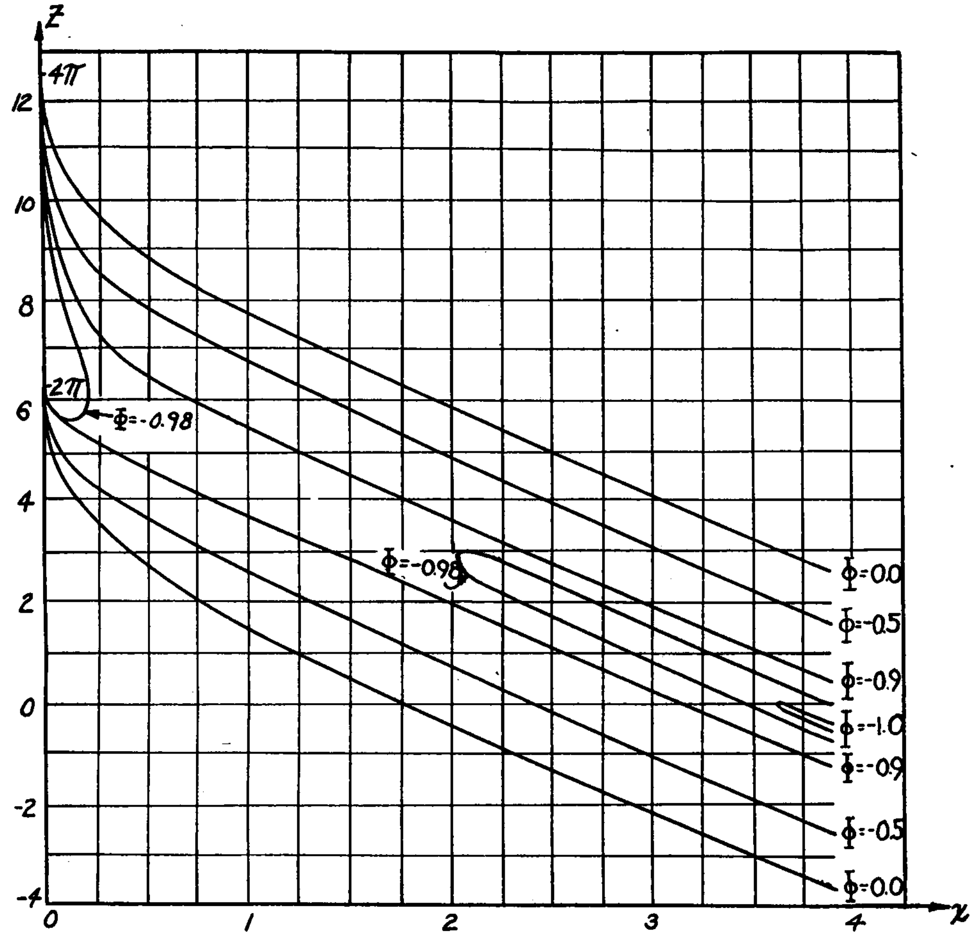

FIG. 16. Level lines for a wave approaching a vertical cliff at an angle.

zero as $x \to 0$ on any such curve. That the contours are all tangent to the z-axis at the points $z=2\pi n$, n an integer, is also readily seen. It is interesting to observe that the height of the wave crest is lower at some points near to the cliff than it is at ∞ (where the value is minus one), so that a saddle point occurs. We observe also that a contour for $\eta=-1$ occurs at the right hand edge of Fig. 16. It is believed that the numerical calculations are sufficiently accurate to guarantee the existence of such a contour for

$\eta = -1$; if so, then this contour is likely to be a closed curve, since $\eta \to 1$ at ∞. It may be that the wave crest is a ridge with a number of saddle points.[32]

Appendix I. Solution of the complex differential equation in the general case

In this appendix we give some of the details of the methods used to obtain explicit solutions of the differential equation (5.11) for the analytic function $f(z)$ which satisfies the boundary conditions (5.1) and (5.2). The differential equation (5.11) is

$$\prod_{k=1}^{n} (e^{-i(k/n+1/2)\pi}D - 1)\cdot f = A \frac{i}{z^n}, \qquad A \text{ real.} \tag{I.1}$$

The symbol $\prod$ means, as usual, a continued product, and $D = d/dz$. The boundary conditions are

$$\Re e(iD - 1)f = 0 \quad \text{for} \quad z \text{ real, and} \tag{I.2}$$

$$\Re e(ie^{-i\pi/2n}D)f = 0 \quad \text{on} \quad z = re^{-i\pi/2n}, \tag{I.3}$$

in conformity with (5.1) and (5.2) for a beach sloping at the angle $\pi/2n$.

In order to obtain all phases at ∞ it is sufficient to obtain solutions $f_1(z)$ and $f_2(z)$ for $A = 0$ and for $A \neq 0$, respectively.

We begin with the simpler case $A = 0$. In this case the general solution of (I.1) is obviously

$$f_1(z) = \sum_{k=1}^{n} c_k e^{z\beta_k}, \quad \text{with} \tag{I.4}$$

with

$$\beta_k = e^{i\pi(k/n+1/2)}, \qquad k = 1, 2, \cdots, n. \tag{I.5}$$

The c_k are arbitrary constants. (The quantities β_k are the same as those used in Secton, 5, cf. (5.13).)

We wish to determine the c_k so that the boundary conditions (I.2) and (I.3) are satisfied. That this really can be done for all points on these lines is far from obvious a priori. To satisfy the condition (I.2) we first write

$$H(z) = (iD - 1)f_1 = \sum_{k=1}^{n} (i\beta_k - 1)c_k e^{z\beta_k} = -\sum_{k=1}^{n} (e^{i\pi k/n} + 1)c_k e^{z\beta_k},$$

and observe that in this relation the coefficient c_n may be chosen arbitrarily since $e^{i\pi} + 1 = 0$. We note also that $\beta_k = \bar{\beta}_{n-k}$, in which the bar over a number denotes the complex conjugate of the number so that $e^{z\beta_k} = \overline{e^{z\beta_{n-k}}}$ if z is real. In order to ensure that $\Re eH(z)$ vanishes for *all real* z, it is therefore necessary to choose the constants c_k in such a way that

$$H(\bar{z}) = -\overline{H(z)}, \quad \text{for } z \text{ real.}$$

[32] It should be pointed out that we are no more able to decide in the present case than we were in the two-dimensional cases whether the waves are reflected back to infinity from the shore, and if so to what extent. Our numerical solution was obtained on the assumption that no reflection takes place, which is probably not well justified for the case of a vertical cliff, but would be for a beach of small slope.

To satisfy the above condition, we equate the pair of terms

$$(e^{i\pi k/n}+1)c_k e^{z\beta_k}; \qquad \overline{-[e^{i\pi(n-k)/n}+1]c_{n-k}e^{z\beta_{n-k}}}, \qquad (k=1,2,\cdots,n-1)$$

Since $e^{z\beta_k}$ and $\overline{e^{z\beta_{n-k}}}$ are conjugate for z real, we need only require

$$(e^{i\pi k/n}+1)c_k = -\overline{[e^{i\pi(n-k)/n}+1]}\bar{c}_{n-k}, \quad \text{or} \quad c_k = \frac{[e^{i\pi k/n}-1]}{[e^{i\pi k/n}+1]}\bar{c}_{n-k}.$$

Therefore, if the c_k are chosen so that

$$c_k = i\tan(\pi k/2n)\bar{c}_{n-k}, \qquad (k=1,2,\cdots,n-1) \tag{I.6}$$

the boundary condition (I.2) will be satisfied.

The condition at the bottom requires a similar analysis of the expression

$$K(z) = e^{i\pi(n-1)/2n}Df_1 = \sum_{k=1}^{n} e^{i\pi(n-1)/2n}\beta_k c_k e^{z\beta_k} \quad \text{for} \quad z = re^{-i\pi/2n}$$

or, upon insertion of the special values of z:

$$K(re^{-i\pi/2n}) = -\sum_{k=1}^{n} e^{i\pi(2k-1)/2n}c_k e^{re^{i\pi(2k-1+n)/2n}} = L(r).$$

The real part of $L(r)$ should vanish for all $r>0$. We observe that $e^{re^{i\pi(2k-1+n)/2n}}$ is conjugate to $e^{re^{i\pi[2(n+1-k)-1+n]/2n}}$. Hence our requirement that $L(r)$ be real leads to

$$e^{i\pi(2k-1)/2n}c_k = -\overline{e^{i\pi(2(n+1-k)-1+n)/2n}c_{n+1-k}}$$

or, as one readily sees

$$c_k = \bar{c}_{n+1-k}, \quad \text{for} \quad k=1,2,\cdots,n. \tag{I.7}$$

Thus, in order to satisfy the boundary conditions at the free surface as well as at the bottom we must impose on the c_k the conditions

$$\begin{aligned} c_k &= i\bar{c}_{n-k}\tan(\pi k/2n), \qquad & k&=1,2,\cdots,n-1 \\ c_k &= \bar{c}_{n-k+1}, & k&=1,2,\cdots,n. \end{aligned} \tag{I.8}$$

We proceed to show that these relations can be satisfied by a set of values c_k which are uniquely determined within a real multiplying factor.

From (I.8) we easily obtain the following recurrence relation by taking conjugates and eliminating $\bar{c}_k$

$$c_{n-k} = ic_{n-k+1}\cot\frac{\pi k}{2n}, \qquad k=1,2,\cdots,n-1,$$

which may also be expressed in the form

$$c_{n-k} = (i)^k c_n \cot\frac{\pi}{2n}\cot\frac{2\pi}{2n}\cdots\cot\frac{k\pi}{2n}, \quad \text{for} \quad k=1,2,\cdots,n-1. \tag{I.9}$$

For $k=n$ we have the additional relation

$$c_1 = \bar{c}_n. \tag{I.10}$$

If we set $k=n-1$ in (I.9) the cotangents cancel each other and the relation $c_1=(i)^{n-1}c_n$ results; this relation is easily seen to be compatible with (I.10) only if c_n is given as follows:

$$c_n = re^{-i\pi(n-1)/4} \tag{I.11}$$

in which r is any real number. In order to fix the c_k in such a way as to satisfy the boundary conditions it is therefore only necessary to choose c_n in accordance with (I.11) and calculate the remaining quantities by using (I.9). The following somewhat more convenient form might also be used to calculate the c_k:

$$c_k = re^{i\pi[(n+1)/4-k/2]} \cot\frac{\pi}{2n}\cdots\cot\frac{\pi(k-1)}{2n}, \qquad k = 2, 3, \cdots, n, \tag{I.12}$$
$$c_1 = \bar{c}_n$$

the first relation resulting through combining the first of (I.8) with (I.9) and noting afterwards that the relation holds for $k=n$ since the product of the cotangents has the value one in this case and c_k for $k=n$ thus has the value given by (I.11).

We turn next to the case $A\neq 0$ and, in fact, set $A=1$ for the purpose of the present investigation. It is clear that the results for any other value of A are obtained simply by multiplication by A. A solution $f_2(z)$ of the non-homogeneous equations can be obtained without difficulty, though the calculations are somewhat laborious. Instead of proceeding constructively we prefer to give the solution and then verify that it satisfies all conditions, particularly the boundary conditions. The solution $f_2(z)$ is*

$$f_2(z) = \sum_{k=1}^{n} a_k\left\{e^{z\beta_k}\int_{+\infty}^{z\beta_k}\frac{e^{-t}}{t}\,dt - \pi i e^{z\beta_k}\right\}. \tag{I.13}$$

In this solution the β_k are given by (I.5) and the a_k turn out to be multiples of the c_k:

$$a_k = c_k/(n-1)!\sqrt{n}. \tag{I.14}$$

These values of the a_k are of course required in order that $f_2(z)$ should satisfy the inhomogeneous equation. The path of integration for the complex integrals in (I.13) has already been given by Fig. 2 of Sec. 4. The path of integration comes from $+\infty$ along the real axis, then goes along a circular arc with center at the origin (leaving the origin to the left), and finally along a ray from the origin to the point $z\beta_k$. Since z lies in the sector for which $-\pi/2n\leqq \arg z\leqq 0$ and the β_k are given by (I.5) it follows that $z\beta_k$ always lies in the left half-plane. We consider first the condition (I.2). We write

$$(iD-1)f_2 = \sum_{k=1}^{n}(i\beta_k-1)a_k\{A_k\} + \frac{i}{z}\sum_{k=1}^{n}a_k = M(z) + \frac{i}{z}\sum_{k=1}^{n}a_k$$

in which $\{A_k\}$ has an obvious significance. Since $\sum_{k=1}^{n}a_k$ is real, (as one sees from (I.7) and (I.14)) the last term is pure imaginary when z is real. We must verify that the a_k are so chosen that the real part of the remaining terms, $M(z)$, vanishes for z real.

* Compare with (5.15) of Sec. 5.

As with the similar problem with the c_k above, we must verify that $M(\bar{z}) = -\overline{M(z)}$ for z real. In this case it is the terms

$$(i\beta_k - 1)a_k\{A_k\} \quad \text{and} \quad (i\beta_{n-k} - 1)a_{n-k}\{A_{n-k}\}$$

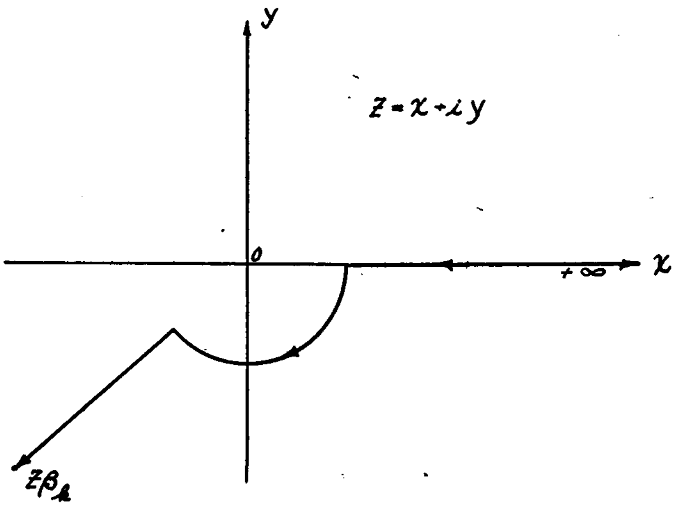

Fig. 17

for which the real parts cancel on addition. By evaluating the residue of $\oint e^{-t}dt/t$ at the origin we find readily that

$$a_k\{A_k\} = a_k\left\{e^{z\beta_k}\,{}_C\!\int_{+\infty}^{z\beta_k} \frac{e^{-t}}{t}\,dt - \pi i e^{z\beta_k}\right\} = a_k\left\{e^{z\beta_k}\,{}_{\overline{C}}\!\int_{+\infty}^{z\beta_k} \frac{e^{-t}}{t}\,dt + \pi i e^{z\beta_k}\right\}$$

in which the integral on the right hand side of this equation is taken along the path $\overline{C}$ shown in Fig. 17, while the integral on the left is taken over the path C shown in Fig. 2. The origin is kept to the right on $\overline{C}$ instead of to the left. We observe that the exponential terms $e^{z\beta_k}$ and $e^{z\beta_{n-k}}$ are complex conjugates for z real, so that $-ie^{-z\beta_k}$ and $+ie^{z\beta_{n-k}}$ are complex conjugates, since the same is true for β_k and β_{n-k}. In addition, the integrals in $\{A_k\}$ and $\{A_{n-k}\}$ are complex conjugates, as one sees from the above relation: if we replace k by $n-k$ in the integral along $\overline{C}$ it is clear that it becomes the conjugate of the integral along the path conjugate to $\overline{C}$, since the integrands are the same but the paths are complex conjugates of each other. Consequently we have only to verify that

$$(i\beta_k - 1)a_k = \frac{1}{-(i\beta_{n-k} - 1)a_{n-k}}\,.$$

But this is the same as the corresponding relation for the c_k given above, and hence is satisfied by the a_k since the a_k are the same as the c_k except for a real multiplying factor.

To check the condition (I.3) at the bottom we consider the expression

$$\begin{aligned} e^{i\pi(n-1)/2n}Df_2(z) &= \sum_{k=1}^{n} e^{i\pi/2}e^{-i\pi/2n}\beta_k a_k\{A_k\} + z^{-1}e^{i\pi/2}e^{-i\pi/2n}\sum_1^n a_k \\ &= N(z) + z^{-1}e^{i\pi/2}e^{-i\pi/2n}\sum_1^n a_k \end{aligned}$$

evaluated for $z=re^{i\pi/2n}$. The last term is pure imaginary for $z=re^{-i\pi/2n}$ since $\sum_1^n a_k$ is real. It can be shown that $\Re e N(z)=0$ for $z=re^{-i\pi/2n}$ by proceeding in the same way as above, except that the terms should be paired in a slightly different manner. In fact, the terms

$$e^{i\pi/2}e^{-i\pi/2n}\beta_k a_k\{A_k\} \quad \text{and} \quad e^{i\pi/2}e^{-i\pi/2n}\beta_{n+1-k}a_{n+1-k}\{A_{n+1-k}\}$$

are negative conjugates in this case, as one can readily verify.

APPENDIX II: ASYMPTOTIC BEHAVIOR OF $\int_\infty^{z\beta_k}(e^{-t}/t)dt$

In this appendix we prove a number of assertions made in Sec. 5 with regard to the behavior of the integrals

$$\int_{+\infty}^{z\beta_k} \frac{e^{-t}}{t}\,dt, \tag{II.1}$$

$$\beta_k = e^{i\pi(k/n+1/2)}, \qquad k = 1, 2, \cdots, n \tag{II.2}$$

when $z\to\infty$ in the sector S defined by

$$S:\quad 0 \geqq \arg z \geqq -\frac{\pi}{2n}\cdot$$

The path of integration is given by Fig. 2 of Sec. 4.

From (II.2) and the definitions of the sector S it follows that

$$\pi/2 \leqq \arg z\beta_k \leqq 3\pi/2. \tag{II.3}$$

It might also be noted that $\arg z\beta_k=3\pi/2$ only for $k=n$, i.e. for $\beta_k=-i$, and z real. We shall show that the integral (II.1) behaves for z in S and $|z|$ large as follows:

$$\int_{+\infty}^{z\beta_k} \frac{e^{-t}}{t}\,dt \sim \begin{cases} -\dfrac{e^{-z\beta_k}}{\beta_k}\left(\dfrac{1}{z}+\cdots\right), & \dfrac{\pi}{2} < \arg z\beta_k \leqq \pi, \\ 2\pi i - \dfrac{e^{-z\beta_k}}{\beta_k}\left(\dfrac{1}{z}+\cdots\right), & \pi < \arg z\beta_k \leqq \dfrac{3\pi}{2}, \end{cases} \tag{II.4}$$

in which the dots refer to terms of higher order in $1/z$.

We consider first the case in which $\pi/2 \leqq \arg z\beta_k \leqq \pi$ and begin by integrating twice by parts to obtain

$$\int_{+\infty}^{z\beta_k} \frac{e^{-t}}{t}\,dt = -\frac{e^{-z\beta_k}}{z\beta_k} + \frac{e^{-z\beta_k}}{(z\beta_k)^2} + 2\int_{\infty}^{z\beta_k} \frac{e^{-t}}{t^3}\,dt. \tag{II.5}$$

We shall show that

$$\left|\int_{\infty}^{z\beta_k} \frac{e^{-t}}{t^3} dt\right| \leqq c \left|\frac{e^{-z\beta_k}}{z^2}\right|$$

for z in the sector S, with c a positive number independent of z. Clearly, this would suffice to show that our integral behaves at ∞ like $-e^{-z\beta_k}/z\beta_k$.

We consider the integral $\oint(^{-t}/t^3)dt$ along the closed path indicated in Fig. 17. Since the integral over the closed path vanishes, it is clear that the behavior of our integral as $z \to \infty$ can be reduced to the investigation of the behavior of $\int(e^{-t}/t^3)dt$ over the circular arc PQR as $r \to \infty$. The point P corresponds to $z\beta_k$ of course. Upon setting $t = r(\cos\theta + i\sin\theta)$ the integral becomes

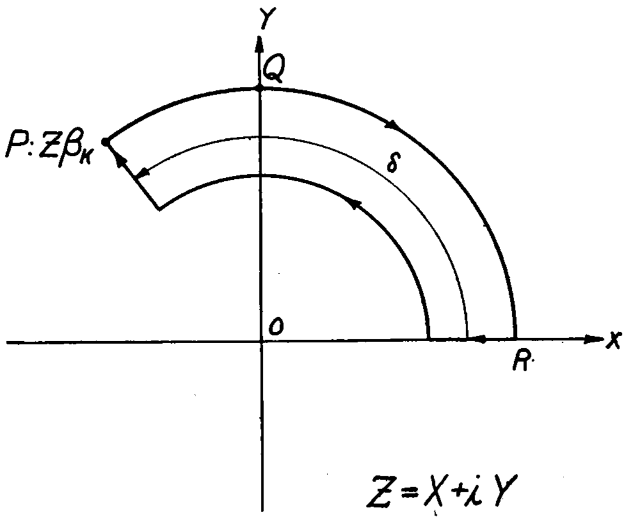

FIG. 18

$$I = i\int_{\delta}^{0} e^{-r\cos\theta} e^{-ri\sin\theta} \cdot \frac{d\theta}{r^2 e^{2i\theta}} \cdot \tag{II.6}$$

For $|I|$ we have, obviously

$$|I| \leqq 2\left|\int_0^{\delta} \frac{1}{r^2} \cdot e^{-r\cos\theta} d\theta\right|,$$

and we may write

$$|I| \leqq 2\frac{e^{-r\cos\delta}}{r^2}\int_0^{\delta} d\theta = 2\delta\frac{e^{-r\cos\delta}}{r^2}, \quad \text{with} \quad \delta = \arg z\beta_k,$$

since $e^{-r\cos\theta} \leqq e^{-r\cos\delta}$ for $0 \leqq \theta \leqq \delta \leqq \pi$. We observe that

$$\left| e^{-z\beta_k}/(z\beta_k)^2 \right| = r^{-2}e^{-r}\cos\delta, \qquad r = |z\beta_k|,$$

and this establishes (II.4) for $\pi/2 \leqq \arg z\beta_k \leqq \pi$.

To establish (II.4) for $\arg z\beta_k > \pi$, we write

$$_{C}\int_{+\infty}^{z\beta_k} \frac{e^{-t}}{t} dt = 2\pi i + {}_{\bar{C}}\int_{+\infty}^{z\beta_k} \frac{e^{-t}}{t} dt,$$

by evaluating the residue at the origin. The integral over $\bar{C}$ (see Fig. 17) can be treated in the same fashion as above. The only difference is that $\int(e^{-t}/t^3)dt$ is taken over a circular arc such that $\delta \leqq \theta \leqq 0$ with $-\pi < \delta \leqq -\pi/2$. The integral I' for this case corresponding to I in (II.6) is the same as that for I except that δ is replaced by $-\delta$. The inequalities for $|I'|$ are thus exactly the same as for $|I|$ since $\cos\theta$ is an even function and δ lies in the range $\pi < \delta \leqq \pi/2$. Thus (II.4) is established in general.

APPENDIX III: TABLE OF $E(z) = e^z \int_z^\infty (e^{-u}/u) du$ for $z = re^{i\beta_k}$; $\beta_k = e^{i\pi(k/15+1/2)}$; $k = 1, \cdots, 7$

The following table of the function $E(z) = e^z \int e^{-u} du/u$ is an extension of the short table of this function which has been prepared by the Mathematical Tables Project [8]. The path of integration is that of Fig. 2, but taken in the opposite direction.

The calculations were begun with fourteen six-decimal place values of the function $E(z)$ obtained from the above mentioned source, M.T.P. [8]. [We then computed* the rest of the numbers by means of power series expansions along each of the seven rays in the second quadrant defined by $re^{i\beta_k}$; $k = 1, \cdots, 7$; $1 \leqq r \leqq 7$.] The values are believed to be correct to within one unit in the last figure.

Real part of $E(z)$

r \ k	1	2	3	4	5	6	7
1.00	+.2650	+.1731	+.0657	−.0597	−.2063	−.3784	−.5817
.25	.1901	.1013	−.0023	−.1232	−.2644	−.4304	−.6271
.50	.1398	.0551	−.0434	−.1580	−.2912	−.4473	−.6315
.75	.1046	.0242	−.0688	−.1763	−.3006	−.4449	−.6139
2.00	+.0790	+.0031	−.0844	−.1849	−.2999	−.4322	−.5849
.25	.0601	−.0117	−.0939	−.1875	−.2936	−.4140	−.5507
.50	.0457	−.0222	−.0993	−.1864	−.2841	−.3933	−.5149
.75	.0345	−.0296	−.1020	−.1831	−.2730	−.3719	−.4795
3.00	+.0258	−.0348	−.1029	−.1784	−.2612	−.3507	−.4457
.25	.0190	−.0385	−.1026	−.1731	−.2494	−.3304	−.4141
.50	.0135	−.0411	−.1015	−.1673	−.2377	−.3112	−.3849
.75	.0090	−.0428	−.0998	−.1615	−.2266	−.2932	−.3582
4.00	+.0054	−.0439	−.0978	−.1556	−.2159	−.2766	−.3339
.25	.0025	−.0445	−.0956	−.1499	−.2059	−.2612	−.3119
.50	.0001	−.0448	−.0933	−.1444	−.1965	−.2470	−.2920
.75	−.0019	−.0448	−.0909	−.1391	−.1877	−.2340	−.2739
5.00	−.0036	−.0446	−.0885	−.1340	−.1794	−.2221	−.2576
.25	−.0050	−.0443	−.0861	−.1292	−.1718	−.2111	−.2428
.50	−.0061	−.0438	−.0838	−.1246	−.1646	−.2010	−.2295
.75	−.0071	−.0433	−.0815	−.1203	−.1579	−.1916	−.2173
6.00	−.0079	−.0427	−.0793	−.1162	−.1517	−.1830	−.2062
.25	−.0085	−.0421	−.0771	−.1123	−.1459	−.1751	−.1962
.50	−.0091	−.0414	−.0750	−.1087	−.1404	−.1678	−.1870
.75	−.0095	−.0407	−.0731	−.1052	−.1353	−.1610	−.1785
7.00	−.0099	−.0400	−.0711	−.1019	−.1306	−.1546	−.1708

* See Sec. 6 for a description of the procedure employed.

Imaginary part of $E(z)$

r \ k	1	2	3	4	5	6	7
1.00	−.6999	−.7765	−.8510	−.9230	−.9925	−1.0593	−1.1239
.25	−.6109	−.6723	−.7294	−.7811	−.8263	−.8633	−.8906
.50	−.5411	−.5908	−.6345	−.6705	−.6969	−.7110	−.7092
.75	−.4849	−.5253	−.5586	−.5826	−.5946	−.5913	−.5675
2.00	−.4386	−.4718	−.4968	−.5115	−.5127	−.4963	−.4565
.25	−.3999	−.4272	−.4458	−.4532	−.4463	−.4204	−.3691
.50	−.3672	−.3897	−.4031	−.4050	−.3919	−.3592	−.3002
.75	−.3391	−.3577	−.3670	−.3645	−.3469	−.3094	−.2456
3.00	−.3148	−.3301	−.3361	−.3303	−.3094	−.2688	−.2022
.25	−.2935	−.3062	−.3095	−.3011	−.2778	−.2353	−.1676
.50	−.2748	−.2853	−.2864	−.2761	−.2511	−.2075	−.1399
.75	−.2582	−.2668	−.2662	−.2543	−.2283	−.1843	−.1177
4.00	−.2435	−.2504	−.2484	−.2354	−.2087	−.1649	−.0998
.25	−.2302	−.2358	−.2327	−.2188	−.1917	−.1485	−.0853
.50	−.2183	−.2227	−.2187	−.2042	−.1770	−.1345	−.0735
.75	−.2074	−.2109	−.2061	−.1912	−.1641	−.1225	−.0639
5.00	−.1976	−.2003	−.1948	−.1796	−.1528	−.1123	−.0559
.25	−.1886	−.1906	−.1846	−.1693	−.1427	−.1034	−.0494
.50	−.1803	−.1817	−.1753	−.1599	−.1338	−.0956	−.0440
.75	−.1728	−.1736	−.1669	−.1515	−.1259	−.0889	.0395
6.00	−.1658	−.1661	−.1592	−.1439	−.1188	−.0829	−.0357
.25	−.1593	−.1593	−.1521	−.1369	−.1124	−.0777	−.0325
.50	−.1534	−.1529	−.1457	−.1306	−.1066	−.0730	−.0297
.75	−.1478	−.1471	−.1397	−.1247	−.1013	−.0688	−.0274
7.00	−.1426	−.1416	−.1341	−.1194	−.0965	−.0651	−.0254

TABLE OF $e^{i\beta_k}$

k	Real	Imaginary
1	−.20791	+.97815
2	−.40674	+.91355
3	−.58779	+.80902
4	−.74315	+.66913
5	−.86603	+.50000
6	−.95106	+.30902
7	−.99452	+.10453

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