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SURFACE WAVES ON WATER OF VARIABLE DEPTH

Lecture Notes, Fall Semester, 1950-51

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LIST OF SYMBOLS

A, B	wave amplitudes; constants of integration.
$\mathcal{A}, \mathcal{B}$	see pp. 66, 67.
D/Dt	partial differentiation $= \partial/\partial t + u\, \partial/\partial x + v\, \partial/\partial y + w\, \partial/\partial z$
E	total energy per cm^2.
E_x, E_y	components of energy flow.
$F(a,b,c,x)$	hypergeometric series.
G_n	Jacobi's polynomial.
H	differential operator, defined p. 84.
I	angle of incidence after refraction.
J_n	Bessel's function of order n .
K	kinetic energy density.
L	p. 73: spring-length.
L_n	Laguerre's polynomial of degree n .
L, M, N	differential operators: p. 31; p. 48; p. 52; p. 12.
N_n	Neumann's function of order n .
P	p. 13: $P = p_o h^{1/2}$; p. 88: $P = p_o \mu^{1/2}$
Q	wave amplitude, p. 47; p. 61.
R	radius.
S	phase function.
T	period of waves $= 2\pi/\omega$; kinetic energy per cm^2.
X	see p. 26 for definition.
X, Y, Z	components of vector potential: see p. 97.
a, b	components of wave number vector: $a^2 + b^2 = k^2$
c	phase velocity: ω/k
c_g	group velocity: $d\omega/dk$

g	acceleration of gravity; auxiliary function defined p. 83
h	water depth.
k	local wave number, radians/cm; $k = 2\pi/\lambda$
n, m	integers.
p	excess pressure.
p_0	excess pressure at $z = 0$
q	auxiliary variable: p. 31; p. 48; p. 52; p. 61.
r, ϕ	polar coordinates.
s	slope of bottom.
t	time.
u, v, w	components of velocity.
v	auxiliary variable, p. 33.
x, y	horizontal coordinates.
z	vertical coordinate; positive upward.
α, β	numerical parameters: p. 62; p. 93.
γ	Euler's constant = 0.577... .
θ	auxiliary variable, p. 27.
ψ	angle of incidence in deep water; also, see p. 65.
ϕ	phase constant; in Appendix: scalar potential; polar coordinate.
κ	ω^2/g ; $2\pi/\kappa$ = deep water wavelength.
λ	local wavelength.
μ	$1 - exp(-2\kappa h)$.
ν	integer.
ρ	density in general; p. 53 et seq., a parameter.
σ	parameter in $h = h_\infty[1 - exp(-\sigma x)]$.
τ	parameter for rays.

ξ	p. 62: $\exp(-\sigma x)$; p. 93: $\xi = \exp(-2\kappa s x)$.
ζ	auxiliary depth coordinate.
ω	frequency, radians/sec $= 2\pi/T$
Γ	gamma function.
Δ	$(\partial/\partial x)^2 + (\partial/\partial y)^2$

PREFACE

These lecture notes are, in one sense, systematic. It was the intention to show that, starting from the linearized equations of hydrodynamics, it is possible to give a logically consistent treatment of the propagation of waves in water of variable depth. This theory does not, in all respects, conform to observation; it was the intention to exhibit the assumptions that underlie the theory, so that later investigators might alter them with a view to improving the theory. The later parts of Section 7 show in detail how the theory fails when the water depth becomes very small.

In another sense, the lectures are eclectic: two phenomena are emphasized at the expense of others. The earlier lectures are directed toward the study of "edge waves," (Section 9). These seem closely related to the surf beat phenomena described by Munk* and to recent speculations concerning the reflection of surface waves by deep water**. In the later lectures, a new wave equation is derived (Section 14), that appears able to account for the dispersion and refraction that occur simultaneously in water of moderate depth. This equation is consistent with the "energy theory" previously used to deal with this case, but appears to have systematic advantages over the earlier theory.

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*Munk, W. H., Trans. A.G.U., 30, 849 (1949).

**Isaacs, J.D., Williams, E. A., and Eckart, C., Trans. AGU, 32, 37 (1951).

SURFACE WAVES ON WATER OF VARIABLE DEPTH

1. The General Equations.

Let the z-axis be vertically upward, the x- and y-axes in the horizontal plane; let u, v, w be the three components of velocity and the pressure be $-\rho g z + p(x y z t)$. Then the linearized equations of motion are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 ; \qquad 1.$$

$$\rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0 ,$$

$$\rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = 0 , \qquad 2.$$

$$\rho \frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} = 0 ,$$

provided one neglects all density gradients.

Differentiating Eq. (1) with respect to t and substituting from Eq. (2), we obtain Laplace's equation:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0. \qquad 3.$$

If the atmospheric pressure is constant (say zero), the equation of the free surface is

$$-\rho g z + p(x y z t) = 0. \qquad 4.$$

However, the free surface must also move with the water: this is expressed by the equation

$$\frac{D}{Dt}\left[-\rho g z + p(x y z t)\right] = 0,$$

or, since

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z},$$

$$-\rho g w + \frac{\partial p}{\partial t} + \text{quadratic terms} = 0. \qquad 4a.$$

Neglecting the quadratic terms and eliminating w between this equation and the third of Eq. (2):

$$g \frac{\partial p}{\partial z} + \frac{\partial^2 p}{\partial t^2} = 0. \qquad 5.$$

The Eq. (4) and (5) must both be satisfied at the free surface; since p is small, we may approximate Eq. (4) by

$$\rho g z = p(x\, y\, 0\, t), \qquad 4b.$$

and consider Eq. (5) to be imposed at $z = 0$, rather than at $z = p(x\, y\, 0\, t)/\rho g$.

If the bottom of the body of water is impermeable and has the equation

$$z = -h(x\, y) \qquad 6.$$

it is necessary to formulate the condition of impermeability in mathematical form. The vector

$$\frac{\partial h}{\partial x}, \quad \frac{\partial h}{\partial y}, \quad 1$$

is normal to the bottom surface, and hence the condition that no water flows through it is

$$u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} + w = 0 \qquad \text{6a.}$$

$$\text{at} \quad z = -h(xy).$$

By differentiating this with respect to t and eliminating u, v, w by means of Eq. (2), we obtain the equation

$$\frac{\partial p}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial p}{\partial y}\frac{\partial h}{\partial y} + \frac{\partial p}{\partial z} = 0, \qquad \text{7.}$$

$$\text{at} \quad z = -h$$

The mathematical problem is now one of a single dependent variable p : that solution of Laplace's equation (Eq.(3)) which satisfies the boundary conditions Eq. (5) and (7) is required.

To simplify matters, we seek only those solutions that are periodic in time, so that Eq. (5) becomes

$$g \frac{\partial p}{\partial z} - \omega^2 p = 0 \qquad 5a.$$

where $2\pi/\omega$ is the period of the waves.

2. The Shallow Water Approximation.

Since the function p is to be without singularities, one may expand it in powers of z:

$$p(x\,y\,z) = p_0(x\,y) + \frac{1}{1!}\,p_1(x\,y)\,z + \frac{1}{2!}\,p_2(x\,y)\,z^2 + \cdots \qquad 8.$$

If this series is substituted into Laplace's equation, one can collect terms having the same power of z, and separately equate the coefficient of each power of z to zero. In this way one obtains the set of equations

$$\begin{aligned} \Delta p_0 + p_2 &= 0 \\ \Delta p_1 + p_3 &= 0 \\ \Delta p_2 + p_4 &= 0 \\ \text{etc.}, & \end{aligned} \qquad 9.$$

where the abbreviation

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

has been used.

If the series is substituted into Eq. (5a) and evaluated for $z = 0$, only the terms independent of z remain, and these are

$$g p_1 - \omega^2 p_0 = 0. \qquad 10.$$

With Eq. (9) and (10), all the functions p_n can be expressed in terms of p_0 and its derivatives: using the abbreviation $\kappa = \omega^2/g$ (Note that $2\pi/\kappa$ is the deep water wave-length) we have

$$\begin{aligned} p_1 &= \kappa p_0 \\ p_2 &= -\Delta p_0 \\ p_3 &= -\kappa \Delta p_0 \\ p_4 &= \Delta^2 p_0 \\ p_5 &= \kappa \Delta^2 p_0 \\ &\text{etc} \end{aligned} \qquad 11.$$

In this way, one obtains one solution of Eq. (3) and (5a) for every choice of the function p_0. It remains to determine p_0 in such a way that Eq. (7) is also atisfied. Parenthetically, it may be remarked that Eq. (4b) becomes

$$z = p_0(xy)/\rho g \qquad 4c.$$

so that, except for the factor $1/\rho g$, p_0 is the height of the displaced free surface above its undisturbed level. The problem has thus been reduced to that of finding the shape of the free surface; once this has been found, the distribution of pressure and velocity can be calculated from Eq. (11), (8) and (2).

Substituting Eq. (8) into Eq. (7), one finds that

$$
\begin{aligned}
0 = & \left(\frac{\partial p_0}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial p_0}{\partial y} \frac{\partial h}{\partial y} + p_1 \right) \\
& + \frac{1}{1!}\left(\frac{\partial p_1}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial p_1}{\partial y} \frac{\partial h}{\partial y} + p_2 \right) z \\
& + \frac{1}{2!}\left(\frac{\partial p_2}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial p_2}{\partial y} \frac{\partial h}{\partial y} + p_3 \right) z^2 \\
& + \text{etc.}
\end{aligned}
$$

Setting $z = -h$ and using Eq. (11), this becomes

$$
\begin{aligned}
0 = &\left(\frac{\partial p_0}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial p_0}{\partial y}\frac{\partial h}{\partial y} + \kappa p_0 \right) \\
&- \frac{1}{1!}\left(\kappa \frac{\partial p_0}{\partial x}\frac{\partial h}{\partial x} + \kappa \frac{\partial p_0}{\partial y}\frac{\partial h}{\partial y} - \Delta p_0 \right) h \qquad 12. \\
&+ \frac{1}{2!}\left(- \frac{\partial \Delta p_0}{\partial x}\frac{\partial h}{\partial x} - \frac{\partial \Delta p_0}{\partial y}\frac{\partial h}{\partial y} - \kappa \Delta p_0 \right) h^2 \\
&- \cdots
\end{aligned}
$$

Eq. (12) is a partial differential equation of infinite order, to be solved for p_0. While such equations have been discussed at intervals since the time of Euler, no systematic method of solving them has been developed. In this case, we suppose that the dimensionless qualities

$$\kappa h, \quad \frac{\partial h}{\partial x}, \quad \frac{\partial h}{\partial y}$$

are small for all x and y. If we then neglect products of these small qualities, we arrive at the approximate differential equation of second order:

$$0 = \frac{\partial p_0}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial p_0}{\partial y}\frac{\partial h}{\partial y} + \kappa p_0 + h \Delta p_0$$

or

$$\frac{\partial}{\partial x}\left(h \frac{\partial p_0}{\partial x}\right) + \frac{\partial}{\partial y}\left(h \frac{\partial p_0}{\partial y}\right) + \kappa p_0 = 0 . \qquad 13.$$

3. Water of Constant Depth.

It is worth considering the case of constant h, since in this case, we know the rigorous solution.

In this case, Eq. (13) becomes

$$\Delta p_0 = -\kappa p_0/h = -k^2 p_0 \qquad 14.$$

where $2\pi/k = 2\pi\sqrt{gh}/\omega$ is the shallow water wave-length.
Using this result in Eq. (11) it becomes

$$p_1 = \kappa p_0 \qquad p_2 = k^2 p_0$$

$$p_3 = \kappa k^2 p_0 \qquad p_4 = k^4 p_0$$

$$p_5 = \kappa k^4 p_0 \qquad p_6 = k^6 p_0$$

so that

$$p = p_0\left\{\left[1 + \frac{1}{2!}k^2z^2 + \frac{1}{4!}k^4z^4 \cdots\right] + \kappa/k\left[kz + \frac{1}{3!}k^3z^3 + \frac{1}{5!}k^5z^5 \cdots\right]\right\},$$

$$= p_0\left[\cosh kz + kh \sinh kz\right]. \qquad 15.$$

Now the rigorous solution is known to be

$$p = p_0 \cosh k(z+h) \qquad 16.$$

where p_0 is any solution of Eq. (14). Since Eq. (16) may be written

$$p = p_0\left[\cosh kz \cosh kh + \sinh kz \sinh kh\right],$$

it is seen that the errors in Eq. (15) are of the order of magnitude $k^2h^2 = \kappa h$, as was to be expected.

4. Canonic Form of the Shallow Water Equation.

The introduction of the new variable

$$P = p_0 h^{1/2} \qquad 17.$$

reduces the Eq. (13) to the form

$$\Delta P + k^2 P = 0 \qquad 18.$$

where

$$k^2 = \kappa / h - h^{-1/2} \Delta h^{1/2} \qquad 19.$$

The expression for the wave number in water of constant depth is $k^2 = \omega^2 / gh = \kappa / h$. Consequently, the second term of Eq. (19) may be expected to function as a correction factor to the usual formula. To estimate its magnitude, let $h = s x$, s being the constant slope of the bottom; then Eq. (19) becomes

$$k^2 = \omega^2 / g s x + 1/4 x^2.$$

Consequently, the correction term is negligible when

$$x \gg g s / 4 \omega^2 = s / 4 \kappa$$

Taking $g = 10^3$, $s = 10^{-2}$, $2\pi/\omega = 10$ sec, this becomes $x \gg 6$ cm. Thus, we may neglect this term everywhere except very close to the water's edge. It appears that it is justifiable to use the ordinary formula in almost all, if not in all, cases of interest. The small region near the water's edge is in the surf, and we cannot expect our linearized theory to give a good account of this phenomenon anyway.

It is possible that the correction term may be important at the edge of the continental shelf, in the case of the 3, 6, and 12 hr. components of the tide. Moreover, in some problems, the neglect of the correction term complicates, rather than simplifies the analysis.

5. The Approximation of Ray Theory.

A traditional way of obtaining an approximate solution of Eq. (18) leads to Huyghens' construction for the wave fronts, and to other well-known formulae. It is assumed that there are solutions of the equation that have the forms

$$P = A\cos(S-\omega t),$$
$$P = A\sin(S-\omega t), \qquad 20.$$
$$P = Ae^{i(S-\omega t)},$$

the functions A and S being the same in all these solutions and being real.

If we substitute the exponential form for P in Eq. (18), the result is

$$e^{i(S-\omega t)}\Big\{\Delta A + k^2A - A\Big[\Big(\frac{\partial S}{\partial x}\Big)^2 + \Big(\frac{\partial S}{\partial y}\Big)^2\Big] + i\Big[2\Big(\frac{\partial A}{\partial x}\frac{\partial S}{\partial x} + \frac{\partial A}{\partial y}\frac{\partial S}{\partial y}\Big) + A\,\Delta S\Big]\Big\} = 0.$$

Since A and S are both real, this equation is equivalent to the two equations

$$\Big(\frac{\partial S}{\partial x}\Big)^2 + \Big(\frac{\partial S}{\partial y}\Big)^2 - k^2 = A^{-1}\Delta A \qquad 21.$$

and

$$A\,\Delta S + 2\Big(\frac{\partial A}{\partial x}\frac{\partial S}{\partial x} + \frac{\partial A}{\partial y}\frac{\partial S}{\partial y}\Big) = 0 \qquad 22.$$

which will be considered successively.

In Eq. (21), the term $A^{-1}\Delta A$ is known as the diffraction term. It has the same dimensions as k^2, and is somewhat similar to the term $h^{-1/2}\Delta h^{1/2}$ in Eq. (19). However, instead of depending on the known bottom topography, the former depends on the unknown amplitude function. The method of solution now under discussion is useful only when the diffraction term is negligible compared to k^2, the square of the wave number.

Under these conditions, we may write

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = k^2; \qquad 21a.$$

this equation is known as the geometric wave equation, because it can be solved graphically by a simple method. This method will first be described and then proven.

Since Eq. (21a) is a partial differential equation, it will involve an arbitrary function of integration: this may be taken as the curve C_0 in the x-y plane on which S takes on some constant numerical value — say the curve $S = a$. The construction for finding the curve C_1: $S = a + \delta a$ (δa being small) is as follows: at points of C_0, one erects the normals. On these normals, one locates points at the distance $\delta s = \delta a / k(x,y)$ from C_0. This is possible since k is a known function of x, y and δa is given. The curve C_1 passes through these points. Having obtained C_1, one can repeat the procedure to obtain C_2: $S = a + 2\delta a$, ,etc., thus obtaining the contour lines of the function S in succession. In addition to obtaining the contours, one also obtains a family of curves that intersect the contours at right angles; these are the rays.

To prove this, let $x\ y$ be a point on C_0 and $x+\delta x,\ y+\delta y$ the point on C_1 where the normal to C_0 intersects the latter. Then because $S(x+\delta x, y+\delta y) = S(xy) + \delta a$ we have

$$\delta x \frac{\partial S}{\partial x} + \delta y \frac{\partial S}{\partial y} = \delta a. \qquad \text{21b.}$$

Because the vector $\delta x, \delta y$ is normal to C_0, we have

$$\delta x = \epsilon \frac{\partial S}{\partial x}, \qquad \delta y = \epsilon \frac{\partial S}{\partial y}$$

where ϵ is a factor of proportionality. It is related to δs by the equation.

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2 = \epsilon^2 \left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 \right]$$

$$= k^2 \epsilon^2 \qquad \text{by Eq } (21^a).$$

Hence

$$\delta x = \frac{\delta s}{k}\left(\frac{\partial S}{\partial x}\right), \quad \delta y = \frac{\delta s}{k}\left(\frac{\partial S}{\partial y}\right);$$

substituting into Eq. (21b), we have

$$\frac{\delta\rho}{k}\left[\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2\right]=\delta a$$

or by Eq. (21a), $\delta\rho = \delta a/k$, which was to be shown.

We can therefore consider the function S, which is known as the phase function, or as the Hamilton-Jacobi function, to have been determined.

Turning to Eq. (22), it may be written

$$\frac{\partial}{\partial x}\left(A^2\frac{\partial S}{\partial x}\right)+\frac{\partial}{\partial y}\left(A^2\frac{\partial S}{\partial y}\right)=0 \qquad 23.$$

which has the form of an equation of conservation. It suggests that the quantities

$$E_x=\frac{1}{2\rho\omega}A^2\frac{\partial S}{\partial x}, \quad E_y=\frac{1}{2\rho\omega}A^2\frac{\partial S}{\partial y} \qquad 24.$$

may be interpreted as the vector flow of energy in the wave-train. (The constant of proportionality, $1/2\rho\omega$, will be justified below.)

To prove this, we return to Eq. (1) and (2); multiplying the first by p, the others by u, v and w respectively, and then adding, one obtains

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho\left(u^2+v^2+w^2\right)\right]+\frac{\partial up}{\partial x}+\frac{\partial vp}{\partial y}+\frac{\partial wp}{\partial z}=0 \qquad 25.$$

Since $\frac{1}{2}\rho(u^2+v^2+w^2)$ is the density of kinetic energy, we identify

$$u = \left(h^{-1/2} A / \rho\omega\right) \cos(S - \omega t) \frac{\partial S}{\partial x} .$$

Hence

$$\overline{u p} = \frac{h^{-1} A^2}{2\rho\omega} \frac{\partial S}{\partial x}$$

and hence

$$\int_{-h}^{0} \overline{u p}\, dz = \frac{A^2}{2\rho\omega} \frac{\partial S}{\partial x} := E_x$$

We have had to make many approximations to obtain this interpretation of E_x ; it is therefore important to notice that Eq. (23) was derived without making these approximations, and has a higher degree of validity than our interpretation.

To summarize these results: it has been shown that a certain approximmation, Huyghens' wave-front construction is part of the approximate solution of the Eq. (18). Then it has been shown that the "energy flow" is normal to the wave fronts, and is proportional to the square of the amplitude function, and to the gradient of the phase function.

This section has been entitled "Ray Theory," but thus far, rays have scarcely been mentioned. They are defined as those curves that intersect all of the curves $S =$ const. at right angles: they are the orthogonals of the family of curves $S =$ const. They derive their importance from a very elegant, though somewhat inobvious, analytic method for solving Eq. (21a).

Consider any curve, Γ , joining two points A and B in the x-y plane, and let ds be its element of length. Then we can calculate the integral

$$I_\Gamma = \int_A^B k(xy)\, ds$$

If we keep A and B fixed, but change the curve Γ , I_Γ will change its value. Among all the possible curves, there will often be one for which I_Γ has a smaller value than any other; alternatively, there may be one for which it has a larger value than for any other; these "extremals" will be shown to be the rays: this is Fermat's principle of least times. The extreme value of I_Γ $(= I)$ is related to the phase function in a simple way.

First, we obtain Euler's differential equation of the rays. For this purpose, suppose that the equation of the rays is

$$x = x(\tau), \qquad y = y(\tau)$$

where τ is any parameter, such that the point A corresponds to $\tau = \tau_A$ and the point B to $\tau = \tau_B$. Then

$$ds = u\, d\tau, \quad \text{where} \quad u^2 = (dx/d\tau)^2 + (dy/d\tau)^2.$$

and hence

$$I = \int_{\tau_A}^{\tau_B} k u \, d\tau .$$

If we consider a neighboring curve joining A to B, we may write

$$x = x(\tau) + \delta x(\tau), \quad y = y(\tau) + \delta y(\tau)$$

provided that

$$\delta x(\tau_A) = \delta x(\tau_B) = \delta y(\tau_A) = \delta y(\tau_B) = 0.$$

The value of I must be, to a first approximation, unaltered, since the curve is an extremal. But, value for the neighboring path is

$$I = \int_{\tau_A}^{\tau_B} \left\{ ku + u\left(\frac{\partial k}{\partial x}\delta x + \frac{\partial k}{\partial y}\delta y\right) + \frac{k}{u}\left(\frac{dx}{d\tau}\frac{d\delta x}{d\tau} + \frac{dy}{d\tau}\frac{d\delta y}{d\tau}\right)\right\} d\tau.$$

If we integrate the last two terms by parts, and remember that the δx, δy vanish for $\tau = \tau_A$ and τ_B, we get

$$I = \int_{\tau_A}^{\tau_B} \left\{ ku + \delta x\left[u\frac{\partial k}{\partial x} - \frac{d}{d\tau}\left(\frac{k}{u}\frac{dx}{d\tau}\right)\right] + \delta y\left[u\frac{\partial k}{\partial y} - \frac{d}{d\tau}\left(\frac{k}{u}\frac{dy}{d\tau}\right)\right]\right\} d\tau$$

Since δx and δy are arbitrary functions, their coefficients must vanish in order that the two expressions for I be equal:

$$u \frac{\partial k}{\partial x} - \frac{d}{d\tau}\left(\frac{k}{u}\frac{dx}{d\tau}\right) = 0$$

$$u \frac{\partial k}{\partial y} - \frac{d}{d\tau}\left(\frac{k}{u}\frac{\partial y}{d\tau}\right) = 0$$

These are Euler's differential equations. By solving them, we obtain the rays; through any point, there is one ray in every direction, since these equations are of second order.

Having thus determined what we mean by "rays," we now proceed to determine the function S. As before, we suppose the one contour $S = a$ to be given, as the arbitrary element that enters into the solution of Eq. (21a). Then through each point of this curve, we construct that ray which intersects the contour at right angles. Then, in general, if x, y is any point, one of these rays will pass through it, and will intersect the surface $S = a$ in the point $x_a y_a$. The required function is now to be calculated by the formula

$$S(xy) = a + \int_{x_a y_a}^{xy} k\, ds,$$

the integration being along the ray in question.

There are various ways of proving this; perhaps the simplest is to take $x\, y$ very near to $x_a\, y_a$: then this formula reduces to the previous Huyghens' construction. Repeating the construction for successive elements ds, and adding, we obtain the result just given. However, this argument does not make it clear that the rays must be determined by Euler's equations, and it cannot be considered to be entirely satisfactory for this reason.

A more satisfactory proof was given by W. R. Hamilton: consider two points $B = x\,y$ and $B' = x+\delta x,\ y+\delta y$. These will in general lie on two different rays, which intersect $S = a$ perpendicularly in the points $A = x_a\,y_a$ and $A' = x_a+\delta x_a,\ y_a+\delta y_a$. Let the equations of the two rays be

$$x = x(\tau), \quad y = y(\tau)$$

and

$$x = x(\tau) + \delta x(\tau), \quad y = y(\tau) + \delta y(\tau).$$

Now, however, δx, δy are <u>not</u> zero at $\tau = \tau_A$ and τ_B. We can calculate

$$S(x+\delta x, y+\delta y) - S(x\,y) = \delta x \frac{\partial S}{\partial x} + \delta y \frac{\partial S}{\partial y}$$

$$= \int_{A'}^{B'} k\,u\,d\tau - \int_{A}^{B} k\,u\,d\tau$$

by the same method as was previously used in deriving Euler's equations:

$$\frac{\partial S}{\partial x}\delta x + \frac{\partial S}{\partial y}\delta y = \int_{\tau_A}^{\tau_B}\left\{\delta x\left[u\frac{\partial k}{\partial x} - \frac{d}{d\tau}\left(\frac{k}{u}\frac{dx}{d\tau}\right)\right]\right.$$

$$\left. + \delta y\left[u\frac{\partial k}{\partial y} - \frac{d}{d\tau}\left(\frac{k}{u}\frac{dy}{d\tau}\right)\right]\right\}d\tau$$

$$+\left[\delta x\frac{k}{u}\frac{dx}{d\tau} + \delta y\frac{k}{u}\frac{dy}{d\tau}\right]_{\tau_B} - \left[\delta x\frac{k}{u}\frac{dx}{d\tau} + \delta y\frac{k}{u}\frac{dy}{d\tau}\right]_{\tau_A}$$

Because $x(\tau)$, $y(\tau)$ is a solution of Euler's equation, the integral vanishes; the last bracket also vanishes, since δx_a δy_a is a vector parallel to $S = a$, while $\left(\frac{dx}{d\tau}, \frac{dy}{d\tau}\right)_{\tau_A}$ is a vector normal to the same curve; this leaves

$$\frac{\partial S}{\partial x}\delta x + \frac{\partial S}{\partial y}\delta y = \frac{k}{u}\left(\frac{dx}{d\tau}\delta x + \frac{dy}{d\tau}\delta y\right)$$

it being understood that $\tau = \tau_B$ on the right. Since δx and δy are arbitrary increments, we have

$$\frac{\partial S}{\partial x} = \frac{k}{u}\frac{dx}{d\tau}, \quad \frac{\partial S}{\partial y} = \frac{k}{u}\frac{dy}{d\tau}$$

but, because of the definition of u, this leads to

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = k^2$$

which was to be shown.

6. Straight, Parallel Contours. (Ray Theory)

If h and k are constants, the geometric wave equation can be solved by inspection: if we set

$$S = ax + by$$

the equation reduces to

$$a^2 + b^2 = k^2 \quad \text{or} \quad a = k\cos\psi;\ b = k\sin\psi$$

and

$$P = A(x,y)\cos(ax + by - \omega t).$$

Moreover, one solution of Eq. (23) is then $A(x,y) =$ constant. In this solution, the wave-fronts are straight lines, and the amplitude of the waves is everywhere the same. However, even when k is constant, Eq. (21a) has solutions that are more complicated.

If h and k are independent of y, we have the case of straight bottom contours, parallel to the y-axis. In this case, we try to find a function $X(x)$ such that

$$S = X(x) + by \qquad 26.$$

This leads to the equation

$$\left(\frac{\partial X}{\partial x}\right)^2 + b^2 = k^2(x) \qquad 27.$$

or

$$X = \int \sqrt{k^2(x) - b^2}\, dx\,. \qquad 28.$$

A particular solution for the amplitude function can also be found: if A is a function of x only, Eq. (23) reduces to

$$\frac{d}{dx}\left(A^2 \frac{dX}{dx}\right) = 0$$

or

$$A^2 = \text{const} / \sqrt{k^2 - b^2}\,. \qquad 29.$$

Now, the vector

$$\frac{\partial S}{\partial x} = \sqrt{k^2 - b^2}\,, \qquad \frac{\partial S}{\partial y} = b$$

is normal to the wave-fronts: consequently, if I is the angle these make with the y-axis

$$\frac{\partial S}{\partial x} = k \cos I\,, \qquad \frac{\partial S}{\partial y} = k \sin I\,.$$

Hence

$$A^2 = \text{const} / k \cos I$$

and hence the wave height is proportional to

$$A / h^{1/2} = \text{const} / (h k \cos I)^{1/2}$$

30.

$$= \text{const} / [h^2 (k^2 - b^2)]^{1/4}.$$

Nearshore, $\cos I \sim 1$ and we have proven the known result that the wave height is proportional to $1/h^{1/4}$

To illustrate these calculations in more detail, let

$$h = h_\infty (1 - e^{-\sigma x}) = h_\infty (1 - \xi) \qquad 31$$

where h_∞ and σ are constants. Then

$$k^2 = \frac{\omega^2}{g h_\infty} \frac{1}{1 - \xi} = \frac{a^2 + b^2}{1 - \xi} \qquad 32$$

where $a^2 + b^2 = \omega^2 / g h_\infty = k_\infty^2$

Then

$$X = -\frac{1}{\sigma} \int \sqrt{\frac{a^2 + b^2 \xi}{1 - \xi}} \, \frac{d\xi}{\xi}.$$

The introduction of a new variable, θ, defined by

$$\sin \theta = \sqrt{1 - \xi} \sin \psi, \qquad \sin \psi = b / k_\infty \qquad 32a.$$

the vector $u p$, $v p$, $w p$ with the energy flow in three dimensions. We then show that, to an adequate approximation,

$$E_x = \int_{-h}^{0} \overline{up}\, dz$$

$$E_y = \int_{-h}^{0} \overline{vp}\, dz$$

where the bar indicates a time average.

Taking

$$P = A \cos(S - \omega t)$$

we have, by Eq. (17),

$$p_0 = h^{-1/2} A \cos(S - \omega t) = p$$ to a rough approximation.

Hence, by Eq. (2),

$$\rho \frac{\partial u}{\partial t} = - \frac{\partial p}{\partial x}$$

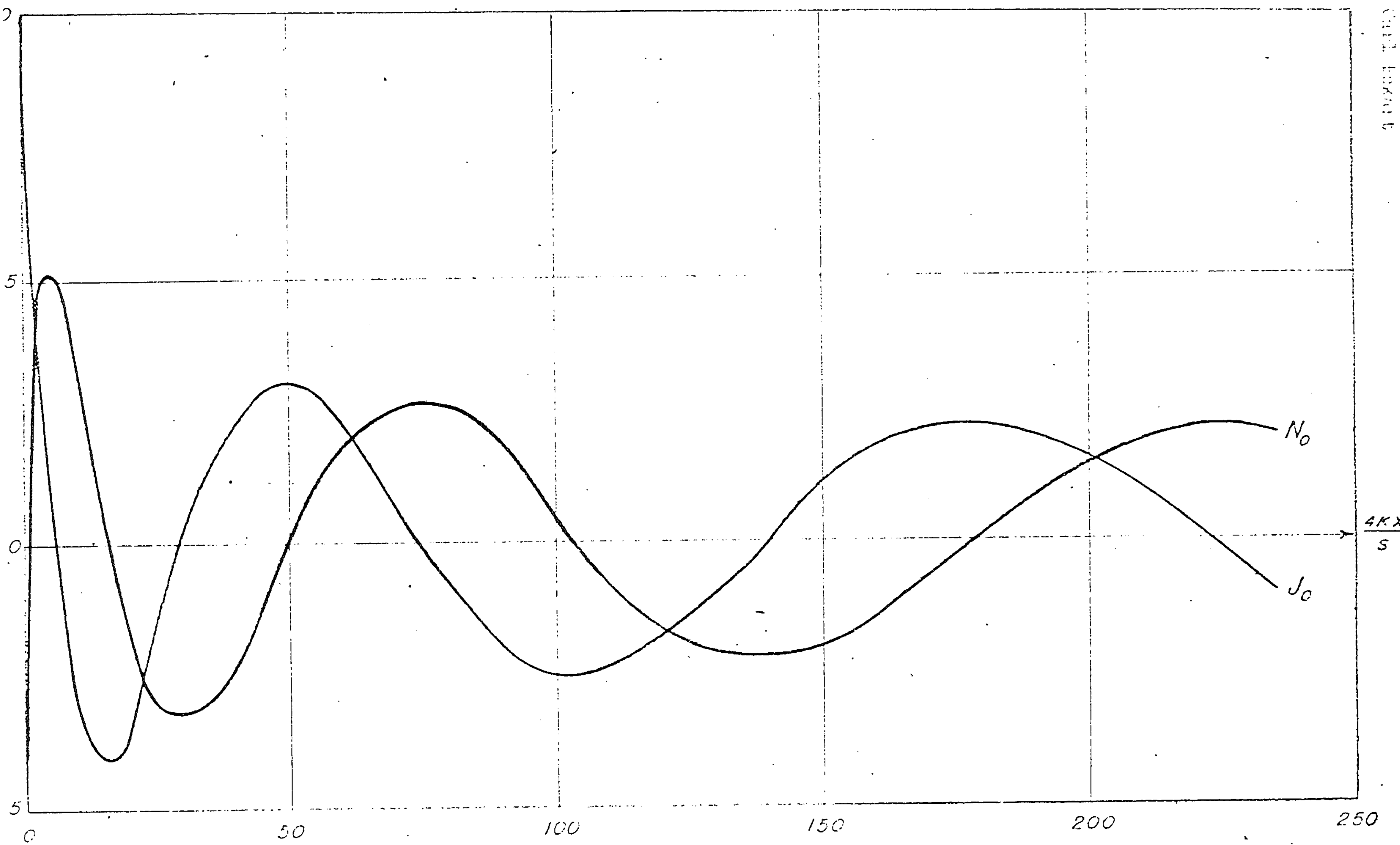
$$= h^{-1/2} A \sin(S - \omega t) \frac{\partial S}{\partial x} + \cdots$$

where the dots indicate terms in $\partial h/\partial x$ and $\partial A/\partial x$. We have already supposed that such terms are small, in deriving Eq. (13) and (21a), so we are perhaps justified in neglecting them here. In that case, makes it possible to evaluate the integral:

$$X = \frac{2b}{\sigma}\theta + \frac{a}{\sigma}\log\frac{\sin(\psi+\theta)}{\sin(\psi-\theta)} \qquad 33.$$

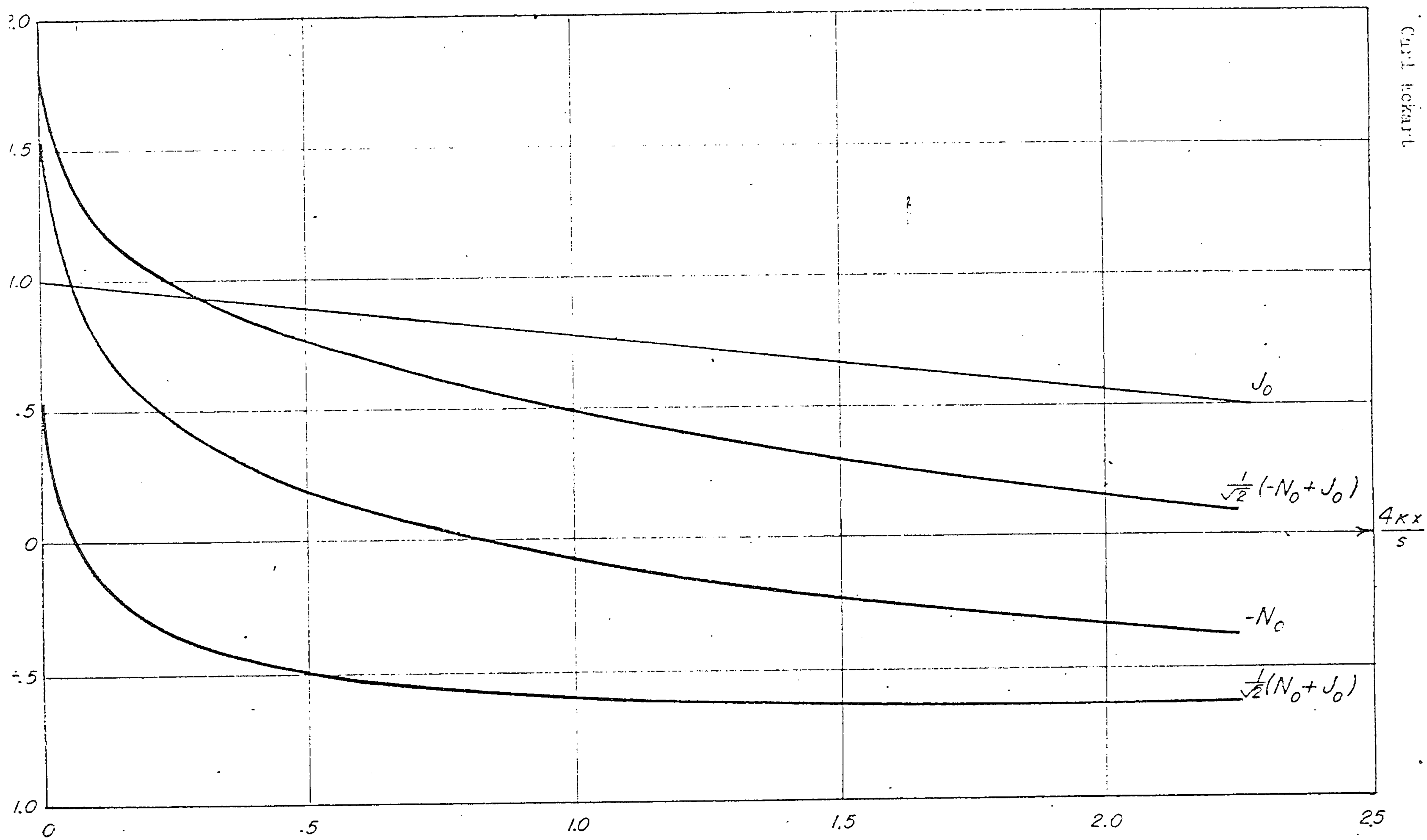
which is well suited for numerical evaluation, and will be useful below. The amplitude function A is proportional to $1/(\sin\psi \operatorname{ctn}\theta)^{1/2}$ and the height of the waves to $(\sin\psi/\sin 2\theta)^{1/2}$.

Note: If $\psi = 0$, this formula becomes indeterminate, but the result $X = (k_\infty/\sigma)\log[(1+\sqrt{1-\xi})/(1-\sqrt{1-\xi})]$ is easily obtained.



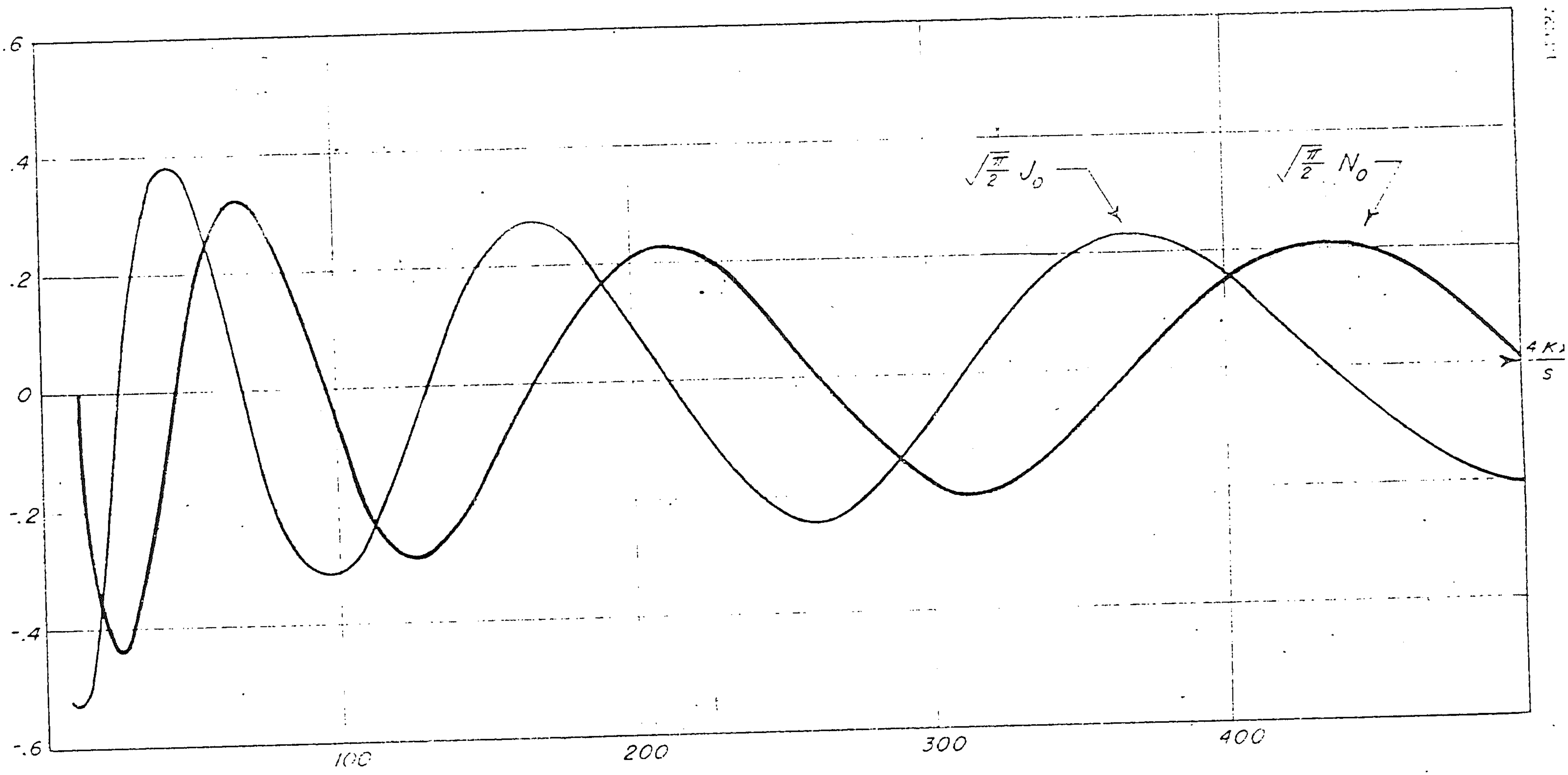
<u>Note</u>: For $T = 20$ sec., $s = 0.02$, $s/4\kappa = 50$ cm.

FIGURE 1.



Note: For T = 15 min., $s = 0.02$, $s/4\kappa$ = 1 km.
For T = 20 sec., $s = 0.02$, $s/4\kappa$ = 50 cm.

FIGURE 2.



.6
.4
.2
0
-.2
-.4
-.6
100
200
300
400
$\sqrt{\frac{\pi}{2}}\, J_0$
$\sqrt{\frac{\pi}{2}}\, N_0$
$\frac{4K\lambda}{s}$

FIGURE 3.

7. The Solutions Near the Water's Edge

We return to Eq. (13) and consider the case*

$$h = s x \qquad 31^*.$$

which represents a bottom with constant slope s . This will not be justified for large values of x, since the depth of water will ultimately become greater than one wave-length. However, one can obtain useful conclusions concerning the wave-motion near the water's edge, and results that can be used later in other connections.

Substituting Eq. (31*) into Eq. (13), and making the simplifying assumption that p_o is independent of y (normal incidence of the waves) the wave equation becomes

$$s \frac{d}{dx}\left(x \frac{dp_o}{dx}\right) + \kappa p_o = 0. \qquad 32^*.$$

Define the operator

$$M = \frac{d}{dx} x \frac{d}{dx} + \kappa / s \qquad 33^*.$$

and expand

$$p_o = c_o + c_1 x + c_2 x^2 + \cdots .$$

where $c_o \neq 0$.

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* This problem has been discussed by J. J. Stoker, Quarterly of Applied Mathematics 5 31 (1947).

Then, since

$$M x^{\nu} = \nu^2 x^{\nu-1} + (\kappa/s)\, x^{\nu},$$

we obtain

$$0 = M p_0 = C_0 \left[0^2 + (\kappa/s) \right]$$
$$+ C_1 \left[1^2 + (\kappa/s)\, x \right]$$
$$+ C_2 \left[2^2 x + (\kappa/s)\, x^2 \right]$$
$$+ C_3 \left[3^2 x^2 + (\kappa/s)\, x^3 \right]$$
$$+ \ldots .$$

In order that this equation shall be valid, the coefficient of each power of x must vanish, so that

$$1^2 C_1 + (\kappa/s)\, C_0 = 0.$$
$$2^2 C_2 + (\kappa/s)\, C_1 = 0$$
$$3^2 C_3 + (\kappa/s)\, C_2 = 0$$
$$\ldots .$$

or

$$p_0 = C_0 \left\{ 1 - \frac{1}{1^2}\left(\frac{\kappa x}{s}\right) + \frac{1}{1^2 2^2}\left(\frac{\kappa x}{s}\right)^2 - \frac{1}{1^2 2^2 3^2}\left(\frac{\kappa x}{s}\right)^3 \cdots \right\} \quad 34.$$

This is an infinite series for p_0, and, as such would be rather difficult to use for numerical computation. Fortunately, however, the function

$$J_0(v) = 1 - \frac{1}{1^2}\left(\frac{v}{2}\right)^2 + \frac{1}{1^2 2^2}\left(\frac{v}{2}\right)^4 - \frac{1}{1^2 2^2 3^2}\left(\frac{v}{2}\right)^6 \cdots$$

(known as Bessel's function) has been tabulated,* and we may write Eq. (34)

$$p_0 = J_0\left(\sqrt{\frac{4\kappa x}{s}}\right) \qquad 35.$$

so that is is simple to construct a graph of this solution. (Fig. 1 and 2.)

Unfortunately, this is not the most general solution of the Eq. (32*), and it is rather difficult to obtain the most general one. It is much easier to obtain the general solution of the equation

$$Mq = \frac{d}{dx}\left(x\frac{dq}{dx}\right) + \left[\frac{\kappa}{s} - \frac{n^2}{4x}\right]q = 0 \qquad 32^{**}.$$

It will be instructive to consider this case; to obtain the result, it is necessary to set

$$q = c_0 x^m + c_1 x^{m+1} + c_2 x^{m+2} + \ldots$$

- - - - - - - - - -

* References: Magnes & Oberhettinger "Special Functions of Mathematical Physics," p. 16 (Chelsea, 1949) (hereafter cited as S.F.M.P.).

Jahnke & Ende, "Tables of Functions" p. 128 et seq. (Dover, 1943).

where m is a constant that must be determined. Since

$$M x^{\nu} = (\nu^2 - \tfrac{1}{4}n^2)\, x^{\nu-1} + (\kappa/s)\, x^{\nu}$$

we now get

$$\begin{aligned} 0 = M q = \; & c_0 \left[(m^2 - \tfrac{1}{4}n^2)\, x^{m-1} + (\kappa/s)\, x^{m} \right] \\ & + c_1 \left[\left\{ (m+1)^2 - \tfrac{1}{4}n^2 \right) x^{m} + (\kappa/s)\, x^{m+1} \right] \\ & + c_2 \left[\left\{ (m+2)^2 - \tfrac{1}{4}n^2 \right) x^{m+1} + (\kappa/s)\, x^{m+2} \right] \\ & + \cdots \end{aligned}$$

whence

$$c_0 \left\{ m^2 - \tfrac{1}{4}n^2 \right\} = 0$$

$$c_1 \left\{ (m+1)^2 - \tfrac{1}{4}n^2 \right\} + \frac{\kappa}{s} c_0 = 0$$

$$c_2 \left\{ (m+2)^2 - \tfrac{1}{4}n^2 \right\} + \frac{\kappa}{s} c_1 = 0$$

etc.

The first equation could be satisfied if $c_0 = 0$, but then all the c's would vanish and the trivial solution $\varphi \equiv 0$ would result. Hence $m = \pm\frac{1}{2}n$ are the only other possibilities. Corresponding to these, we can determine two sets of c's and if n is not an integer, get two solutions; which may be written

$$\varphi_1 = J_n\left(\sqrt{\frac{4\kappa x}{s}}\right), \quad \varphi_2 = J_{-n}\left(\sqrt{\frac{4\kappa x}{s}}\right) \qquad 36.$$

where

$$J_n(\sigma) = c_0\left(\frac{\sigma}{2}\right)^n\left[1 - \frac{1}{1(n+1)}\left(\frac{\sigma}{2}\right)^2 + \frac{1}{1\cdot 2\cdot(n+1)(n+2)}\left(\frac{\sigma}{2}\right)^4 \cdots\right] \qquad 37.$$

For systematic reasons the arbitrary constant c_0 is set equal to $1/\Gamma(n+1)$. It is seen that, when $n = 0$, these two solutions become identical, thus explaining the peculiar difficulty with Eq. (32*). The general solution of Eq. (32**) is $\varphi = A\varphi_1 + B\varphi_2$; consequently the function $N_n(\sigma)$ where

$$N_n(\sigma) = \left[\cos\pi n\, J_n(\sigma) - J_{-n}(\sigma)\right] / \sin\pi n \qquad 38.$$

is also a solution of this equation;* it is known as Neumann's function. When $n \to 0$, both numerator and denominator become zero, but the function can be determined by evaluating the indeterminate form:

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*S.R.M.P., p. 16

$$N_0(\sigma) = \frac{2}{\pi}\left(\frac{\partial J_n(\sigma)}{\partial n}\right)_{n=0} .$$

From Eq. (37),

$$\frac{\partial J_n}{\partial n} = \left[\log\frac{\sigma}{2} - \frac{1}{\Gamma(n+1)}\frac{\partial\Gamma(n+1)}{dn}\right] J_n(\sigma)$$

$$+ \frac{1}{\Gamma(n+1)}\left(\frac{\sigma}{2}\right)^n \Big[0 + \frac{1}{1(n+1)}\left[\frac{1}{n+1}\right]\left(\frac{\sigma}{2}\right)^2$$

$$- \frac{1}{1\cdot 2\,(n+1)(n+2)}\left[\frac{1}{n+1} + \frac{1}{n+2}\right]\left(\frac{\sigma}{2}\right)^4 + \cdots$$

Hence

$$\frac{\pi}{2} N_0(\sigma) = J_0(\sigma)\log(\gamma\sigma/2) + \Big[\frac{1}{1^2}\left(\frac{1}{1}\right)\left(\frac{\sigma}{2}\right)^2$$

$$+ \frac{1}{1^2 2^2}\left[\frac{1}{1} + \frac{1}{2}\right]\left(\frac{\sigma}{2}\right)^4 + \frac{1}{1^2 2^2 3^2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right)\left(\frac{\sigma}{2}\right)^6 \cdots$$

where $\Gamma(1) = 1$ and

$$-\log\gamma = \left[\frac{d}{dn}\log\Gamma(n+1)\right]_{n=0} = 0.577 .$$

It is now clear that the two independent solutions of Eq. (32*) are

$$\phi_0 = J_0\left(\sqrt{\frac{4\kappa\chi}{s}}\right) \quad \text{and} \quad \phi_0 = N_0\left(\sqrt{\frac{4\kappa\chi}{s}}\right) \qquad 39.$$

and we proceed to consider them in more detail.

Figure 1 shows a graph of these two functions, plotted with $4\kappa\chi/s$ as abscissa. It is seen that both are oscillatory, and that their wavelength diminishes with diminishing χ. As χ approaches zero, $J_0 \to 1$, while $N_0 \to -\infty$. However, N_0 has its last root so close to the origin that it is difficult to represent this feature already on the scale of Fig. 1. Figure 2 is plotted with the χ-scale enlarged by a factor of 100, and shows, among others, a graph of $-N_0$. This graph has its smallest root at $4\kappa\chi/s = 0.81$, and rises to infinity between this point and $\chi = 0$. The function J_0, on this same scale, is represented by a straight line that has approximately the same slope as the graph of $\int f_0$ for larger values of χ.

The question arises, whether the infinity of the function N_0 can possibly represent the breakers. This must be answered negatively. This is most simply seen by a numerical example: let the period of the waves be 20 sec., and the slope of the bottom 2 %; then $s/4\kappa = 50$ cm. For this example, therefore, the whole graph of Fig. 2 represents only 125 cm, the water's edge being at the left. The maximum depth of water, at the right hand edge, is only 2.5 cm. The smallest root of N_0 occurs 40 cm from the water's edge, and the depth here is only 0.8 cm. It is obvious that there is little physical significance to be ascribed to the solution in this range of χ-values. It is scarcely necessary to enter into all the reasons: one is that the wave heights in which we are interested are all very much greater than the water depths involved. The contrary has been assumed in deriving the boundary condition at the free surface.

A second example is of interest: the periods of tsunamis are of the order of 15 minutes. For this period and $s = 2\%$, $s/4\kappa = 1$ km. Thus, in this case, Fig. 2 covers 2.5 km, and water depths up to 50 meters. Even so, the function $-N_0$ does not exceed J_0 except for $x < 60$ meters, where the water depth is less than 1.2 meters. Again, it is scarcely possible to ascribe any physical significance to the different mathematical properties of the two solutions at $x = 0$. We shall return to this matter below.

These examples make it clear that, for waves of less than 1 minute period, on beaches that do not slope very steeply, we shall be interested primarily in values of $4\kappa x/s$ that are greater than 100. For such large values of the argument, it is known that*

$$\begin{aligned} J_0(\sigma) &\sim \cos(\sigma - \pi/4)\Big/\sqrt{\pi\sigma/2} \\ N_0(\sigma) &\sim \sin(\sigma - \pi/4)\Big/\sqrt{\pi\sigma/2} \end{aligned} \qquad 40.$$

These equations have been used to construct Fig. 3, which extends the graphs of J_0 and N_0 to larger values of x.

It is interesting to note that Eq. (40) is exactly the result we should have obtained from the ray theory approximation: for, if

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* S.F.M P. p. 22.

$$X = \sigma = 2\sqrt{\kappa x/\rho}$$

$$\frac{dX}{dx} = \sqrt{\frac{\kappa}{\rho x}} = \frac{\omega}{\sqrt{gh}} = k$$

Moreover, the height of the waves, from Eq. (40), is proportional to $1/\sqrt{\sigma}$ or $1/h^{1/4}$, again in complete agreement with ray theory.

The phase constants, $\pi/4$, in Eq. (40) are rather typical, and will recur in other applications. It may be noted that, for large σ,

$$\frac{1}{\sqrt{2}}\left(-N_0 + J_0\right) = \cos(\sigma)/\sqrt{\pi\sigma/2}$$

$$\frac{1}{\sqrt{2}}\left(N_0 + J_0\right) = \sin(\sigma)/\sqrt{\pi\sigma/2}$$

These two combinations are plotted, for small χ, on Fig. 2.

The different behavior of the two solutions J_0 and N_0 cannot be dismissed without further investigation. It may serve to clarify the problem if it is known that these functions also occur in other physical problems; a typical one is that of alternating electromagnetic fields. In that case, the function N_0 represents fields near a very thin wire that is located at the logarithmic singularity and carries a current; the function J_0 represents fields in the absence of such a wire. In this case there is a clear physical reason for the mathematical difference between the two solutions.

In the present case, it would seem that no physical reason is to be expected, since the equations we are solving cease to be valid for such small values of χ. It is therefore suggested that we somehow exclude these regions of very shallow water from consideration, in order to obtain an understanding of the problem. Let us suppose that the water depth is still given by $h = s x$ for $x > x_0$, but that, at $x = x_0$, a vertical wall exists. Moreover, suppose the wave-height is small enough so that the waves do not break against this wall, but are reflected without breaking. Then all our previous equations remain valid for $x > x_0$, but a new equation enters the problem; this equation states that no water flows past the point $x = x_0$, and is $u(x_0, y, z) = 0$. This will be satisfied if and only if

$$\frac{dp_0}{dx} = 0 \quad \text{for } x = x_0 \qquad 41.$$

Now, the most general solution for p_0 is (A, B constant)

$$p_0 = (A J_0 + B N_0) \exp\left[-i(\omega t + \phi)\right] \qquad 42.$$

and the Eq. (41) will be satisfied if

$$A = C N_0'\left(\sqrt{\frac{4\kappa x_0}{s}}\right), \quad B = -C J_0'\left(\sqrt{\frac{4\kappa x_0}{s}}\right), \qquad 42a.$$

where C is an arbitrary constant, and the accent indicates differentiation with respect to the argument of the function. If x_0 is large enough so that Eq. (40) may be used, this becomes

$$p_0 = C \cos\left[2\sqrt{\frac{\kappa}{s}}\left(\sqrt{x} - \sqrt{x_0}\right)\right] \Big/ \sqrt{\pi\sqrt{\kappa x/s}} \qquad 43.$$
$$\cdot \exp -i(\omega t + \phi)$$

and represents a wave that has been totally reflected at $x = x_0$.

The important thing to be noted is that the additional boundary condition, Eq. (41) has reduced the number of solutions from two to one; but it has not eliminated the Neumann function from consideration. If we wish to let χ_0 become small, we cannot use Eq. (43) any longer; more important is the physical fact, that in order to keep the waves from breaking, the constant C must be made ever smaller. We can assure this if we set

$$C = 1 \Big/ \sqrt{\left\{ [N_0']^2 + [J_0']^2 \right\}_{\chi = \chi_0}} ,$$

if we do this, then, when $\chi_0 \to 0$ $A \to 1$ $B \to 0$, and we do (rather artificially) eliminate the function N_0 from our considerations.

This example has definitely excluded surf or the breaking of waves. We consider another example that has one characteristic in common with the surf: the removal of energy from the orderly motion of the wave. In the case of surf, this energy is converted into disorderly motion. We cannot treat this within the framework of our present theory, but, if we suppose the sea wall to be movable, and that its motion is resisted by friction, we can treat the removal of energy from the waves.

Let U be the velocity of the wall, M its mass, R the frictional constant and F the force exerted on it by the waves (M, R, and F are all to be calculated per unit length): then

$$M \frac{dU}{dt} + RU = F ,$$

or, if $d/dt = -i\omega$

$$F = (R - i\omega M) U = ZU .$$

By analogy to electrical and acoustical quantities, we may call Z the complex impedance of the wall.

Now, since the wall moves with the wave,

$$\rho i \omega U = \left(\frac{\partial p_0}{\partial x}\right)_{x = x_0}$$

and (to a certain approximation)

$$F = -\Delta x_0 \, p_0(x_0) \, .$$

These equations reduce to the boundary condition

$$Z\left(\frac{\partial p_0}{\partial x}\right) + i \rho \omega \Delta x_0 p_0 = 0 \quad \text{at } x = x_0 . \qquad 44.$$

Substituting the general solution for p_0, we find

$$A = C\left[Z\sqrt{\frac{\kappa}{\Delta x_0}}\, N_0' + i\rho\omega\Delta x_0 N_0\right]$$

$$B = -C\left[Z\sqrt{\frac{\kappa}{\Delta x_0}}\, J_0' + i\rho\omega\Delta x_0 J_0\right] , \qquad 45.$$

the argument in the Bessel functions being $\sqrt{4\kappa x_0/s}$ throughout.

Again, we obtain an additional boundary condition, and this again reduces the number of independent solutions from two to one but, again, we do not eliminate the Neumann function.

At the expense of some algebra, one may write the expression for p_o in the form

$$p_o = \frac{1}{\sqrt{\pi}}\left(\frac{s}{\kappa x}\right)^{1/4}\Big\{A_i\, e^{i[-(\sigma-\sigma_o)-\omega t-\phi]} \qquad 46.$$

$$+ A_r\, e^{i[+(\sigma-\sigma_o)-\omega t-\phi]}$$

where

$$A_i = \left[(R-iM\omega)\sqrt{\frac{\kappa}{\Delta x_o}} + \rho\omega\Delta x_o\right]C \qquad 47.$$

$$A_r = \left[(R-iM\omega)\sqrt{\frac{\kappa}{\Delta x_o}} - \rho\omega\Delta x_o\right]C$$

and χ is supposed large. The equation (46) represents an incoming wave of amplitude proportional to A_i, and a reflected wave of amplitude proportional to A_r. The fraction of the incoming energy that is reflected is

$$|A_r/A_i|^2 = \{[R-\rho(\Delta x_o)^{3/2} g^{1/2}]^{2} + M^2\omega^2\} \qquad 48.$$

$$\div \{[R+\rho(\Delta x_o)^{3/2} g^{1/2}]^2 + M^2\omega^2\}.$$

These examples are highly artificial but serve several purposes. One is the introduction of the quantity $\rho(\Delta x_o)^{3/2}(g)^{1/2}$. If one may argue by analogy to electrical and acoustical problems (always a dangerous procedure) this quantity will be of importance in any theory of surf. It may be called the characteristic impedance of the surf-zone. The value of x_0 is the distance from shore at which the waves begin to break.

The other purpose is to show that our previous equations have been physically incomplete, and that one additional boundary condition is needed. If it is imposed at some point other than $x = 0$ or $x = \infty$, it may take the general form of Eq. (41), or (44), depending on the physical nature of the problem. However, it may also be imposed at $x = 0$: then, in most physical problems, it takes the form

$$\lim_{x \to 0} p_0 = \text{finite} \qquad 49.$$

In our present case, this seems rather artificial (but so are the previous possibilities); if we accept it, then we can restrict our attention to the solution J_0 and exclude N_0. The last possibility is that the boundary condition be imposed at great distances, and is

$$\lim_{x \to \infty} \left(\frac{\partial p_0}{\partial x} + i k p_0 \right) = 0 . \qquad 50.$$

This says, essentially, that there are no waves running away from shore at great distances. All of the energy of the incoming waves is somehow absorbed in the surf zone. Perhaps this equation makes the best sense; this is certainly true of waves whose periods and amplitudes correspond to ordinary swell. However, if we deal with very long period, low amplitude waves, such as the surf beat, the tides or the seiches of lakes and harbors, it is unlikely that Eq. (50)

will be reasonable, since these waves do not break. For such problems, therefore, one of the other boundary conditions would seem more reasonable. Because of its mathematical simplicity, we will choose Eq. (49) in such cases; this definitely constitutes an assumption, however.

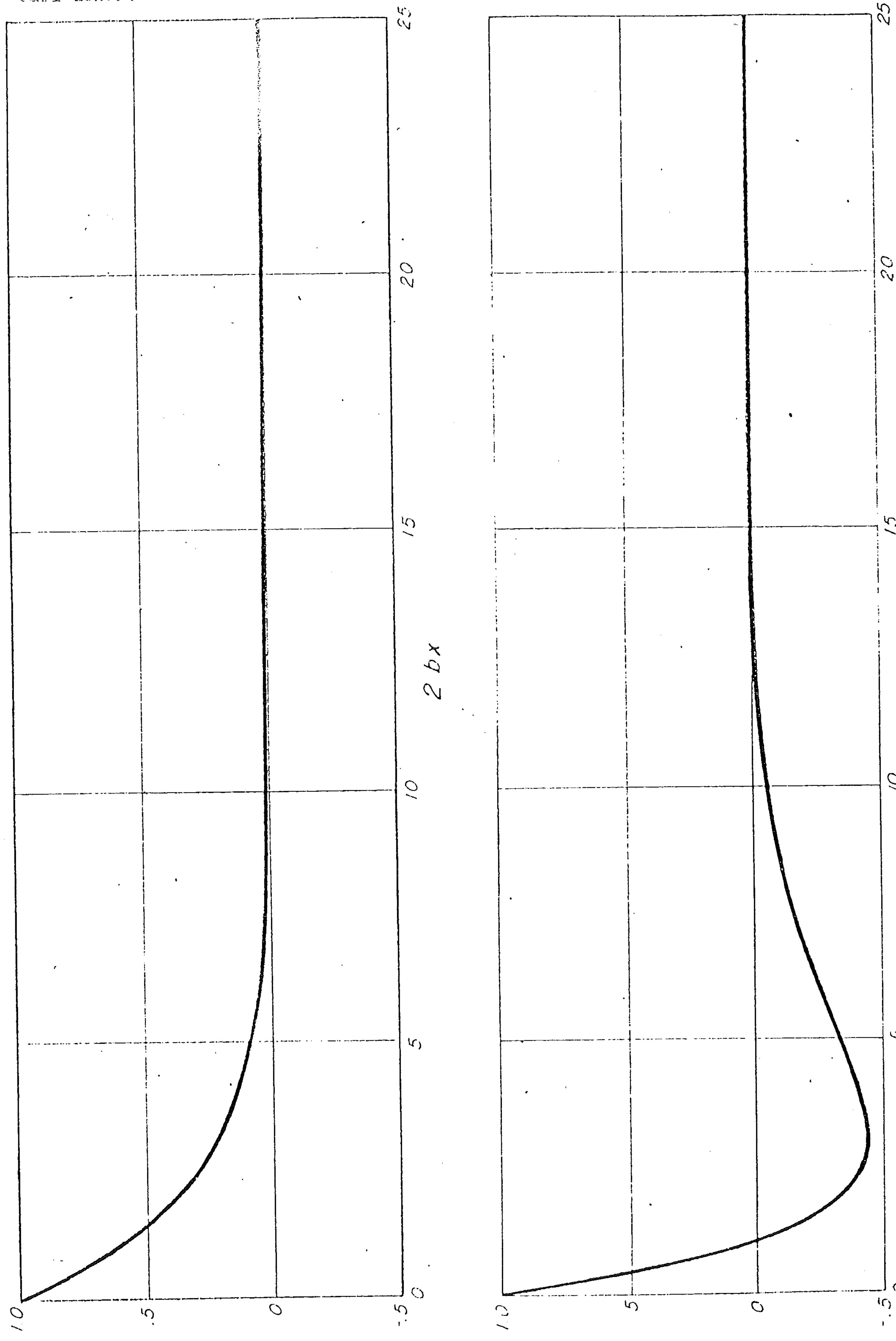

FIGURE 4.

Lecture, Fall Semester, 1950-51,
Carl Eckart

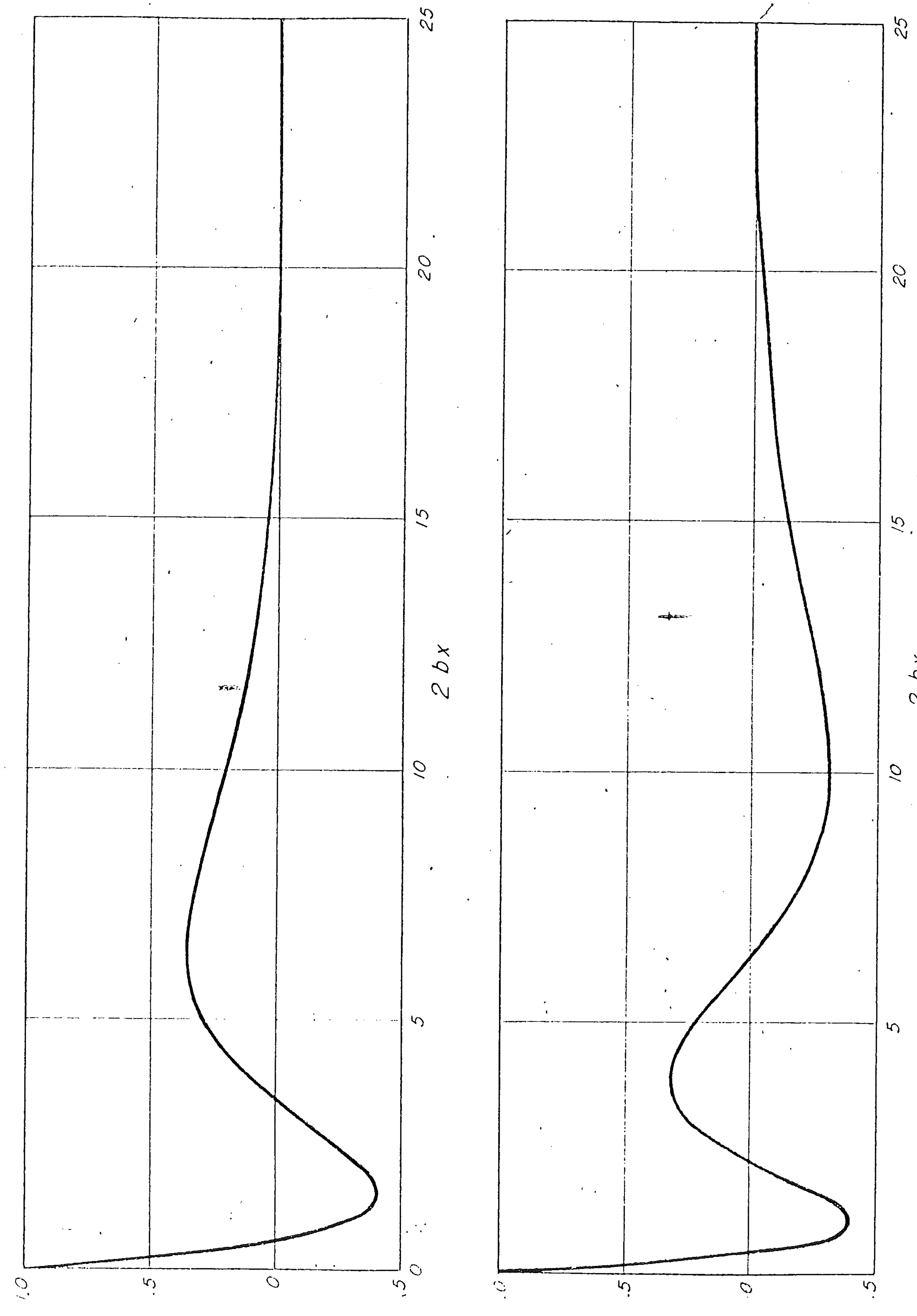

FIGURE 5.

Lecture, Fall Semester, 1950-51
Carl Eckart

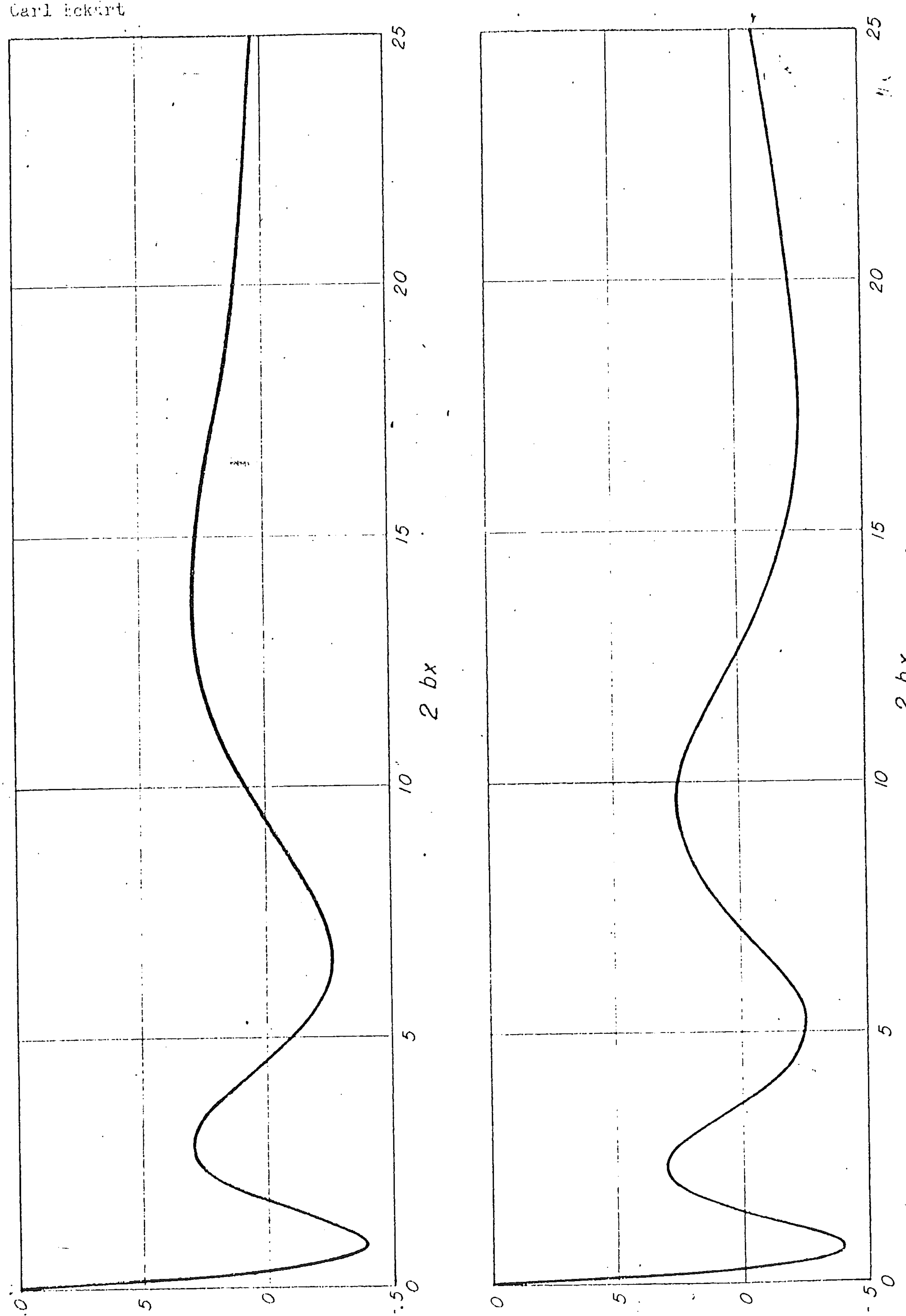

FIGURE 6.

8. The Solutions near the Water's Edge (continued)

We return to Huyghen's construction, but continue to take $h = s\chi$. Since $k^2 = \kappa / h$, it follows that the separation between successive phase lines will be

$$\delta s = \delta a / k = \delta a \sqrt{\frac{s\chi}{\kappa}}$$

and will thus increase with distance from shore. If the initial, arbitrary, phase line is not parallel to shore (i.e., to the y-axis) the successive phase lines will be more and more inclined as they leave the shore.

This can be verified from the geometric wave equation:

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = \frac{\kappa}{s\chi} . \qquad 50.$$

As before, let $S = X(x) + by$ Then

$$X = \int \sqrt{\frac{\kappa}{s x} - b^2}\, dx . \qquad 51.$$

The substitutions

$$v = \sqrt{x} = \sqrt{2R}\cos\tfrac{1}{2}\theta , \quad 2R = \kappa / s b^2 . \qquad 52.$$

enable one to evaluate the integral, and obtain

$$X = bR\,(-\theta + \sin\theta) . \qquad 53.$$

Since Eq. (52) is equivalent to

$$x = R(1 + \cos\theta) \qquad 53^a.$$

it is easily seen that the phase lines $S = X + by =$ const. are cycloids, generated by a circle of radius R rolling on the line $x = 2R$. They therefore have cusps on the line $x = 2R$, and are tangent to the y-axis at points separated by the distance $2\pi R$. It can be shown that the rays are similar cycloids, generated by rolling the circle of radius R on the y-axis. For convenience, the distance $2\pi R$ will be called the spring-length of the rays.

This indicates that waves originating in shallow water may be refracted until they run parallel to shore, and then turned even more, so that they ultimately return to the beach at a distance from their point of origin. If such waves do not break, they may repeat this history indefinitely, unless, because of the topography (curved shore line, etc.) they find an open route to sea. (Our present simple assumption that $h = sx$ does not enable us to deduce all details, and this last remark is really based on an example that will be considered later.)

The amplitude function A can be easily calculated (see Eq. (29)):

$$A^2 = b\Big/\sqrt{\frac{\kappa}{s x} - b^2}$$

$$= \tan\tfrac{1}{2}\theta .$$

The amplitude of the waves is proportional to

$$(sR)^{\frac{1}{2}} A / h^{1/2} = 1/(\sin\theta)^{\frac{1}{2}} .$$

Hence the ray theory approximation yields

$$p_0 = \frac{\cos\left[bR(\sin\theta - \theta) + \phi\right]}{(\sin\theta)^{1/2}} \cos by \cos \omega t \, ,$$

This expression becomes infinite for $\theta = 0$ and π , which values correspond to $x = 2R$ and 0 . We shall see that these infinities do not occur in the more exact physical theory; however, the above equation for p_0 can be made reasonably accurate for values of θ not too near the singular points, if the constant phase angle ϕ is given the right value.

Very little attention has been given this phenomenon until recently. Stokes and Lamb[1] considered it sufficiently to bestow the name "edge waves" on it. However, because there appeared to be no observational evidence for their existence, little more was done. Munk[2] recently has found that very long period (about 4 min.) waves of low amplitude (about 10 cm.) occur near shore, and has shown that they are generated by groups of high breakers which occur at about such time intervals. Later Isaacs[3] adduced reasons for supposing that the phase lines of this "surf beat" are inclined to the shore. The two data: generation near shore and inclined phase lines seem to equate edge waves to surf beat.

When the edge waves are examined from the point of view of the physical wave equation, the phenomenon appears somewhat different than described above. However, the principal reason for this is that ray theory is valid only when the diffraction terms are negligible. The appearance of infinities in the approximate solution is sufficient evidence to show that this is not the case here. However, we may still take it as a working hypothesis that edge waves and surf beat are identical.

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1 Lamb, Hydrodynamics, p. 447 (N.Y. 1945).

2 W. H. Munk, Trans. A.G.U., 30, 849 (1949).

3 Isaacs, Williams, and Eckart, Trans. A.G.U., 32, 37 (1951)

The canonic form of the physical wave equation becomes

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\kappa}{\alpha x} P = 0 .$$

The assumption that the phase lines are inclined to the shore takes the form

$$P(x, y) = Q(x) \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} (by)$$

where $b \neq 0$. (We have considered the case $b = 0$ in the last section.) This reduces the partial differential equation to the ordinary one:

$$\frac{d^2 Q}{dx^2} + \left(\frac{\kappa}{\alpha x} - b^2 \right) Q = 0 . \qquad 54.$$

By inspection of this equation, we immediately see that, when $x < 2R$, Q and d^2Q/dx^2 have opposite signs. Interpreting this graphically, the graph of Q must everywhere be concave toward the x-axis: i.e., the function Q oscillates for $x < 2R$. When $x > 2R$ on the other hand, Q and Q'' have the same sign, so that its graph is convex toward the x-axis. Typical functions having this property are e^{x}, e^{-x}, sinh x, cosh x. Consequently, we may expect that, in general, Q will become infinite for large values of x; only in special cases will it behave like e^{-x} or $1/x$ and approach zero. Now, since we are interested in waves generated near shore, we shall certainly exclude those solutions for which $Q(\infty) = \pm\infty$; this gives us one boundary condition; the other is given by the condition that the waves are so low that they do not break; this gives us $Q(0) =$ finite.

We proceed to study the physical wave equation in order to find these solutions, and the conditions for their existence. The Eq. (13), with

$$p_0(x,y) = q(x)\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}(by)$$

becomes

$$Mq \equiv \frac{d}{dx}\left(x\frac{dq}{dx}\right) + \left(\frac{\kappa}{s} - b^2 x\right)q = 0 \qquad 55.$$

which is rather more general than Eq. (33*). This is an awkward equation to study because $M x^\nu$ turns out to be a trinomial. The further substitution

$$q = e^{-bx} f \qquad 56.$$

reduces Eq. (55) to the seemingly more elaborate equation (known as Laguerre's equation)

$$Nf = x\frac{d^2 f}{dx^2} + (1-2bx)\frac{df}{dx} + \left(\frac{\kappa}{s} - b\right)f = 0 \qquad 57.$$

but, since

$$N x^\nu = \nu^2 x^{\nu-1} - 2b(\nu+\alpha)x^\nu, \quad \alpha = \frac{1}{2}\left(1 - \frac{\kappa}{sb}\right) \qquad 58.$$

is a binomial, Eq. (57) is actually much more tractable than Eq. (55).

We easily obtain the one solution

$$f = 1 + \frac{\alpha}{1^2}(2bx) + \frac{\alpha(\alpha+1)}{1^2 2^2}(2bx)^2 + \cdots\cdots \qquad 59.$$

which is a generalization of the Bessel function J_0. We also see that we again have to do with an exceptional case. However, since we are interested only in those solutions that remain finite at the water's edge, we need not trouble to find the analogue of Neumann's function. The series of Eq. (59) is well suited to determining the value of f for small values of $2bx$, but we wish to find its behavior for large values. To do this, we note that the first few terms form an increasing progression when $2bx$ is large, but that later terms diminish. The largest term is that for which the exponent of x is the integer nearest ν_0, where

$$2bx(\alpha + \nu_0)/\nu_0^2 = 1.$$

If $2bx \gg \alpha$ this is $\nu_0 = 2bx$. Under these conditions, we can approximate the most important terms in the series by setting $\alpha = 0$: this will make an appreciable error in the early terms, but not in the largest, and results in

$$f = 1 + \frac{1}{1!}(2bx) + \frac{1}{2!}(2bx)^2 + \cdots$$

$$= e^{2bx} \quad \text{when} \quad x \gg \alpha/2b.$$

Actually, this crude calculation has led us to a quite correct result: in general f becomes infinite for large x, and increases so rapidly that q also becomes infinite like $\exp(bx)$.

This shows that, in general, there is no solution for q that remains finite both at $x = 0$ and $x = \infty$. But our argument is subject to exception: when α is a negative integer (say $\alpha = -n$), the coefficient of $(2bx)^\nu$ is zero when $\nu > n$. The infinite series reduces to a polynomial — called the Laguerre polynomial $L_n(2bx)$. There are several notations for these polynomials: one is

$$L_n(x) = (-1)^n n! \left\{ 1 - \frac{n}{1^2} x + \frac{n(n-1)}{1^2 2^2} x^2 + \cdots \right\}. \qquad 59^a.$$

As can be verified by trial, or proven in other ways,*

$$L_n(x) = e^{x}\left(\frac{d}{dx}\right)^{n}\left(x^{n}e^{-x}\right). \qquad 59^{b}.$$

The first few of these polynomials are

$$L_0(x) = 1, \qquad -L_1(x) = -x + 1$$

$$L_2(x) = x^2 - 4x + 2$$

$$-L_3(x) = -x^3 + 9x^2 - 18x + 6$$

The graphs** of the functions , (Figs. 4, 5, 6)

$$\mathfrak{z}_n(2bx) = e^{-bx} L_n(2bx)/n!$$

verify the conclusions we drew concerning the function P: the $\mathfrak{z}_n$ oscillate for small values of x , but approach zero asymptotically for large x .

The function

$$p_0(x,y) = \mathfrak{z}_n(x) \sin by \cos \omega t$$

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* Courant and Hilbert, Methoden der mathem. Physik., p. 77 (Berlin, 1924).

** Brief tables of $\mathfrak{z}_n(x)$ are to be found in Physical Review 45 853 (1934).

represents standing waves. Their amplitude vanishes on the "nodal lines" determined by the roots of the factors q_n and $\sin by$. Consequently, there are n nodal lines parallel to shore, while those perpendicular to shore have the spacing $\pi / b = \frac{1}{2} \lambda_y$.

The distance λ_y may also conveniently be called the longitudinal wave length. One may also have solutions of the form

$$p_0 = q_n(x) \cos (by - \omega t).$$

These represent running waves, of length λ_y, propagated parallel to shore; their amplitude depends on x, because of the factor $q_n(x)$ and vanishes on the n nodal lines that run parallel to shore.

If we consider the equation $\alpha = -n$; it may be written

$$\frac{\kappa}{2 s b^2} = (n + \tfrac{1}{2}) / b$$

or

$$2 \pi R = (n + \tfrac{1}{2}) \lambda_y. \qquad 60.$$

In words, the spring-length of the rays is an odd integral multiple of half the longitudinal wave-length.

The period of the waves is $T = 2\pi/\omega$ whence

$$T = \sqrt{2\pi \lambda_y / (2n+1) g s} \qquad 61.$$

and thus depends on λ_y and the integer n : for every λ_y there are many possible periods, of which the longest is $\sqrt{2\pi\lambda_y/g}$. Conversely, for every period, there are many possible λ_y, of which the smallest is $g\sigma T^2/2\pi$.

9. The Seiches of a Circular Lake

The slow oscillations of a lake or other closed body of water are called seiches. These motions are so slow that no turbulence or breaking results. Consequently, we expect them to be reflected from the shore and therefore to be represented by those solutions of the wave equation that are everywhere finite.

This is not the only analogy to the edge waves; since the water deepens toward the center of the lake, the rays will be curved back toward the shore, resulting in arches, perhaps of variable spring length, depending on the topography. If the lake is circular and its bottom has no irregularities, the ray arches will be uniform in length. Perhaps one can immediately foresee that their vertices must coincide with those of a regular polygon; at any rate, this is one conclusion we shall reach.

Lamb* has treated the case

$$h = h_0(1 - r^2/R^2), \qquad r = \sqrt{x^2+y^2} \qquad 62.$$

which represents a lake of radius R in a paraboloidal basin filled to the height h_0 above its lowest point. Using polar coordinates

$$x = r\cos\phi \qquad y = r\sin\phi$$

the physical wave equation (Eq. (13)) becomes

$$h\left(\frac{\partial^2 p_0}{\partial r^2} + \frac{1}{r}\frac{\partial p_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p_0}{\partial \phi^2}\right) + \frac{dh}{dr}\frac{\partial p_0}{\partial r} + \kappa p_0 = 0. \qquad 63.$$

The simplest solutions of this equation have the forms

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* Hydrodynamics, p. 291, (N.Y. 1945).

$$p_0 = q(r) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (m\phi) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (\omega t); \tag{64}$$

since $p(r, \phi) = p(r, \phi + 2\pi)$, it follows that m can have any of the values 0, 1, 2 .... but no others.

The Eq. (64) shows that the nodal lines include the 2m radii $\phi = \pi/m, 2\pi/m, 3\pi/m,$ etc., in the case of the factor $\sin m\phi$, or, in the case of $\cos m\phi$, the 2m radii obtained from these by rotation through the angle $\pi/2m$.

We might also have solutions of the form $q(r) \sin m(\phi - \phi_0) \cos \omega t$, where ϕ_0 is an arbitrary constant; for these the 2m nodal radii are again equally spaced, and one is $\phi = \phi_0$. A slightly different type of solution is

$$p = q(r) \sin(m\phi - \omega t). \tag{63a}$$

This has no nodal radii, but represents waves that travel around the lake with the angular velocity ω/m. The waves therefore complete one revolution in m periods.

All of these solutions may have other nodal lines: the circles determined by the equation $q(r) = 0$, and we now proceed to determine this function. It must satisfy the equation

$$Mq \equiv \left(1 - \frac{r^2}{R^2}\right)\left(\frac{d^2q}{dr^2} + \frac{1}{r}\frac{dq}{dr} - \frac{m^2}{r^2}q\right) \tag{65}$$

$$- \frac{2r}{R^2}\frac{dq}{dr} + k_0^2 q = 0$$

where k_0, defined by

$$k_0^2 = \kappa/h_0 = \omega^2/gh_0, \tag{66}$$

is the wave number calculated for the deepest point of the lake.

As always, we first calculate

$$M r^{\nu} = (\nu^2 - m^2) r^{\nu-2} + \left[k_0^2 R^2 - (\nu+m+1)(\nu-m+1) + 1\right] r^{\nu}/R^2.$$

This can be somewhat simplified: on defining ρ by the equation

$$k_0^2 R^2 + 1 = (\rho + m + 1)(\rho - m + 1) \qquad 67.$$

it becomes

$$M r^{\nu} = (\nu - m)(\nu + m) r^{\nu-2} - (\nu - \rho)(\rho + \nu + 2) r^{\nu}/R^2.$$

Thus, letting

$$q = r^m (c_0 + c_1 r^2 + c_2 r^4 + \cdots$$

we find that

$$q = c_0 r^m F\left(\tfrac{1}{2}m - \tfrac{1}{2}\rho,\ \tfrac{1}{2}m + \tfrac{1}{2}\rho + 1,\ m+1,\ r^2/R^2\right) \qquad 68.$$

where

$$F(a, b, c, x) = 1 + \frac{a\,b}{1 \cdot c} x + \frac{a(a+1)\,b(b+1)}{1 \cdot 2\; c(c+1)} x^2 + \cdots$$

is the hypergeometric series, which will be encountered again later and then studied in more detail.

It might seem that we could get another solution of the form

$$q = r^{-m}\left(c_0 + c_1 r^2 + c_2 r^4 + \cdots\right)$$

and this would be true if m were not an integer. Being an integer, we should be led to conclude that $c_0 = c_1 = \cdots c_{m-1} = 0$, and end with the same solution as given by Eq. (68). The Eq. (65) is thus again an exceptional case; but, the second solution would become infinite at $r = 0$ i.e., in the center of the lake. Since we have already concluded that such a solution is not of interest we are spared the trouble of finding it.

We must, however, consider the behavior of the solution at the water's edge, $r = R$. The experience of the previous section leads us to expect difficulties at that point. And difficulties immediately appear, for the series of Eq. (68), in general, diverges when $r^2/R^2 \gg 1$. There is thus no easy way to calculate q for $r = R_0$, but there is the justified suspicion that it will become infinite there.

However, there are again special cases in which the series $F(a, b, c, x)$ is not infinite but represents a finite polynomial. These polynomial cases occur whenever either a or b is a negative integer, say $-n$. Then the hypergeometric function becomes a polynomial degree n in x. In these cases, q certainly remains finite for $r^2/R^2 = 1$.

We consequently conclude that the only solutions that remain finite everywhere are obtained when

$$\frac{1}{2}\rho - \frac{1}{2} m = n, \qquad n = 0, 1, 2 \cdots,$$

that is, when

$$\omega^2 R^2/g h_0 = k_0^2 R^2$$
$$= (2n+2m+1)(2n+1)-1$$
$$= 4n(n+m+1)+2m.$$
69.

In these cases*

$$\phi_{01} = \phi_{m,n} = (r/R)^m F(-n, n+m+1, m+1, r^2/R^2).$$ 70.

We consider a few of these solutions: let $m = n = 0$, $\phi_{00} = 1$, $\psi_{00} = 1$. Thus this solution is a static one, and represents the case of a permanent change in level of the lake. If $m = 0, n = 1$, $\omega_{01}^2 = 8 g h_0 / R^2$ and $\phi_{01} = 1 - 2 r^2/R^2$. Consequently, in this case there are no nodal radii, but a nodal circle at $r = R/\sqrt{2}$. If $m = 0$, $n = 2$, $\omega_{02}^2 = 24 g h_0 / R^2$, $\phi_{02} = 1 - 6(r/R)^2 + 6(r/R)^4$ and there are two nodal circles at $r = R(1 \pm 1/\sqrt{3})^{1/2}/\sqrt{2}$. In general, the (m, n) modes have n nodal circles, and the normal frequencies increase with both m and n, although not linearly.

This is scarcely an exhaustive treatment of seiches, but will serve to indicate their relation to edge waves, and to suggest the solution of other problems.

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*See Courant-Hilbert, METHODEN DER MATH., Physik, p. 75.

10. General Survey of the influence of Topography on Surface Waves.

Several examples of surface waves on water of variable depth have now been studied. The variety of solutions of the equations is apparent, and it becomes desirable to obtain some insight into the conditions under which each phenomenon occurs. This is best accomplished by a device that makes use of the analogy between the rays and the motion of a particle. It will first be described, and then a proof of the analogy will be presented.

The wave number, k, is a function of the water depth; in the cases we are studying

$$k^2 = \omega^2/gh.$$

Consequently, the contours of a constant depth on a chart of the sea floor are also the contours of constant k. We can also construct a relief map of k^2 -- or imagine it to have been constructed -- so that the equation of its surface is (to some scale)

$$z = -1/h\,(xy). \qquad 71.$$

This may be called the reciprocal relief map of the bottom. The plane $z = 0$ will be called the datum plane of the map; all of its points are below the datum plane. A submarine mound or ridge will appear as a depression on the reciprocal relief, and a submarine canyon will appear as a ridge. The water's edge is a steep cliff that drops off to an infinite distance beneath the datum plane.

If a ball is allowed to roll without friction on the reciprocal relief map, it will trace out one of the possible rays, provided that it has a certain total energy. This energy can be imparted to it by starting it from rest at the datum level and allowing it to roll down a chute until it reaches the surface of the map.

The proof of this is simple: the differential equations of the rays are given on p. 23; the parameter τ was not specifically determined. If it is determined so that

$$\frac{1}{2} u^2 = \frac{1}{2}\left[\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2\right] = g^2k^2/\omega^2 = g/h \qquad 72.$$

The equations on p. 23 reduce to

$$\frac{d^2x}{d\tau^2} = \frac{\partial}{\partial x}(g/h)$$

$$\frac{d^2y}{d\tau^2} = \frac{\partial}{\partial y}\cdot(g/h). \qquad 73.$$

The Eq. (73) are the equations of motion of a sphere rolling on the surface $z = -1/h$, while Eq. (72) specifies that the total energy is zero, when measured from the datum level.

It should be noted, however, that the time required for the sphere to trace out a given segment of the ray is not equal to the time required for the wave-front to travel the same distance. In fact, the sphere's velocity, u, will be inversely proportional to the wave velocity, $c = \omega/k$.

Since it is easy to visualize the reciprocal topography, and the rolling of a sphere, this analogy affords an easy and rapid method of analysing wave problems.

Since an actual ridge becomes a trough on the reciprocal map, it follows that rays will curve into the trough and tend to follow it . . . . However if the particle approaches the trough at an angle, its momentum may be sufficient to carry it over to the other side.

A submarine canyon is represented by a ridge on the reciprocal map. Hence, rays will tend to be deflected (refracted) away from canyons. However, again, the sphere may have sufficient energy to surmount the barrier, and the corresponding ray will traverse the canyon.

The cycloidal rays discussed in the preceding section are represented by a particle started up a slope, but eventually being turned back by the increasing elevation and returning to its original level. The analogy runs into difficulty near the water's edge: the sphere will eventually leave the surface and fall freely through the air. These difficulties with the analogy correspond to the mathematical problems already encountered with the physical wave equation.

The problem of the paraboloidal lake takes on a simple aspect in this analogy. The reciprocal map has its highest point at $r = 0$. To avoid the difficulties with the vertical slope at $r = a$, we may imagine the reflecting sea wall of p. 38 built at a radius slightly less than a. In the analogy, this will reflect the sphere, so that, as it rolls away from the high central region of the map, it encounters the wall and has its direction of motion reversed, climbs up the slope, and repeats the cycle. In this way, the path of the sphere will circle the center; only in certain cases, however, will it retrace its path exactly. These closed paths, and only these, correspond to the solutions we studied.

One may expect such closed paths to occur with other kinds of topography. For example, consider a circular sea mound, rising from an otherwise flat bottom. On the reciprocal map, this will correspond to a circular depression or cup in a flat region, and the rolling sphere <u>may</u> describe a closed path in the cup. However, its energy may be so great that the cup cannot restrain it. In this case, no closed paths will occur.

This can be seen by investigating possible circular paths. Let u, r, be the velocity and radius of the circle. Then the centrifugal force will be u^2/r, and must be balanced by the horizontal component of the force exerted by gravity. This is $g(dz/dr)$, so that

$$u^2/r = (g/h^2)(dh/dr).$$

However, by Eq. (72)

$$u^2 = 2g/h$$

so that

$$\frac{1}{r} = \frac{-1}{2k^2}\frac{dk^2}{dr} = \frac{1}{h}\frac{dh}{dr}.$$

or

$$2h/r = +\,dh/dr. \qquad 74.$$

Graphically, one may solve for r by plotting the two graphs, $y = 2h/r$ and $y = +\,dh/dr$. Their intersection will determine the radius of the possible circular ray. However, the two graphs may not intersect: then there will be no circular ray. If they have several points of intersection, there will be several circular rays. It can be shown that if there are no circular rays, the depression cannot ever trap a sphere whose total energy is zero. It can trap a sphere rolling with less energy -- i.e., started below the datum level. But the motion of such spheres has nothing

11. The Case $h = h_{\infty}(1 - e^{-\sigma x})$.

There is a great difference between the solutions of the physical wave equation when $h(x)$ does or does not become infinite for large x. This is best seen by considering the ray-particle analogy. If $h(x) \to \infty$ as $x \to \infty$, the inverse relief surface will approach the datum plane asymptotically. Therefore, if the ray-particle is started at a great distance from shore, it will start practically from rest. On the other hand, if $h(x)$ approaches a finite value h_{∞} as $x \to \infty$, the relief surface will become nearly a plane for large x, but the particle will start with the finite velocity $\sqrt{2g/h_{\infty}}$.

If h is a function of x only, the particle, in the first case, will immediately start, slowly, in a direction parallel to the x-axis, and travel normally toward shore. That is, at great distances from shore, all rays are incident normally. On the other hand, if the depth is finite, the ray-particle will start with a finite velocity, and this can be directed at any angle to the x-axis that one cares to choose. Thus, in this case, there are rays in every direction, even at very great distances from shore.

If the particle is started close to shore, this difference disappears, but another takes its place. The particle will then start with a finite velocity, determined by the depth of water at the starting point. In either case, it may be started at any angle I, to the x-axis. Suppose this angle is so chosen that the particle initially moves up the slope. Then, in the first case $(h(\infty) = \infty)$, it has just enough energy to climb up high enough to get far from shore. But, if it moves in any direction except parallel to the x-axis, not all of this energy is available for climbing. The component of its velocity parallel to the contours will remain constant, and the kinetic energy associated with this component cannot be converted into potential energy. Consequently, the particle will stop climbing when it reaches a certain height, and therefore, begin descending the slope again. We have seen that this results in the cycloidal rays.

In the second case, ($h(\infty)$ = finite), the particle has more than enough energy to climb onto the plateau. Consequently, it will do so unless the longshore component of its velocity is too great. For small angles, I , the rays will not be arched, but will extend up onto the plateau, becoming straighter, and eventually making an angle ψ with the x-axis. For larger angles, the amount of unavailable kinetic energy becomes too great, and the particle will no longer climb onto the plateau, but will be turned back as before; the rays will in this case have a qualitative resemblance to the cycloidal arches of the preceding case.

The same facts can be derived in other ways. For example, we may appeal to Snell's law of refraction, which states that, along the ray,

$$K(x) \sin I = constant = b . \qquad 75.$$

If the ray is one of those that extends to infinity , the value of the constant will be

$$b = K(x) \sin I = k_\infty \sin \psi < k_\infty .$$

However, there is no reason why the constant cannot have larger values than k_∞ : the ray can start anywhere at any angle, and near shore, $K(x) \gg k_\infty$. In this case, however, $\sin I$ must become unity (or the ray parallel to the y-axis) when $K(x) = b$: the root of this equation is the quantity we called $2R$ in the preceding section. At any point, there is a critical angle, I_c , which separates those rays that