Surfaces Parametrised by the Normals

Jens Gravesen
Technical University of Denmark

September 27, 2006

1 Introduction

Usually a local parametrisation of a surface is a map $\mathbb{R}^2 \supseteq U \to \mathbb{R}^3$, but in this paper we will consider maps $\mathbf{r} : S^2 \supseteq U \to \mathbb{R}^3$. We will furthermore demand that the normal at $\mathbf{r}(\mathbf{n})$ is $\mathbf{n}$, in other words the surface is locally parametrised by the inverse Gauss map. The Gauss map is not always a local diffeomorphism, but if it is, then its differential, which up to a sign is the Weingarten map, is invertible too. So the determinant, i.e., the Gaussian curvature, has to be non vanishing.

In the solution to the classical Minkowski problem the surface is given by the inverse Gauss map, see [2, 3, 4, 10]. Also in the study of minimal surfaces the inverse Gauss map has been used and already Gauss himself considered surfaces given this way. So it is certainly not a new idea to use the inverse Gauss map. When I wrote the paper I believed the idea was new in geometric design, but I have just learned that the idea was suggested by Malcolm Sabin already in 1974, see [14].

One way of obtaining the inverse Gauss map is to specify the support function as a function of the normal, i.e., as a function on the unit sphere. The support function is the distance from the origin to the tangent plane and the surface is simply considered as the envelope of its family of tangent planes. The latter point of view has been used previously, eg., in [11, 12].

The advantages of this surface representation is that geometric properties are easily calculated when the inverse Gauss map is known. The motivation for this paper is similar work on planar curves, [6, 7], where parametrisation by tangent direction was essentially in the geometric modelling of a scroll compressor.

2 The Gauss Map and Support Function

Let Σ be a surface embedded in $\mathbb{R}^3$ and let S^2 denote the unit sphere in $\mathbb{R}^3$. The *Gauss map* of the surface Σ is the map $\Sigma \to S^2$ which to a point $\mathbf{x} \in \Sigma$ assigns the normal $\mathbf{N}(\mathbf{x}) \in S^2$. The differential of the Gauss map at a point $\mathbf{x} \in \Sigma$ is a linear map $\mathrm{d}_{\mathbf{x}}\mathbf{N} : T_{\mathbf{x}}\Sigma \to T_{\mathbf{N}(\mathbf{x})}S^2 \cong T_{\mathbf{x}}\Sigma$. The *Weingarten map* or *shape operator* at $\mathbf{x} \in \Sigma$ is defined as $W(\mathbf{x}) = -\mathrm{d}_{\mathbf{x}}\mathbf{N}$. The *second fundamental form* is the quadratic form $\mathrm{I\!I}(\mathbf{x})\mathbf{v} = \mathbf{v} \cdot W(\mathbf{x})\mathbf{v}$, where '$\cdot$' is the usual inner product in $\mathbb{R}^3$. The *principal curvatures* and *principal directions* are the eigenvalues and eigenvectors of $W(\mathbf{x})$. If the Gauss curvature $K = \det W(\mathbf{x}) = \det \mathrm{d}_{\mathbf{x}}\mathbf{N}$ is non vanishing then the inverse function theorem tells us that the Gauss map can be inverted locally. Thus, *any surface with non vanishing Gauss curvature is locally the image of the inverse Gauss map.*

The *support function* is the signed distance from the tangent plane to the origin. If we denote the inverse Gauss map by $\mathbf{r}$ then the support function can be written as a function of the normal as

$$h = \mathbf{n} \cdot \mathbf{r}(\mathbf{n}), \tag{1}$$

where $\mathbf{n} \in S^2$. If (u, v) are coordinates on S^2 then differentiation of (1) yields

$$h_u = \mathbf{n}_u \cdot \mathbf{r}(\mathbf{n}) + \mathbf{n} \cdot \mathrm{d}\mathbf{r}_{\mathbf{n}}\,\mathbf{n}_u = \mathbf{n}_u \cdot \mathbf{r}(\mathbf{n}) \quad \text{and} \quad h_v = \mathbf{n}_v \cdot \mathbf{r}(\mathbf{n}), \tag{2}$$

where a lower index denotes partial differentiation and where we have used that $\mathrm{d}\mathbf{r}_{\mathbf{n}}\,\mathbf{n}_u$ and $\mathrm{d}\mathbf{r}_{\mathbf{n}}\,\mathbf{n}_v$ are in the tangent space and hence are orthogonal to $\mathbf{n}$. As h_u and h_v are the directional derivatives of h in the directions $\mathbf{n}_u$ and $\mathbf{n}_v$ respectively equation (2) shows that the gradient $\nabla_{S^2} h$ of h is the projection of $\mathbf{r}$ onto the tangent plane. By (1) this projection is $\mathbf{r} - h\,\mathbf{n}$ and we obtain

$$\mathbf{r}(\mathbf{n}) = h(\mathbf{n})\mathbf{n} + \nabla_{S^2} h(\mathbf{n}). \tag{3}$$

Thus, if the inverse Gauss map exists then we have an explicit expression for it in terms of the support function. Not any function h is the support function of a surface with non vanishing curvature, it has to satisfy a regularity condition. Most and probably all of the following lemma can be found in the literature, but the proof is a straightforward calculation which we give here.

Lemma 1. *Let $h : S^2 \supseteq U \to \mathbb{R}$ be a twice times differentiable function, let $H_{S^2}(h)$ be the Hessian of h, i.e., the second covariant derivative of h. Then h is the support function of regular surface if and only if $\det(H_{S^2}(h) + h\,\mathrm{id}) \neq 0$ at all points of U. In this case the Weingarten map is $W = -(H_{S^2}(h) + h\,\mathrm{id})^{-1}$, the principal directions are the eigenvectors of $H_{S^2}(h)$ and if λ is an eigenvalue then the corresponding principal curvature is $\kappa = -(\lambda + h)^{-1}$.*

Proof. Consider a point in $\mathbf{n}_0 \in U$ which we after a rotation of the coordinate system can assume is the north pole $\mathbf{n}_0 = [0,0,1]^T$. If we use the parametrisation $\mathbf{n} = [x, y, \sqrt{1-(x^2+y^2)}]^T$ in a neighbourhood of the north pole, then the inverse of the metric tensor is

$$\left[g_{S^2}^{ij}\right] = \begin{bmatrix} 1-x^2 & -xy \\ -xy & 1-y^2 \end{bmatrix}.$$

If we write $h = a + b_1 x + b_2 y + \frac{1}{2}\left(c_0 x^2 + 2c_1 xy + c_2 y^2\right) + o\left(|(x,y)|^2\right)$, then the gradient of h is

$$\begin{aligned}\nabla_{S^2} h &= \left((1-x^2)\frac{\partial h}{\partial x} - xy\frac{\partial h}{\partial y}\right)\frac{\partial \mathbf{n}}{\partial x} + \left(-xy\frac{\partial h}{\partial x} + (1-y^2)\frac{\partial h}{\partial y}\right)\frac{\partial \mathbf{n}}{\partial y} \\ &= \begin{bmatrix} b_1 + c_0 x + c_1 y + o\left(|(x,y)|\right) \\ b_2 + c_1 x + c_2 y + o\left(|(x,y)|\right) \\ -(b_1 x + b_2 y + c_0 x^2 + 2c_1 xy + c_2 y^2) + o\left(|(x,y)|^2\right) \end{bmatrix},\end{aligned}$$

and (3) becomes

$$\mathbf{r} = \begin{bmatrix} b_1 + (c_0 + a)x + c_1 y + o\left(|(x,y)|\right) \\ b_2 + c_1 x + (c_2 + a)y + o\left(|(x,y)|\right) \\ a - \frac{1}{2}\left(c_0 x^2 + 2c_1 xy + c_2 y^2\right) + o\left(|(x,y)|^2\right) \end{bmatrix}.$$

At the north pole $\mathbf{n}_0$ the tangent plane is the xy-plane and the differential of $\mathbf{r}$ has the matrix expression

$$\mathrm{d}_{\mathbf{n}_0}\mathbf{r} = \begin{bmatrix} c_0 + a & c_1 \\ c_1 & c_2 + a \end{bmatrix} = \begin{bmatrix} c_0 & c_1 \\ c_1 & c_2 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = H_{S^2}(h) + h\,\mathrm{id}.$$

So $\mathbf{r}$ is regular at the north pole if and only if $\det(H_{S^2}(h) + h\,\mathrm{id}) \neq 0$ and as $\mathbf{r}$ in the affirmative case is the inverse Gauss map, the Weingarten map is $W = -(H_{S^2}(h) + h\,\mathrm{id})^{-1}$. Finally we only have to note that the principal curvatures and directions are the eigenvalues and eigenvectors of W. □

If h is a C^k function with $\det(H_{S^2}(h) + h\,\mathrm{id}) \neq 0$ then the corresponding surface is parametrised by $\mathbf{r}$ which obviously is of class C^{k-1}, but the projection from the tangent plane is of class C^k, so we have a C^k surface.

Theorem 2. *Let $h : S^2 \supseteq U \to \mathbb{R}$ be a C^k function, let $\mathbf{r} : U \to \mathbb{R}^3$ be defined by (3), and let $\pi_{\mathbf{n}} : \mathbb{R}^3 \to T_{\mathbf{n}}S^2$ be the orthogonal projection to the tangent plane at $\mathbf{n}$.*

If $k = 1$ and we for each $\mathbf{n}_0 \in U$ have that $\pi_{\mathbf{n}_0} \circ \mathbf{r}$ is local homeomorphism around $\mathbf{n}_0$ such that $\frac{|\mathbf{r}(\mathbf{n}) - \pi_{\mathbf{n}_0} \circ \mathbf{r}(\mathbf{n})|}{|\mathbf{n} - \mathbf{n}_0|} \to 0$ for $\mathbf{n} \to \mathbf{n}_0$, then $\mathbf{r}(U)$ is a C^1 surface.

If $k \geq 2$, and $\det(H_{S^2}(h) + h\,\mathrm{id}) \neq 0$ at all points of U, then $\mathbf{r}(U)$ is a C^k surface.

Proof. We immediately have that the tangent plane depends C^{k-1} on the points of $\mathbf{r}(U)$. The case $k = 1$ is now simply Theorem 10.1 in [5], and the proof generalises verbatim to the case $k \geq 2$. $\square$

The regularity condition in the C^1 case is not easy to verify, but the map $h : U \to \mathbb{R}$ will in practical application be given as a piecewise C^∞ function and then it is possible to give a more manageable condition.

Corollary 3. *Let $h : S^2 \supseteq U \to \mathbb{R}$ be a C^1 function which is piecewise C^2, and let $\mathbf{r} : U \to \mathbb{R}^3$ be defined by* (3)*. If the possible multi valued function* $\det(H_{S^2}(h) + h\,\mathrm{id})$ *is either strictly positive or strictly negative, then* $\mathbf{r}(U)$ *is a C^1 surface.*

Proof. Clearly $\mathbf{r}(U)$ is a collection of C^2 patches and if two patches meet along a curve γ then they either form a C^1 surface or meet at a cuspoidal edge. The two differentials agree in the direction of γ so we only need to check that the two differentials maps a direction orthogonal to γ to the same side of γ, i.e., the orientations induced by the two differentials has to agree and that in turn is determined by the sign of $\det(H_{S^2}(h) + h\,\mathrm{id})$. $\square$

It is important to be able to rotate, translate, scale, and offset surfaces. Straight forward calculations proves the following theorem.

Theorem 4. *Let $Q \in SO(3)$, $\mathbf{a} \in \mathbb{R}^3$, and $c, d \in \mathbb{R}$. Then we have*

$$\begin{aligned}
&\textit{rotation:} && \mathbf{r}^*(\mathbf{n}) = Q\mathbf{r}(Q^{-1}\mathbf{n}) \iff h^*(\mathbf{n}) = h(Q^{-1}\mathbf{n}), && (4)\\
&\textit{translation:} && \mathbf{r}^*(\mathbf{n}) = \mathbf{r}(\mathbf{n}) + \mathbf{a} \iff h^*(\mathbf{n}) = h(\mathbf{n}) + \mathbf{a}\cdot\mathbf{n}, && (5)\\
&\textit{scaling:} && \mathbf{r}^*(\mathbf{n}) = c\mathbf{r}(\mathbf{n}) \iff h^*(\mathbf{n}) = ch(\mathbf{n}), && (6)\\
&\textit{offsetting:} && \mathbf{r}^*(\mathbf{n}) = \mathbf{r}(\mathbf{n}) + d\mathbf{n} \iff h^*(\mathbf{n}) = h(\mathbf{n}) + d. && (7)
\end{aligned}$$

2.1 Extending to $\mathbb{R}^3$

In order to define the support function independent of a parametrisation of S^2 we let it be the restriction of a function defined on an open subset of $\mathbb{R}^3$. In order to use (3) we need the gradient of $h|_{S^2}$, but this is simply the projection of the ordinary gradient in $\mathbb{R}^3$ on the tangent plane of S^2. So we consider the restriction to S^2 of the map

$$\mathbf{r}(\mathbf{x}) = h(\mathbf{x})\mathbf{x} + |\mathbf{x}|^2\nabla h(\mathbf{x}) - (\mathbf{x}\cdot\nabla h(\mathbf{x}))\mathbf{x}. \tag{8}$$

Observe that (8) is linear in h. The factor $|\mathbf{x}|^2$ is of no importance on S^2, but when it is included then homogeneity of h implies homogeneity of $\mathbf{r}$ and $d\mathbf{r}$ preserves the tangent planes of spheres with centre $\mathbf{0}$.

Theorem 5. *If a map* $\mathbf{r} : \mathbb{R}^3 \supseteq U \to \mathbb{R}^3$ *is given by* (8) *then the differential* $\mathrm{d}\mathbf{r}$ *is given by*

$$\mathrm{d}_{\mathbf{x}}\mathbf{r}\,\mathbf{v} = (h - \mathbf{x}\cdot\nabla h)\mathbf{v} + 2(\mathbf{x}\cdot\mathbf{v})\,\nabla h + |\mathbf{x}|^2 H(h)\mathbf{v} - (\mathbf{x}\cdot H(h)\mathbf{v})\mathbf{x}, \qquad (9)$$

where h, ∇h, *and the* Hessian $H(h)$ *are evaluated at* $\mathbf{x}$. *Furthermore,* $\mathrm{d}\mathbf{r}$ *preserves the tangent planes of all spheres centred at* $\mathbf{0}$ *or equivalently*

$$\mathbf{x}\cdot\mathbf{v} = 0 \implies \mathbf{x}\cdot(\mathrm{d}_{\mathbf{x}}\mathbf{r}\,\mathbf{v}) = 0. \qquad (10)$$

Similar

$$\mathbf{x}\cdot(\mathrm{d}_{\mathbf{x}}\mathbf{r}\,\mathbf{x}) = |\mathbf{x}|^2(h + \mathbf{x}\cdot\nabla h). \qquad (11)$$

Finally, two of the eigenvalues, λ_1, λ_2 *say, of* $\mathrm{d}_{\mathbf{x}}\mathbf{r}$ *has corresponding eigenvectors* $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{x}^\perp$ *and if* λ_3 *is the third eigenvalue then*

$$\lambda_1 + \lambda_2 = 2(h - \mathbf{x}\cdot\nabla h) + |\mathbf{x}|^2 \operatorname{tr} H(h) - \mathbf{x}\cdot H(h)\mathbf{x}, \qquad (12)$$

$$\lambda_1\lambda_2 = \big(h - \mathbf{x}\cdot\nabla h\big)^2 + \big(h - \mathbf{x}\cdot\nabla h\big)\big(|\mathbf{x}|^2 \operatorname{tr} H(h) - \mathbf{x}\cdot H(h)\mathbf{x}\big) + |\mathbf{x}|^2\,\mathbf{x}\cdot H(h)^C\mathbf{x}, \qquad (13)$$

$$\lambda_3 = h + \mathbf{x}\cdot\nabla h, \qquad (14)$$

where $H(h)^C$ *is the co-factor matrix of* $H(h)$. *If* $\mathbf{x} \in S^2$ *then the* $\mathbf{v}_1, \mathbf{v}_2$ *are the principal directions and the corresponding principal curvatures are* $\kappa_1 = -1/\lambda_1$ *and* $\kappa_2 = -1/\lambda_2$.

Proof. By a straightforward calculation,

$$\mathrm{d}_{\mathbf{x}}\mathbf{r}\,\mathbf{v} = (\mathbf{v}\cdot\nabla h)\mathbf{x} + h\mathbf{v} + 2(\mathbf{x}\cdot\mathbf{v})\,\nabla h + |\mathbf{x}|^2 H(h)\mathbf{v} - (\mathbf{v}\cdot\nabla h)\mathbf{x} - (\mathbf{x}\cdot H(h)\mathbf{v})\mathbf{x} - (\mathbf{x}\cdot\nabla h)\mathbf{v},$$

which simplifies to (9). Taking the inner product with $\mathbf{x}$ yields $\mathbf{x}\cdot\mathrm{d}_{\mathbf{x}}\mathbf{r}\,\mathbf{v} = (h + \mathbf{x}\cdot\nabla h)(\mathbf{x}\cdot\mathbf{v})$. Letting $\mathbf{x}\cdot\mathbf{v} = 0$ proves (10) and letting $\mathbf{v} = \mathbf{x}$ proves (11). As $\mathrm{d}_{\mathbf{x}}\mathbf{r}$ preserves $\mathbf{x}^\perp$, an eigenvector for $\mathrm{d}_{\mathbf{x}}\mathbf{r}|_{\mathbf{x}^\perp}$ is also an eigenvector for $\mathrm{d}_{\mathbf{x}}\mathbf{r}$ with the same eigenvalue. So if $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues for $\mathrm{d}_{\mathbf{x}}\mathbf{r}$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are the corresponding eigenvectors then we may assume that $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{x}^\perp$. We can then write $\mathbf{x} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3$ where $\gamma\mathbf{v}_3\cdot\mathbf{x} = |\mathbf{x}|^2$. Hence

$$\mathbf{x}\cdot\mathrm{d}_{\mathbf{x}}\mathbf{r}\,\mathbf{x} = \mathbf{x}\cdot(\alpha\lambda_1\mathbf{v}_1 + \beta\lambda_2\mathbf{v}_2 + \gamma\lambda_3\mathbf{v}_3) = \gamma\lambda_3\mathbf{x}\cdot\mathbf{v}_3 = \lambda_3|\mathbf{x}|^2,$$

and now (11) proves (14). In matrix notation we have

$$\mathrm{d}_{\mathbf{x}}\mathbf{r} = (h - \mathbf{x}\cdot\nabla h)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + 2\begin{bmatrix}\frac{\partial h}{\partial x}\\ \frac{\partial h}{\partial y}\\ \frac{\partial h}{\partial z}\end{bmatrix}\begin{bmatrix}x & y & z\end{bmatrix}$$
$$+ |\mathbf{x}|^2 \begin{bmatrix}\frac{\partial^2 h}{\partial x^2} & \frac{\partial^2 h}{\partial x\partial y} & \frac{\partial^2 h}{\partial x\partial z}\\ \frac{\partial^2 h}{\partial y\partial x} & \frac{\partial^2 h}{\partial y^2} & \frac{\partial^2 h}{\partial y\partial z}\\ \frac{\partial^2 h}{\partial z\partial x} & \frac{\partial^2 h}{\partial z\partial y} & \frac{\partial^2 h}{\partial z^2}\end{bmatrix} - \begin{bmatrix}x\\ y\\ z\end{bmatrix}\begin{bmatrix}x & y & z\end{bmatrix}\begin{bmatrix}\frac{\partial^2 h}{\partial x^2} & \frac{\partial^2 h}{\partial x\partial y} & \frac{\partial^2 h}{\partial x\partial z}\\ \frac{\partial^2 h}{\partial y\partial x} & \frac{\partial^2 h}{\partial y^2} & \frac{\partial^2 h}{\partial y\partial z}\\ \frac{\partial^2 h}{\partial z\partial x} & \frac{\partial^2 h}{\partial z\partial y} & \frac{\partial^2 h}{\partial z^2}\end{bmatrix}$$

and a short calculation shows that the trace of $\mathrm{d}_{\mathbf{x}}\mathbf{r}$ is

$$\operatorname{tr}\mathrm{d}_{\mathbf{x}}\mathbf{r} = 3h - \mathbf{x}\cdot\nabla h + |\mathbf{x}|^2 \operatorname{tr} H(h) - \mathbf{x}\cdot H(h)\mathbf{x}. \tag{15}$$

A longer calculation (using Maple) shows that the determinant of $\mathrm{d}_{\mathbf{x}}\mathbf{r}$ is

$$\det \mathrm{d}_{\mathbf{x}}\mathbf{r} = (h + \mathbf{x}\cdot\nabla h)\Big[\big(h - \mathbf{x}\cdot\nabla h\big)^2 + |\mathbf{x}|^2\,\mathbf{x}\cdot H(h)^C\mathbf{x} \\ + \big(h - \mathbf{x}\cdot\nabla h\big)\big(|\mathbf{x}|^2 \operatorname{tr} H(h) - \mathbf{x}\cdot H(h)\mathbf{x}\big)\Big]. \tag{16}$$

As $\lambda_3 = h + \mathbf{x}\cdot\nabla h$ we immediately get (12) and (13). □

If h is homogeneous of degree n, i.e., $h(t\mathbf{x}) = t^n h(\mathbf{x})$ then

$$\mathbf{x}\cdot\nabla h(\mathbf{x}) = \frac{\mathrm{d}}{\mathrm{d}t}h(\mathbf{x}+t\mathbf{x})\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(1+t)^n h(\mathbf{x})\Big|_{t=0} = nh(\mathbf{x}), \tag{17}$$

and ∇h is homogeneous of degree $n-1$ so

$$\mathbf{x}\cdot H(h)(\mathbf{x})\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}s}h(\mathbf{x}+t\mathbf{x}+s\mathbf{v})\Big|_{s=0}\right)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}\cdot\nabla h(\mathbf{x}+t\mathbf{x})\Big|_{t=0} \\ = \frac{\mathrm{d}}{\mathrm{d}t}(1+t)^{n-1}\mathbf{v}\cdot\nabla h(\mathbf{x})\Big|_{t=0} = (n-1)\mathbf{v}\cdot\nabla h(\mathbf{x}). \tag{18}$$

These two equations immediately yield

Corollary 6. *If h is homogeneous of degree n then*

$$\mathbf{r} = (1-n)h\mathbf{x} + |\mathbf{x}|^2\nabla h, \tag{19}$$
$$\mathrm{d}_{\mathbf{x}}\mathbf{r}\,\mathbf{v} = (1-n)h\mathbf{v} + (1-n)(\mathbf{v}\cdot\nabla h)\mathbf{x} + 2(\mathbf{x}\cdot\mathbf{v})\nabla h + |\mathbf{x}|^2 H(h)\mathbf{v}, \tag{20}$$
$$= (1-n)h\mathbf{v} + 2(\mathbf{x}\cdot\mathbf{v})\nabla h + |\mathbf{x}|^2 H(h)\mathbf{v} - (\mathbf{x}\cdot H(h)\mathbf{v})\mathbf{x}, \tag{21}$$
$$\lambda_1 + \lambda_2 = (n+2)(1-n)h + |\mathbf{x}|^2 \operatorname{tr} H(h), \tag{22}$$
$$\lambda_1\lambda_2 = (n+1)(1-n)^2h^2 + (1-n)|\mathbf{x}|^2 h \operatorname{tr} H(h) + |\mathbf{x}|^2\,\mathbf{x}\cdot H(h)^C\mathbf{x}, \tag{23}$$
$$\lambda_3 = (1+n)h. \tag{24}$$

Clearly if h is a polynomial of degree n then the inverse Gauss map has degree $n+1$ and by choosing a rational parametrisation of the sphere we obtain a rational parametrisation of the surface of degree $2n+2$ and the support function has degree $2n$ in this parametrisation. The offsets are rational too, of the same degree, so we have a rational surface with rational offsets, see [8, 9, 11, 12, 13, 15]. We certainly do not obtain all rational surfaces with rational offsets, e.g., in [12] Pottmann exhibit a 8-parameter family of rational surfaces of degree 4 and in our setup they should correspond to a support function of degree $4/2 - 1 = 1$, i.e., the restriction of linear functions to S^2, but such a linear function $h = \mathbf{a} \cdot \mathbf{x}$ yields the constant $\mathbf{r} = \mathbf{a}$, i.e., not even a surface.

3 Interpolation

Suppose we are given a set of points and normals $(\mathbf{x}_i, \mathbf{n}_i)$ in $\mathbb{R}^3$ and that we want a surface with non vanishing Gaussian curvature that fits these data. The value of the support function at the point $\mathbf{x}_i$ is $h_i = \mathbf{x}_i \cdot \mathbf{n}_i$ and the gradient is $\nabla_{S^2} h(\mathbf{n}_i) = \mathbf{x}_i - h(\mathbf{n}_i)\mathbf{n}_i$. The surface with inverse Gauss map $\mathbf{r}(\mathbf{n}) = (1-n)h(\mathbf{n})\mathbf{n} + \nabla_{S^2} h$ fits the data if $h : \mathbb{R}^3 \supseteq U \to \mathbb{R}$ is a homogeneous map of degree n such that

$$h(\mathbf{n}_i) = \mathbf{x}_i \cdot \mathbf{n}_i, \tag{25}$$

and $\nabla_{S^2} h(\mathbf{n}_i) = \mathbf{x}_i - h(\mathbf{n}_i)\mathbf{n}_i$ which is equivalent to

$$\nabla h(\mathbf{n}_i) = \mathbf{x}_i + (n-1)h(\mathbf{n}_i)\mathbf{n}_i, \tag{26}$$

where we have used (17) and that $\nabla_{S^2} h(\mathbf{n}_i) = \nabla h(\mathbf{n}_i) - (\mathbf{n}_i \cdot \nabla h(\mathbf{n}_i))\mathbf{n}_i$. In other words we have transformed the G^k surface interpolation problem to a C^k interpolation problem on the unit sphere.

Now suppose, in addition to points and normal we are given principal directions $\mathbf{e}_{1,i}, \mathbf{e}_{2,i}$ and corresponding principal radii of curvature $\rho_{1,i}, \rho_{2,i}$. The requirement that the surface fits this new second order data is a condition on the differential of the map $\mathbf{r}(\mathbf{x}) = (1-n)h(\mathbf{x})\mathbf{x} + |\mathbf{x}|^2 \nabla h$. If we work in the basis $\mathbf{e}_{1,i}, \mathbf{e}_{2,i}, \mathbf{n}_i$ a short calculation shows that the conditions can be written

$$\mathbf{e}_{1,i} \cdot H(h)\mathbf{e}_{1,i} = \rho_{1,i} + (n-1)\mathbf{x}_i \cdot \mathbf{n}_i, \tag{27}$$

$$\mathbf{e}_{2,i} \cdot H(h)\mathbf{e}_{2,i} = \rho_{2,i} + (n-1)\mathbf{x}_i \cdot \mathbf{n}_i, \tag{28}$$

$$\mathbf{e}_{1,i} \cdot H(h)\mathbf{e}_{2,i} = 0, \tag{29}$$

where $H(h)$ is the Hessian of h as a function on $\mathbb{R}^3$.

One way of solving a C^1 interpolation problem on the sphere is given in [1] where the data is triangulated and then a macro element, eg. a Powell-Sabin element is used on each triangle. We consider two examples of G^1 interpolation on a single spherical triangle. The first with positive Gaussian curvature and the second with negative Gaussian curvature. In both cases the given normals are the three standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of $\mathbb{R}^3$. In the first case we consider the points $(2,0,0)$, $(0,3,0)$, and $(-1,0,4)$ and in the second case we consider the points $(-1,0,1)$, $(0,1,-1)$, and $(0,0,0)$. We first use a single Powell-Sabin element on the spherical triangle spanned by the three basis vectors. The results are shown to the left and in the middle of Figure 1. In the first example $1/K \geq 2$ and the surface is smooth. In the

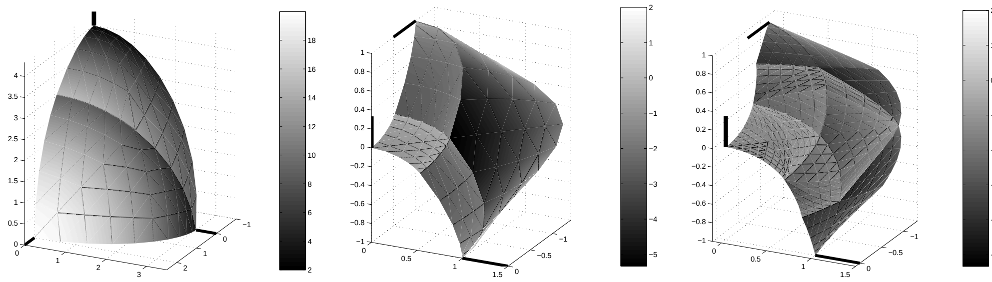

Figure 1: To the left a smooth positively curved surface with the support function given as a single Powell-Sabin element. In the middle a singular (mostly negatively curved) surface with the support function given as a single Powell-Sabin element. To the right a smooth negatively curved surface with the support function given as four Powell-Sabin elements, the apparent corners are all contained in the tangent plane. The data are indicated by the thick black lines and the grey scale corresponds to $1/K$.

second example the value of $1/K$ goes between -5.33 to 2.00 and it vanishes near the right hand edge. At these points the surface has a cuspoidal edge. This is a general problem with the approach in this paper. First of all the data needs to admit a surface with non vanishing curvature and if this is the case then we have to make sure that $1/K = \rho_1\rho_2 \neq 0$. In the present case we obviously want $1/K$ to be strictly negative.

We now subdivide the spherical triangle using a four split, i.e., we introduce the three points $\frac{\mathbf{e}_1+\mathbf{e}_2}{|\mathbf{e}_1+\mathbf{e}_2|}$, $\frac{\mathbf{e}_2+\mathbf{e}_3}{|\mathbf{e}_2+\mathbf{e}_3|}$, and $\frac{\mathbf{e}_3+\mathbf{e}_1}{|\mathbf{e}_3+\mathbf{e}_1|}$ on the sphere, and nine free parameters in form of corresponding points on the surface. By choosing a suitable objective function we can formulate an optimisation problem with $1/K < 0$ as a constraint. For simplicity we consider the following quadratic optimisation problem with quadratic constraints,

$$\text{minimise} \int_{\triangle} (\rho_1 + \rho_2)^2 \, \mathrm{d}A_{S^2},$$
$$\text{such that } \rho_1 \rho_2 < 0 \text{ for all } \mathbf{n} \in \triangle.$$

The choice $(\rho_1 + \rho_2)^2$ is arbitrary, another choice could be $|\nabla_{S^2} h|^2$ which also leads to a quadratic problem. Using the optimisation toolbox of Matlab we obtain the result to the right in Figure 1, where the value of $1/K$ is in the interval $[-4.32, -0.57]$. It may look as though the surface has a cusp or corner between the two given normals to the left and a similar problem at the far right. The two 'suspicious' points corresponds to corners in the middle of the four domain triangles. Recall that the parametrisation by the normals need not be C^1 so the boundary curve may indeed have corners at vertices of the domain triangles. But such a corner is contained in the tangent plane and does not contradict the smoothness of the surface.

4 Conclusion

Any surface with non vanishing Gaussian curvature can locally be given as the image of the inverse Gauss map. If the support function is given as a C^k function of the normal then the inverse Gauss map is a C^{k-1} map and is given as a linear functional of the support function, but the surface is of class C^k. Furthermore, the sum and the product of the principal radii of curvature is a linear and quadratic functional of the support function, respectively. Rotation, translation, scaling, and offsetting of a surface is easily expressed in terms of the support function. Finally, the problem of fitting a G^k surface with non vanishing Gaussian curvature is transferred to a constrained C^k interpolation problem on the sphere.

References

[1] Alfeld, P., Neamtu, M., Schumaker, L. L., Fitting scattered data on sphere-like surfaces using spherical splines, *J. Comput. Appl. Math.*, 73, 5–43 (1996).

[2] Bonnesen, T. and Fenchel, W., *Theorie der konvexen Körper*, Ergebnisse, vol. 3, Springer, Berlin, 1934.

[3] Bonnesen, T. and Fenchel, W., *Theory of convex bodies.* Transl. from the German and ed. by L. Boron, C. Christenson and B. Smith, with the collab. of W. Fenchel, BCS Associates, Moscow, Idaho, 1987.

[4] Fenchel, W. and Jessen, B., Mengenfunktionen und konvexe Körper, *Danske Vid. Selsk. Math.-Fys. Medd.* **16**, no. 3 1–31, (1938).

[5] Gluck, H., Geometric characterization of differentiable manifolds in Euclidean space II, *Michigan Math. J.* **15**, 33–50 (1968).

[6] Gravesen, J. and Henriksen, C., The geometry of the scroll compressor, *SIAM Review* **43**, 113–126 (2001).

[7] Gravesen, J., The Intrinsic Equation of Planar Curves and G^2 Hermite Interpolation, in *Seattle Geometric Design Proceedings*, M. Lucian and M. Neamtu (eds.), Nashboro Press, Brentwood, 2004, pp. 295–310.

[8] Jüttler, B., Triangular Bézier surface patches with a linear normal vector field, in: *The Mathematics of Surfaces VIII*, Cripps, R. (ed), Information Geometers, 431–446. (1998).

[9] Jüttler, B. and Sampoli, M.L., Hermite interpolation by piecewise polynomial surfaces with rational offsets, *Comp. Aided Geom. Design* **17**, 361–385, (2000).

[10] Lewy, H. On differential geometry in the large. I. Minkowski's problem. *Trans. Amer. Math. Soc.* **43**, 258–270 (1938).

[11] Peternell, M., and Pottmann, H., A Laguerre geometric approach to rational offsets, *Computer Aided Geometric Design* **15**, 223–249 (1998).

[12] Pottmann, H., Rational curves and surfaces with rational offsets, *Computer Aided Geometric Design* **12**, 175–192 (1995).

[13] Pottmann, H. and Wallner, J., *Computational Line Geometry*, Springer Verlag, Heidelberg, Berlin, 2001.

[14] Sabin, M., A Class of Surfaces Closed under Five Important Geometric Operations. VTO/MS report 207, 1974. `http://www.damtp.cam.ac.uk/user/na/people/Malcolm/vtoms/vtoms207.ps.gz`

[15] Sampoli, M.L., Peternell, M., and Jüttler, B., Exact Parameterization of Convolution Surfaces and Rational Surfaces with Linear Normals, *Comput. Aided Geom. Design* **22** (2005), to appear.