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Surfaces of Minimal Area Enclosing a Given Body in $\mathbb{R}^3$

GIOVANNI MANCINI - ROBERTA MUSINA

Given a body $\overline{\Omega}$ in $\mathbb{R}^3$, we consider a class of surfaces, parametrized by S^2, which enclose, in a weak sense, $\overline{\Omega}$. To "enclose" means, under some regularity assumption on the surface under consideration, that such a surface is not contractible in $\mathbb{R}^3 \backslash \Omega$.

The first problem we deal with, is concerned with the existence of surfaces which minimize the area integral in such a class. In case $\partial\Omega$ is of class C^2, this will lead to finding a $C^{1,\alpha}$ surface parametrized by a map $U^\infty : \mathbb{R}^2 \to \mathbb{R}^3 \backslash \Omega$ which satisfies

$$\begin{cases} -\Delta U^\infty = \chi_{(U^\infty)^{-1}(\partial\Omega)} \left(-b(U^\infty)(\nabla U^\infty, \nabla U^\infty)\right) \ \nu(U^\infty) \quad \text{a.e. in } \mathbb{R}^2 \\ |U_x^\infty|^2 = |U_y^\infty|^2, \ U_x^\infty \cdot U_y^\infty = 0 \text{ in } \mathbb{R}^2, \end{cases}$$

and which is *not contractible* in $\mathbb{R}^3 \backslash \Omega$ (i.e. "encloses" $\overline{\Omega}$ in a strong sense). Here b is the second fundamental form of ∂C (see Section 1), ν is the inner normal at ∂C, and χ_A is the characteristic function of the set $A \subseteq \mathbb{R}^2$.

This problem, which at our knowledge was not previously considered in the framework of parametric surfaces, is somehow related to the problem of minimal boundaries with obstacles (see for example [12]).

We attack our problem by means of a Dirichlet's Principle, i.e. we look for extremals of the Dirichlet integral over a suitable class of maps from S^2 into $\mathbb{R}^3 \backslash \Omega$. In a more regular setting, this problem amounts to finding not homotopically trivial minimal spheres in a Riemannian manifold N with boundary. In case N has empty boundary, striking results have been obtained in a celebrated paper by Sacks and Uhlenbeck ([17], see also [10], [19]). Here we perform a blow-up technique introduced in this context by Sacks and Uhlenbeck. But, in order to avoid estimates on the solutions of Euler-Lagrange equations related to approximated problems, we follow a more direct approach based on a lemma by Brézis, Coron and Lieb [4].

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We also consider the case of disk-type minimal surfaces spanned by a given wire Γ over the obstacle $\overline{\Omega}$. The existence and regularity of an area minimizing surface $\underline{u}_\Gamma$, spanned over $\overline{\Omega}$, was proved by Tomi [20] (see also [9]). We answer here the rather natural question whether it exists a second minimal surface u_Γ which, jointly with $\underline{u}_\Gamma$, "encloses" $\overline{\Omega}$. While this is not the case in general, we prove that this occurs provided

$$\operatorname*{Inf}_{u\in X_\Gamma^e} \int |\nabla u|^2 < \int |\nabla U_\infty|^2 + \int |\nabla \underline{u}_\Gamma|^2.$$

Here X_Γ is a suitable class of surfaces which, jointly with $\underline{u}_\Gamma$, "enclose" $\overline{\Omega}$.

In the first section of this paper we present preliminary remarks on the functional setting and we define precisely the class of surfaces enclosing $\overline{\Omega}$.

In Section 2 we describe a Dirichlet's principle for minimal surfaces enclosing a given body $\overline{\Omega}$ and we prove the existence of a closed regular S^2-type minimal surface spanned over the obstacle $\overline{\Omega}$.

In Section 3 we give an existence result for pairs of minimal surfaces spanned by the same wire Γ over an obstacle $\overline{\Omega}$ and enclosing it.

In an Appendix we present a result concerning continuous dependence, upon boundary data, of minimizers for the Dirichlet integral in presence of obstacles. This result, which we did not find in the literature, turns out to be a key tool in proving the basic inequality (see Proposition 3.4) on which our existence results rely.

Notations. $D_r(z)$ denotes the open disk of radius r and center z in $\mathbb{R}^2$, $|\cdot|$, $\cdot$ denote the norm and the scalar product in $\mathbb{R}^3$, $\rightharpoonup$ denotes weak convergence in various spaces, $|\cdot|_\infty$ and $|\cdot|_2$ denote L^∞ and L^2 norms respectively.

1. - Preliminary remarks and statement of the Problem

Let C be the closure of the unbounded connected component of $\mathbb{R}^3\backslash\overline{\Omega}$, where Ω is a given bounded open connected set in $\mathbb{R}^3$. We will assume throughout the paper that

$$\text{(1.1)}\qquad \begin{array}{l}\text{there is an open neighbourhood } \mathcal{O} \text{ of } C\\ \text{and a Lipschitz retraction } \pi : \mathcal{O} \to C.\end{array}$$

We shall denote

$$X := \left\{ U \in L^\infty(\mathbb{R}^2,\mathbb{R}^3) \;\middle|\; \int_{\mathbb{R}^2} |\nabla U|^2 < +\infty \right\}$$

$$X(C) := \left\{ U \in X \mid U(z) \in C \text{ for a.e. } z \in \mathbb{R}^2 \right\}.$$

Using a smoothing - by averaging - method (see [18], and [1], Appendix), one easily obtains a density result which will be useful in the sequel.

LEMMA 1.1. *For every $U \in X(C)$ there exists a sequence $U_n \in C^\infty \cap X$ such that*

$$\operatorname{Sup} |U_n|_\infty < +\infty, \quad |\nabla U_n - \nabla U|_2 \to 0, \quad U_n \to U \text{ a.e. and}$$
$$\lim_n \operatorname*{Sup}_{z \in \mathbb{R}^2} \mathrm{d}(U_n(z), C) = 0.$$

Furthermore, for each n, U_n can be taken constant far away.

In order to give our notion of "mappings enclosing Ω", we define the Volume Functional (see for example [21]):

$$V(U) := \int_{\mathbb{R}^2} U \cdot U_x \wedge U_y, \qquad U \in X,$$

which is well defined since, by Hölder inequality,

$$|V(U)| \leq \frac{1}{2} |U|_\infty |\nabla U|_2^2. \tag{1.2}$$

Notice that if $U_n \in X$, $\operatorname{Sup} |U_n|_\infty < \infty$ and $\nabla U_n \to \nabla U$ in L^2 for some $U \in X$, then $V(U_n) \to V(U)$.

Now, assuming for simplicity, $0 \notin C$, we define the map

$$p\xi = \frac{\xi}{|\xi|} \text{ for } \xi \in \mathbb{R}^3 \backslash \{0\}.$$

Since p is Lipschitz continuous far away from 0, we have $pU \in X$ if $U \in X(C)$. Moreover, if a sequence $(U_n)_n \subseteq X(C)$ is bounded in L^∞ and $\nabla U_n \to \nabla U$ in L^2, $U_n \to U$ a.e., then $\nabla(pU_n) \to \nabla(pU)$ in L^2 and $V(pU_n) \to V(pU)$. We recall that, if $U \in X(C) \cap C^0(\mathbb{R}^2, \mathbb{R}^3)$ and U is regular at infinity, that is

$$\text{there exists } \lim_{|z| \to \infty} U(z) = U(\infty), \tag{1.3}$$

then

$$\frac{1}{4\pi} V(pU) \in \mathbb{Z}$$

and gives the degree of $pU \circ \Pi \in C^0(S^2, S^2)$, where Π denotes the stereographic projection of S^2 onto $\mathbb{R}^2$ (see [15], and [1], Lemma 1). We notice that, because of the density Lemma, and continuity properties of the volume functional, we have

$$\frac{1}{4\pi} V(pU) \in \mathbb{Z} \qquad \text{for every } U \in X(C). \tag{1.4}$$

This integer still denotes the degree of pU, so that $X^e(C)$ is the set of maps which have a non-zero degree with respect to the sphere $|\xi| = 1$. In particular, if $U \in X(C) \cap C^0(\mathbb{R}^2, \mathbb{R}^3)$ is regular at infinity, and $V(pU) \neq 0$, then U, as a map from S^2 into C, is not contractible.

Accordingly, we set

$$X^e(C) := \{U \in X(C) \mid V(pU) \neq 0\}.$$

REMARK 1.2. If $p : C \to S^2$ induces an isomorphism between the second homotopy groups of C and S^2, then

$$\{U \in X^e(C) \mid U \text{ is continuous and regular at infinity}\},$$

can be identified with the closure of the set of smooth, non-contractible maps from S^2 into C, by Hopf's Theorem.

In the following section, we will study

PROBLEM 1. *Find $U_\infty \in X^e(C)$, continuous and regular at infinity, in the sense of* (1.3), *which has minimal area among all the surfaces in $X^e(C)$.*

In view of the previous remarks, U_∞ will be a closed non-contractible surface in C.

Before ending this section, we wish to state a problem concerning disk-type minimal surfaces spanned by a wire Γ over the obstacle $\overline{\Omega}$.

We first recall a well known result (see [20], [9]). Let $\Gamma \subseteq C$ be a closed Jordan curve, and suppose that the class $X_\Gamma(C)$ of maps $u \in H^1(D, C)$, whose trace on ∂D is a continuous, weakly monotone parametrization of the curve Γ, is not empty. Then, if ∂C is of class C^2, there is

$$\underline{u} \in H^{2,p}_{\text{loc}}(D) \cap C^0(\overline{D}) \cap X_\Gamma(C)$$

which has minimal area among all surfaces in $X_\Gamma(C)$, and which satisfies the conformality conditions

$$|\underline{u}_x|^2 - |\underline{u}_y|^2 = 0 = \underline{u}_x \cdot \underline{u}_y \quad \text{ in } D,$$

i.e. its area is given exactly by $\frac{1}{2} \int |\nabla \underline{u}|^2$.

We wish to find a second surface

$$\overline{u} \in H^{2,p}_{\text{loc}}(D) \cap C^0(\overline{D}) \cap X_\Gamma(C)$$

satisfying the conformality conditions, which is harmonic where it does not touch $\partial\Omega$, and which "encloses, jointly with $\underline{u}$", the obstacle $\overline{\Omega}$. To make more precise the last statement, let us write

$$V_D(u) := \int_D u \cdot u_x \wedge u_y, \quad u \in H^1 \cap L^\infty(D, \mathbb{R}^3)$$

and set

$$X^e_\Gamma(C) := \{u \in X_\Gamma(C) \mid V_D(pu) \neq V_D(p\underline{u})\},$$

$$\text{where } p(\xi) = \frac{\xi}{|\xi|}, \text{ if } \xi \neq 0.$$

In order to describe the geometric property of surfaces in $X^e_\Gamma(C)$, let us first recall (see [1]) that

$$\frac{1}{4\pi}\,[V_D(pu) - V_D(pv)] \in \mathbb{Z}, \tag{1.5}$$

for every $u,\ v \in X_\Gamma(C)$ with $u - v \in H^1_0(D)$. Actually, (1.5) holds for every $u, v \in X_\Gamma(C)$ (Corollary B.4).

Furthermore, the integer in (1.5) gives the degree of $p \circ U \circ \Pi \in H^1(S^2, S^2)$, where

$$U(z) := \begin{cases} u(z) & \text{if } |z| \leq 1 \\ v\left(\dfrac{z}{|z|^2}\right) & \text{if } |z| > 1 \qquad (\text{for } u - v \in H^1_0(D)). \end{cases} \tag{1.6}$$

Thus, if $u,\ v \in X_\Gamma(C) \cap C^0(\overline{D}, \mathbb{R}^3)$, $u = v$ on ∂D, the condition $V_D(pu) \neq V_D(pv)$ is equivalent to the non-contractibility of $p \circ U \circ \Pi$, U given by (1.6).

In addition, if $\underline{u} \in C^1(\overline{D})$ (which occurs, e.g., if $\Gamma \cap \partial C = \emptyset$, see [8]), one can build, for every $u \in X_\Gamma(C)$ (see Lemma B.3), a change of variables $g_u \in C^0(\overline{D}, \overline{D})$ such that

$$\underline{u} \circ g_u = u \text{ on } \partial D \quad \text{and} \quad V_D(p \circ \underline{u} \circ g_u) = V(p\underline{u}).$$

In this case, if $u \in X^e_\Gamma(C) \cap C^0(\overline{D}, \mathbb{R}^3)$, then $p \circ U \circ \Pi$ is not contractible, where U corresponds here to the pair $u,\ \underline{u} \circ g_u$.

After this preliminaries we are ready to state

PROBLEM 2. *Find $u \in X^e_\Gamma(C)$ of class $C^{1,\alpha}(D) \cap C^0(\overline{D})$ which has minimal area in the class $X^e_\Gamma(C)$.*

2. - Closed minimal surfaces spanned over obstacles

As a standard procedure, we are going to replace the minimization Problem 1,

$$\operatorname*{Min}_{U \in X^e(C)} \int_{\mathbb{R}^2} |U_x \wedge U_y|,$$

with the simpler problem:

Find $U_\infty \in X^e(C)$ *such that*

$$\int_{\mathbb{R}^2} |\nabla U_\infty|^2 = I_\infty := \operatorname*{Inf}_{U \in X^e(C)} \int_{\mathbb{R}^2} |\nabla U|^2. \tag{2.1}$$

First we prove the following

THEOREM 2.1. *Let C be as in the above Section, and assume in addition that $\partial C \in C^2$.*

Let $U_\infty \in X^e(C)$ be a solution of (2.1). *Then*

(i) $U_\infty \in C^{1,\alpha}(\mathbb{R}^2, \mathbb{R}^3 \backslash \Omega)$, *and $U_\infty(z)$ has a limit as $|z| \to +\infty$;*

(ii) $|(U_\infty)_x|^2 - |(U_\infty)_y|^2 = 0 = (U_\infty)_x \cdot (U_\infty)_y$, *in* $\mathbb{R}^2$;

(iii) $\Delta U_\infty = 0$, *in* $\{(x,y) \mid U_\infty(x,y) \notin \overline{\Omega}\}$;

(iv) $\int |(U_\infty)_x \wedge (U_\infty)_y| \leq \int |U_x \wedge U_y|$, *for every* $U \in X^e(C)$.

PROOF. (i) In view of a result by Duzaar [7], it is enough to prove

$$\begin{cases} \forall\ a \in \mathbb{R}^2\ \exists\ r > 0 & \text{such that} \\ \displaystyle\int_{D_r(a)} |\nabla U_\infty|^2 \leq \int_{D_r(a)} |\nabla \Phi|^2 & \forall\ \Phi \in H^1(D_r(a), C), \text{ with} \\ & \Phi - U \in H_0^1(D_r(a), \mathbb{R}^3). \end{cases} \tag{2.2}$$

According to Duzaar's result, this will imply $U_\infty \in H^{2,p}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^3)$ for every $p \in [1, \infty[$, and hence $U_\infty \in C^{1,\alpha}$ by Sobolev imbedding Theorem.

Given $a \in \mathbb{R}^2$, let $r > 0$ be such that

$$\int_{D_r(a)} |\nabla U_\infty|^2 \leq \frac{I_\infty}{2} \tag{2.3}$$

and let $\Phi \in H^1(D_r(a), C)$, with $\Phi = U_\infty$ on $\partial D_r(a)$. Let us consider

$$\Psi(z) := \begin{cases} \Phi(z) & \text{in } D_r(a) \\ U_\infty(z) & \text{in } \mathbb{R}^2 \backslash D_r(a). \end{cases}$$

Since truncation decreases the Dirichlet integral, we can assume $|\Phi|_\infty \leq |U_\infty|_\infty$, so that in particular, if Ψ is admissible (i.e. $\Psi \in X^e(C)$),

$$\int_{\mathbb{R}^2} |\nabla U_\infty|^2 \leq \int_{\mathbb{R}^2} |\nabla \Psi|^2 = \int_{\{|z-a|>r\}} |\nabla U_\infty|^2 + \int_{D_r(a)} |\nabla \Phi|^2.$$

If $\Psi \notin X^e(C)$, i.e.

$$0 = \int_{\mathbb{R}^2} \det(p\Psi, \nabla(p\Psi)) = \int_{D_r(a)} \det(p\Phi, \nabla(p\Phi)) + \int_{\mathbb{R}^2 \backslash D_r(a)} \det(pU_\infty, \nabla(pU_\infty))$$

and hence

$$\int_{D_r(a)} \det(p\Phi, \nabla(p\Phi)) = \int_{D_r(a)} \det(pU_\infty, \nabla pU_\infty) - V(pU_\infty), \tag{2.4}$$

let us consider

$$W(z) := \begin{cases} \Phi(z) & \text{in } D_r(a) \\ U_\infty\left(a + r^2 \dfrac{z-a}{|z-a|^2}\right) & \text{for } |z-a| \geq r. \end{cases}$$

Since $\int_{\{|z-a|>r\}} \det(pW, \nabla(pW))dz = -\int_{\{|z-a|<r\}} \det(pU_\infty, \nabla(pU_\infty))dz$, from (2.4) we deduce

$$\begin{aligned} V(pW) &= \int_{D_r(a)} \det(p\Phi, \nabla(p\Phi)) - \int_{\{|z-a|<r\}} \det(pU_\infty, \nabla(pU_\infty)) \\ &= -V(pU_\infty) \neq 0 \end{aligned}$$

and hence $W \in X^e(C)$. Thus

$$I_\infty = \int |\nabla U_\infty|^2 \leq \int |\nabla W|^2 = \int_{D_r(a)} |\nabla \Phi|^2 + \int_{D_r(a)} |\nabla U_\infty|^2 \leq \int_{D_r(a)} |\nabla \Phi|^2 + \frac{I_\infty}{2}$$

and (2.2) follows from (2.3). Finally, being

$$U^*_\infty(z) := U_\infty\left(\frac{z}{|z|^2}\right) \tag{2.5}$$

again a minimizer, it is continuous at $z = 0$ and we find

$$\lim_{|z|\to\infty} U_\infty(z) = \lim_{z\to 0} U^*_\infty(z),$$

i.e. U_∞ is regular at infinity.

(ii) - (iii) Here we rely on the "Euler Equation" for the "energy minimizing maps" (i.e. for minima of (2.1)) established by Duzaar [7]:

$$-\Delta U_\infty = \chi_{U_\infty^{-1}(\partial\Omega)}\left(-b(U_\infty)(\nabla U_\infty, \nabla U_\infty)\right)\ \nu(U_\infty), \quad \text{a.e. in } \mathbb{R}^2.$$

Here b is the second fundamental form of ∂C, ν is the inner normal at ∂C, and χ_A is the characteristic function of the set $A \subseteq \mathbb{R}^2$. As a consequence, U_∞ is harmonic in the open set $\{z \in \mathbb{R}^2 \mid U_\infty(z) \notin \overline{\Omega}\}$. Also

$$\Delta U_\infty \cdot (U_\infty)_x = 0 = \Delta U_\infty \cdot (U_\infty)_y, \quad \text{a.e. on } \mathbb{R}^2.$$

This easily implies that, setting (in complex notation),

$$\varphi + i\psi = |(U_\infty)_x|^2 - |(U_\infty)_y|^2 - 2i(U_\infty)_x \cdot (U_\infty)_y,$$

then $0 = \int \varphi \Delta\eta = \int \psi \Delta\eta$, $\forall \eta \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)$, i.e. φ and ψ are harmonic. Since $\varphi + i\psi \in L^1(\mathbb{R}^2)$, this implies $\varphi = \psi = 0$.

(iv) From Morrey's ε - conformality result ([13], see also [16], §226), we have that for every $U \in C^\infty \cap X^e(C)$ constant far away, and for every $\varepsilon > 0$, there exists a $V_\varepsilon \in X^e(C)$ such that $\frac{1}{2}\int |\nabla V_\varepsilon|^2 \leq \int |U_x \wedge U_y| + \varepsilon$. Thus, for every $U \in C^\infty \cap X^e(C)$ constant far away, we have

$$\frac{1}{2}\int |\nabla U_\infty|^2 = \int |(U_\infty)_x \wedge (U_\infty)_y| \leq \int |U_x \wedge U_y|,$$

and the conclusion follows from the density Lemma. ∎

The main result in this Section is

THEOREM 2.2.

$$I_\infty := \operatorname*{Inf}_{U \in X^e(C)} \int_{\mathbb{R}^2} |\nabla U|^2 \qquad \textit{is achieved.}$$

The proof is based on a blow up technique, introduced in this class of problems by Sacks und Uhlenbeck [17]. A crucial step in the proof of Theorem 2.2 is the description of the behaviour of sequences $(U_n)_n \subseteq X^e(C)$ which, in the limit, jump out of the class. To this extent, we first recall a result in [4] (see also [23]).

Let $(U_n)_n \subseteq X(C)$ satisfy

$$\nabla U_n \rightharpoonup \nabla U \text{ in } L^2 \text{ and } \operatorname{Sup} |U_n|_\infty < \infty. \tag{2.6}$$

Then, eventually passing to a subsequence,

$$\det\,(pU_n,\ \nabla(pU_n)) \rightharpoonup \det\,(pU, \nabla(pU)) + 4\pi \sum_{i=1}^{n} d_i \delta_{a_i}, \tag{2.7}$$

weakly in the sense of measures, for some $d_i \in \mathbb{Z}$, $a_i \in \mathbb{R}^2$. Here δ_{a_i} denotes the Dirac measure concentrated at a_i.

We first show that, if det $(pU_n, \nabla(pU_n))$ "concentrates" at some a_i, then U_n loses, in the limit, at least as much energy as I_∞.

PROPOSITION 2.3. *Let* $(U_n)_n \subseteq X^e(C)$ *satisfy* (2.6). *Assume* $U \notin X^e(C)$ *and* $d_i \neq 0$ *for some index* i *in* (2.7). *Then for every* $\rho > 0$ *small enough,*

$$\liminf \int_{B_\rho(a_i)} |\nabla U_n|^2 \geq \int_{B_\rho(a_i)} |\nabla U|^2 + I_\infty.$$

PROOF. Fix $\rho > 0$ such that $a_j \notin D_{2\rho}(a_i)$ if $j \neq i$. We can assume, eventually passing to a subsequence, that there exists

$$\lim_n \int_{D_\rho(a_i)} |\nabla U_n|^2.$$

For almost every $r < \rho$, there exists a subsequence U_{n_k} (depending on r) such that

$$\operatorname*{Sup}_k \int_{\partial D_r(a_i)} \left[|\nabla U_{n_k}|^2 + |U_{n_k}|^2\right] < +\infty$$

and hence $U_{n_k} \rightharpoonup U$ weakly in $H^1(\partial D_r(a_i), \mathbb{R}^3)$. Now, we denote by h_k a solution of

$$\operatorname{Inf}\left\{ \int_{D_r(a_i)} |\nabla v|^2 \;\middle|\; v \in H^1(D_r(a_i), C),\ v - U_{n_k}|_{D_r(a_i)} \in H^1_0(D_r(a_i)) \right\}.$$

Because of the good behaviour of h_k on $\partial D_r(a_i)$, one can prove (see Proposition A.1) that up to subsequences

$$h_k \to h \text{ in } H^1(D_r(a_i)), \tag{2.8}$$

where h minimizes the Dirichlet integral with constraint C and boundary data U. In particular

$$\int_{D_r(a_i)} |\nabla h|^2 \leq \int_{D_r(a_i)} |\nabla U|^2. \tag{2.9}$$

Now, let us define

$$\tilde{U}_k(z) := \begin{cases} U_{n_k}(z) & \text{if } |z - a| \leq r \\ h_k\left(a_i + \dfrac{z - a_i}{|z - a_i|^2}\, r^2\right) & \text{if } |z - a| > r. \end{cases}$$

We claim that, if r is chosen small enough, then $\tilde{U}_k \in X^e(C)$, for $k \geq k(r)$ big enough. In fact, using (1.2), (2.8), (2.9), we find

$$\left|V(p\tilde{U}_k)\right| = \left| \int_{D_r(a_i)} \det\ (pU_{n_k}, \nabla(pU_{n_k})) - \int_{D_r(a_i)} \det\ (ph_k, \nabla(ph_k)) \right|$$

$$\geq o(1) + 4\pi|d_i| - \left| \int_{D_r(a_i)} \det\ (pU, \nabla(pU)) \right| - \frac{1}{2} \int_{D_r(a_i)} |\nabla h|^2 > 0,$$

for $k \geq k(r)$. Hence, by using again (2.8), (2.9), we get

$$I_\infty \leq \liminf \int_{\mathbb{R}^2} |\nabla \tilde{U}_k|^2 = \liminf \left[\int_{D_r(a_i)} |\nabla U_{n_k}|^2 + \int_{D_r(a_i)} |\nabla h_k|^2 \right]$$

$$\leq \liminf \int_{D_r(a_i)} |\nabla U_{n_k}|^2 + \int_{D_r(a_i)} |\nabla U|^2.$$

Thus, for such good r's, we have

$$\lim_n \int_{D_\rho(a_i)} |\nabla U_n|^2 = \lim_k \left[\int_{D_r} |\nabla U_{n_k}|^2 + \int_{D_\rho \setminus D_r} |\nabla U_{n_k}|^2 \right]$$

$$\geq I_\infty - \int_{D_r} |\nabla U|^2 + \int_{D_\rho \setminus D_r} |\nabla U|^2$$

$$\geq I_\infty + \int_{D_\rho} |\nabla U|^2 - 2 \int_{D_r} |\nabla U|^2$$

and, letting r go to zero, we conclude the proof of Proposition 2.3. ∎

In particular, we get

PROPOSITION 2.4. *Let* $(U_n)_n \subseteq X^e(C)$ *satisfy* (2.6). *Assume* $U \notin X^e(C)$. *Then*

$$\liminf \int_{\mathbb{R}^2} |\nabla U_n|^2 \geq I_\infty + \int_{\mathbb{R}^2} |\nabla U|^2.$$

PROOF. Let us first remark that, in case $d_i = 0$ for every i, then $U_n^* \rightharpoonup U^*$ (U_n^*, U^* defined as in (2.5)) satisfy the same assumptions as U_n, U and (compare with (2.7))

$$\det(pU_n^*, \nabla(pU_n^*)) \rightharpoonup \det(pU^*, \nabla(pU^*)) + 4\pi d \delta_0, \tag{2.10}$$

where $d = -\frac{V(pU_n)}{4\pi} \in \mathbb{Z} \setminus \{0\}$ (since $U \notin X^e(C)$ by assumption) does not depend on n, for n large. Since $\int |\nabla U_n^*|^2 = \int |\nabla U_n|^2$, eventually replacing U_n by U_n^*, we can assume U_n satisfies the assumptions in Proposition 2.3 and hence, for some a_i,

$$\liminf \int_{\mathbb{R}^2} |\nabla U_n|^2 \geq \liminf \int_{D_r(a_i)} |\nabla U_n|^2 + \liminf \int_{\{|z-a_i|>r\}} |\nabla U_n|^2$$

$$\geq I_\infty + \int_{D_r(a_i)} |\nabla U|^2 + \int_{\{|z-a_i|>r\}} |\nabla U|^2. \quad ∎$$

In case U_n is, in addition, minimizing, i.e. $\int |\nabla U_n|^2 \to I_\infty$, we can say more.

PROPOSITION 2.5. *Let* $(U_n)_n \subseteq X^e(C)$ *satisfy* (2.6) *and, in addition,* $\int |\nabla U_n|^2 \to I_\infty$.

If $U \notin X^e(C)$, *then* U *is a constant, and either*

(i) *There is (exactly one)* $a \in \mathbb{R}^2$ *such that*

$$\int_{D_r(a)} |\nabla U|^2 \to I_\infty, \ \forall\ r > 0 \text{ and } \nabla U_n \to 0 \text{ in } L^2_{\text{loc}}(\mathbb{R}^2 \backslash \{a\}, \mathbb{R}^6), \text{ or}$$

(ii) $\nabla U_n \to 0$ *in* $L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^6)$.

PROOF. First, $U =$ const. by Proposition 2.4. Furthermore, if $d_i \neq 0$ for some i in (2.7), there is just one $d_i \neq 0$, by Proposition 2.3, which, at the same time, implies

$$I_\infty \geq \lim \int_{D_r(a_i)} |\nabla U_n|^2 \geq I_\infty$$

and hence

$$\lim \int_{\{|z-a_i|>r\}} |\nabla U_n|^2 = \lim \int_{\mathbb{R}^2} |\nabla U_n|^2 - \lim \int_{D_r(a_i)} |\nabla U_n|^2 = 0.$$

In case $\det(pU_n, \nabla(pU_n)) \rightharpoonup 0$, we have (see (2.10))

$$\det\ (pU_n^*, \nabla(pU_n)) \rightharpoonup 4\pi d\delta_0, \qquad \text{with } d \in \mathbb{Z} \backslash \{0\}.$$

Again by Proposition 2.3, we have, as in the previous case,

$$\int_{\{|z|<R\}} |\nabla U_n|^2 = \int_{\{|z|>1/R\}} |\nabla U_n^*|^2 \to 0, \text{ for every } R > 0.$$

∎

PROOF OF THEOREM 2.2. Let $(U_n)_n \subseteq X^e(C)$ be minimizing:

$$\int |\nabla U_n|^2 \to I_\infty.$$

Since truncation does not increase the Dirichlet integral and Ω is bounded, we can assume the U_n have a common L^∞ bound and, passing to a subsequence, $\nabla U_n \rightharpoonup \nabla U$ in L^2, $U_n \to U$ a.e. for some $U \in X(C)$. If $U \in X^e(C)$, U is a minimizer by the lower semicontinuity of the Dirichlet integral. If $U \notin X^e(C)$, we want to show that, after rescaling and translating U_n, we can construct a new minimizing sequence weakly converging in $X^e(C)$. Let us introduce the concentration function (see for example [11], [3]):

$$Q_n(t) = \operatorname{Sup}_{z\in\mathbb{R}^2} \int_{D_t(z)} |\nabla U_n|^2.$$

This is a continuous, non-decreasing function, with $Q_n(0) = 0$ and

$$\lim \operatorname{Sup}\ \{Q_n(t) : t > 0\} = I_\infty.$$

Thus, given $\delta \in]0, I_\infty[$, there are, for n large, $t_n > 0, z_n \in \mathbb{R}^2$ such that

$$\delta = \int_{D_{t_n}(z_n)} |\nabla U_n|^2 = Q_n(t_n).$$

Set $\tilde{U}_n(z) := U_n(t_n z + z_n)$. Notice that $\tilde{U}_n \in X^e(C)$, $\int |\nabla \tilde{U}_n|^2 = \int |\nabla U_n|^2 \to I_\infty$ and $\operatorname{Sup}\ |\tilde{U}_n|_\infty = \operatorname{Sup}\ |U_n|_\infty < +\infty$. Again we can find a subsequence $(\tilde{U}_n)_n$ and $U_\infty \in X(C)$ such that

$$\nabla \tilde{U}_n \rightharpoonup \nabla U_\infty \text{ in } L^2,\ \tilde{U}_n \to U_\infty \text{ a.e.}$$

We claim that $U_\infty \in X^e(C)$. Otherwise, by Proposition 2.5, either

$$\int_{D_r(a)} |\nabla \tilde{U}_n|^2 \to I_\infty \qquad \text{for some } a \in \mathbb{R}^2 \text{ and } \forall\ r > 0, \text{ or}$$

$$\int_{D_R(0)} |\nabla \tilde{U}_n|^2 \to 0,\ \forall R > 0.$$

But the first alternative cannot occour, since

$$\int_{D_r(a)} |\nabla \tilde{U}_n|^2 = \int_{D_{t_n r}(t_n a + z_n)} |\nabla U_n|^2 \leq Q_n(t_n) = \delta < I_\infty,$$

for $r \leq 1$. Finally, the second alternative cannot occour either, because

$$\int_{D_1(0)} |\nabla \tilde{U}_n|^2 = \int_{D_{t_n}(z_n)} |\nabla U_n|^2 = \delta > 0.$$

∎

REMARK 2.6. It may happen that the "coincidence set" $\{U_\infty \in \partial\Omega\}$ is the all plane $\mathbb{R}^2$. Since projections on convex sets reduce the Dirichlet integral, this is for example the case when Ω is a convex set. Moreover, in this case, it results that the image through the map U_∞ is exactly $\partial\Omega$ (otherwise the map

U_∞ would be contractible in C) and, identifying U_∞ with its composition with the stereographic projection, our solution U_∞ is in fact a non-constant harmonic map from the sphere onto $\partial\Omega$.

In order to avoid this phenomena, we could use an observation by Duzaar [7]: since the minimizer U_∞ satisfies

$$-b(U_\infty(z)) \ (\nabla U_\infty(z), \nabla U_\infty(z)) \geq 0,$$

for almost every $z \in U_\infty(\partial\Omega)$, the obstacle $\partial\Omega$ has to satisfy a "concavity condition" (when viewed from C) in order to be "essentially touched" by the enveloping surface U_∞. In other words, if b is positive defined somewhere on $\partial\Omega$, U_∞ cannot lie entirely on $\partial\Omega$, and as we have previously noticed, it is harmonic outside the coincidence set.

3. - Pairs of solutions of the Plateau Problem for disk-type minimal surfaces with obstructions

Given the obstacle $\overline{\Omega}$, we assume as in the previous Sections that C, the unbounded connected component of $\mathbb{R}^3\backslash\overline{\Omega}$, is of class C^2 and satisfies (1.1).

Let $\Gamma \subseteq C$ be a given Jordan curve, parametrizable with a diffeomorphism $\gamma^0 : \partial D \to \mathbb{R}^3$. Let us denote by $\mathcal{A}_\Gamma$ the class of $H^{1/2} \cap C^0(\partial D, \mathbb{R}^3)$ - weakly monotone parametrizations of Γ which are normalized by a three-point condition. We suppose that the class of "admissible functions":

$$X_\Gamma(C) := \{u \in H^1(D, \mathbb{R}^3) | \ u_{|\partial D} \in \mathcal{A}_\Gamma, \ u(z) \in C \text{ for a.e. } z \in D\},$$

is not empty.

The "small solution" $\underline{u}$, obtained by Tomi [20], is just a solution of the minimum problem:

$$d_\Gamma := \operatorname*{Min}_{u \in X_\Gamma(C)} \int_D |\nabla u|^2$$

and its existence is easily proved using weakly lower semicontinuity of the Dirichlet integral and the Courant-Lebesgue Lemma [5]. In order to find a second solution as an extremal for the Dirichlet integral, we will first prove that the set

$$X_\Gamma^e(C) := \{u \in X_\Gamma(C) \mid V_D(pu) \neq V_D(p\underline{u})\}$$

(see Section 1), is not empty whenever $X_\Gamma(C) \neq \emptyset$. Then we will consider the following minimization problem:

Find $u \in X_\Gamma^e(C)$ *such that*

$$\int_D |\nabla \overline{u}|^2 = I_\Gamma := \operatorname*{Inf}_{u \in X_\Gamma^e(C)} \int_D |\nabla u|^2. \tag{3.1}$$

As in Section 2, one can prove the following Dirichlet's Principle:

THEOREM 3.1. *Let C be as above, and $\partial C \in C^2$. Let $\overline{u} \in X^{e}_{\Gamma}(C)$ be a solution of* (3.1). *Then*

(i) $\overline{u} \in C^0(\overline{D}, \mathbb{R}^3) \cap C^{1,\alpha}(D, \mathbb{R}^3)$, *for every* $\alpha \in]0,1[$;

(ii) $\overline{u}$ *is conformal:* $\overline{u}_x \cdot \overline{u}_y = 0 = |\overline{u}_x| - |\overline{u}_y|$;

(iii) $\Delta \overline{u} = 0$ *in* $\{(x,y) \in D \mid \overline{u}(x,y) \notin \partial\Omega\}$;

(iv) $\int |\overline{u}_x \wedge \overline{u}_y| \leq \int |u_x \wedge u_y|$, *for every* $u \in X^{e}_{\Gamma}(C)$.

The regularity result in (i) follows from a Theorem by Hildebrandt ([9], see also Tomi [20], Satz [6] and Duzaar [7]), via arguments similar to those used in the proof of Theorem 2.1. Propositions (ii), (iii), (iv) can be obtained in a standard way (see [5], pg. 105 for (ii) and the Morrey's ε-conformality result - [13], Theorem 1.2 - for (iv)), using the invariance of the volume functional under reparametrizations of the domain.

REMARK 3.2. We notice that in case $\Gamma \subseteq \overset{\circ}{C}$, the conformal map $\overline{u}$ is harmonic in a neighbourhood of ∂D, and thus $\overline{u} \in C^1(\overline{D}, \mathbb{R}^3)$ (see [8]). At our knowledge, it is not known a C^1-regularity result up to the boundary in the general case.

In order to solve (3.1), we first prove

LEMMA 3.3. *The set $X^{e}_{\Gamma}(C)$ is not empty and*

$$I_\Gamma \leq \int |\nabla \underline{u}|^2 + I_\infty .$$

Actually, we want to prove a more general result:

$$\tag{3.2} \begin{cases} \textit{for every } u \in H^1(D,C) \cap C^0(D,C) \textit{ and for every } \varepsilon > 0, \\ \textit{there exists } v_\varepsilon \in H^1(D,C) \textit{ such that } v_\varepsilon = u \textit{ on } \partial D, \\ V_D(pv_\varepsilon) \neq V_D(pu) \textit{ and} \\ \displaystyle\int |\nabla v_\varepsilon|^2 \leq \int |\nabla u|^2 + I_\infty + o(1), \qquad \textit{as } \varepsilon \to 0. \end{cases}$$

PROOF OF STATEMENT (3.2). Denoted by U a solution of Problem 1, notice that, under our regularity assumptions on ∂C, U is continuous and regular at infinity, that is

$$\text{there exists } \lim_{|z| \to \infty} U(z) =: U(\infty).$$

We set

$$U^\varepsilon(z) := U\left(\frac{z}{\varepsilon^5}\right), \text{ for } \varepsilon > 0.$$

Let $\lambda : [0,1] \to C$ be a Lipschitz map with $\lambda(0) = u(0)$, $\lambda(1) = U(\infty)$. Our map v_ε is given by

$$v_\varepsilon(z) := \begin{cases} u(z) & \text{if } \varepsilon \le |z| \le 1 \\ \pi\left(u(z) - u(0) + \lambda(\Phi_\varepsilon(|z|))\right) & \text{if } \varepsilon^2 \le |z| \le \varepsilon \\ \pi(\Phi_{\varepsilon^2}(|z|)\ [U^\varepsilon(z) - u(z) + u(0) - U(\infty)] & \\ +u(z) - u(0) + U(\infty)) & \text{if } \varepsilon^4 \le |z| \le \varepsilon^2 \\ U^\varepsilon(z) & \text{if } |z| \le \varepsilon^4, \end{cases}$$

where π is the Lipschitz retraction in (1.1) and $\Phi_\varepsilon = \dfrac{\log r - \log \varepsilon}{\log \varepsilon}$, if $\varepsilon < 1$, $\varepsilon^2 \le r \le \varepsilon$. For ε small, $u(z) - u(0)$ is small, if $|z| \le \varepsilon$, since u is continuous, and hence $u(z) - u(0) + \lambda\left(\Phi_\varepsilon(|z|)\right)$ belongs to a small neighbourhood of C. Similarly, if $\varepsilon^4 \le |z| \le \varepsilon^2$, $U^\varepsilon(z) = U\left(\frac{z}{\varepsilon^5}\right)$ is close to $U(\infty)$ and hence $\Phi_{\varepsilon^2}(|z|)\ [U^\varepsilon(z) - U(\infty) + u(0) - u(z)] + u(z) - u(0) + U(\infty)$ belongs to a neighbourhood of C as well. Thus v_ε is well defined and $v_\varepsilon \in H^1(D, C)$, $v_\varepsilon = u$ on ∂D. A direct computation shows that

$$\int_{\{\varepsilon^4 < |z| < \varepsilon\}} |\nabla v_\varepsilon|^2 \to 0, \quad \text{as } \varepsilon \to 0, \tag{3.3}$$

while

$$\int_{\{\varepsilon < |z| < 1\}} |\nabla v_\varepsilon|^2 \to \int_D |\nabla u|^2,$$

$$\int_{\{|z| < \varepsilon^4\}} |\nabla v_\varepsilon|^2 = \int_{\{|z| < \frac{1}{\varepsilon}\}} |\nabla U|^2 \to I_\infty,$$

and thus

$$\int_D |\nabla v_\varepsilon|^2 = I_\infty + \int_D |\nabla u|^2 + o(1).$$

To end the proof, it is enough to observe that

$$\begin{aligned} V_D(pv_\varepsilon) - V_D(pu) = &- \int_{D_\varepsilon} \det\,(pu, \nabla(pu)) \\ &+ \int_{\{\varepsilon^4 < |z| < \varepsilon\}} \det\,(pv_\varepsilon, \nabla(pv_\varepsilon)) \\ &+ \int_{\{|z| < \frac{1}{\varepsilon}\}} \det\,(pU, \nabla(pU)) = V(pU) + o(1), \end{aligned}$$

by (3.3), and hence, if ε is small enough, $V_D(pv_\varepsilon) - V_D(pu) \in 4\pi\mathbb{Z}\setminus\{0\}$, since $V(pU) \neq 0$. ∎

REMARK 3.4. Equality in Lemma 3.3 cannot be excluded, in general. Moreover, the sequence $(v_\varepsilon)_\varepsilon$ in the proof of (3.2) shows that, whenever equality occurs, there exist minimizing sequences for Problem (3.1) which weakly converge to the small solution $\underline{u}$ and hence do not have strongly convergent subsequences.

Equality occours, for example, in the "degenerate case", i.e. when Γ reduces to a point z^0. In this case the set of "admissible functions" is $X_\Gamma^e(C) = \{u \in H^1(D, C) | u = z^0 \text{ on } \partial D,\ V_D(pu) \neq 0\}$ and $I_\Gamma \leq I_\infty$. Actually, $I_\Gamma = I_\infty$, since in this case $X_\Gamma^e(C)$ is embedded in a natural way in $X^e(C)$.

It is quite likely that I_Γ is not achieved whenever equality holds. This is the case if Ω is the unit ball and Γ reduces to a point, e.g. in $\partial\Omega = S^2$. In fact a minimizer for the Dirichlet integral would be a non-constant harmonic map from the disk into S^2 with constant boundary data; but this cannot occour in view of a uniqueness result due to Lemaire [10].

Notice also that such a minimizer would also be an extremal for the Bononcini-Wente isoperimetric inequality:

$$|V_D(v)|^{\frac{2}{3}} \leq \frac{1}{(32\pi)^{\frac{1}{3}}} \int |\nabla v|^2, \qquad \text{for every } v \in H^1(D, \mathbb{R}^3) \text{ constant on } \partial D,$$

which is known not to exist ([22]).

The main result in this Section is

THEOREM 3.5. *Let C, Γ be as above. Then I_Γ is achieved, provided*

$$I_\Gamma < \int |\nabla \underline{u}|^2 + I_\infty. \tag{3.4}$$

PROOF. We split the proof into two steps.

STEP 1. There is $v^0 \in X_\Gamma(C)$ such that

$$I_\Gamma = \operatorname*{Inf}_{\substack{u \in X_\Gamma^e(C) \\ u - v^0 \in H_0^1}} \int |\nabla u|^2.$$

STEP 2. $I_{v^0} := \operatorname{Inf}\left\{\int |\nabla u|^2 : u \in X_\Gamma^e(C),\ u - v^0 \in H_0^1\right\}$ is achieved in $X_\Gamma^e(C)$.

PROOF OF STEP 1. Here we do not make use of assumption (3.4). Let $(u_n)_n \subseteq X_\Gamma^e(C)$ be such that $\int |\nabla u_n|^2 \to I_\Gamma$. We can assume $\operatorname{Sup} |u_n|_\infty < \infty$, and since $(u_{n|\partial D})_n$ is equicontinuous on ∂D, by Courant-Lebesgue Lemma, we can also assume $u_n \rightharpoonup v^0$ weakly in H^1 and $u_n \to v^0$ uniformly on ∂D for some $v^0 \in X_\Gamma(C)$. Thus, if $\Delta h_n = 0$, $h_n = u_n - v^0$ on ∂D, $h_n \to 0$ uniformly and weakly in H^1. As a consequence $w_n := \pi(u_n - h_n)$ is well defined for large n (here π is the retraction of some neighbourhood of C onto C) and, with easy computations

$$\liminf \int |\nabla w_n|^2 \leq \liminf \int |\nabla u_n|^2 = I_\Gamma. \tag{3.5}$$

Since $w_n - v^0 \in H_0^1$, it is enough to prove, in view of (3.5), that $V_D(pw_n) \neq V_D(p\underline{u})$. But this readily follows, because $|pw_n - pu_n|_\infty \leq$ const. $|h_n|_\infty \to 0$ and $pw_{n|\partial D} = v^0$, so that, by Lemma B.1, $V_D(pu_n) = V_D(pu_n - pw_n + pw_n) = V_D(pw_n) + o(1)$. Since

$$V_D(pu_n) - V_D(p\underline{u}) \in 4\pi\mathbb{Z}\backslash\{0\}$$

(see Corollary B.4), the proof is complete. ∎

PROOF OF STEP 2. The argument we present here applies to the solvability of Dirichlet problems, and hence we give it in this more general form. Let $v \in X_\Gamma(C)$ and let

$$w_n - v \in H_0^1, \ \ w_n \in X_\Gamma^e(C), \ \ w_n \rightharpoonup w.$$

If $w \notin X_\Gamma^e(C)$ then, applying Proposition 2.4 to the sequence

$$U_n(z) := \begin{cases} w_n(z) & \text{if } |z| \leq 1 \\ w\left(\frac{z}{|z|^2}\right) & \text{if } |z| > 1, \end{cases}$$

we immediately get

$$\liminf \int_D |\nabla w_n|^2 \geq I_\infty + \int_D |\nabla w|^2. \tag{3.6}$$

From (3.6), it follows that the infimum

$$I_v := \operatorname{Inf}\left\{\int |\nabla w|^2 : w - v \in H_0^1, \ w \in X_\Gamma^e(C)\right\}$$

is achieved provided (compare with (3.2)):

$$I_v < I_\infty + \operatorname*{Inf}_{\substack{w \in X_\Gamma(C) \\ w - v \in H_0^1}} \int |\nabla w|^2. \tag{3.7}$$

This ends the proof of Step 2 since (3.7) holds, with $v = v^0$ given by Step 1, in view of assumption (3.4). ∎

REMARK 3.6. It is interesting to reformulate Theorem 3.5 from the point of view of *relaxation*. If we define the energy associated to the minimum problem

$$\operatorname*{Min}_{u\in X_\Gamma^\varepsilon(C)} \int |\nabla u|^2,$$

as

$$E(u) := \begin{cases} \int |\nabla u|^2 & \text{if } u \in X_\Gamma^\varepsilon(C) \\ +\infty & \text{otherwise in } X_\Gamma(C), \end{cases}$$

then the relaxed functional, in the weak H^1-topology, is defined by

$$(\mathrm{sc}^- E)(u) := \operatorname{Inf} \left\{ \liminf \int |\nabla u_n|^2 : u_n \in X_\Gamma(C),\ u_n \rightharpoonup u \text{ weakly in } H^1 \right\}.$$

Slight modifications in our arguments show that

$$(\mathrm{sc}^- E)(u) := \begin{cases} \int |\nabla u|^2 & \text{if } u \in X_\Gamma^\varepsilon(C) \\ \int |\nabla u|^2 + I_\infty & \text{otherwise in } X_\Gamma(C). \end{cases}$$

Now, let $\overline{u}$ be a minimum point for the functional $\mathrm{sc}^- E$, that is

$$(\mathrm{sc}^- E)(\overline{u}) = \operatorname*{Inf}_{X_\Gamma(C)} E = \inf_{X_\Gamma^\varepsilon(C)} \int |\nabla u|^2.$$

If (3.4) holds, then necessarily $\overline{u} \in X_\Gamma^\varepsilon(C)$, and hence $\overline{u}$ is also a solution of our minimization Problem 2.

REMARK 3.7. A simple variant of Problem 1 arises if we drop the connectivity assumption on the obstacle Ω; related results are presented in [14], where are also considered extensions to higher dimensions.

It would be of interest to describe the limit problem as the connected components of Ω become infinite while their size go to zero. This could also be a way to deal with a much deeper variant of Problem 1, namely the case of thin obstacles. Problems of this kind have been considered in the framework of minimal boundaries (see [6]).

Appendix A

We present here a result concerning continuous dependence of minimizers for the Dirichlet integral, subjected to obstacle conditions, with respect to H^1 weak convergence of boundary values.

Let $C \subseteq \mathbb{R}^3$ be a closed set satisfying

(A.1) There is $\delta > 0$ and a Lipschitz retraction
$$\pi : \{\xi \in \mathbb{R}^3 | \mathrm{d}(\xi, C) < \delta\} \to C.$$

Let us denote

$$H^1(D,C) = \{u \in H^1(D,\mathbb{R}^3) | u(z) \in C \text{ for a.e. } z \in D\}.$$

Let $h_n \in H^1(D,C)$ satisfy

$$\text{(A.2)} \qquad \int |\nabla h_n|^2 = \operatorname*{Min}_{\substack{v \in H^1(D,C) \\ v - h_n \in H_0^1}} \int |\nabla v|^2 \quad \text{and}$$

$$\operatorname*{Sup}_n \|h_n\|_{H^1(D)} < +\infty,$$

$$\text{(A.3)} \qquad h_{n|\partial D} \in H^1(\partial D, \mathbb{R}^3) \quad \text{and} \quad \operatorname*{Sup}_n \left\|h_{n|\partial D}\right\|_{H^1(\partial D)} < +\infty.$$

PROPOSITION A.1. *Let h_n satisfy* (A.2), (A.3). *Then if $h_n \rightharpoonup h$, we have*

(i) $$\int |\nabla h|^2 = \operatorname*{Min}_{\substack{v \in H^1(D,C) \\ v - h \in H_0^1}} \int |\nabla v|^2;$$

(ii) $$\int |\nabla h_n|^2 \to \int |\nabla h|^2.$$

PROOF. Since $\int |\nabla h|^2 \leq \liminf \int |\nabla h_n|^2$, it is enough to prove

$$\text{(A.4)} \qquad \limsup \int |\nabla h_n|^2 \leq \operatorname*{Inf}_{\substack{v \in H^1(D,C) \\ v - h \in H_0^1}} \int |\nabla v|^2.$$

To prove (A.4), let us consider, for $r \in]0,1[$:

$$v_n^r(s,\vartheta) := \frac{\log r/s}{\log r} [h_n(1,\vartheta) - h(1,\vartheta)] + h(1,\vartheta)$$

(in polar coordinates). Since

$$\operatorname*{Sup}_{\substack{0 \leq \vartheta < 2\pi \\ r \leq s \leq 1}} |v_n^r(s,\vartheta) - h(1,\vartheta)| \leq |h_n - h|_{L^\infty(\partial D)} \to 0, \text{ as } n \to \infty,$$

πv_n is well defined on $D \backslash D_r$, where π is the retraction given by (A.1). Moreover

$$\text{(A.5)} \qquad \int_{D \backslash D_r} |\nabla(\pi v_n^r)|^2 \leq L^2 \int_{D \backslash D_r} |\nabla v_n^r|^2,$$

if L denotes the Lipschitz constant for π. Now, let $\hat{h} \in H^1(D, C)$ be such that

$$\hat{h} - h \in H_0^1 \text{ and } \int |\nabla \hat{h}|^2 = \operatorname*{Min}_{\substack{v \in H^1(D,C) \\ v-h \in H_0^1}} \int |\nabla v|^2$$

and we define

$$\omega_n(z) := \begin{cases} \pi v_n^r(z) & \text{if } z \in D \backslash D_r \\ \hat{h}\left(\frac{z}{r}\right) & \text{if } z \in D_r. \end{cases}$$

Since $\omega_n \in H^1(D, C)$ and $\omega_n = h_n$ on ∂D, we have

$$\int_D |\nabla h_n|^2 \leq \int_D |\nabla \omega_n|^2, \text{ while } \int_D |\nabla \omega_n|^2 \leq L^2 \int_{D \backslash D_r} |\nabla v_n^r|^2 + \int_D |\nabla \hat{h}|^2.$$

An easy computation gives, using (A.3),

$$\int_{D \backslash D_r} |\nabla v_n^r|^2 = O(|\log\ r|)$$

and hence lim sup $\int |\nabla h_n|^2 \leq O(|\log\ r|) + \int |\nabla \hat{h}|^2$, for every $r \in]0, 1[$, i.e. (A.4). ∎

COROLLARY A.2. *Let h_n satisfy* (A.2). *If $h_n \rightharpoonup h$, then*

(i) $\displaystyle\int |\nabla h|^2 = \operatorname*{Min}_{\substack{v \in H^1(D,C) \\ v-h \in H_0^1}} \int |\nabla v|^2;$

(ii) $h_n \to h$, in $H^1_{\text{loc}}(D, \mathbb{R}^3)$.

PROOF. For a.e. $r < 1$, we have Sup $\left\|h_{n|\partial D_r}\right\|_{H^1(\partial D_r)} < +\infty$. Since clearly

$$\int_{D_r} |\nabla h_n|^2 = \operatorname*{Min}_{\substack{v \in H^1(D_r,C) \\ v-h \in H_0^1(D_r)}} \int |\nabla v|^2,$$

Proposition A.1 applies to obtain $h_n \to h$ in $H^1(D_r)$ and $\int_D |\nabla h|^2 \leq \int_D |\nabla v|^2$, for every $v \in H^1(D_r, C)$, with $v - h \in H_0^1(D_r)$. Thus, if $\omega \in H^1(D, C)$, $\omega - h \in H_0^1(D)$, setting

$$\omega_r(z) := \begin{cases} h\left(\frac{z}{|z|^2}\ r^2\right) & \text{if } r^2 \leq |z| \leq r \\ \omega\left(\frac{z}{r^2}\right) & \text{if } |z| \leq r^2, \end{cases}$$

we see that, for a.e. $r < 1$:

$$\int_{D_r} |\nabla h|^2 \leq \int_{D_r} |\nabla \omega_r|^2 \tag{A.6}$$

because $\omega_r - h \in H_0^1(D_r)$. But

$$\int_{D_r} |\nabla \omega_r|^2 = \int_{D_r} |\nabla \omega|^2 + \int_{\{r<|z|<1\}} |\nabla h|^2$$

and hence, sending r to 1 in (A.6), we get (i). ∎

REMARK A.3. To complete these continuous dependence results, it would be of interest to prove that, if in addition to the assumptions in Corollary A.2, one also assumes $h_n \to h$ uniformly on ∂D, then $h_n \to h$ uniformly on $\overline{D}$. Since we do not need this result, we do not go into details.

Appendix B

For convenience of the reader, we list here a few simple properties of the volume functional (see [21] and [2], Appendix).

LEMMA B.1. *Let $k^n, \psi^n, \psi \in H^1 \cap L^\infty(D, \mathbb{R}^3)$. Assume that $k^n \to 0$ in L^∞ and weakly in H^1, and $\psi^n - \psi \rightharpoonup 0$ in $H_0^1(D, \mathbb{R}^3)$. Then*

$$\int_D (\psi^n + k^n) \cdot (\psi^n + k^n)_x \wedge (\psi^n + k^n)_y = \int_D \psi^n \cdot (\psi^n)_x \wedge (\psi^n)_y + o(1).$$

PROOF. From (1.2), one sees that

$$\begin{aligned}\int_D (\psi^n + k^n) \cdot (\psi^n + k^n)_x \wedge (\psi^n + k^n)_y &= \int_D \psi^n \cdot (\psi^n)_x \wedge (\psi^n)_y \\ &+ \int_D \psi^n \cdot [(\psi^n)_x \wedge (k^n)_y + (k^n)_x \wedge (\psi^n)_y] \\ &+ \int_D (\psi^n - \psi) \cdot (k^n)_x \wedge (k^n)_y + o(1),\end{aligned}$$

because $k^n \to 0$ in L^∞ and weakly in H^1. Now, the second integral in the right hand side goes to zero by Lemma A.7 in [2], while the third one goes to zero by Lemma A.6 in [2]. ∎

LEMMA B.2. *Let $g \in C^0(\overline{D}, \overline{D})$ be an orientation preserving bilipschitz homeomorphism. Then*

$$V_D(u) = V(u \circ g), \qquad \text{for every } u \in H^1 \cap L^\infty(D, \mathbb{R}^3).$$

This follows from the chain rule:

$$\int \det\ (u \circ g,\ \nabla(u \circ g)) = \int \det\ (u(g(z)), (\nabla u)(g(z)))\ \det J_g\ dz.$$

∎

LEMMA B.3. *Let $\alpha : [0, 2\pi] \to [0, 2\pi]$ be a nondecreasing function, with $\alpha(0) = 0$, $\alpha(2\pi) = 2\pi$. Let $g(r, \vartheta) := re^{i\alpha(\vartheta)}$. Then*

$$V_D(u \circ g) = V_D(u), \qquad \text{for every } u \in C^1(\overline{D}).$$

PROOF. Since $J_g = r\alpha'$, $\det\ (u \circ g, \nabla(u \circ g)) \in L^1$. After properly extending α, we can regularize it to get $\hat{\alpha}_n \in C^\infty$, $\hat{\alpha}_n \to \alpha$ uniformly, $\hat{\alpha}'_n \to \alpha'$ in L^1, $\hat{\alpha}_n(2\pi) = 2\pi + \hat{\alpha}_n(0)$ and $\hat{\alpha}'_n \geq 0$ in $[0, 2\pi]$. Then we set

$$\alpha_n(\vartheta) := \frac{n}{n+1}\ \hat{\alpha}_n(\vartheta) + \frac{1}{n+1}\ \vartheta,$$
$$g_n(r, \vartheta) = re^{i\alpha_n(\vartheta)}.$$

By the previous Lemma we get $V_D(u \circ g_n) = V_D(u)$. But

$$\begin{aligned} V_D(u \circ g_n) &= \int_D \det\ (u \circ g_n, \nabla(u \circ g_n)) \\ &= \int_D \det\ (u(g_n(z)), (\nabla u)(g_n(z)))\ J_{g_n} dz \\ &= \int_D \det\ (u(g(z)), (\nabla u)(g(z)))\ \det J_g + o(1) \\ &= \int_D \det\ (u \circ g, \nabla(u \circ g)) + o(1), \end{aligned}$$

because $\alpha_n \to \alpha$ uniformly and $u \in C^1(\overline{D})$ imply

$$\det\ (u(g_n(z)), (\nabla u)(g_n(z))) \to \det\ (u(g(z)), (\nabla u)(g(z)))$$

uniformly, while

$$\det J_{g_n} \to \det J_g, \qquad \text{in } L^1.$$

∎

COROLLARY B.4. *Assume* Γ *is parametrizable with a diffeomorphism* $\gamma^0 : \partial D \to \mathbb{R}^3$. *Then*

$$\frac{1}{4\pi}\ \{V_D(pu) - V_D(pv)\} \in \mathbb{Z}, \qquad \textit{for every } u, v \in X_\Gamma(C).$$

PROOF. Given $\delta > 0$, let $h \in C^1(\overline{D}, \mathbb{R}^3 \backslash B_\delta)$, with $h_{|\partial D} = \gamma^0$ (assuming for simplicity $0 \notin C$). It is enough to prove

$$\frac{1}{4\pi}\ \{V_D(pu) - V_D(ph)\} \in \mathbb{Z}, \qquad \text{for every } u \in X_\Gamma(C).$$

Since, for a given $u \in X_\Gamma(C)$, $u_{|\partial D}$ is a weakly monotone reparametrization of Γ, there is a map $\alpha_u : [0, 2\pi] \to [0, 2\pi]$, continuous and nondecreasing, with $\alpha_u(0) = 0$, $\alpha_u(2\pi) = 2\pi$, such that

$$u\left(e^{i\vartheta}\right) = \gamma^0\left(e^{i\alpha_u(\vartheta)}\right).$$

By Lemma B.3, setting $g_u(r, \vartheta) = re^{i\alpha_u(\vartheta)}$, we have $V_D(p \circ h) = V_D(p \circ h \circ g_u)$ and hence

$$\frac{1}{4\pi}\ \{V_D(pu) - V_D(ph)\} = \frac{1}{4\pi}\ \{V_D(pu) - V_D(ph \circ g_u)\} \in \mathbb{Z},$$

because $u(z) = h(g_u(z))$, for every $z \in \partial D$, so that Lemma 1 in [1] applies. ∎

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