Annali di Matematica (2014) 193:529–550
DOI 10.1007/s10231-012-0289-3

# Surfaces in three-dimensional space forms with divergence-free stress-bienergy tensor

**R. Caddeo · S. Montaldo · C. Oniciuc · P. Piu**

Received: 26 April 2012 / Accepted: 5 September 2012 / Published online: 23 September 2012


**Abstract** We introduce the notion of *biconservative* hypersurfaces, that is hypersurfaces with conservative *stress-energy* tensor with respect to the bienergy. We give the (local) classification of biconservative surfaces in three-dimensional space forms.

**Keywords** Biharmonic maps · Stress-energy tensor · Space forms

**Mathematics Subject Classification (2000)** 58E20 · 53A05 · 53C42

## 1 Introduction

A hypersurface $M^m$ in an $(m+1)$-dimensional Riemannian manifold $N^{m+1}$ is called *biconservative* if

$$2A(\operatorname{grad} f) + f \operatorname{grad} f = 2f \operatorname{Ricci}^N(\eta)^\top, \tag{1}$$

where $A$ is the shape operator, $f = \operatorname{trace} A$ is the mean curvature function, and $\operatorname{Ricci}^N(\eta)^\top$ is the tangent component of the Ricci curvature of $N$ in the direction of the unit normal $\eta$ of $M$ in $N$.

R. Caddeo · S. Montaldo (✉) · P. Piu
Dipartimento di Matematica e Informatica, Università degli Studi di Cagliari,
Via Ospedale 72, 09124 Cagliari, Italy
e-mail: montaldo@unica.it

R. Caddeo
e-mail: caddeo@unica.it

P. Piu
e-mail: piu@unica.it

C. Oniciuc
Faculty of Mathematics, "Al.I. Cuza" University of Iasi,
Bd. Carol I no. 11, 700506 Iasi, Romania
e-mail: oniciucc@uaic.ro

The name biconservative, as we shall describe in Sect. 2, comes from the fact that condition (1) is equivalent to the conservativeness of a certain *stress-energy* tensor $S_2$, that is div $S_2 = 0$ if and only if the hypersurface is biconservative. The tensor $S_2$ is associated with the bienergy functional. In general, a submanifold is called *biconservative* if div $S_2 = 0$.

Moreover, the class of biconservative submanifolds includes that of biharmonic submanifolds, which have been of large interest in the last decade (see, for example, [1–4,9,19,20]). Biharmonic submanifolds are characterized by the vanishing of the bitension field and they represent a generalization of harmonic (minimal) submanifolds. In fact, as detailed in Sect. 2, a submanifold is biconservative if the tangent part of the bitension field vanishes. It is worth to point out that, thinking at the energy functional instead of the bienergy functional, the notion of *conservative* submanifolds is not useful as all submanifolds are conservative (see Remark 2.1). We also would like to point out that submanifolds with vanishing tangent part of the bitension field have been considered by Sasahara in [22] where he studied certain three-dimensional submanifolds in $\mathbb{R}^6$.

In this paper, we consider biconservative surfaces in a three-dimensional space form $N^3(c)$ of constant sectional curvature $c$. In this case, (1) becomes

$$2A(\operatorname{grad} f) + f \operatorname{grad} f = 0. \tag{2}$$

From (2), we see that CMC surfaces, that is, surfaces with constant mean curvature, in space forms are biconservative. Thus, our interest will be on NON CMC biconservative surfaces.

As a general fact, we first prove that the mean curvature function $f$ of a biconservative surface in a three-dimensional space form satisfies the following PDE

$$f\Delta f + |\operatorname{grad} f|^2 - \frac{16}{9}K(K-c) = 0,$$

where $K$ denotes the Gauss curvature of the surface, while $\Delta$ is the Laplace–Beltrami operator on $M$.

Then, the paper is completely devoted to the local classification of biconservative surfaces in three-dimensional space forms. This is done in three sections where we examine, separately, the cases of: surfaces in the three-dimensional Euclidean space; surfaces in the three-dimensional sphere; surfaces in the three-dimensional hyperbolic space.

For biconservative surfaces in $\mathbb{R}^3$, we shall reprove a result of Hasanis and Vlachos contained in [13], where they call $H$-surfaces the biconservative surfaces.

**Theorem 4.5** *Let $M^2$ be a biconservative surface in $\mathbb{R}^3$ with $f(p) > 0$ and grad $f(p) \neq 0$ for any $p \in M$. Then, locally, $M^2$ is a surface of revolution.*

In fact, we give the explicit parametrization of the profile curve of a biconservative surface of revolution (see Proposition 4.1), which is not in [13]. In their paper, the authors also studied the case of biconservative hypersurfaces in $\mathbb{R}^4$ obtaining a similar result to Theorem 4.5.

Our approach is slightly different and allows us to go further and classify the biconservative surfaces in $\mathbb{S}^3$ and in $\mathbb{H}^3$. Moreover, the notion of biconservative submanifolds is more general than the notion of $H$-hypersurfaces in $\mathbb{R}^n$.

Considering $\mathbb{S}^3$ as a submanifold of $\mathbb{R}^4$, the biconservative surfaces in $\mathbb{S}^3$ are characterized by the following

**Theorem 5.2** *Let $M^2$ be a biconservative surface in $\mathbb{S}^3$ with $f(p) > 0$ and grad $f(p) \neq 0$ at any point $p \in M$. Then, locally, $M^2 \subset \mathbb{R}^4$ can be parametrized by*

$$X_C(u,v)=\sigma(u)+\frac{4}{3\sqrt{C}k(u)^{3/4}}\big(C_1(\cos v-1)+C_2\sin v\big),$$

*where $C$ is a positive constant of integration, $C_1, C_2 \in \mathbb{R}^4$ are two constant orthonormal vectors such that*

$$\langle\sigma(u), C_1\rangle=\frac{4}{3\sqrt{C}k(u)^{3/4}},\quad \langle\sigma(u), C_2\rangle=0,$$

*while $\sigma=\sigma(u)$ is a curve lying in the totally geodesic $\mathbb{S}^2=\mathbb{S}^3\cap\Pi$ ($\Pi$ the linear hyperspace of $\mathbb{R}^4$ orthogonal to $C_2$), whose geodesic curvature $k=k(u)$ is a positive non-constant solution of the following ODE*

$$k''k=\frac{7}{4}(k')^2+\frac{4}{3}k^2-4k^4.$$

Geometrically, Theorem 5.2 means that, locally, the surface $M^2$ is given by a family of circles of $\mathbb{R}^4$, passing through the curve $\sigma$ and belonging to a pencil of planes which are parallel to the linear space spanned by $C_1$ and $C_2$. Now, these circles must be the intersection of the pencil with the sphere $\mathbb{S}^3$. Let $G$ be the one-parameter group of isometries of $\mathbb{R}^4$ generated by the Killing vector field

$$T=\langle\mathbf{r}, C_2\rangle C_1+\langle\mathbf{r}, C_1\rangle C_2,$$

where $\mathbf{r}$ represents the position vector of a point in $\mathbb{R}^4$. Then, $G$ acts also on $\mathbb{S}^3$ by isometries and it can be identified with the group $SO(2)$. Since the orbits of $G$ are circles of $\mathbb{S}^3$, we deduce that $X_C(u,v)$, in Theorem 5.2, describes an SO(2) invariant surface of $\mathbb{S}^3$ obtained by the action of $G$ on the curve $\sigma$. Moreover, as we shall explain in Remark 5.3, there exist solutions of the ODE in Theorem 5.2 for the corresponding profile curve $\sigma$. Although we are not able to give explicit solutions for $\sigma$, as we have done for the biconservative surfaces in $\mathbb{R}^3$, using Mathematica, we give a plot of a numerical solution of the ODE in Theorem 5.2, which describes the behavior of the curvature of $\sigma$.

Let us consider the following model for the hyperbolic space

$$\mathbb{H}^3=\{(x_1,x_2,x_3,x_4)\in\mathbb{L}^4\ :\ x_1^2+x_2^2+x_3^2-x_4^2=-1,\ x_4>0\},$$

where $\mathbb{L}^4$ is the four-dimensional Lorentz–Minkowski space. Then, we have the following description of biconservative surfaces in $\mathbb{H}^3$.

**Theorem 6.2** *Let $M^2$ be a biconservative surface in $\mathbb{H}^3$ with $f(p)>0$ and grad $f(p)\neq 0$ at any point $p\in M$. Put $W=9|\mathrm{grad}\, f|^2/(16f^2)+9f^2/4-1$. Then, locally, $M^2\subset\mathbb{L}^4$ can be parametrized by:*

*(a) if $W>0$*

$$X_C(u,v)=\sigma(u)+\frac{4}{3\sqrt{C}k(u)^{3/4}}\big(C_1(\cos v-1)+C_2\sin v\big),$$

*where $C$ is a positive constant of integration, $C_1, C_2\in\mathbb{L}^4$ are two constant vectors such that*

$$\langle C_i, C_j\rangle=\delta_{ij},\quad \langle\sigma(u), C_1\rangle=\frac{4}{3\sqrt{C}k(u)^{3/4}},\quad \langle\sigma(u), C_2\rangle=0,$$

*while $\sigma=\sigma(u)$ is a curve lying in the totally geodesic $\mathbb{H}^2=\mathbb{H}^3\cap\Pi$ ($\Pi$ the linear hyperspace of $\mathbb{L}^4$ defined by $\langle\mathbf{r}, C_2\rangle=0$), whose geodesic curvature $k=k(u)$ is a* *positive non-constant solution of the following ODE*

$$k''k = \frac{7}{4}(k')^2 - \frac{4}{3}k^2 - 4k^4.$$

*(b) if* $W < 0$

$$X_C(u,v) = \sigma(u) + \frac{4}{3\sqrt{-C}k(u)^{3/4}}\big(C_1(e^v - 1) + C_2(e^{-v} - 1)\big),$$

*where* $C$ *is a negative constant of integration,* $C_1, C_2 \in \mathbb{L}^4$ *are two constant vectors such that*

$$\langle C_i, C_i\rangle = 0, \quad \langle C_1, C_2\rangle = -1, \quad \langle \sigma(u), C_1\rangle = \langle \sigma(u), C_2\rangle = -\frac{2\sqrt{2}}{3\sqrt{-C}k(u)^{3/4}},$$

*while* $\sigma = \sigma(u)$ *is a curve lying in the totally geodesic* $\mathbb{H}^2 = \mathbb{H}^3 \cap \Pi$ *(*$\Pi$ *the linear hyperspace of* $\mathbb{L}^4$ *orthogonal to* $C_1 - C_2$*), whose geodesic curvature* $k = k(u)$ *is a positive non-constant solution of the same ODE in* (a).

We note that a surface in a three-dimensional space form for which both tangent and normal part of its bitension field vanish, that is, a biharmonic surface, must be CMC (see [6,8]). Therefore, the assumption that only the tangent part of the bitension field vanishes does not imply that the surface is CMC.
Conventions. Throughout this paper, all manifolds, metrics, and maps are assumed to be smooth, that is, of class $C^\infty$. All manifolds are assumed to be connected. The following sign conventions are used

$$\Delta^\varphi V = -\text{trace}\,\nabla^2 V, \qquad R^N(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

where $V \in C(\varphi^{-1}(\text{TN}))$ and $X, Y \in C(\text{TN})$.

By a *submanifold* $M$ in a Riemannian manifold $(N,h)$ we mean an isometric immersion $\varphi : M \to (N,h)$.

## 2 Biharmonic maps and the stress-energy tensor

As described by Hilbert in [14], the *stress-energy* tensor associated with a variational problem is a symmetric 2-covariant tensor $S$ conservative at critical points, that is, with $\operatorname{div} S = 0$.

In the context of harmonic maps $\varphi : (M,g) \to (N,h)$ between two Riemannian manifolds, that by definition are critical points of the energy

$$E(\varphi) = \frac{1}{2}\int_M |\mathrm{d}\varphi|^2\, v_g,$$

the stress-energy tensor was studied in detail by Baird and Eells in [5] and Sanini in [21]. Indeed, the Euler–Lagrange equation associated with the energy is equivalent to the vanishing of the tension field $\tau(\varphi) = \text{trace}\,\nabla \mathrm{d}\varphi$ (see [11]), and the tensor

$$S = \frac{1}{2}|\mathrm{d}\varphi|^2 g - \varphi^* h$$

satisfies $\operatorname{div} S = -\langle \tau(\varphi), \mathrm{d}\varphi\rangle$. Therefore, $\operatorname{div} S = 0$ when the map is harmonic.

*Remark 2.1* We point out that, in the case of isometric immersions, the condition $\operatorname{div} S = 0$ is always satisfied, since $\tau(\varphi)$ is normal.

A natural generalization of harmonic maps, first proposed in [12], can be obtained considering the *bienergy* of $\varphi : (M, g) \to (N, h)$ which is defined by

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 \, v_g.$$

The map $\varphi$ is *biharmonic* if it is a critical point of $E_2$ or, equivalently, if it satisfies the associated Euler–Lagrange equation

$$\tau_2(\varphi) = -\Delta\tau(\varphi) - \operatorname{trace} R^N(\mathrm{d}\varphi, \tau(\varphi))\mathrm{d}\varphi = 0.$$

The study of the stress-energy tensor for the bienergy was initiated in [15] and afterward developed in [17]. Its expression is

$$\begin{aligned} S_2(X, Y) = &\frac{1}{2}|\tau(\varphi)|^2\langle X, Y\rangle + \langle \mathrm{d}\varphi, \nabla\tau(\varphi)\rangle\langle X, Y\rangle \\ &-\langle \mathrm{d}\varphi(X), \nabla_Y\tau(\varphi)\rangle - \langle \mathrm{d}\varphi(Y), \nabla_X\tau(\varphi)\rangle, \end{aligned}$$

and it satisfies the condition

$$\operatorname{div} S_2 = -\langle \tau_2(\varphi), \mathrm{d}\varphi\rangle, \tag{3}$$

thus conforming to the principle of a stress-energy tensor for the bienergy.

If $\varphi : (M, g) \to (N, h)$ is an isometric immersion, then (3) becomes

$$\operatorname{div} S_2 = -\tau_2(\varphi)^\top.$$

This means that isometric immersions with $\operatorname{div} S_2 = 0$ correspond to immersions with vanishing tangent part of the corresponding bitension field. The decomposition of the bitension field with respect to its normal and tangent components was obtained with contributions of [1,7,16,18,19] and for hypersurfaces, it can be summarized in the following theorem.

**Theorem 2.2** *Let $\varphi : M^m \to N^{m+1}$ be an isometric immersion with mean curvature vector field $H = f\eta$. Then, $\varphi$ is biharmonic if and only if the normal and the tangent components of $\tau_2(\varphi)$ vanish, that is, respectively,*

$$\Delta f - f|A|^2 + f\,Ricci^N(\eta, \eta) = 0 \tag{4a}$$

*and*

$$2A(grad\, f) + f\, grad\, f - 2f\,Ricci^N(\eta)^\top = 0, \tag{4b}$$

*where $A$ is the shape operator, $f = trace\, A$ is the mean curvature function, and $Ricci^N(\eta)^\top$ is the tangent component of the Ricci curvature of $N$ in the direction of the unit normal $\eta$ of $M$ in $N$.*

Finally, from (4b), an isometric immersion $\varphi : M^m \to N^{m+1}$ satisfies $\operatorname{div} S_2 = 0$, that is, it is biconservative, if and only if

$$2A(\operatorname{grad} f) + f \operatorname{grad} f - 2f \operatorname{Ricci}^N(\eta)^\top = 0,$$

which is Eq. (1) given in the introduction.

## 3 Biconservative surfaces in the three-dimensional space forms

In this section, we consider the case of biconservative surfaces $M^2$ in a three-dimensional space form $N^3(c)$ of sectional curvature $c$. In this setting, (1) becomes

$$A(\operatorname{grad} f) = -\frac{f}{2}\operatorname{grad} f. \tag{5}$$

If $M^2$ is a CMC surface, that is $f$ = constant, then $\operatorname{grad} f = 0$ and (5) is automatically satisfied. Thus, biconservative surfaces include the class of CMC surfaces whether compact or not.

We now assume that $\operatorname{grad} f \neq 0$ at a point $p \in M$ and, therefore, there exists a neighborhood $U$ of $p$ such that $\operatorname{grad} f \neq 0$ at any point of $U$. On the set $U$, we can define an orthonormal frame $\{X_1, X_2\}$ of vector fields by

$$X_1 = \frac{\operatorname{grad} f}{|\operatorname{grad} f|}, \quad X_2 \perp X_1, \quad |X_2| = 1. \tag{6}$$

From (5), we have

$$A(X_1) = -\frac{f}{2}X_1,$$

thus $X_1$ is a principal direction corresponding to the principal curvature $\lambda_1 = -f/2$. Since $X_2 \perp X_1$, $X_2$ is a principal direction with eigenvalue $\lambda_2$ such that

$$f = \operatorname{trace} A = \lambda_1 + \lambda_2 = -\frac{f}{2} + \lambda_2$$

and therefore $\lambda_2 = 3f/2$. From this, using the Weingarten equation, we immediately see that the Gauss curvature of the surface is

$$K = \det A + c = -3f^2/4 + c \tag{7}$$

and the norm of the shape operator is $|A|^2 = 5f^2/2$. Moreover, by the definition of $X_1$, we obtain

$$(X_1 f)X_1 = \langle \operatorname{grad} f, X_1\rangle X_1 = \operatorname{grad} f.$$

Thus,

$$\operatorname{grad} f = (X_1 f)X_1 + (X_2 f)X_2 = \operatorname{grad} f + (X_2 f)X_2,$$

which implies that

$$X_2 f = 0. \tag{8}$$

We are now in the right position to state the main result of this section.

**Theorem 3.1** *Let $M^2$ be a biconservative surface in $N^3(c)$ which is not CMC. Then, there exists an open subset $U$ of $M$ such that the restriction of $f$ in $U$ satisfies the following equations*

$$K = \det A + c = -3f^2/4 + c \tag{9}$$

*and*

$$f\Delta f + |grad\, f|^2 - \frac{16}{9}K(K-c) = 0, \tag{10}$$

*where* $\Delta$ *is the Laplace–Beltrami operator on* $M$.

*Proof* Since $M^2$ is not CMC, there exists a point $p$ with grad $f(p) \neq 0$. Thus, grad $f \neq 0$ in a neighborhood $V$ of $p$. Now, since $f$ cannot be zero for all $q \in V$, there exists an open set $U \subset V$ with $f(q) \neq 0$ for all $q \in U$. Let us define on $U$ the local orthonormal frame $\{X_1, X_2\}$ as in (6) and let $\{\omega^1, \omega^2\}$ be the dual 1-forms of $\{X_1, X_2\}$ with $\omega_i^j$ the connection 1-forms given by $\nabla X_i = \omega_i^j X_j$. Since $f \neq 0$ on $U$, we can assume that $f > 0$ on $U$.

Equation (9) is just (7). We shall prove (10).

Since $A(X_1) = -(f/2)X_1$ and $A(X_2) = (3f/2)X_2$, from the Codazzi equation

$$\nabla_{X_1} A(X_2) - \nabla_{X_2} A(X_1) = A([X_1, X_2]),$$

we obtain

$$\left(4f\omega_2^1(X_1) + X_2 f\right) X_1 + \left(3X_1 f + 4f\omega_1^2(X_2)\right) X_2 = 0.$$

Since $X_2 f = 0$ and $f(p) \neq 0$ for all $p \in U$, we deduce that

$$\begin{cases} \omega_2^1(X_1) = 0 \\ \omega_2^1(X_2) = \frac{3}{4}\frac{X_1 f}{f}. \end{cases} \tag{11}$$

Next, using (11), the Gauss curvature of $M^2$ is

$$K = \langle R(X_1, X_2)X_2, X_1\rangle = X_1(\omega_2^1(X_2)) - (\omega_2^1(X_2))^2,$$

that, together with (7), gives

$$-\frac{3f^2}{4} + c = X_1(\omega_2^1(X_2)) - (\omega_2^1(X_2))^2$$

which is equivalent, taking into account (11), to

$$(X_1 X_1 f)f = \frac{7}{4}(X_1 f)^2 + \frac{4c}{3}f^2 - f^4. \tag{12}$$

Now, a straightforward computation gives

$$-\Delta f = X_1 X_1 f - \frac{3}{4f}(X_1 f)^2,$$

that, substituted in (12), taking into account (7), yields the desired equation

$$f\Delta f + |\text{grad}\, f|^2 - \frac{16}{9}K(K-c) = 0.$$

□

## 4 Biconservative surfaces in $\mathbb{R}^3$

We shall now consider the case of biconservative surfaces in $\mathbb{R}^3$. We start our study investigating in detail the case of surfaces of revolution. Without loss of generality, we can assume that the surface is (locally) parametrized by

$$X(u,v) = (\rho(u)\cos v, \rho(u)\sin v, u) \tag{13}$$

where the real-valued function $\rho$ is assumed to be positive. The induced metric is $\mathrm{d}s^2 = (1+\rho'^2)\mathrm{d}u^2 + \rho^2\mathrm{d}v^2$, and a routine calculation gives

$$A = \begin{pmatrix} -\frac{\rho''}{(1+\rho'^2)^{3/2}} & 0 \\ 0 & \frac{1}{\rho(1+\rho'^2)^{1/2}} \end{pmatrix}.$$

Thus,

$$f = \frac{1}{(1+\rho'^2)^{1/2}}\left(\frac{1}{\rho} - \frac{\rho''}{(1+\rho'^2)}\right),$$

and

$$\operatorname{grad} f = \frac{1}{(1+\rho'^2)} f' \frac{\partial}{\partial u}.$$

Then, (5) becomes

$$\frac{f'}{2(1+\rho'^2)^{3/2}}\left(\frac{3\rho''}{1+\rho'^2} - \frac{1}{\rho}\right) = 0. \tag{14}$$

**Proposition 4.1** *Let $M^2$ be a biconservative surface of revolution in $\mathbb{R}^3$ with non-constant mean curvature. Then, locally, the surface can be parametrized by*

$$X_C(\rho, v) = (\rho\cos v, \rho\sin v, u(\rho)),$$

*where*

$$u(\rho) = \frac{3}{2C}\left(\rho^{1/3}\sqrt{C\rho^{2/3}-1} + \frac{1}{\sqrt{C}}\ln\left[2(C\rho^{1/3} + \sqrt{C^2\rho^{2/3}-C})\right]\right),$$

*with $C$ a positive constant and $\rho \in (C^{-3/2}, \infty)$. The parametrization $X_C$ consists of a family of biconservative surfaces of revolution any two of which are not locally isometric.*

*Proof* If $f$ is not constant, then from (14), we must have that $\rho$ is a solution of the following ODE

$$3\rho\,\rho'' = 1 + (\rho')^2. \tag{15}$$

We shall now integrate (15). Using the change of variables $y = \rho'^2$, we get

$$3\frac{\mathrm{d}y}{1+y} = 2\frac{\mathrm{d}\rho}{\rho}.$$

Integration yields

$$\rho'^2 = C\rho^{2/3} - 1,$$

where $C$ is a positive constant. Thus,

$$\frac{\mathrm{d}\rho}{\sqrt{C\rho^{2/3}-1}} = \pm\mathrm{d}u.$$

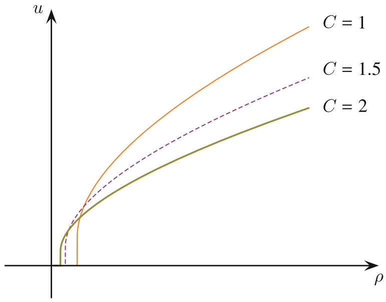


**Fig. 1** Plots of the function $u(\rho)$ for $C = 1$, $C = 1.5$ and $C = 2$

Now, using the change of variable $y = \rho^{1/3}$, we obtain

$$\frac{3y^2}{\sqrt{Cy^2 - 1}}\mathrm{d}y = \pm \mathrm{d}u.$$

The latter equation can be integrated and, up to a symmetry with respect to the $xy$-plane, followed by a translation along the vertical $z$-axis, gives the following solution

$$u = u(\rho) = \frac{3}{2C}\left(\rho^{1/3}\sqrt{C\rho^{2/3} - 1} + \frac{1}{\sqrt{C}}\ln\left[2(C\rho^{1/3} + \sqrt{C^2\rho^{2/3} - C})\right]\right),$$

where $\rho \in (C^{-3/2}, \infty)$. Since the derivative of $u(\rho)$ is

$$u'(\rho) = \frac{1}{\sqrt{C\rho^{2/3} - 1}},$$

we deduce that $u(\rho)$ is invertible for $\rho \in (C^{-3/2}, \infty)$ and its inverse function produces the desired solution of (15). For a plot of the function $u(\rho)$ see Fig. 1. □

*Remark 4.2* If we denote by $\sigma(u) = (\rho(u), 0, u)$ the profile curve of the surface described in Proposition 4.1 and we reparametrize it by arc-length, then its curvature function $k$ satisfies the ODE

$$kk'' = \frac{7}{4}(k')^2 - 4k^4.$$

Moreover, the Gauss curvature and mean curvature functions of the surface are

$$K(\rho, v) = -\frac{1}{3C\rho^{8/3}}, \quad f(\rho, v) = \frac{2}{3\sqrt{C}\rho^{4/3}}.$$

It is worth remarking that $f$ is non-constant (as assumed in the Proposition 4.1) and that the values of $K$ and $f$ are in accord with (7).

### 4.1 The general case

We shall now prove that, essentially, the family described in Proposition 4.1 gives, locally, all non-CMC biconservative surfaces. To achieve this, we assume that $f$ is positive and that grad $f \neq 0$ at any point. We define the local orthonormal frame $\{X_1, X_2\}$ as in (6) and from the calculations in the proof of Theorem 3.1, we have

$$\begin{cases} \nabla_{X_1} X_1 = 0, & \nabla_{X_1} X_2 = 0, \\ \nabla_{X_2} X_1 = -\frac{3(X_1 f)}{4f} X_2, & \nabla_{X_2} X_2 = \frac{3(X_1 f)}{4f} X_1. \end{cases} \tag{16}$$

Let $\eta$ be a unit vector field normal to the surface $M$. Then, if we denote by $\overline{\nabla}$ the connection of $\mathbb{R}^3$, a straightforward computation gives

$$\begin{cases} \overline{\nabla}_{X_1} X_1 = -\frac{f}{2}\eta, & \overline{\nabla}_{X_1} X_2 = 0, \\ \overline{\nabla}_{X_2} X_1 = -\frac{3(X_1 f)}{4f} X_2, & \overline{\nabla}_{X_2} X_2 = \frac{3(X_1 f)}{4f} X_1 + \frac{3f}{2}\eta, \\ \overline{\nabla}_{X_1} \eta = \frac{f}{2} X_1, & \overline{\nabla}_{X_2} \eta = -\frac{3f}{2} X_2. \end{cases} \tag{17}$$

Put

$$\kappa_2 \xi = \frac{3(X_1 f)}{4f} X_1 + \frac{3f}{2}\eta = \overline{\nabla}_{X_2} X_2, \tag{18}$$

where

$$\kappa_2 = \sqrt{\frac{9(X_1 f)^2}{16 f^2} + \frac{9 f^2}{4}}. \tag{19}$$

We have the following lemma.

**Lemma 4.3** *The function $\kappa_2$ and the vector field $\xi$ satisfy*

*(a)* $X_2 \kappa_2 = 0$;
*(b)* $\overline{\nabla}_{X_2} \xi = -\kappa_2 X_2$;
*(c)* $4(X_1 \kappa_2)/\kappa_2 = 3(X_1 f)/f$;
*(d)* $\overline{\nabla}_{X_1} \xi = 0$.

*Proof* From $X_2 f = 0$ and $[X_1, X_2] = 3(X_1 f) X_2/(4f)$, if follows that

$$X_2 X_1 f = X_1 X_2 f - [X_1, X_2] f = 0.$$

Since $\kappa_2$ depends only on $f$ and $X_1 f$, (a) follows. To prove (b), using (a) and (17), we have

$$\begin{aligned} \overline{\nabla}_{X_2} \xi &= \frac{1}{\kappa_2} \overline{\nabla}_{X_2} \left( \frac{3(X_1 f)}{4f} X_1 + \frac{3f}{2}\eta \right) \\ &= \frac{1}{\kappa_2} \left( -\frac{9(X_1 f)^2}{16 f^2} X_2 - \frac{9 f^2}{4} X_2 \right) \\ &= -\frac{1}{\kappa_2} \kappa_2^2 X_2 = -\kappa_2 X_2. \end{aligned}$$

To prove (c), first observe that a direct computation gives

$$4 \frac{X_1 \kappa_2}{\kappa_2} = \frac{1}{4 f^4} \frac{9 f^2 (X_1 f)(X_1 X_1 f) - 9 f (X_1 f)^3 + 36 f^5 (X_1 f)}{\frac{9(X_1 f)^2}{16 f^2} + \frac{9 f^2}{4}}.$$

Then, (c) is equivalent to

$$3 \frac{X_1 f}{f} = \frac{1}{4 f^4} \frac{9 f^2 (X_1 f)(X_1 X_1 f) - 9 f (X_1 f)^3 + 36 f^5 (X_1 f)}{\frac{9(X_1 f)^2}{16 f^2} + \frac{9 f^2}{4}}$$

which is itself equivalent to

$$f(X_1X_1f) - \frac{7}{4}(X_1f)^2 + f^4 = 0.$$

Now, the latter equation is (10) with $c = 0$ (see also (12)).

We now prove (d). First, from a direct computation, taking into account (17), we have

$$\overline{\nabla}_{X_1}\xi = \frac{3}{4}\left(X_1\left(\frac{X_1f}{f\kappa_2}\right) + \frac{f^2}{\kappa_2}\right)X_1 + \frac{3}{2}\left(X_1\left(\frac{f}{\kappa_2}\right) - \frac{1}{4}\frac{X_1f}{\kappa_2}\right)\eta.$$

We have to show that both components are zero. First,

$$X_1\left(\frac{f}{\kappa_2}\right) - \frac{1}{4}\frac{X_1f}{\kappa_2} = 0$$

if and only if

$$4\frac{X_1\kappa_2}{\kappa_2} = 3\frac{X_1f}{f},$$

which is identity (c). Similarly, using (c),

$$X_1\left(\frac{X_1f}{f\kappa_2}\right) + \frac{f^2}{\kappa_2} = 0$$

if and only if

$$f(X_1X_1f) - \frac{7}{4}(X_1f)^2 + f^4 = 0,$$

which is identity (12). □

*Remark 4.4* It is useful to observe that, from Lemma 4.3, (a)–(b), the integral curves of the vector field $X_2$ are circles in $\mathbb{R}^3$ with curvature $\kappa_2$.

We are now in the right position to state the main result of this section.

**Theorem 4.5** (see also Proposition 3.1 in [13]) *Let $M^2$ be a biconservative surface in $\mathbb{R}^3$ with $f(p) > 0$ and grad $f(p) \neq 0$ for any $p \in M$. Then, locally, $M^2$ is a surface of revolution.*

*Proof* Let $\gamma$ be an integrable curve of $X_2$ parametrized by arc-length. From Lemma 4.3, (a)–(b), $\gamma$ is a circle in $\mathbb{R}^3$ with curvature $\kappa_2$, that can be parametrized by

$$\gamma(s) = c_0 + c_1\cos(\kappa_2 s) + c_2\sin(\kappa_2 s), \quad c_0, c_1, c_2 \in \mathbb{R}^3 \tag{20}$$

with

$$|c_1| = |c_2| = \frac{1}{\kappa_2}, \quad \langle c_1, c_2\rangle = 0.$$

Let $p_0 \in M$ be an arbitrary point and let $\sigma(u)$ be an integral curve of $X_1$ with $\sigma(0) = p_0$. Consider the flow $\phi$ of the vector field $X_2$ near the point $p_0$. Then, for all $u \in (-\delta, \delta)$ and for all $s \in (-\varepsilon, \varepsilon)$,

$$\phi_{\sigma(u)}(s) = c_0(u) + c_1(u)\cos(\kappa_2(u)s) + c_2(u)\sin(\kappa_2(u)s),$$

where the vectorial functions $c_0(u), c_1(u), c_2(u)$, which are uniquely determined by their initial conditions, satisfy

$$\sigma(u) = c_0(u) + c_1(u), \quad |c_1(u)| = |c_2(u)| = \frac{1}{\kappa_2(u)}, \quad \langle c_1(u), c_2(u)\rangle = 0,$$

while $\kappa_2(u) = \kappa_2(\sigma(u))$. Thus, locally, the surface can be parametrized by

$$X(u,s) = \phi_{\sigma(u)}(s).$$

Now, since $\kappa_2(0) > 0$, there exists $\delta' > 0$ such that for $u \in (-\delta', \delta')$ we have $\kappa_2(u) > \kappa_2(0)/2$. Then, we can reparametrize $X(u,s)$ using the change of parameter

$$(u,s) \to (u, v = \kappa_2(u)s),$$

where $v$ is defined in a interval which includes $(-\kappa_2(0)\varepsilon/2, \kappa_2(0)\varepsilon/2)$. With respect to the above change of parameters, the parametrization of the surface becomes

$$X(u,v) = c_0(u) + \frac{1}{\kappa_2(u)}\left(C_1(u)\cos(v) + C_2(u)\sin(v)\right),$$

where

$$C_1(u) = \kappa_2(u)c_1(u), \quad C_2(u) = \kappa_2(u)c_2(u).$$

Since the integral curves of $X_2$ start (at $v = 0$) from $\sigma$, we have

$$\sigma(u) = X(u,0) = c_0(u) + \frac{1}{\kappa_2(u)}C_1(u).$$

From this

$$X(u,v) = \sigma(u) + \frac{1}{\kappa_2(u)}\big(C_1(u)(\cos v - 1) + C_2(u)\sin v\big). \tag{21}$$

Using (20), we find

$$C_2 = \kappa_2 c_2 = \gamma'(0) = X_2(\gamma(0)),$$

which implies that $C_2(u) = X_2(\sigma(u))$. Using (20) again, we get

$$-\kappa_2^2\, c_1 = \gamma''(0) = \kappa_2(\gamma(0))\,\xi(\gamma(0)) = \kappa_2(u)\,\xi(\sigma(u)),$$

which implies that $C_1(u) = -\xi(\sigma(u))$. Now, we shall prove that $C_1(u)$ and $C_2(u)$ are, in fact, constant vectors. Indeed, taking into account Lemma 4.3,(d),

$$\frac{\mathrm{d}C_1}{\mathrm{d}u} = -\overline{\nabla}_{\sigma'}\xi = -\overline{\nabla}_{X_1}\xi = 0.$$

Moreover, using (17),

$$\frac{\mathrm{d}C_2}{\mathrm{d}u} = \overline{\nabla}_{\sigma'}X_2 = \overline{\nabla}_{X_1}X_2 = 0.$$

Thus, the image of the parametrization (21) is given by a one-parameter family of circles passing through the points of $\sigma(u)$ lying in affine planes parallel to the space spanned by $C_1$ and $C_2$.

To finish the proof, we need to show that the curve of the centers of the circles is a line orthogonal to $C_1$ and $C_2$. The parametrization (21) can be written as

$$X(u,v) = \beta(u) + \frac{1}{\kappa_2(u)}\big(C_1 \cos v + C_2 \sin v\big),$$

where

$$\beta(u) = \sigma(u) - \frac{C_1}{\kappa_2(u)}$$

is the curve of the centers. Let show that $\beta$ is a line. For this, we prove that $\beta' \wedge \beta'' = 0$. Since

$$\sigma''(u) = -\frac{f(u)}{2}\,\eta(\sigma(u)),$$

where $f(u) = f(\sigma(u))$ and $X_1 \wedge X_2 = \eta$, we have

$$\begin{aligned}
\beta' \wedge \beta'' &= \left(\sigma' - \left(\frac{1}{\kappa_2}\right)' C_1\right) \wedge \left(\sigma'' - \left(\frac{1}{\kappa_2}\right)'' C_1\right) \\
&= -\frac{f}{2} X_1 \wedge \eta + \left(\frac{1}{\kappa_2}\right)'' X_1 \wedge \xi - \frac{f}{2}\left(\frac{1}{\kappa_2}\right)' \xi \wedge \eta \\
\text{(using (18))} &= \left(\frac{f}{2} - 3\frac{f}{2}\left(\frac{1}{\kappa_2}\right)''\left(\frac{1}{\kappa_2}\right) + \frac{3}{4}\frac{X_1 f}{2}\left(\frac{1}{\kappa_2}\right)\left(\frac{1}{\kappa_2}\right)'\right) X_2.
\end{aligned}$$

Now, replacing (19) in

$$\left(\frac{f}{2} - 3\frac{f}{2}\left(\frac{1}{\kappa_2}\right)''\left(\frac{1}{\kappa_2}\right) + \frac{3}{4}\frac{X_1 f}{2}\left(\frac{1}{\kappa_2}\right)\left(\frac{1}{\kappa_2}\right)'\right)$$

and using the identities (12) and Lemma 4.3, (c), we find zero.

Finally, $\beta'$ is clearly orthogonal to $C_2$ and

$$\begin{aligned}
\langle \beta', C_1 \rangle &= \langle X_1, C_1 \rangle - \left(\frac{1}{\kappa_2}\right)' \\
&= -\langle X_1, \xi \rangle - \left(\frac{1}{\kappa_2}\right)' \\
\text{(using (18))} &= -\frac{1}{\kappa_2}\left(\frac{3}{4}\frac{X_1 f}{f} - \frac{\kappa_2'}{\kappa_2}\right) \\
\text{(using Lemma 4.3(c))} &= 0.
\end{aligned}$$

□

## 5 Biconservative surfaces in $\mathbb{S}^3$

In this section, we consider biconservative surfaces in 3-dimensional sphere $\mathbb{S}^3$. We assume that the surface is not CMC and thus we can choose $f$ to be positive and grad $f \neq 0$ at any point of the surface. We define the local orthonormal frame $\{X_1, X_2\}$ as in (6) and we look at $\mathbb{S}^3$ as a submanifold of $\mathbb{R}^4$. With this in mind and denoting by $\nabla$, $\nabla^{\mathbb{S}^3}$ and $\overline{\nabla}$ the connections of $M$, $\mathbb{S}^3$ and $\mathbb{R}^4$, respectively, we have, at a point $\mathbf{r} \in M \subset \mathbb{S}^3 \subset \mathbb{R}^4$,

$$\begin{cases} \nabla^{\mathbb{S}^3}_{X_1} X_1 = -\frac{f}{2}\eta, & \nabla^{\mathbb{S}^3}_{X_1} X_2 = 0, \\ \nabla^{\mathbb{S}^3}_{X_2} X_1 = -\frac{3(X_1 f)}{4f} X_2, & \nabla^{\mathbb{S}^3}_{X_2} X_2 = \frac{3(X_1 f)}{4f} X_1 + \frac{3f}{2}\eta, \end{cases} \tag{22}$$

and

$$\begin{cases} \overline{\nabla}_{X_1} X_1 = -\frac{f}{2}\eta - \mathbf{r}, & \overline{\nabla}_{X_1} X_2 = 0, \\ \overline{\nabla}_{X_2} X_1 = -\frac{3(X_1 f)}{4f} X_2, & \overline{\nabla}_{X_2} X_2 = \frac{3(X_1 f)}{4f} X_1 + \frac{3f}{2}\eta - \mathbf{r}, \\ \overline{\nabla}_{X_1}\eta = \frac{f}{2} X_1, & \overline{\nabla}_{X_2}\eta = -\frac{3f}{2} X_2, \end{cases} \tag{23}$$

where $\eta$ is a unit vector field normal to the surface $M$ in $\mathbb{S}^3$. Put

$$\kappa_2 \xi = \frac{3(X_1 f)}{4f} X_1 + \frac{3f}{2}\eta - \mathbf{r} = \overline{\nabla}_{X_2} X_2, \tag{24}$$

where

$$\kappa_2 = \sqrt{\frac{9(X_1 f)^2}{16 f^2} + \frac{9 f^2}{4} + 1}. \tag{25}$$

We have the following analogue of Lemma 4.3.

**Lemma 5.1** *The function $\kappa_2$ and the vector field $\xi$ satisfy*

*(a)* $X_2\kappa_2 = 0$;
*(b)* $\overline{\nabla}_{X_2}\xi = -\kappa_2 X_2$;
*(c)* $4(X_1\kappa_2)/\kappa_2 = 3(X_1 f)/f$;
*(d)* $\overline{\nabla}_{X_1}\xi = 0$.

Now, let $M^2$ be a biconservative surface in $\mathbb{S}^3$ with $f > 0$ and grad $f \neq 0$ at any point. Then, using the same argument as in the proof of Theorem 4.5, we find that, locally, $M^2 \subset \mathbb{R}^4$ can be parametrized by

$$X(u, v) = \sigma(u) + \frac{1}{\kappa_2(u)}\big(C_1(u)(\cos v - 1) + C_2(u)\sin v\big), \tag{26}$$

where $\sigma(u)$ is an integral curve of $X_1$, $\kappa_2(u) = \kappa_2(\sigma(u))$ is the curvature of the integral curves of $X_2$, which are circles in $\mathbb{R}^4$, and $C_1, C_2$ are two vector functions such that $|C_1| = |C_2| = 1$ and $\langle C_1, C_2\rangle = 0$. Moreover,

$$C_1(u) = -\xi(\sigma(u)), \quad C_2(u) = X_2(\sigma(u)). \tag{27}$$

Further, it is easy to see that $C_1$ and $C_2$ are constant vectors. Then, it is clear from (26) that locally the surface $M^2$ is given by a family of circles of $\mathbb{R}^4$, passing through the curve $\sigma$, and belonging to a pencil of planes which are parallel to the linear space spanned by $C_1$ and $C_2$. Now, these circles must be the intersection of the pencil with the sphere $\mathbb{S}^3$. Let $G$ be the one-parameter group of isometries of $\mathbb{R}^4$ generated by the Killing vector field

$$T = \langle \mathbf{r}, C_2\rangle C_1 + \langle \mathbf{r}, C_1\rangle C_2.$$

Then, $G$ acts also on $\mathbb{S}^3$ by isometries and it can be identified with the group SO(2). Since the orbits of $G$ are circles of $\mathbb{S}^3$, we deduce that $X(u, v)$, in (26), describes an SO(2) invariant surface of $\mathbb{S}^3$ obtained by the action of $G$ on the curve $\sigma$. Moreover, we can give the following explicit construction.

**Theorem 5.2** *Let $M^2$ be a biconservative surface in $\mathbb{S}^3$ with $f > 0$ and grad $f \neq 0$ at any point. Then, locally, $M^2 \subset \mathbb{R}^4$ can be parametrized by*

$$X_C(u, v) = \sigma(u) + \frac{4}{3\sqrt{C}k(u)^{3/4}}\big(C_1(\cos v - 1) + C_2 \sin v\big), \tag{28}$$

*where $C$ is a positive constant of integration, $C_1, C_2 \in \mathbb{R}^4$ are two constant orthonormal vectors such that*

$$\langle \sigma(u), C_1\rangle = \frac{4}{3\sqrt{C}k(u)^{3/4}}, \quad \langle \sigma(u), C_2\rangle = 0, \tag{29}$$

*while $\sigma = \sigma(u)$ is a curve lying in the totally geodesic $\mathbb{S}^2 = \mathbb{S}^3 \cap \Pi$ ($\Pi$ the linear hyperspace of $\mathbb{R}^4$ orthogonal to $C_2$), whose geodesic curvature $k = k(u)$ is a positive non-constant solution of the following ODE*

$$k''k = \frac{7}{4}(k')^2 + \frac{4}{3}k^2 - 4k^4. \tag{30}$$

*Proof* From (26), we know that

$$X(u, v) = \sigma(u) + \frac{1}{\kappa_2(u)}\big(C_1(\cos v - 1) + C_2 \sin v\big),$$

Since

$$\langle \sigma(u), C_2\rangle = \langle \sigma(u), X_2(\sigma(u))\rangle = 0,$$

we deduce that $\sigma \subset \Pi$, where $\Pi$ is the hyperplane of $\mathbb{R}^4$ defined by the equation $\langle \mathbf{r}, C_2\rangle = 0$. Thus $\sigma$ is a curve in $\mathbb{S}^3 \cap \Pi = \mathbb{S}^2$, where $\mathbb{S}^2$ is a totally geodesic 2-sphere of $\mathbb{S}^3$. Now, let $k$ denotes the geodesic curvature of $\sigma$ in $\mathbb{S}^2$. Then, taking into account (22), we have

$$\nabla^{\mathbb{S}^2}_{\sigma'}\sigma' = \nabla^{\mathbb{S}^3}_{\sigma'}\sigma' = -\frac{f(u)}{2}\,\eta(\sigma(u)),$$

where $f(u) = f \circ \sigma(u)$. We deduce that $k(u) = |\nabla^{\mathbb{S}^2}_{\sigma'}\sigma'| = f(u)/2$. From (12), with $c = 1$, we know that $f = f(u)$ is a solution of

$$f''f = \frac{7}{4}(f')^2 + \frac{4}{3}f^2 - f^4,$$

which implies that $k = k(u)$ is a solution of (30). To finish, we have to compute $\kappa_2(u)$ as a function of $k(u)$. First, by a standard argument, we find that (30) has the prime integral

$$(k')^2 = -\frac{16}{9}k^2 - 16k^4 + Ck^{7/2}, \quad C \in \mathbb{R},\ C > 0. \tag{31}$$

Substituting (31) in (25), we find

$$\kappa_2(u) = \frac{3}{4}\sqrt{C}k(u)^{3/4}.$$

Finally, using the value of $C_1$ in (27) and that of $\xi$ in (24), we get

$$\langle \sigma(u), C_1\rangle = \langle \sigma(u), -\xi(\sigma(u))\rangle = \frac{1}{\kappa_2(u)} = \frac{4}{3\sqrt{C}k(u)^{3/4}}.$$

□

*Remark 5.3* Theorem 5.2 asserts that if $M^2$ is a biconservative surface of $\mathbb{S}^3$, then, locally, it is an SO(2)-invariant surface whose profile curve $\sigma$ satisfies (29) and (30). It is worth to show that such a curve exists.

First, the condition in Theorem 5.2 that $k$ is a positive non-constant solution of (30) is not restrictive. In fact, choosing the initial condition $k(u_0) > 0$ and $k'(u_0) > 0$, from Picard's theorem, there is a unique solution of (30) which is positive and non-constant in an open interval containing $u_0$.

Next, let assume that $C_1 = e_3$ and $C_2 = e_4$, where $\{e_1, \ldots, e_4\}$ is the canonical basis of $\mathbb{R}^4$. Then, using (29), $\sigma$ can be explicitly described as

$$\sigma(u) = \left(x(u), y(u), \frac{4}{3\sqrt{C}}\, k(u)^{-3/4}, 0\right), \tag{32}$$

for some functions $x(u)$ and $y(u)$. Since $\sigma$ is parametrized by arc-length and its curvature must be the given function $k$ (i.e., $\sigma'' = -k\,\eta - \mathbf{r}$), the functions $x = x(u)$ and $y = y(u)$ must satisfy the system

$$\begin{cases} x^2 + y^2 + \frac{16}{9C}\, k^{-3/2} = 1 \\ (x')^2 + (y')^2 + \frac{16}{9C}\, \left(\left(k^{-3/4}\right)'\right)^2 = 1 \\ (x'')^2 + (y'')^2 + \frac{16}{9C}\, \left(\left(k^{-3/4}\right)''\right)^2 = 1 + k^2. \end{cases} \tag{33}$$

Taking the derivative and using (30)–(31), system (33) becomes

$$\begin{cases} x^2 + y^2 + \frac{16}{9C}\, k^{-3/2} = 1 \\ (x')^2 + (y')^2 = \frac{16}{9C}\, (1 + 9k^2)\, k^{-3/2} \\ (x'')^2 + (y'')^2 + \frac{16}{9C}\, (1 - 3k^2)^2\, k^{-3/2} = 1 + k^2. \end{cases} \tag{34}$$

Now, since $k' \neq 0$, we can locally invert the function $k = k(u)$ and write $u = u(k)$. Then, system (34) becomes

$$\begin{cases} x^2 + y^2 + \frac{16}{9C}\, k^{-3/2} = 1 \\ (k')^2 \left(\frac{\mathrm{d}x}{\mathrm{d}k}\right)^2 + (k')^2 \left(\frac{\mathrm{d}y}{\mathrm{d}k}\right)^2 = \frac{16}{9C}\, (1 + 9k^2)\, k^{-3/2} \\ \left(\frac{\mathrm{d}^2x}{\mathrm{d}k^2}(k')^2 + \frac{\mathrm{d}x}{\mathrm{d}k} k''\right)^2 + \left(\frac{\mathrm{d}^2y}{\mathrm{d}k^2}(k')^2 + \frac{\mathrm{d}y}{\mathrm{d}k} k''\right)^2 + \frac{16}{9C}\, \frac{(1-3k^2)^2}{k^{3/2}} = 1 + k^2, \end{cases} \tag{35}$$

where, according to (31),

$$(k')^2 = -\frac{16}{9}k^2 - 16k^4 + Ck^{7/2}, \quad k'' = -\frac{16}{9}k - 32k^3 + \frac{7}{4}Ck^{5/2}.$$

From the first equation of (35), we get

$$y(k) = \pm\sqrt{1 - x(k)^2 - \frac{16}{9C}\, k^{-3/2}},$$

that substituted in the second gives

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}k} = &\frac{12x(k)}{k(9Ck^{3/2} - 16)} \\ &\pm \frac{36\sqrt{-9Ck^{3/2}x(k)^2 + 9Ck^{3/2} - 16}}{(9Ck^{3/2} - 16)\sqrt{9Ck^{3/2} - 144k^2 - 16}}. \end{aligned} \tag{36}$$

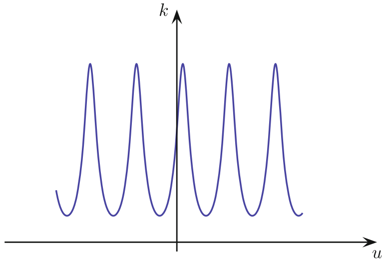


**Fig. 2** Plot of a numerical solution of (30) with $k(0) = 1$ and $k'(0) = 1$. The constant of integration is, in this case, $C = 169/9$

We note that $\mathrm{d}x/\mathrm{d}k \neq 0$. In fact, if it were zero, from (36), we should have $x(k) = \pm 3k/\sqrt{1+9k^2}$ which is not constant. Taking the derivative of (36) with respect to $k$ and replacing in it the value $dx/dk$ given in (36), we find that $\mathrm{d}^2x/\mathrm{d}k^2$ depends only on $x(k)$ and $k$. In the same way, we find that $\mathrm{d}y/\mathrm{d}k$ and $\mathrm{d}^2y/\mathrm{d}k^2$ depend only on $x(k)$ and $k$. Finally, substituting in the third equation of system (35), the values of $\mathrm{d}x/\mathrm{d}k$, $\mathrm{d}y/\mathrm{d}k$, $\mathrm{d}^2x/\mathrm{d}k^2$, $\mathrm{d}^2y/\mathrm{d}k^2$, $k'$ and $k''$, we find an identity. This means that the solution $x(k)$ of (36) and the corresponding $y(k)$ give a curve $\sigma$, as described in (32), which satisfies all the desired conditions.

Now, although we could not find an explicit solution of (30), which would give the curvature of the profile curve $\sigma$, using Mathematica, we were able to plot a numerical solution as shown in Fig. 2.

## 6 Biconservative surfaces in the hyperbolic space

Let $\mathbb{L}^4$ be the four-dimensional Lorentz–Minkowski space, that is, the real vector space $\mathbb{R}^4$ endowed with the Lorentzian metric tensor $\langle , \rangle$ given by

$$\langle , \rangle = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 - \mathrm{d}x_4^2,$$

where $(x_1, x_2, x_3, x_4)$ are the canonical coordinates of $\mathbb{R}^4$. The three-dimensional unitary hyperbolic space is given as the following hyperquadric of $\mathbb{L}^4$,

$$\mathbb{H}^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{L}^4 \,:\, x_1^2 + x_2^2 + x_3^2 - x_4^2 = -1,\ x_4 > 0\}.$$

As it is well known, the induced metric on $\mathbb{H}^3$ from $\mathbb{L}^4$ is Riemannian with constant sectional curvature $-1$. In this section, we shall use this model of the hyperbolic space. For convenience, we shall recall that, if $X$, $Y$ are tangent vector fields to $\mathbb{H}^3$, then

$$\overline{\nabla}_X Y = \nabla^{\mathbb{H}^3}_X Y + \langle X, Y \rangle \mathbf{r},$$

where $\overline{\nabla}$ is the connection on $\mathbb{L}^4$, $\nabla^{\mathbb{H}^3}$ is that of $\mathbb{H}^3$, while $\mathbf{r}$ is the position vector of a point $\mathbf{r} \in M \subset \mathbb{H}^3 \subset \mathbb{L}^4$.

Let $M^2$ be a biconservative surface in the 3-dimensional hyperbolic space $\mathbb{H}^3$. We assume that the surface is not CMC and thus we can choose $f$ to be positive and grad $f \neq 0$ at any point of the surface. We define again the local orthonormal frame $\{X_1, X_2\}$ as in (6). We have

$$\begin{cases} \nabla^{\mathbb{H}^3}_{X_1} X_1 = -\frac{f}{2}\eta, & \nabla^{\mathbb{H}^3}_{X_1} X_2 = 0, \\ \nabla^{\mathbb{H}^3}_{X_2} X_1 = -\frac{3(X_1 f)}{4f} X_2, & \nabla^{\mathbb{H}^3}_{X_2} X_2 = \frac{3(X_1 f)}{4f} X_1 + \frac{3f}{2}\eta, \end{cases} \tag{37}$$

and

$$\begin{cases} \overline{\nabla}_{X_1} X_1 = -\frac{f}{2}\eta + \mathbf{r}, & \overline{\nabla}_{X_1} X_2 = 0, \\ \overline{\nabla}_{X_2} X_1 = -\frac{3(X_1 f)}{4f} X_2, & \overline{\nabla}_{X_2} X_2 = \frac{3(X_1 f)}{4f} X_1 + \frac{3f}{2}\eta + \mathbf{r}, \\ \overline{\nabla}_{X_1}\eta = \frac{f}{2} X_1, & \overline{\nabla}_{X_2}\eta = -\frac{3f}{2} X_2, \end{cases} \tag{38}$$

where $\eta$ is a unit vector field normal to the surface $M$ tangent to $\mathbb{H}^3$.

Put

$$\kappa_2\, \xi = \overline{\nabla}_{X_2} X_2 = \frac{3(X_1 f)}{4f} X_1 + \frac{3f}{2}\eta + \mathbf{r}, \tag{39}$$

where

$$\kappa_2 = \sqrt{\left| \frac{9(X_1 f)^2}{16 f^2} + \frac{9f^2}{4} - 1 \right|}. \tag{40}$$

Differently from the case of surfaces in $\mathbb{R}^3$ or in $\mathbb{S}^3$, in this case, the quantity

$$W = \frac{9(X_1 f)^2}{16 f^2} + \frac{9f^2}{4} - 1 = \frac{9|\operatorname{grad} f|^2}{16 f^2} + \frac{9f^2}{4} - 1$$

can take both positive and negative values. Taking this in consideration, we have the following analogue of Lemma 4.3.

**Lemma 6.1** *The function $\kappa_2$ and the vector field $\xi$ satisfy*

*(a)* $X_2\kappa_2 = 0$;
*(b)* $\overline{\nabla}_{X_2}\xi = -\varepsilon\kappa_2\, X_2$;
*(c)* $4(X_1\kappa_2)/\kappa_2 = 3(X_1 f)/f$;
*(d)* $\overline{\nabla}_{X_1}\xi = 0$,

*where $\varepsilon$ is 1 when $W > 0$ and is $-1$ when $W < 0$.*

As in the case of biconservative surfaces in $\mathbb{S}^3$, we can give the following explicit construction.

**Theorem 6.2** *Let $M^2$ be a biconservative surface in $\mathbb{H}^3$ with $f > 0$ and $\operatorname{grad} f \neq 0$ at any point. Then, locally, $M^2 \subset \mathbb{L}^4$ can be parametrized by:*

*(a) if $W > 0$,*

$$X_C(u, v) = \sigma(u) + \frac{4}{3\sqrt{C}k(u)^{3/4}}\big(C_1(\cos v - 1) + C_2 \sin v\big), \tag{41}$$

*where $C$ is a positive constant of integration, $C_1, C_2 \in \mathbb{L}^4$ are two constant vectors such that*

$$\langle C_i, C_j\rangle = \delta_{ij}, \quad \langle\sigma(u), C_1\rangle = \frac{4}{3\sqrt{C}k(u)^{3/4}}, \quad \langle\sigma(u), C_2\rangle = 0, \tag{42}$$

*while $\sigma = \sigma(u)$ is a curve lying in the totally geodesic $\mathbb{H}^2 = \mathbb{H}^3 \cap \Pi$ ($\Pi$ the linear hyperspace of $\mathbb{L}^4$ defined by $\langle\mathbf{r}, C_2\rangle = 0$), whose geodesic curvature $k = k(u)$ is a positive non-constant solution of the following ODE*

$$k''k = \frac{7}{4}(k')^2 - \frac{4}{3}k^2 - 4k^4\,; \tag{43}$$

*(b) if* $W < 0$,

$$X_C(u,v) = \sigma(u) + \frac{4}{3\sqrt{-C}k(u)^{3/4}}\big(C_1(e^v - 1) + C_2(e^{-v} - 1)\big), \tag{44}$$

*where* $C$ *is a negative constant of integration,* $C_1, C_2 \in \mathbb{L}^4$ *are two constant vectors such that*

$$\langle C_i, C_i\rangle = 0, \quad \langle C_1, C_2\rangle = -1, \quad \langle \sigma(u), C_1\rangle = \langle \sigma(u), C_2\rangle = -\frac{2\sqrt{2}}{3\sqrt{-C}k(u)^{3/4}}, \tag{45}$$

*while* $\sigma = \sigma(u)$ *is a curve lying in the totally geodesic* $\mathbb{H}^2 = \mathbb{H}^3 \cap \Pi$ *(*$\Pi$ *the linear hyperspace of* $\mathbb{L}^4$ *defined by* $\langle \mathbf{r}, C_1 - C_2\rangle = 0$*), whose geodesic curvature* $k = k(u)$ *is a positive non-constant solution of* (43)*.*

*Proof* (a) In this case, $W > 0$. Define the local orthonormal frame $\{X_1, X_2\}$ as in (6). Let $\gamma(s)$ be an integral curve of $X_2$ parametrized by arc-length. Then, from

$$\gamma''(s) = \overline{\nabla}_{\gamma'}\gamma' = \kappa_2(s)\xi(s)$$

and

$$\gamma'''(s) = \overline{\nabla}_{\gamma'}\gamma'' = -\kappa_2^2(s)\gamma'(s),$$

it follows that the parametrization $\gamma(s)$ satisfies the following ODE

$$\gamma''' + \kappa_2^2\gamma' = 0.$$

Then, as we have proceeded in the proof of Theorem 4.5, we find that, locally, $M^2 \subset \mathbb{L}^4$ can be parametrized by

$$X(u,v) = \sigma(u) + \frac{1}{\kappa_2(u)}\big(C_1(\cos v - 1) + C_2 \sin v\big), \tag{46}$$

where $\sigma(u)$ is and integral curve of $X_1$, $\kappa_2(u) = \kappa_2(\sigma(u))$ is the curvature of the integral curves of $X_2$ and $C_1, C_2 \in \mathbb{L}^4$ are two constant vectors such that

$$\langle C_i, C_j\rangle = \delta_{ij}, \quad C_1 = -\xi(\sigma(u)), \quad C_2 = X_2(\sigma(u)). \tag{47}$$

Since

$$\langle \sigma(u), C_2\rangle = \langle \sigma(u), X_2(\sigma(u))\rangle = 0,$$

we deduce that $\sigma \subset \Pi$, where $\Pi$ is the hyperspace of $\mathbb{L}^4$ defined by the equation $\langle \mathbf{r}, C_2\rangle = 0$. Thus, $\sigma$ is a curve in $\mathbb{H}^3 \cap \Pi = \mathbb{H}^2$, where $\mathbb{H}^2$ is totally geodesic in $\mathbb{H}^3$. Now, let $k = k(u)$ denote the geodesic curvature of $\sigma$ in $\mathbb{H}^2$. Then, as in the proof of Theorem 5.2, we find that $k$ is a solution of (43). In order to conclude, we have to compute $\kappa_2(u)$ as a function of $k(u)$. First, by a standard argument, we find that (43) has the prime integral

$$(k')^2 = \frac{16}{9}k^2 - 16k^4 + Ck^{7/2}, \quad C \in \mathbb{R},\ C > 0. \tag{48}$$

Substituting (48) in (40) and recalling that $k(u) = |\nabla^{\mathbb{H}^3}_{\sigma'}\sigma'| = f(u)/2$, we find

$$\kappa_2(u) = \frac{3}{4}\sqrt{C}k(u)^{3/4}.$$

Finally, by using the value of $C_1$ in (47) and that of $\xi$ in (39), we get

$$\langle \sigma(u), C_1 \rangle = \langle \sigma(u), -\xi(\sigma(u)) \rangle = \frac{1}{\kappa_2(u)} = \frac{4}{3\sqrt{C} k(u)^{3/4}}.$$

(b) In this case, $W < 0$ and the curve $\gamma(s)$ satisfies the following ODE

$$\gamma''' - \kappa_2^2 \gamma' = 0.$$

Thus, $\gamma(s) = c_o + c_1 \, \mathrm{e}^{\kappa_2 s} + c_2 \, \mathrm{e}^{-\kappa_2 s}$, where, since $\langle \gamma', \gamma' \rangle = 1$, $c_1$ and $c_2$ are vectorial functions such that $\langle c_1, c_1 \rangle = \langle c_2, c_2 \rangle = 0$ and $\langle c_1, c_2 \rangle = -1/(2\kappa_2^2)$. It follows that, locally, $M^2 \subset \mathbb{L}^4$ can be parametrized by

$$X(u, s) = c_0(u) + c_1(u) \, \mathrm{e}^{\kappa_2(u)s} + c_2(u) \, \mathrm{e}^{-\kappa_2(u)s},$$

where $\kappa_2(u) = \kappa_2(\sigma(u))$, $\sigma = \sigma(u)$ being an integral curve of $X_1$. Now, if we perform the change of variables $v = \kappa_2(u)s$ and use the condition $X(u, 0) = \sigma(u)$, we obtain that the parametrization of $M^2$ in $\mathbb{L}^4$ is

$$X(u, v) = \sigma(u) + \frac{1}{\sqrt{2}\kappa_2(u)} \big( C_1(\mathrm{e}^v - 1) + C_2(\mathrm{e}^{-v} - 1) \big), \tag{49}$$

where $C_1, C_2 \in \mathbb{R}^4$ are two constant vectors such that

$$\langle C_i, C_i \rangle = 0, \quad \langle C_1, C_2 \rangle = -1, \quad C_1 + C_2 = \sqrt{2}\,\xi(\sigma(u)), \quad C_1 - C_2 = \sqrt{2}\, X_2(\sigma(u)).$$

Since

$$\langle \sigma(u), C_1 - C_2 \rangle = \sqrt{2} \langle \sigma(u), X_2(\sigma(u)) \rangle = 0,$$

we deduce that $\sigma \subset \Pi$, where $\Pi$ is the hyperspace of $\mathbb{L}^4$ defined by the equation $\langle \mathbf{r}, C_1 - C_2 \rangle = 0$. Thus, $\sigma$ is a curve in $\mathbb{H}^3 \cap \Pi = \mathbb{H}^2$, where $\mathbb{H}^2$ is totally geodesic in $\mathbb{H}^3$. Now, let $k(u)$ denotes the geodesic curvature of $\sigma(u)$ in $\mathbb{H}^2$. Then, $k = k(u)$ is a solution of (43) and, in this case, we find the same prime integral (48) but with the constant $C < 0$. Next, as we have done in case (a), we get the value of $\kappa_2(u)$ as a function of $k(u)$ as well as $\langle \sigma(u), C_1 \rangle$ and $\langle \sigma(u), C_2 \rangle$ as indicated in (45).

□

*Remark 6.3* If we assume that $C_1 = e_2$ and $C_2 = e_1$, where $\{e_1, \ldots, e_4\}$ is the canonical basis of $\mathbb{L}^4$, using an argument as in Remark 5.3, we can check that the curve $\sigma(u)$ in Theorem 6.2 (a) must be of the form

$$\sigma(u) = \left( 0, \frac{4}{3\sqrt{C}} \, k(u)^{-3/4}, x(u), y(u) \right),$$

for some functions $x(u)$ and $y(u)$ which are solution of the system

$$\begin{cases} x^2 - y^2 + \frac{16}{9C} \, k^{-3/2} = -1 \\ (x')^2 - (y')^2 = \frac{16}{9C} \, (9k^2 - 1) \, k^{-3/2} \\ (x'')^2 - (y'')^2 + \frac{16}{9C} \, (1 + 3k^2)^2 \, k^{-3/2} = k^2 - 1. \end{cases}$$

By a direct computation, one can show that this system has a solution.

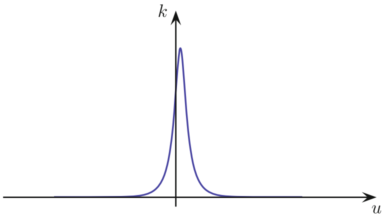


**Fig. 3** Plot of a numerical solution of (43) with $k(0) = 1$ and $k'(0) = 1$ and integration constant $C = 137/9$. Choosing $k(0) = 1/4$ and $k'(0) = 1/5$, we obtain a negative integration constant $C = -248/225$ (thus, a solution to the case (b) of Theorem 6.2) but the qualitative behavior of $k$ is similar to the case $C > 0$

For the curve $\sigma(u)$ in Theorem 6.2 (b), we have that, choosing $C_1 = e_1 + e_4$ and $C_2 = e_2 + e_4$,

$$\sigma(u) = \left( y(u) - \frac{\sqrt{2}}{2\kappa_2(u)}, y(u) - \frac{\sqrt{2}}{2\kappa_2(u)}, x(u), y(u) \right),$$

where, in this case, $x(u)$ and $y(u)$ are solution of the system

$$\begin{cases} 2\left(y - \frac{\sqrt{2}}{2\kappa_2}\right)^2 + x^2 - y^2 = -1 \\ 2\left(\left(y - \frac{\sqrt{2}}{2\kappa_2}\right)'\right)^2 + (x')^2 - (y')^2 = 1 \\ 2\left(\left(y - \frac{\sqrt{2}}{2\kappa_2}\right)''\right)^2 + (x'')^2 - (y'')^2 = k^2 - 1. \end{cases}$$

Again, using the same machineries as in Remark 5.3, we can check that this system has a solution.

Moreover, also in this case, as we have noticed in Remark 5.3, we can plot a numerical solution of (43) as shown in Fig. 3.

*Remark 6.4* We have the following geometric interpretation of the surfaces described in Theorem 6.2 (a). As we have already observed, choosing $C_1 = e_2$ and $C_2 = e_1$, where $\{e_1, \ldots, e_4\}$ is the canonical basis of $\mathbb{L}^4$, the curve $\sigma(u)$ is of the form

$$\sigma(u) = \left(0, \frac{1}{\kappa_2(u)}, x(u), y(u)\right),$$

and the corresponding biconservative surface is parametrized by

$$X(u, v) = \left(\frac{1}{\kappa_2(u)} \sin v, \frac{1}{\kappa_2(u)} \cos v, x(u), y(u)\right).$$

Therefore, the surface is clearly given by the action, on the curve $\sigma$, of the group of isometries of $\mathbb{L}^4$ which leaves the plane $P^2$ generated by $e_3$ and $e_4$ fixed. These surfaces, following the terminology given by do Carmo and Dajczer (see [10]), are called rotational surfaces of spherical type. In fact, the metric of $\mathbb{L}^4$ restricted on $P^2$ is Lorentzian and when this happens, as described in [10, pag. 688], the orbits are circles.

**Acknowledgments** The authors would like to thank Ye-Lin Ou for reading a first draft of the paper and making some helpful suggestions. The third author was supported by a grant of the Romanian National Authority for Scientific Research, CNCS – UEFISCDI, project number PN-II-RU-TE-2011-3-0108; and by Regione Autonoma della Sardegna, Visiting Professor Program.

## References


1. Balmuş, A., Montaldo, S., Oniciuc, C.: Biharmonic PNMC submanifolds in spheres. Ark. Mat. (to appear)
2. Balmuş, A., Montaldo, S., Oniciuc, C.: Properties of biharmonic submanifolds in spheres. J. Geom. Symmetry Phys. **17**, 87–102 (2010)
3. Balmuş, A., Montaldo, S., Oniciuc, C.: Biharmonic hypersurfaces in 4-dimensional space forms. Math. Nachrichten **283**, 1696–1705 (2010)
4. Balmuş, A., Montaldo, S., Oniciuc, C.: Classification results for biharmonic submanifolds in spheres. Israel J. Math. **168**, 201–220 (2008)
5. Baird, P., Eells, J.: A conservation law for harmonic maps. In: Geometry Symposium, Utrecht 1980, pp. 1–25, Lecture Notes in Mathematics, 894. Springer, Berlin (1981)
6. Caddeo, R., Montaldo, S., Oniciuc, C.: Biharmonic submanifolds in spheres. Israel J. Math. **130**, 109–123 (2002)
7. Chen, B.-Y.: Total Mean Curvature and Submanifolds of Finite Type. Series in Pure Mathematics 1. World Scientific, Singapore (1984)
8. Chen, B.-Y.: Some open problems and conjectures on submanifolds of finite type. Soochow J. Math. **17**, 169–188 (1991)
9. Chen, B.-Y., Ishikawa, S.: Biharmonic surfaces in pseudo-Euclidean spaces. Mem. Fac. Sci. Kyushu Univ. Ser. A **45**, 323–347 (1991)
10. do Carmo, M., Dajczer, M.: Rotation hypersurfaces in spaces of constant curvature. Trans. Am. Math. Soc. **277**, 685–709 (1983)
11. Eells, J., Sampson, J.H.: Harmonic mappings of Riemannian manifolds. Am. J. Math. **86**, 109–160 (1964)
12. Eells, J., Lemaire, L.: Selected topics in harmonic maps. In: CBMS Regional Conference Series in Mathematics, 50. American Mathematical Society, Providence (1983)
13. Hasanis, Th, Vlachos, Th: Hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field. Math. Nachrichten **172**, 145–169 (1995)
14. Hilbert, D.: Die grundlagen der physik. Math. Ann. **92**, 1–32 (1924)
15. Jiang, G.Y.: The conservation law for 2-harmonic maps between Riemannian manifolds. Acta Math. Sin. **30**, 220–225 (1987)
16. Loubeau, E., Montaldo, S.: Biminimal immersions. Proc. Edinb. Math. Soc. **51**, 421–437 (2008)
17. Loubeau, E., Montaldo, S., Oniciuc, C.: The stress-energy tensor for biharmonic maps. Math. Z. **259**, 503–524 (2008)
18. Oniciuc, C.: Biharmonic maps between Riemannian manifolds. An. Ştiinţ. Univ. Al. I. Cuza. Iaşi. Mat. (N.S.) **48**, 237–248 (2002)
19. Ou, Y.-L.: Biharmonic hypersurfaces in Riemannian manifolds. Pac. J. Math. **248**, 217–232 (2010)
20. Ou, Y.-L., Wang, Z.-P.: Constant mean curvature and totally umbilical biharmonic surfaces in 3-dimensional geometries. J. Geom. Phys. **61**, 1845–1853 (2011)
21. Sanini, A.: Applicazioni tra varietà riemanniane con energia critica rispetto a deformazioni di metriche. Rend. Mat. **3**, 53–63 (1983)
22. Sasahara, T.: Surfaces in Euclidean 3-space whose normal bundles are tangentially biharmonic. Arch. Math. (Basel), to appear