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SURFACES OF REVOLUTION WITH CONSTANT MEAN CURVATURE IN LORENTZ-MINKOWSKI SPACE

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In an old paper [1] Delaunay proved that the profile curve of a surface of revolution with nonzero constant mean curvature in Euclidean 3-space can be described as the locus of a focus when a quadratic curve is rolled along the axis of revolution. This result was restudied by Kenmotsu [4] and Hsiang and Yu [3] for generalizations in various directions. In the present paper we shall examine the problem in Lorentz-Minkowski 3-space by studying spacelike surfaces of revolution.

In Lorentz-Minkowski 3-space the axis of revolution is either spacelike or timelike or null. In the first two cases, we can prove the results of the same kind as Delaunay's except that the nature of quadrics needs special attention. In the third case, we can determine the profile curves completely without giving a geometric interpretation. In any case, we are interested in the surfaces up to congruence by a Lorentz transformation.

In what follows, $\{x, y, z\}$ is a Lorentz coordinate system for which the metric of the space is $dx^2 + dy^2 - dz^2$. Our starting point is the lemma in [2], Section 1.

1. Surfaces of revolution with spacelike axis. Here we deal with a surface of the form

$$(x(s),\ z(s)sh\, t,\ z(s)ch\, t)\ ,$$

where s is an arc-length parameter of the profile curve $(x(s), z(s))$ in the xz-plane and $z < 0$. The principal curvatures of the surface are given by $\ddot{x}/\dot{z}$ and $\dot{x}/z$, where $\dot{x} = dx/ds$, $\ddot{x} = d^2x/ds^2$, etc.

Adopting the method in [3] we now discuss the rolling of a curve in the xz-plane. Let Γ be a smooth curve in the xz-plane given by a timelike vector-valued function

$$(1)\qquad x = r(\theta)sh\,\theta\ ,\quad z = r(\theta)ch\,\theta$$

We assume that $r > 0$ and that the tangent vector of Γ is always spacelike, that is, $r^2 - r'^2 > 0$, where the prime denotes $d/d\theta$. Let Ω

be the locus of the origin when Γ is rolled along the x-axis in such a way that Ω appears below the x-axis. Then Ω is written as

$$(2) \qquad x = \xi - \xi_0 - r\,sh\,\phi\,, \quad z = -r\,ch\,\phi\,,$$

where $\phi = \phi(\theta)$ is determined by the fact that $r\,sh\phi$ is equal to the Lorentz inner product of the position vector (1) of Γ and the unit tangent vector of Γ. Thus

$$sh\,\phi = -r'/\sqrt{r^2 - r'^2} \quad \text{and} \quad ch\,\phi = r/\sqrt{r^2 - r'^2}\,.$$

It follows that the tangent vector (x', z') is perpendicular to the vector $(sh\,\phi, ch\,\phi)$: $x'sh\,\phi - z'ch\,\phi = sh\,\phi\xi' + r' = 0$. It is easily verified that Ω is regular and hence spacelike if and only if the curvature of Γ is non-vanishing. We henceforth assume this condition.

We choose an arc-length parameter s for Ω in such a way that $(\dot{x}, \dot{z}) = (ch\phi, sh\phi)$, where the dot denotes d/ds. Then we obtain

$$\dot{r} = -th\,\phi(1 + r\dot{\phi})\,, \quad \dot{\xi} = \frac{1}{ch\,\phi}(1 + r\dot{\phi})$$

and from $\xi' = \sqrt{r^2 - r'^2}$

$$(3) \qquad \dot{\theta} = (1 + r\dot{\phi})/r\,.$$

Since $\dot{\xi} > 0$, we have $1 + r\dot{\phi} > 0$ and hence the center of curvature of Ω never lies on the x-axis.

Conversely, if Ω is a spacelike curve in the lower half-plane $z < 0$ whose center of curvature lies on the x-axis, then there exists a curve Γ of the form (1) such that Ω is the locus of the origin when Γ is rolled along the x-axis. This can be verified as follows. Denote by $(ch\phi, sh\phi)$ the unit tangent vector of Ω and let $(\xi, 0)$ be the intersection of the normal line of Ω with the x-axis. Then Ω is written in the form (2) with $\xi_0 = 0$. Return to the equation (3). The right hand side is determined by Ω as a function of s and never vanishes by virtue of the assumption on Ω. Thus the solution $\theta = \theta(s)$ is a monotone function of s and we obtain a function $r = r(\theta)$. It is easily seen that Ω is indeed the locus of the origin when Γ as in (1) is rolled along the x-axis.

Summarizing the discussions above, we obtain

LEMMA. *Let Γ be a spacelike curve given in the form* (1) *and let Ω be the locus of the origin when Γ is rolled along the x-axis. If the curvature of Γ never vanishes, then Ω is a spacelike curve for which the center of curvature never lies on the x-axis. Conversely, such a curve Ω is obtained as the locus of the origin for the rolling of a certain* *spacelike curve Γ, which is uniquely determined up to a Lorentz transformation of the xz-plane.*

Now let S be the surface of revolution with the profile curve Ω as above. The principal curvatures are

$$\ddot{x}/\dot{z} = \dot{\phi} \quad \text{and} \quad \dot{x}/z = -1/r \, .$$

By a computation similar to that in the proof of Theorem 1 in [3], we easily find that S has constant mean curvature H if and only if the function $r(\theta)$ in (1) satisfies the differential equation

$$(4) \qquad d^2 \log r/d\theta^2 = \left[(d \log r/d\theta)^2 - 1\right]\frac{1 + 2rH}{2 + 2rH} \, .$$

The general solution of (4) is

$$1/r = a\, ch\, \theta + b\, sh\, \theta + c \, , \quad r > 0 \, ,$$

where

$$2Hc = a^2 - b^2 - c^2 \, .$$

When $c = 0$, the solutions $r = ae^{\pm\theta}$ are null lines (namely, $y \pm x = 0$) so we exclude them.

Thus a curve Γ gives rise to a surface of revolution with constant mean curvature if and only if r is one of the following forms:

(i) $r = 1/c$ with $H = -c/2 \quad (c \neq 0)$

(ii) $1/r = \pm\lambda ch(\theta + \mu) + c$ with $\lambda > 0$, $H = (\lambda^2 - c^2)/2c \quad (c \neq 0)$

(iii) $1/r = \pm\lambda sh(\theta + \mu) + c$ with $\lambda > 0$, $H = -(c^2 + \lambda^2)/2c \quad (c \neq 0)$

(iv) $1/r = ae^{\theta} + c$ or $ae^{-\theta} + c$ with $H = -c/2 \quad (c \neq 0)$.

If two curves Γ_1 and Γ_2 simply differ by a Lorentzian transformation of the xz-plane fixing the origin, then the resulting curves Ω_1 and Ω_2 generate congruent surfaces of revolution. Thus in the list above, we may assume $\mu = 0$ in (ii) and (iii), consider only the $+$ sign in (iii) and one or the other (say, $ae^{-\theta} + c$) in (iv). We shall describe these curves in Section 2.

Finally, we take up the case where the center of curvatue of the profile curve Ω is always on the x-axis. If we write Ω in the form (2) with $\xi_0 = 0$, our present assumption means that $1 + r\dot{\phi}$ is identically 0. On the other hand, constancy of the mean curvature H implies $1 - r\dot{\phi} = -2Hr$. Obviously $H \neq 0$ and $H = \dot{\phi} = -1/r$. We now get $(\dot{x}, \dot{z}) = (ch(Hs + \phi_0), sh(Hs + \phi_0))$ and Ω is given by

$$(x, z) = (x_0 + sh(Hs + \phi_0)/H, ch(Hs + \phi_0)/H) \, .$$

The resulting surface of revolution is congruent to the standard hyperboloid $x^2 + y^2 - z^2 = -1/H^2$, $z > 0$.

2. Quadrics in the Lorentz-Minkowski plane. In this section we shall show that each of the curves Γ given by the polar equations (i)-(iv) in Section 1 is part of a certain quadratic curve. The theory of quadratic curves in the Lorentz-Minkowski plane is little known. We only sketch the essence that is necessary for the understanding of the geometric nature of these curves Γ.

Because of the reduction we already made, we consider the following polar equations (I)-(IV).

$$\text{(I)} \qquad r = 1/c\,, \quad c > 0\,.$$

This describes the spacelike curve $x = sh\theta/c$, $z = ch\theta/c$ with timelike position vectors. It is an analogue of a Euclidean circle, and the profile curve resulting as the locus of the origin is the line $z = -1/c$. The surface of revolution is an isometric imbedding of the Euclidean plane given by $(s, t) \to (s, -sh\, t/c, -ch\, t/c)$ with constant mean curvature c.

$$\text{(II)} \qquad 1/r = \pm\lambda\, ch\,\theta + c \quad \text{with} \quad c > 0 \text{ or } c < 0\,.$$

In order to classify all these curves, we let $d = 1/\lambda > 0$ *and* $e = \lambda/|c| > 0$.

We first consider the case where the polar equation is

$$(5) \qquad 1/r = ch\,\theta/d + 1/ed \quad \text{with} \quad 0 < e < 1\,.$$

This equation can be rewritten as $r = e(d - r\,ch\,\theta)$, whose geometric meaning is the following. Denote the origin $(0, 0)$ by F and consider the spacelike line L: $z = d$. Our equation shows that the curve Γ consists of all points $P = (r\,sh\,\theta, r\,ch\,\theta)$ below the line L with the property that the distance to F is e times the distance to the line L (which is, of course, the norm of the vector $\overrightarrow{PA}$, where A is a point on L such that PA is perpendicular to L). Now by making a coordinate transformation, we assume that F has coordinates $(0, be)$ and the line L has equation $z = b/e$, where $b(1/e - e) = d$. Then our curve is part of the hyperbola

$$(6) \qquad -x^2/a^2 + z^2/b^2 = 1 \quad \text{with} \quad a = b\sqrt{1 - e^2}\,,$$

subject to $z > 0$ (namely, the upper branch) and strictly between the intersections $G_1 = (-d, b/e)$ and $G_2 = (d, b/e)$ of the hyperbola with the line L. The lines FG_1 and FG_2 are actually null lines tangent to the hyperbola and Γ is spacelike. See Figure 1. We can say that (6) is the *standard equation of the hyperbola defined by a focus F and a directrix L.* Note the difference of the appearance of F and L relative to the hyperbola in comparison with Euclidean geometry.

The equation $1/r = ch\,\theta/d - 1/ed$ describes part of the hyperbola (6) subject to the restriction $x < -d$ or $x > d$. These curves have timelike tangent vectors.

The equation $1/r = -ch\,\theta/d + 1/ed$ describes the same curve as the case $1/r = ch\,\theta/d + 1/ed$, *except that we must take* $\overrightarrow{F'P} = (r\,sh\,\theta, r\,ch\,\theta)$, *where* $F' = (0, -be)$. Depending on whether F or F' is used, the resulting profile curves Ω and Ω' are, of course, different. See also Figure 1.

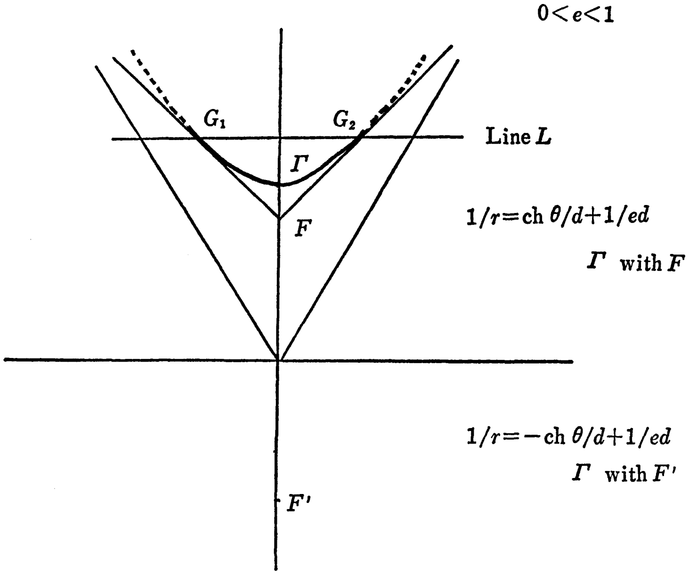

FIGURE 1

If $e > 1$ or $e = 1$ in (5), we get spacelike curves which are part of an ellipse or a parabola. The equation $1/r = ch\,\theta/d - 1/ed$ with $e > 1$ gives another spacelike curve Γ' and a timelike curve on the ellipse, while $1/r = -ch\,\theta/d + 1/ed$ gives an empty set. The equation $1/r = ch\,\theta/d - 1/d$ gives a timelike part on the parabola, and $1/r = -ch\,\theta/d + 1/d$ gives an empty set. See Figures 2 and 3.

Next we consider the equation

(III) $$1/r = sh\,\theta/d \pm 1/ed \quad \text{(no restriction on } e > 0) .$$

Here we have to get a hyperbola

(7) $$-x^2/a^2 + z^2/b^2 = 1 \quad \text{with} \quad b = a\sqrt{1 + e^2}$$

starting with a focus $F = (-ae, 0)$ and a directrix $L: x = a/e$. Note that F is not on the z-axis in this case. We get a spacelike curve Γ for the

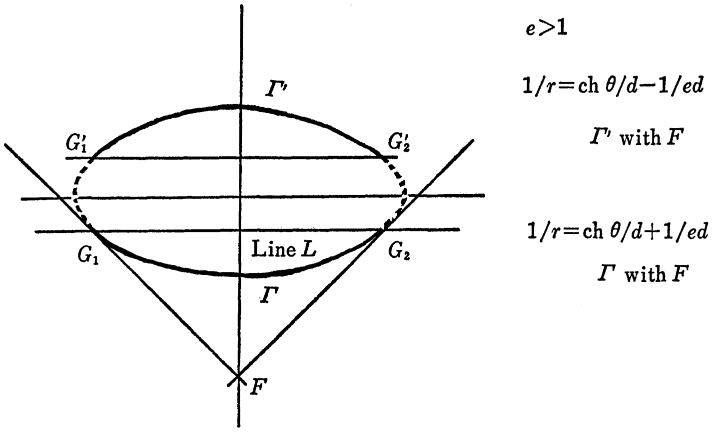

FIGURE 2

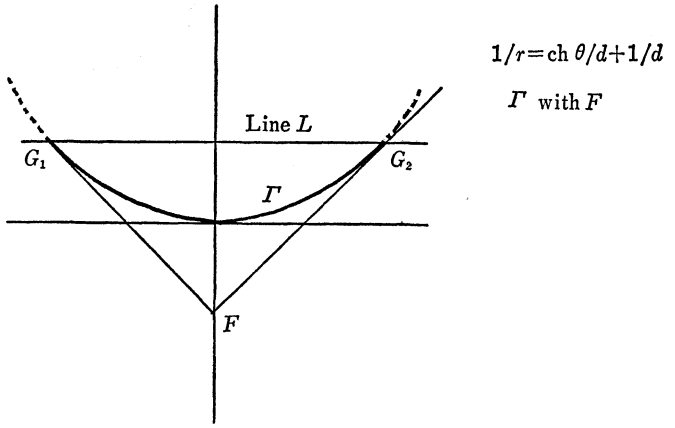

FIGURE 3

plus sign in (III) and a timelike curve for the minus sign. See Figure 4.

Finally, we examine

$$\text{(IV)} \qquad 1/r = ae^{-\theta} + c\ , \quad c \neq 0\ .$$

We have to consider three cases:

$$1/r = Ae^{-\theta} + B \text{ or } Ae^{-\theta} - B \text{ or } -Ae^{-\theta} + B\ , \quad \text{where } A, B > 0\ .$$

In the first case, $r > 0$ and $r^2 - r'^2 > 0$ for all θ. Choose $a > 0$ and $k > 0$ such that $B = 2/k$ and $A = 2\sqrt{2}\,a/k^2$. Then we get

$$(2\sqrt{2}\,ae^{-\theta} + 2k)r = k^2 \quad \text{hence} \quad 2\sqrt{2}\,are^{\theta} + r^2 = (k - r)^2\ .$$

Since $1/r > 1/k$, that is, $k > r$, we get

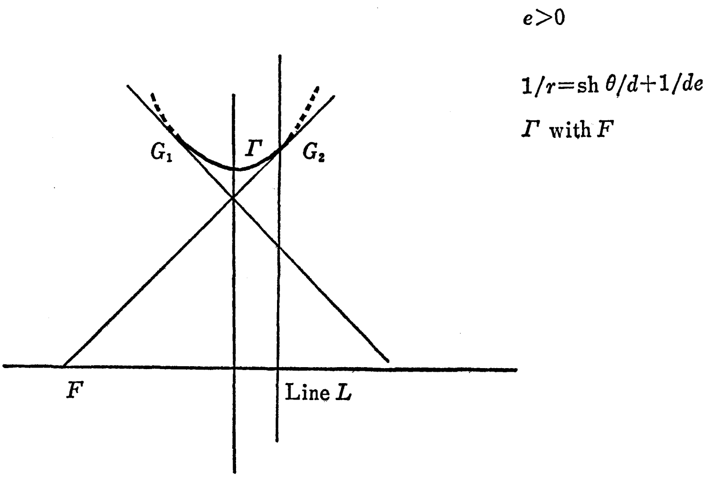

FIGURE 4

$$k = \sqrt{2\sqrt{2}\,are^{-\theta} + r^2} + r\,. \tag{8}$$

The geometric interpretation of (8) is the following. If $P = (r\,sh\,\theta,\ r\,ch\,\theta)$, then $r = \|\overrightarrow{OP}\|$ where $O = (0, 0)$. Consider the point $O' = (-\sqrt{2}\,a,\ -\sqrt{2}\,a)$. Then we find $\overrightarrow{O'P} = (r\,sh\,\theta + \sqrt{2}\,a,\ r\,ch\,\theta + \sqrt{2}\,a)$ and so $\|\overrightarrow{O'P}\| = \sqrt{2\sqrt{2}\,are^{-\theta} + r^2}$. Thus (8) means

$$\|\overrightarrow{OP}\| + \|\overrightarrow{O'P}\| = k\,. \tag{8'}$$

The second case of (IV) leads to

$$\|\overrightarrow{O'P}\| - \|\overrightarrow{OP}\| = k\,. \tag{9'}$$

If we use the null coordinate system $\{u, v\}$ for which $O = (a, 0)$ and $O' = (-a, 0)$, then both (8′) and (9′) are part of a hyperbola (the upper branch where $u, v > 0$)

$$8k^2uv - 16a^2v^2 = k^4 \tag{10}$$

with asymptotes $v = 0$ and $k^2u = 2a^2v$. The equation $1/r = Ae^{-\theta} + B$ is part of the hyperbola that is spacelike. See Figure 5, where the null line OG is tangent to (10). The equation $1/r = Ae^{-\theta} - B$ is the complement (excluding the point G) which is a timelike curve.

Finally, it can be shown that the equation $1/r = -Ae^{-\theta} + B$ represents the same curve as (9′) except that the point O' is used as the origin, that is, $\overrightarrow{O'P} = (r\,sh\,\theta,\ r\,ch\,\theta)$.

The points O and O' are foci of the hyperbola (10).

REMARK. The equation (10), or more generally, an equation of the

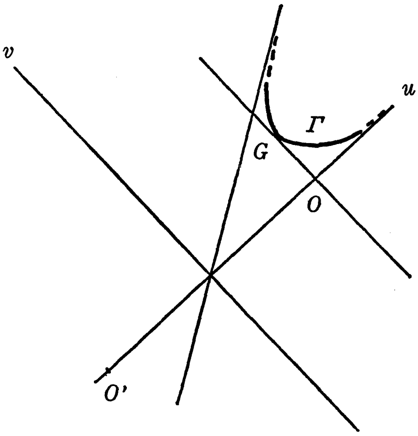

FIGURE 5

form

$$2Auv - Bv^2 = C\ , \quad A, B, C \neq 0 \tag{11}$$

cannot be reduced to a standard form of a hyperbola (6) or (7) relative to a Lorentz coordinate system. From the point of view of linear algebra, this reflects the fact that the Lorentz-symmetric endomorphism represented by $\begin{bmatrix} -A & B \\ 0 & -A \end{bmatrix}$ relative to the null coordinate system has $-A$ as double eigenvalue with null eigenvector.

We can now summarize Sections 1 and 2 in the following theorem.

THEOREM 1. (a) *Let Ω be a spacelike curve in the xz-plane such that its center of curvature never lies on the x-axis. Then the surface of revolution with Ω as profile curve for the rotation around the x-axis has constant mean curvature if and only if Ω is the locus of a focus of a quadric when its spacelike part Γ is rolled along the x-axis; the precise determination of Γ is done as in Figures* 1–5, *except when Ω is a line parallel to the x-axis (equation* I).

(b) *Let Ω be a spacelike curve in the xz-plane such that its center of curvature always lies on the x-axis. Then the surface of revolution with Ω as profile curve has constant mean curvature H if and only if it is congruent to the standard hyperboloid $x^2 + y^2 - z^2 = -1/H^2$, $z > 0$.*

3. Surfaces of revolution with timelike axis. A surface of revolution with the z-axis as axis is written in the form $(x(s)\cos t, x(s)\sin t, z(s))$, where s is an arc-length of the profile curve $(x(s), z(s))$ in the xz-plane. We may assume $x(s) > 0$.

This case is quite similar to that treated in Section 1. If the center of curvature of the profile curve Ω never lies on the z-axis, then there exists a timelike curve Γ with timelike position vectors $(r(\theta) sh\, \theta, r(\theta) ch\, \theta)$ and non-vanishing curvature such that Ω is the locus of the origin when Γ is rolled along the z-axis. Moreover, the function $r(\theta)$ satisfies the same differential equation (4). The determination of (timelike) Γ is also similar to Section 2.

If the center of curvature of Ω is always on the z-axis, then we obtain the standard hyperboloid in the same way as Section 1.

4. **Surfaces of revolution with null axis.** A surface of revolution with the axis $x = 0$, $y = z$, is given by

$$x = -t(f(s) - g(s))$$
$$y = f(s) - (f(s) - g(s))t^2/2$$
$$z = g(s) - (f(s) - g(s))t^2/2 ,$$

where s is an arc-length parameter of the profile curve Ω: $y = f(s)$, $z = g(s)$ in the yz-plane. The curve Ω is spacelike, that is, $\dot{f}^2 - \dot{g}^2 = 1$. The principal curvatures are given by $\ddot{f}/\dot{g}$ and $-(\dot{f} - \dot{g})/(f - g)$, where the former is the curvature of Ω. In fact, the unit tangent vector $(\dot{f}, \dot{g})$ of Ω is written as $\pm(ch\, \phi, sh\, \phi)$ and the unit normal vector is $\pm(sh\, \phi, ch\, \phi)$. Since $(\ddot{f}, \ddot{g})$ is perpendicular to $(\dot{f}, \dot{g})$, we get $(\ddot{f}, \ddot{g}) = k(\dot{g}, \dot{f})$ defining the curvature k. Obviously, $k = \ddot{f}/\dot{g} = \dot{\phi}$. The line $(f, g) + r(\dot{g}, \dot{f})$ intersects the axis of revolution $y = z$ at $r = (f - g)/(\dot{f} - \dot{g})$.

For our convenience, we use the null coordinates (u, v) such that $u = (y + z)/\sqrt{2}, v = (-y + z)/\sqrt{2}$. Then Ω is written as $u = (f + g)/\sqrt{2}$, $v = (-f + g)/\sqrt{2}$, with $v > 0$, and the unit tangent vector is of the form $(u, v) = (e^{\phi}/\sqrt{2}, -e^{-\phi}/\sqrt{2})$ by choosing an arc-length parameter s so that $\dot{v} < 0$. From $\dot{v} = -e^{-\phi}/\sqrt{2}$, it follows that $\dot{\phi} = -\ddot{v}/\dot{v}$. The principal curvatures of the surface are expressed as $-\ddot{v}/\dot{v}$ and $-\dot{v}/v$.

The surface has constant mean curvature H if and only if

$$v\ddot{v} = -2Hv\dot{v} - \dot{v}^2 , \quad v > 0 , \quad \dot{v} < 0 .$$

As usual we put $\dot{v} = p$ and $\ddot{v} = p\, dp/dv$. The equation above becomes an exact equation

$$vdp + (2vH + p)dv = 0$$

and hence

$$vp + Hv^2 = c$$

or equivalently

$$\dot{v} = (c - Hv^2)/v \; .$$

Since $(\dot{u}, \dot{v})$ is a unit vector, we have $-2\dot{u}\dot{v} = 1$ and

$$du/dv = -v^2/2(c - Hv^2)^2 \; .$$

Integrating this, we obtain the profile curve as the graph of the function $u = u(v)$ as follows:

(i) If $H \neq 0$ and $c/H = a^2 > 0$, then

$$u(v) = \frac{1}{4H^2}\left(\frac{v}{v^2 - a^2} - \frac{1}{2a}\log\frac{v - a}{v + a} + b\right),$$

where b is an arbitrary constant.

(ii) If $H \neq 0$ and $c/H = -a^2 < 0$, then

$$u(v) = \frac{1}{4H^2}\left(-\frac{1}{a}\arctan\frac{v}{a} + \frac{v}{v^2 + a^2} + b\right),$$

where b is an arbitrary constant.

(iii) If $H \neq 0$, and $c = 0$, then

$$u(v) = \frac{1}{2H^2}\left(\frac{1}{v} + b\right) \qquad (b\text{: arbitrary constant})\; .$$

(iv) If $H = 0$, then c cannot be 0 as $v > 0$ and $\dot{v} < 0$. In this case, we have

$$u(v) = -\frac{v^3}{6c^2} + b \; .$$

In summary, we have

THEOREM 2. *The profile curve of a surface of revolution with null axis $x = 0$, $y = z$ and of constant mean curvature H is given by*

$$y = (u(v) - v)/\sqrt{2}\;, \quad z = (u(v) + v)/\sqrt{2}\;,$$

where $u = u(v)$ is one of the four functions (i)-(iv) *above. The only complete surface of revolution of this type is generated by the profile curve in case* (i) *and $H \neq 0$.*

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