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SURJECTIVE STABILITY IN DIMENSION 0 FOR K_2 AND RELATED FUNCTORS

BY

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ABSTRACT. This paper continues the investigation of generators and relations for Chevalley groups over commutative rings initiated in [14]. The main result is that if A is a semilocal ring generated by its units, the groups $L(\Phi, A)$ of [14] are generated by the values of certain cocycles on $A^* \times A^*$. From this follows a surjective stability theorem for the groups $L(\Phi, A)$, as well as the result that $L(\Phi, A)$ is the Schur multiplier of the elementary subgroup of the points in A of the universal Chevalley-Demazure group scheme with root system Φ, if Φ has large enough rank. These results are proved via a Bruhat-type decomposition for a suitably defined relative group associated to a radical ideal. These theorems generalize to semilocal rings results of Steinberg for Chevalley groups over fields, and they give an effective tool for computing Milnor's groups $K_2(A)$ when A is semilocal.

Let Φ_l be a reduced irreducible root system of rank l and A a commutative ring with 1. There is an exact sequence

$$(1) \qquad 1 \longrightarrow L(\Phi_l, A) \longrightarrow \mathrm{St}(\Phi_l, A) \longrightarrow E(\Phi_l, A) \longrightarrow 1$$

where $\mathrm{St}(\Phi_l, A)$ is the Steinberg group [14, (3.7)] and $E(\Phi_l, A)$ is the elementary subgroup of the points in A of the universal Chevalley-Demazure group scheme with root system Φ_l [14, (3.3)]. If Φ_m is a second such root system, containing Φ_l as a subsystem generated by a connected subgraph of the Dynkin diagram of Φ_m, there are induced homomorphisms $\theta(l, m)$: $L(\Phi_l, A) \longrightarrow L(\Phi_m, A)$, and Steinberg [17] has shown these are surjective for all $m \geq l \geq 1$ when A is a field. In this paper I will prove that this is true for any semilocal ring A with at most one residue field isomorphic to $\mathbf{F}_2$. I will also show, in this case, that the groups $L(\Phi_l, A)$ are generated by the values of certain cocycles on $A^* \times A^*$ and that (1) is a central extension (and not just stably central; cf. [14, (5.1)]), theorems again due to Steinberg [17] when A is a field. These results were announced in [13].

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In general one conjectures that $\theta(l, m)$ is surjective for all $m \geq l \geq d$, where d is a fixed positive integer related to the dimension of the maximal ideal space of A; the theorem proved here may thus be thought of as the dimension 0 case of a surjective stability theorem for $L(\Phi_l, \ )$. If Φ_l belongs to one of the infinite families A_l, B_l, C_l, D_l, one deduces, under the same hypotheses, the surjectivity of

$$\theta(l, \infty)\colon L(\Phi_l, A) \to L(\Phi_\infty, A) = \lim_{l \to \infty} L(\Phi_l, A).$$

This reveals one motivation of the present research, since $L(A_\infty, \ )$ is Milnor's algebraic K_2 functor [9].

The paper proceeds as follows. Let $\mathfrak{q} \subset A$ be an ideal, and write $(1 + \mathfrak{q})^*$ for the units congruent to 1 modulo $\mathfrak{q}$. In §1 I define pairings ("relative Steinberg symbols")

$$\{\ ,\ \}\colon A^* \times (1 + \mathfrak{q})^* \to L(\Phi_l, \mathfrak{q})$$

and recall some of their properties. In §2 I prove, when $\mathfrak{q} \subset \operatorname{rad} A$, a normal form for the relative group $\mathrm{St}(\Phi, \mathfrak{q})$ analogous to the Bruhat decomposition of the Chevalley groups over fields [17, 7.6]. This implies that the groups $L(\Phi_l, \mathfrak{q})$ are generated by the relative symbols of §1, and, therefore, that $L(\Phi_l, \mathfrak{q}) \to L(\Phi_m, \mathfrak{q})$ is surjective for all $m \geq l \geq 1$. Combining this with Steinberg's theorem for fields yields the above-mentioned results for semilocal rings. In addition the theorems of this section allow one to deduce a presentation for $E(\Phi, A)$ of such a semilocal ring.

In §3 I compute $L(\Phi_l, A)$ for various local rings, using the results of §§1 and 2. In §4 I apply these results to the problem of surjective stability for the maps

$$H_2(\mathrm{SL}_2(A), \mathbf{Z}) \to H_2(E(\Phi_l, A), \mathbf{Z}).$$

The reader primarily interested in K_2 should note the following. Milnor's groups $E_{n+1}(A)$, $\mathrm{St}_{n+1}(A)$ are the groups $E(A_n, A)$, $\mathrm{St}(A_n, A)$ of this paper ($n \geq 2$), and $K_2(A) = L(A_\infty, A)$. The symbols $\{\ ,\ \}_\alpha$ are always bilinear in this case. A positive root $\alpha \in A_n$ is to be identified with a pair (ij), $1 \leq i < j \leq n+1$; $-\alpha$ then corresponds to (ji).

Milnor's K_2 theory exists for noncommutative rings as well, and most of the results of §2 remain true in this case, provided certain elements in A^* lie in $[A^*, A^*]$. I have omitted a discussion of these points since the surjective stability theorem for K_2 of noncommutative semilocal rings has recently been obtained by Dennis [3], based on work of Silvester [12].

When $A = K$ is a field, Matsumoto [8] has shown that the maps $\theta(l, m)$ are *injective* as well. This injective stability theorem remains true for radical ideals in the semilocal rings considered here, and will be the subject of a subsequent paper [15].

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Notation and terminology. The definitions, notations and terminology regarding root systems, Chevalley groups, Steinberg groups and their subgroups and relations are to be found in [14, §3]. However in this paper we always assume that the Chevalley-Demazure group schemes in question are *universal* [14, (3.3)]. If $\Phi_l \subset \Phi_m$ are reduced irreducible root systems, we say they are *of the same type* if they satisfy

(a) Φ_l is generated by a connected subgraph of the Dynkin diagram of Φ_m.

(b) If Φ_m is symplectic, then Φ_l is also symplectic and at least one long root of Φ_m occurs in Φ_l.

The inclusions $D_l \subset B_l$ violate (a) and the inclusions $A_{l-1} \subset C_l$, $l > 2$, violate (b).

The reader is reminded that the relative groups used in this paper differ from those of [9] and [16] (cf. the warnings following [14, (3.13)]). However the results of this paper *do* apply to the relative groups of [16], as follows from [16, (1.1), (2.5) and (2.6)].

All *rings* are commutative with 1; all homomorphisms *preserve* 1. If A is a ring, rad A is its *Jacobson radical* and A^* is its multiplicative *group of units*. A *pair* (A, q) consists of a ring A together with an ideal $q \subset A$; if $q \subset \operatorname{rad} A$ we say (A, q) is a *radical pair*. We write $(1 + q)^* = (1 + q) \cap A^*$. If T is a subset of A, the *subring of A generated by T* is denoted $\mathbf{Z}[T]$.

Let G be a group. For $\sigma, \tau \in G$ we write ${}^\tau\sigma = \tau\sigma\tau^{-1}$, $[\tau, \sigma] = {}^\tau\sigma \cdot \sigma^{-1} = \tau\sigma\tau^{-1}\sigma^{-1}$.

If H, K are subgroups of G, $[H, K]$ is the subgroup generated by $\{[h, k], h \in H, k \in K\}$; in particular the *commutator subgroup* of G is $[G, G]$. We write $G^{ab} = G/[G, G]$. If G is finite, $|G|$ is its *order*.

Finally, $\mathbf{Z}$ denotes the *rational integers* and $\mathbf{F}_q$ a *finite field with q elements*.

1. Relative Steinberg symbols and the subgroup $L(\Phi, A) \cap \hat{K}(\Phi, q)$. Recall [14, (3.12)] that $\hat{H}(\Phi, q)$ is the smallest *normal* subgroup of $\hat{H}(\Phi, A)$ containing all $\hat{h}_\alpha(v)$, $\alpha \in \Phi$, $v \in (1 + q)^*$. $\hat{H}(\Phi, q)$ is a subgroup of $\mathrm{St}(\Phi, q)$ (cf. (2.7)(a)).

Definition. Let $\alpha \in \Phi$, $u, v \in A^*$, and set

$$\{u, v\}_\alpha = \hat{h}_\alpha(uv)\hat{h}_\alpha(u)^{-1}\hat{h}_\alpha(v)^{-1}. \tag{1}$$

The subgroup of $\hat{H}(\Phi, A)$ generated by all $\{u, w\}_\alpha$, $\{w, u\}_\alpha$, where $u \in A^*$, $w \in (1 + q)^*$ and α ranges over Φ is denoted $D(\Phi, q)$. $D(\Phi, q)$ is a subgroup of $\mathrm{St}(\Phi, q)$ (cf. (2.7)(a)).

It follows from relation (R8) that for all $\alpha, \beta \in \Phi$,

(2)
$$\{u^{\langle \beta, \alpha \rangle}, v\}_\beta = [\hat{h}_\alpha(u), \hat{h}_\beta(v)].$$

Thus if there is an $\alpha \in \Phi$ with $\langle \beta, \alpha \rangle = 1$, we have $\{u, v\}_\beta \in [\hat{H}(\Phi, A), \hat{H}(\Phi, \mathfrak{q})] \subset \hat{H}(\Phi, \mathfrak{q})$. This will be the case *except* when Φ is *symplectic* and β is *long*.

The following proposition summarizes various well-known identities satisfied by $\{\ ,\ \}_\alpha$. Proofs may be found in [8, 5.5–5.7], [10, 3.2, 3.9, Appendix] and [18, Lemma 39 and Theorem 12].

(1.1) **Proposition.** *Let* $\alpha \in \Phi$, $u, v, w \in A^*$. *Then* $\{u, v\}_\alpha^{-1} = \{v, u\}_{-\alpha}$. *Writing* $\{\ ,\ \} = \{\ ,\ \}_\alpha$ *the following identities hold in* $D(\Phi, A)$:

(S1) $\{u, 1\} = \{1, u\} = 1$.

(S2) $\{u, v\}\{uv, w\} = \{u, vw\}\{v, w\}$.

(S3) $\{u, v\} = \{u^{-1}, v^{-1}\}$.

(S4) $\{u, v\} = \{u, -uv\}$.

(S5) $\{u, v\} = \{u, (1-u)v\}$ *if* $1 - u \in A^*$.

(S6) $\{u, v^2w\} = \{u, v^2\}\{u, w\}$; $\{u^2, vw\} = \{u^2, v\}\{u^2, w\}$; $\{u^2, v\} = \{u, v^2\}$; $\{u, v\} = \{v^{-1}, u\}$; $\{u, -1\} = \{u, v\}\{u, -v^{-1}\}$.

(S7) *If* u, v *generate a cyclic subgroup of* A^*, *then* $\{u, v\} = \{v, u\}$.

(S8) *If* $\{u, v\} = \{v, u\}$, *then* $\{u, v^2\} = 1$.

Moreover, if Φ *is nonsymplectic or if* α *is short*,

(S°2) $\{u, vw\} = \{u, v\}\{u, w\}$.

(S°3) $\{u, v\} = \{v, u\}^{-1}$.

Remarks. 1. The above identities are not independent. For example, (S1)–(S4) imply (S6)–(S8), and if Φ is nonsymplectic or if α is short, (S1)(S5)(S°2)(S°3) imply the others. (Cf. [10, Appendix].)

2. Identity (S5), which is of great importance for computations when A is a field, is valueless when $u \in (1 + \mathfrak{q})^*$ (since in that case $1 - u \notin A^*$ if $\mathfrak{q} \neq A$). A new identity which can sometimes be used to replace (S5) in such computations when $\mathfrak{q} \subset \operatorname{rad} A$ will be proved in (2.8).

(1.2) **Definition.** A *relative Steinberg symbol* on the pair $(A, \mathfrak{q})$ with values in an abelian group C is a mapping

$$\{\ ,\ \}: A^* \times (1 + \mathfrak{q})^* \to C$$

satisfying (S1)–(S5) of (1.1) and (2.8). When $\mathfrak{q} = A$, we call $\{\ ,\ \}$ a *Steinberg symbol*. If (S°2) holds, we call $\{\ ,\ \}$ a *(relative) bilinear Steinberg symbol.* We sometimes abbreviate "Steinberg symbol" to "symbol."

In this paper the word symbol will always refer to one of the symbols $\{\ ,\ \}$ with values in $D(\Phi, \mathfrak{q})$ constructed above.

Let $\hat{K}(\Phi, \mathfrak{q})$ be the subgroup of $\mathrm{St}(\Phi, \mathfrak{q})$ generated by $D(\Phi, \mathfrak{q})$ and all $\hat{h}_\alpha(v)$, $\alpha \in \Phi$, $v \in (1 + \mathfrak{q})^*$.

(1.3) **Proposition.** (a) *$D(\Phi, q)$ is a central subgroup of* $\mathrm{St}(\Phi, A)$.

(b) $\hat{H}(\Phi, q) \subset \hat{K}(\Phi, q)$, *and*

$$[\hat{H}(\Phi, A), \hat{H}(\Phi, q)] \subset L(\Phi, A) \cap \hat{H}(\Phi, q) \subset L(\Phi, A) \cap \hat{K}(\Phi, q) \subset D(\Phi, q),$$

with equality if Φ is nonsymplectic or if every element of $(1+q)^*$ is a square.

(c) *$D(\Phi, q)$ is generated by all $\{u, v\}_\alpha$, $u \in A^*$, $v \in (1+q)^*$ for any fixed long root α. Hence if $\Phi_l \subset \Phi_m$ are reduced irreducible root systems of the same type, the homomorphism $D(\Phi_l, q) \longrightarrow D(\Phi_m, q)$ is surjective for all $m \geq l \geq 1$, including $m = \infty$ if Φ is classical.*

Since $H(A)$ is an abelian subgroup of $E(\Phi, A)$ [18, Lemma 28(b)], $D(\Phi, q)$ is a subgroup of $\hat{H}(\Phi, A) \cap L(\Phi, A)$, and the latter group is central in $\mathrm{St}(\Phi, A)$ [18, p. 39, Corollary 1]. This also proves $[\hat{H}(\Phi, A), \hat{H}(\Phi, q)] \subset L(\Phi, A) \cap \hat{H}(\Phi, q)$, since $\hat{H}(\Phi, q)$ is normal in $\hat{H}(\Phi, A)$.

If $u \in A^*$, $v \in (1+q)^*$, then

$$\hat{h}_\alpha(u)\hat{h}_\beta(v)\hat{h}_\alpha(u)^{-1} = \hat{h}_\beta(u^{\langle\beta,\alpha\rangle}v)\hat{h}_\beta(u^{\langle\beta,\alpha\rangle})^{-1} = \{u^{\langle\beta,\alpha\rangle}, v\}_\beta \hat{h}_\beta(v) \in \hat{K}(\Phi, q).$$

Since $D(\Phi, q)$ is central in $\mathrm{St}(\Phi, q)$ by (a), this shows that $\hat{K}(\Phi, q)$ is a normal subgroup of $\hat{H}(A)$ containing all $\hat{h}_\alpha(v)$; hence $\hat{H}(\Phi, q) \subset \hat{K}(\Phi, q)$. Thus $L(\Phi, A) \cap \hat{H}(\Phi, q) \subset L(\Phi, A) \cap \hat{K}(\Phi, q)$.

Given $\hat{h} \in \hat{K}(\Phi, q)$, it follows from [17, 7.7] that we may write $\hat{h} = d\hat{h}_1(u_1) \cdots \hat{h}_l(u_l)$ where $d \in D(q)$, $\hat{h}_i(u_i) = \hat{h}_{\alpha_i}(u_i)$, $\alpha_i \in \Delta$, and $u_i \in (1+q)^*$. Then if

$$1 = \pi(\hat{h}) = h_1(u_1) \cdots h_l(u_l)$$

we must have $u_i = 1$ for all i, since $E(\Phi, A)$ is a subgroup of a *universal* Chevalley group [18, Corollary to Lemma 28]. Hence $\hat{h}_i(u_i) = 1$ for all i; that is, $\hat{h} = d \in D(q)$ proving the last inclusion of (b).

Now if Φ is nonsymplectic, it follows from (2) that $D(\Phi, q) \subset [\hat{H}(\Phi, A), \hat{H}(\Phi, q)]$, and the inclusions in (b) are equalities. If Φ is symplectic, we may assume $\langle\beta, \alpha\rangle = 2$ and (2) becomes

$$\{u^2, v\}_\beta = [\hat{h}_\alpha(u), \hat{h}_\beta(v)]. \tag{3}$$

By (1.1), $\{u^2, v\}_\beta = \{u, v^2\}_\beta$; thus it follows from (3) that if every $v \in (1+q)^*$ is a square, again

$$D(\Phi, q) \subset [\hat{H}(\Phi, A), \hat{H}(\Phi, q)]$$

which completes the proof of (b).

For fixed β, let D_β be the subgroup of $D(\Phi, q)$ generated by all $\{u, v\}_\beta$, $u \in A^*$, $v \in (1+q)^*$. Let $\sigma = \sigma_\alpha$ be an element of the Weyl group of Φ. Then relation (R5) and (a) imply

$$
\begin{aligned}
\{u, v\}_\beta &= \hat{w}_\alpha(1) \cdot \{u, v\}_\beta \cdot \hat{w}_\alpha(-1) \\
&= \hat{w}_\alpha(1) \cdot \hat{h}_\beta(uv)\hat{h}_\beta(u)^{-1}\hat{h}_\beta(v)^{-1} \cdot \hat{w}_\alpha(-1) \\
&= \hat{h}_{\sigma\beta}(\eta uv)\hat{h}_{\sigma\beta}(\eta)^{-1}\hat{h}_{\sigma\beta}(\eta)\hat{h}_{\sigma\beta}(\eta u)^{-1}\hat{h}_{\sigma\beta}(\eta)\hat{h}_{\sigma\beta}(\eta v)^{-1} \\
&= \hat{h}_{\sigma\beta}(\eta uv)\hat{h}_{\sigma\beta}(\eta u)^{-1}\hat{h}_{\sigma\beta}(v)^{-1}\hat{h}_{\sigma\beta}(v)\hat{h}_{\sigma\beta}(\eta)\hat{h}_{\sigma\beta}(\eta v)^{-1} \\
&= \{\eta u, v\}_{\sigma\beta}\{\eta, v\}_{\sigma\beta}^{-1}
\end{aligned}
$$

for some $\eta = \pm 1$. This proves $D_\beta \subset D_{\sigma\beta}$, and, by symmetry, $D_\beta = D_{\sigma\beta}$. Since the Weyl group acts transitively on roots of the same length, we have shown that if α and β have the same length, $D_\alpha = D_\beta$.

Suppose then that β is short and choose a long root α such that $\langle\beta, \alpha\rangle = 1$. Then by (2)

$$
\{u, v\}_\beta = [\hat{h}_\alpha(u), \hat{h}_\beta(v)] = [\hat{h}_\beta(v), \hat{h}_\alpha(u)]^{-1} = \{v^{\langle\alpha, \beta\rangle}, u\}_\alpha^{-1} \tag{4}
$$

which proves $D_\beta \subset D_\alpha$. Since by (1.1)(S6) $\{v, u\}_\alpha = \{u^{-1}, v\}_\alpha$, we have shown $D_\alpha = D(\Phi, q)$, proving the first part of (c); the rest of (c) is now an easy corollary.

Remark. In view of (1.3) we will usually write $\{\ ,\ \}$ for $\{\ ,\ \}_\alpha$; in that case it is to be understood that the symbol in question is taken with respect to a fixed long root α.

2. The relative Bruhat decomposition for a radical ideal.

(2.1) **Lemma.** *Let* $\alpha \in \Delta$.

(a) $\hat{U}(\Phi, q) = \hat{U}(\Phi_+ - \{\alpha\}, q) \cdot \hat{U}(\alpha, q)$.

(a$^-$) $\hat{U}^-(\Phi, q) = \hat{U}(\Phi_- - \{-\alpha\}, q) \cdot \hat{U}(-\alpha, q)$.

(b) $\hat{U}(\Phi_+ - \{\alpha\}, q)$ *is normalized by* $\mathrm{St}_\alpha(A)$.

(b$^-$) $\hat{U}(\Phi_- - \{-\alpha\}, q)$ *is normalized by* $\mathrm{St}_\alpha(A)$.

The set of roots $\Phi_+ - \{\alpha\}$ (*resp.* $\Phi_- - \{-\alpha\}$) is an ideal in the closed sets of roots Φ_+ and $(\Phi_+ - \{\alpha\}) \cup \{-\alpha\}$ (*resp.* Φ_- and $(\Phi_- - \{-\alpha\}) \cup \{\alpha\}$). The lemma thus follows from [18, Lemmas 16, 17, 18, 36].

Definition. Set $\hat{M}(\Phi, q) = \hat{U}^-(\Phi, q)\hat{K}(\Phi, q)\hat{U}(\Phi, q)$, a subset of $\mathrm{St}(\Phi, q)$ (cf. (2.7)). Recall from (1.3) that if Φ is nonsymplectic or if $((1+q)^*)^2 = (1+q)^*$, then $\hat{K}(\Phi, q) = \hat{H}(\Phi, q)$, and that in any case, $\hat{K}(\Phi, q)$ is the product of the central subgroup $D(\Phi, q)$ with the group generated by all $\hat{h}_\alpha(v)$, $v \in (1+q)^*$. Thus $\pi(\hat{K}(\Phi, q)) = H(\Phi, q)$.

(2.2) **Lemma.** $\hat{U}^-(\Phi, q)\hat{K}(\Phi, q)\hat{M}(\Phi, q) = \hat{M}(\Phi, q) = \hat{M}(\Phi, q)\hat{K}(\Phi, q)\hat{U}(\Phi, q)$.

This follows from relation (R6) which shows that $\hat{H}(\Phi, q)$, and therefore also $\hat{K}(\Phi, q)$, normalizes $\hat{U}^-(\Phi, q)$ and $\hat{U}(\Phi, q)$.

(2.3) **Theorem.** (a) *The product map*

$$\hat{U}^-(\Phi, \mathfrak{q}) \times \hat{K}(\Phi, \mathfrak{q}) \times \hat{U}(\Phi, \mathfrak{q}) \longrightarrow \mathrm{St}(\Phi, \mathfrak{q})$$

is injective.

(b) $L(\Phi, A) \cap \hat{M}(\Phi, \mathfrak{q}) \subset \hat{K}(\Phi, \mathfrak{q})$.

(c) $\hat{M}(\Phi, \mathfrak{q}) = \mathrm{St}(\Phi, \mathfrak{q})$ *implies* $\mathfrak{q} \subset \mathrm{rad}\, A$.

Suppose $\hat{u}, \hat{u}' \in \hat{U}(\mathfrak{q})$, $\hat{v}, \hat{v}' \in \hat{U}^-(\mathfrak{q})$ and $\hat{k}, \hat{k}' \in \hat{K}(\mathfrak{q})$. Then if $\hat{v}\hat{k}\hat{u} = \hat{v}'\hat{k}'\hat{u}'$, we have

$$\pi(\hat{v}'\hat{v}^{-1}) = \pi(\hat{k}'\hat{u}'\hat{u}^{-1}\hat{k}^{-1}) \in U^-(A) \cap U(A)H(A) = \{1\}$$

by [18, Lemma 21]. Hence $\hat{v} = \hat{v}'$, since $\pi|\hat{U}^-(A)$ is an isomorphism [18, Lemma 36]. Similarly $\hat{u} = \hat{u}'$, and therefore $\hat{k} = \hat{k}'$, proving (a).

Now suppose $\pi(\hat{v}\hat{k}\hat{u}) = 1$. Then $\pi(\hat{v}) = \pi(\hat{u}^{-1}\hat{k}^{-1}) \in U^-(A) \cap U(A)H(A) = \{1\}$ implies $\hat{v} = 1$; hence $\hat{u} = 1$ also, proving (b).

Finally, it is easily checked in $\mathrm{SL}(2, A)$ that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U^{-1}HU$ implies $a \in A^*$. Moreover, $\phi_\alpha^{-1}(U^-HU) \subset U^-HU$, where the decomposition on the right is in $\mathrm{SL}(2, A)$ and $\phi_\alpha\colon \mathrm{SL}(2, A) \longrightarrow E_\alpha(A)$ is the homomorphism of [14, (3.6)].

Applying these remarks to

$$\begin{pmatrix} 1+q & -q \\ q & 1-q \end{pmatrix} \in \phi_\alpha^{-1}(\pi(x_\alpha(1)x_{-\alpha}(q)x_\alpha(-1)))$$

for any $q \in \mathfrak{q}$, we see that $\hat{M}(\mathfrak{q}) = \mathrm{St}(\mathfrak{q})$ implies $(1 + \mathfrak{q}) \subset A^*$ and therefore, $\mathfrak{q} \subset \mathrm{rad}\, A$. This proves (c).

The key result of this section is the following partial converse to (2.3)(c):

(2.4) **Theorem.** *Let* $(A, \mathfrak{q})$ *be a radical pair and assume* $A = \mathbf{Z}[A^*]$. *Then* $\mathrm{St}(\Phi, \mathfrak{q}) = \hat{M}(\Phi, \mathfrak{q})$.

(2.5) **Theorem.** *Let* $(A, \mathfrak{q})$ *be a radical pair with* $A = \mathbf{Z}[A^*]$, *and suppose* $\Phi_l \subset \Phi_m$ *are reduced irreducible root systems of the same type. Then* $L(\Phi_m, \mathfrak{q})$ *is generated by all* $\{u, v\}_\alpha$, $u \in A^*$, $v \in (1 + \mathfrak{q})^*$ *for any fixed long root* α, *and the homomorphisms* $L(\Phi_l, \mathfrak{q}) \longrightarrow L(\Phi_m, \mathfrak{q})$ *are surjective for all* $m \geq l \geq 1$, *including* $m = \infty$ *if* Φ_m *is classical.*

If, in addition, Φ_m *and* A *satisfy one of the hypotheses of* [14, *Theorem* 5.3], $\mathrm{St}(\Phi_m, (0, \mathfrak{q}))$ *is the universal* $E(\Phi_m, A)$*-covering* [14, §2] *of* $E(\Phi_m, \mathfrak{q})$.

This theorem is a corollary of (2.3)(b), (2.4) and (1.3).

Note. The hypothesis $A = \mathbf{Z}[A^*]$ is innocent. It is fulfilled, for example, by semilocal rings having at most one residue field with 2 elements [14, (4.2)] (in particular, by local rings) and by group rings.

The proof of (2.4) will be based on a series of lemmas.

(2.6) **Lemma.** *Let* $\alpha \in \pm\Delta$, $t \in A$. *Then* $x_\alpha(t)$ *normalizes* $M(q)$ *if and only if* $x_\alpha(t)\hat{U}(-\alpha, q)x_\alpha(-t) \subset \hat{M}(q)$.

The "only if" is clear. For the converse, we assume $\alpha \in \Delta$ (the case $\alpha \in -\Delta$ is similar). By (2.1)(a^-), we have

$$\hat{M}(q) = \hat{U}(\Phi_- - \{-\alpha\}, q) \cdot \hat{U}(-\alpha, q) \cdot \hat{K}(q) \cdot \hat{U}(q).$$

Since $x_\alpha(t)$ normalizes $\hat{U}(\Phi_- - \{-\alpha\}, q)$ by (2.1)(b^-) and also normalizes $\hat{U}(q)$, it suffices to prove

$$x_\alpha(t) \cdot \hat{U}(-\alpha, q)\hat{K}(q) \cdot x_\alpha(-t) \subset \hat{M}(q)$$

and, in view of the hypothesis and (2.2), that would follow from

$$x_\alpha(t) \cdot \hat{K}(q) \cdot x_\alpha(-t) \subset \hat{K}(q)U(q)$$

which is true since $\hat{K}(q) \subset \hat{H}(A)$ and $\hat{H}(A)$ normalizes $\hat{U}(q)$ by relation (R6).

(2.7) **Proposition.** *Let* $u, v \in A^*$, $\alpha \in \Phi$. *The following identities hold in* $\mathrm{St}(\Phi, A)$:

(a)
$$\begin{aligned}&\{u, v\}_\alpha \hat{h}_\alpha(v)\\ &= x_{-\alpha}(u^{-1}(1 - v^{-1})) \cdot {}^{x_{-\alpha}(-u^{-1})}x_\alpha(u(v-1)) \cdot x_\alpha(u(v^{-1}-1)),\end{aligned}$$

(b)
$$\begin{aligned}&{}^{x_{-\alpha}(-u^{-1})}x_\alpha(u(v-1))\\ &= x_{-\alpha}(u^{-1}(v^{-1}-1))\{u, v\}_\alpha \hat{h}_\alpha(v) x_\alpha(u(1-v^{-1})),\end{aligned}$$

(c)
$$\begin{aligned}&{}^{x_\alpha(-u)}x_{-\alpha}(u^{-1}(1-v))\\ &= x_{-\alpha}(u^{-1}(v^{-1}-1))\{u, v\}_\alpha \hat{h}_\alpha(v) x_\alpha(u(1-v^{-1})).\end{aligned}$$

Proof. (a)

$$\begin{aligned}\{u, v\}_\alpha \hat{h}_\alpha(v) &= \hat{h}_\alpha(uv)\hat{h}_\alpha(u)^{-1} = \hat{w}_\alpha(uv)\hat{w}_\alpha(-u)\\
&= \hat{w}_{-\alpha}(-u^{-1}v^{-1})\hat{w}_\alpha(-u)\\
&= x_{-\alpha}(-u^{-1}v^{-1})x_\alpha(uv)x_{-\alpha}(-u^{-1}v^{-1})\hat{w}_\alpha(-u)\\
&= x_{-\alpha}(-u^{-1}v^{-1}) \cdot x_\alpha(uv)\hat{w}_\alpha(-u) \cdot {}^{\hat{w}_\alpha(u)}x_{-\alpha}(-u^{-1}v^{-1})\\
&= x_{-\alpha}(-u^{-1}v^{-1}) \cdot x_\alpha(uv)x_\alpha(-u)x_{-\alpha}(u^{-1})x_\alpha(-u) \cdot x_\alpha(uv^{-1})\\
&= x_{-\alpha}(u^{-1}(1-v^{-1}))x_{-\alpha}(-u^{-1}) \cdot x_\alpha(u(v-1))x_{-\alpha}(u^{-1})x_\alpha(u(v^{-1}-1))\\
&= x_{-\alpha}(u^{-1}(1-v^{-1})) \cdot {}^{x_{-\alpha}(-u^{-1})}x_\alpha(u(v-1)) \cdot x_\alpha(u(v^{-1}-1)).\end{aligned}$$

(b) follows immediately from (a).

(c) In (b) exchange α with $-\alpha$ and u with u^{-1}; then take the inverse of each side. The identities $\hat{h}_{-\alpha}(v)^{-1} = \hat{h}_\alpha(v)$ and $\{u^{-1}, v\}_{-\alpha}^{-1} = \{v, u^{-1}\}_\alpha = \{u, v\}_\alpha$ complete the proof.

(2.8) **Corollary.** *Let $\alpha \in \Phi$, $q \in \operatorname{rad} A$. For all $u, v, u', v' \in A^*$ such that $u + v = u' + v'$, the symbol $\{\ ,\ \}_\alpha$ satisfies the identity*

$$\text{(Sq)}\quad \begin{aligned} &\{u, (1+qz)/(1+qv)\}_\alpha \{v, 1+qv\}_\alpha \{1+qv, -(1+qz)\}_\alpha^{-1} \\ &= \{u', (1+qz)/(1+qv')\}_\alpha \{v', 1+qv'\}_\alpha \{1+qv', -(1+qz)\}_\alpha^{-1} \end{aligned}$$

where $z = u + v = u' + v'$. Moreover if $z \in A^*$, both sides of (S9) *equal* $\{z, 1+qz\}_\alpha$.

Since $u + v = u' + v'$, we must have

$$(1)\qquad {}^{x_\alpha(-u)x_\alpha(-v)}x_{-\alpha}(q) = {}^{x_\alpha(-z)}x_{-\alpha}(q) = {}^{x_\alpha(-u')x_\alpha(-v')}x_{-\alpha}(q).$$

We will use (2.7) to put (1) into $\hat{M}(q)$; (S9) will then follow by comparing the terms in $\hat{K}(q)$ which are uniquely determined according to (2.3)(a).

Write $w = 1 - qv \in A^*$. Then $q = v^{-1}(1-w)$ and $w^{-1} - 1 = qvw^{-1}$; applying (2.7)(c) with $u = v$, $v = w$ yields

$$(2)\qquad {}^{x_\alpha(-v)}x_{-\alpha}(q) = x_{-\alpha}(qw^{-1})\{v, w\}_\alpha \hat{h}_\alpha(w) x_\alpha(-qv^2w^{-1}).$$

Similarly write $x = 1 - quw^{-1} = w^{-1}(1-qz) \in A^*$; then $qw^{-1} = u^{-1}(1-x)$, $x^{-1} - 1 = qu(1-qz)^{-1}$ and we have

$$(3)\qquad {}^{x_\alpha(u)}x_{-\alpha}(qw^{-1}) = x_{-\alpha}(q(1-qz)^{-1})\{u, x\}_\alpha \hat{h}_\alpha(x) x_\alpha(-qu^2(1-qz)^{-1}).$$

Combining (2) and (3), and simplifying using relation (R6) and the definition of $\{\ ,\ \}_\alpha$ gives the identity

$$(4)\qquad \begin{aligned} &{}^{x_\alpha(-u)x_\alpha(-v)}x_{-\alpha}(q) \\ &= x_{-\alpha}(q(1-qz)^{-1})\{u, x\}_\alpha \{v, w\}_\alpha \{w, x\}_\alpha^{-1} \hat{h}_\alpha(1-qz) x_\alpha(-qz^2(1-qz)^{-1}). \end{aligned}$$

(It should be noted that in deriving (4) we need only the weaker hypotheses $u, v, 1-qv, 1-qu, 1-qz \in A^*$; this will be important in (2.9) below.) We perform a similar calculation for ${}^{x_\alpha(-u')x_\alpha(-v')}x_{-\alpha}(q)$; the identity follows by comparing the terms in $\hat{K}(q)$ (noting that $\hat{h}_{-\alpha}(1-qz)$ depends only on z) and replacing q by $-q$.

Finally if $z \in A^*$, we may use (2.7)(c) to compute ${}^{x_\alpha(z)}x_{-\alpha}(q)$ directly; comparing $\hat{K}(q)$ terms, we see that $\{z, 1+qz\}_\alpha$ must equal both sides of (S9).

(2.9) **Corollary.** *Let $u, v \in A^*$, $\alpha \in \Phi$ and write $p = u - 1$, $q = v - 1$. Then if $pq = 0$, $\{1+q, 1+p\}_\alpha = [x_{-\alpha}(q), x_\alpha(p)]$.*

We will compute the right-hand side using (4) above. Make the substitutions $-u = u$, $-v = -1$, $q = -q$ in (4); then $z = -p$, $1 = qz = 1 - qp = 1$, $x^{-1} = w = 1 + q$, and

$$^{x_\alpha(p)}x_{-\alpha}(-q) = {}^{x_\alpha(u)x_\alpha(-1)}x_{-\alpha}(-q) = x_{-\alpha}(-q)\{-u, x\}_\alpha\{x^{-1}, x\}_\alpha^{-1}.$$

Therefore

$$[x_{-\alpha}(q), x_\alpha(p)] = \{-u, x\}_\alpha\{x^{-1}, x\}_\alpha^{-1}.$$

But (1.1) implies

$$\{-u, x\}_\alpha\{u^{-1}, x\}_\alpha = \{-1, x\}_\alpha = \{x^{-1}, x\}_\alpha$$

and therefore

$$[x_{-\alpha}(q), x_\alpha(p)] = \{u^{-1}, x\}_\alpha^{-1} = \{x, u^{-1}\}_{-\alpha} = \{x^{-1}, u\}_{-\alpha} = \{1+q, 1+p\}_{-\alpha}$$

which yields the desired result by interchanging α and $-\alpha$.

(2.10) **Proposition.** *Let* $(A, \mathfrak{q})$ *be a radical pair. Then* $\hat{M}(\mathfrak{q})$ *is a normal subgroup of* $\mathrm{St}(\Phi, \mathbf{Z}[A^*])$.

Let us first show that (2.10) completes the proof of (2.4). The hypotheses of (2.4) imply that $\mathrm{St}(\Phi, A) = \mathrm{St}(\Phi, \mathbf{Z}[A^*])$; thus by (2.10), $\hat{M}(\mathfrak{q})$ is a normal subgroup of $\mathrm{St}(\Phi, A)$ containing all $\hat{U}(\alpha, \mathfrak{q})$. Therefore $\mathrm{St}(\Phi, \mathfrak{q}) \subset \hat{M}(\mathfrak{q})$. But $\hat{M}(\mathfrak{q}) \subset \mathrm{St}(\Phi, \mathfrak{q})$, whence (2.4).

Now let us prove (2.10). $\mathrm{St}(\Phi, \mathbf{Z}[A^*])$ is generated by all $x_\alpha(t)$, $\alpha \in \pm\Delta$, $t \in A^*$. By (2.6), the *set* $\hat{M}(\mathfrak{q})$ is normalized by $\mathrm{St}(\Phi, \mathbf{Z}[A^*])$ if and only if $^{x_\alpha(t)}x_{-\alpha}(q) \in \hat{M}(\mathfrak{q})$ for all $\alpha \in \pm\Delta$, $t \in A^*$, $q \in \mathfrak{q}$. Since $\mathfrak{q} \subset \mathbf{Z}[A^*]$, this follows from (2.7)(b) and (c).

Now since $\hat{U}^-(\mathfrak{q}) \subset \mathrm{St}(\Phi, \mathbf{Z}[A^*])$, we have

$$\hat{M}(\mathfrak{q})\hat{M}(\mathfrak{q}) = \hat{M}(\mathfrak{q})\hat{U}^-(\mathfrak{q})\hat{K}(\mathfrak{q})\hat{U}(\mathfrak{q}) = \hat{U}^-(\mathfrak{q})\hat{M}(\mathfrak{q})\hat{K}(\mathfrak{q})\hat{U}(\mathfrak{q}) = \hat{M}(\mathfrak{q})$$

by (2.2). Therefore $\hat{M}(\mathfrak{q})$, being the monoid generated by 3 groups, is a group.

Remark. In showing $\hat{M}(\mathfrak{q}) = \mathrm{St}(\Phi, \mathfrak{q})$ for a radical pair $(A, \mathfrak{q})$, the restriction $A = \mathbf{Z}[A^*]$ was needed only in verifying (2.6). In $\mathrm{SL}(2, A)$, however, it is easy to show that

$$e_\alpha(t)U(-\alpha, \mathfrak{q})e_\alpha(-t) \subset U^-(\mathfrak{q})H(\mathfrak{q})U(\mathfrak{q});$$

this is simply the matrix equation

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ qu^{-1} & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}\begin{pmatrix} 1 & -t^2qu^{-1} \\ 0 & 1 \end{pmatrix}$$

where $u = 1 + tq \in A^*$, since $\mathfrak{q} \subset \mathrm{rad}\, A$. We conclude

(2.11) **Corollary.** *Let* $(A, \mathfrak{q})$ *be a radical pair. Then*

$$E(\Phi, \mathfrak{q}) = U^-(\mathfrak{q})H(\mathfrak{q})U(\mathfrak{q}).$$

(2.12) **Lemma.** *If* $\operatorname{rk}\Phi \geq 2$, $\operatorname{St}(\Phi, \ )$ *preserves finite products. If* $\operatorname{rk}\Phi = 1$, $\operatorname{St}(\Phi, A) \times \operatorname{St}(\Phi, B) \approx \operatorname{St}(\Phi, A \times B)/C$, *where* C *is the normal subgroup generated by all* $[x_\alpha((a, 0)), x_{-\alpha}((0, b))]$.

There is always a surjective homomorphism $p: \operatorname{St}(\Phi, A \times B) \longrightarrow \operatorname{St}(\Phi, A) \times \operatorname{St}(\Phi, B)$ induced by the projections of $A \times B$ onto its factors. Now $\operatorname{St}(\Phi, A) \times \operatorname{St}(\Phi, B)$ is generated by all $(x_\alpha(a), 1)$, $(1, x_\alpha(b))$, and we may define a map s backwards by

$$(x_\alpha(a), 1) \mapsto x_\alpha((a, 0)), \qquad (1, x_\alpha(b)) \mapsto x_\alpha((0, b)).$$

To show this defines an inverse isomorphism to p, we must check that the defining relations of $\operatorname{St}(\Phi, A) \times \operatorname{St}(\Phi, B)$ are preserved by s. These relations are

(i) the defining relations of $\operatorname{St}(\Phi, A)$ applied to the generators $(x_\alpha(a), 1)$,

(ii) the defining relations of $\operatorname{St}(\Phi, B)$ applied to the generators $(1, x_\alpha(b))$,

(iii) $[(x_\alpha(a), 1), (1, x_\beta(b))] = 1$ for all $\alpha, \beta \in \Phi$, $a \in A$, $b \in B$.

It is clear that s preserves (i) and (ii). Moreover relation (R2) in $\operatorname{St}(\Phi, A \times B)$ shows that s preserves (iii) whenever $\beta \neq -\alpha$. Hence the induced map $\bar{s}: \operatorname{St}(A) \times \operatorname{St}(B) \longrightarrow \operatorname{St}(A \times B)/C$ is an isomorphism, since $p(C) = 1$. This completes the proof when $\operatorname{rk}\Phi = 1$.

If $\operatorname{rk}\Phi \geq 2$, there exist $\beta, \gamma \in \Phi$, $\beta, \gamma \neq -\alpha$, such that

$$x_{-\alpha}((0, b)) = [x_\beta((0, 1)), x_\gamma((0, b))]y$$

where $y \in \hat{U}(S, (0, B))$, for some $S \subset \Phi$ with $-\alpha \notin S$. Hence

$$\begin{aligned}&[x_\alpha((a, 0)), x_{-\alpha}((0, b))]\\ &\qquad = [x_\alpha((a, 0)), [x_\beta((0, 1)), x_\gamma((0, b))]y] = 1\end{aligned}$$

which proves $C = 1$ and the lemma.

(2.13) **Theorem.** *Let* A *be a semilocal ring with at most one residue field isomorphic to* $\mathbf{F}_2$, *and suppose* $\Phi_l \subset \Phi_m$ *are reduced irreducible root systems of the same type. Then the homomorphisms* $\theta(l, m)$: $L(\Phi_l, A) \longrightarrow L(\Phi_m, A)$ *are surjective for all* $m \geq l \geq 1$, *including* $m = \infty$ *if* Φ_m *is classical.*

If $l \geq 2$, $L(\Phi_l, A)$ *is the central subgroup generated by all* $\{u, v\}_\alpha$, $u, v \in A^*$, *for any fixed long root* α. *This is also true when* $l = 1$, *provided either that* A *has no residue field isomorphic to* $\mathbf{F}_2$ *or that* A *is a local ring.*

If, in addition, Φ_l *and* A *satisfy one of the hypotheses of* [14, *Theorem* 5.3], $\operatorname{St}(\Phi_l, A)$ *is the universal covering of* $E(\Phi_l, A)$ *and* $L(\Phi_l, A) \approx H_2(E(\Phi_l, A), \mathbf{Z})$.

Write $\bar{A} = A/\operatorname{rad} A$, a finite product of fields. Steinberg [17] has shown that $L(\Phi, k) = D(\Phi, k)$ when k is a field. Since $E(\Phi, \ )$ preserves finite products, it follows from (2.12) that $L(\Phi, \bar{A}) = D(\Phi, \bar{A})$ if $\operatorname{rk}\Phi \geq 2$, and that $L(\Phi, \bar{A})$ is generated by $D(\Phi, \bar{A})$ and C when $\operatorname{rk}\Phi = 1$, where C is the normal subgroup generated by all

$$[x_\alpha((0, \cdots, k_i, \cdots, 0)), x_{-\alpha}((0, \cdots, k_j, \cdots, 0))]$$

(the appropriate generalization of the subgroup C of (2.12) when $\bar{A}$ is a product of more than 2 factors).

Now suppose rk $\Phi = 1$. Then if A is local, $L(\Phi, \bar{A}) = D(\Phi, \bar{A})$ by Steinberg [17]. If A is semilocal but has no residue field isomorphic to $\mathbf{F}_2$, we want to show $C \subset D(\Phi, \bar{A})$, and it clearly suffices to consider the case $\bar{A} = k \times k'$, a product of two fields. Then by (2.9),

$$[x_\alpha((a, 0)), x_{-\alpha}((0, b))] = \{(1 + a, 1), (1, 1 + b)\}_{-\alpha} \in D(\Phi, \bar{A})$$

provided neither a nor b equals -1. But even if $a = -1$,

$$[x_\alpha((-1, 0)), x_{-\alpha}((0, b))] = {}^{x_\alpha((-1,0))}[x_{-\alpha}((0, b)), x_\alpha((1, 0))]$$

$$= \{(1, 1 + b), (2, 1)\}_\alpha \in D(\Phi, \bar{A})$$

and a similar argument applies if $b = -1$. Hence if $-1 \neq 1$, $C \subset D(\Phi, \bar{A})$.

Thus our hypotheses imply $L(\Phi_l, \bar{A}) = D(\Phi_l, \bar{A})$; since $A^* \longrightarrow \bar{A}^*$ is surjective, so is $D(\Phi_l, A) \longrightarrow L(\Phi_l, \bar{A})$. But our hypotheses also imply (2.5) for $\mathfrak{q} = \operatorname{rad} A$; therefore $L(\Phi_l, \mathfrak{q}) = D(\Phi_l, \mathfrak{q})$ and the second part of the theorem follows from the exact sequence

$$1 \longrightarrow L(\Phi_l, \mathfrak{q}) \longrightarrow L(\Phi_l, A) \longrightarrow L(\Phi_l, \bar{A}) \longrightarrow 1$$

together with (1.3).

The first part of the theorem is a consequence of the second and (1.3), and the last part follows from [14, (5.3)].

(2.14) **Corollary.** *Let A be a semilocal ring with at most one residue field isomorphic to* $\mathbf{F}_2$. *If* rk $\Phi = 1$, *assume further that either A is local, or that A has no residue field isomorphic to* $\mathbf{F}_2$. *Then $E(\Phi, A)$ has a presentation by generators $e_\alpha(t)$, $\alpha \in \Phi$, $t \in A$, and relations* (R1), (R2) (*resp.* (R3) *if* rk $\Phi = 1$) *and*

$$(C)\, h_\alpha(u) h_\alpha(v) = h_\alpha(uv), \qquad u, v \in A^*, \alpha \in \Phi.$$

The proof is the same as [18, Theorem 8(b)] in view of (2.13).

Note. Theorems related to (2.14) have been proved by Silvester [11], [12], and Wardlaw [19].

(2.15) **Proposition.** *Let $\mathfrak{p}$, $\mathfrak{q}$ be ideals of A.*

(a) *If* rk $\Phi = 1$, *assume $L(\Phi, \mathfrak{q})$ is central in* $\operatorname{St}(\Phi, A)$. *Then if* $\operatorname{St}(\Phi, \mathfrak{q})$ *is generated by* $\hat{M}(\mathfrak{q})$,

$$[\operatorname{St}(\Phi, A), [\operatorname{St}(\Phi, \mathfrak{q}), \operatorname{St}(\Phi, \mathfrak{p})]] \subset \operatorname{St}(\Phi, \mathfrak{p}\mathfrak{q}).$$

(b) *Suppose* $\operatorname{rk} > 1$ *and that* $2 \in A^*$ *if* $\Phi = C_2$. *If either* $\mathrm{St}(\Phi, \mathfrak{q})$ *is generated by* $\hat{M}(\mathfrak{q})$ *or* $\mathrm{St}(\Phi, \mathfrak{p}^2)$ *is generated by* $\hat{M}(\mathfrak{p}^2)$, *then*

$$[\mathrm{St}(\Phi, \mathfrak{q}), \mathrm{St}(\Phi, \mathfrak{p}^2)] \subset \mathrm{St}(\Phi, \mathfrak{p}\mathfrak{q}).$$

Suppose M, N are normal subgroups of a group G, and define

$$(M:N) = \{g \in G \mid [g, N] \subset M\}.$$

It follows from the commutator formulas of [14, (2.1)] that $(M:N)$ is a normal subgroup of G. The conclusions of the proposition are thus equivalent to

(a$'$) $\mathrm{St}(\mathfrak{p}) \subset ((\mathrm{St}(\mathfrak{p}\mathfrak{q}): \mathrm{St}(A)): \mathrm{St}(\mathfrak{q}))$,

(b$'$) $\mathrm{St}(\mathfrak{p}^2) \subset (\mathrm{St}(\mathfrak{p}\mathfrak{q}): \mathrm{St}(\mathfrak{q}))$.

The groups on the right in (a$'$) and (b$'$) are normal in $\mathrm{St}(\Phi, A)$; therefore by [14, (2.1)] it suffices to prove

(a$''$) $\hat{U}(\alpha, \mathfrak{p}) \subset ((\mathrm{St}(\mathfrak{p}\mathfrak{q}): \mathrm{St}(A)): \mathrm{St}(\mathfrak{q}))$,

(b$''$) $\hat{U}(\alpha, \mathfrak{p}^2) \subset (\mathrm{St}(\mathfrak{p}\mathfrak{q}): \mathrm{St}(\mathfrak{q}))$

for one root α of each length.

If $\beta \neq -\alpha$, (R2) implies that

$$[\hat{U}(\alpha, \mathfrak{p}), \hat{U}(\beta, \mathfrak{q})] \subset \mathrm{St}(\mathfrak{p}\mathfrak{q}). \tag{5}$$

Suppose $\operatorname{rk}\Phi > 1$ and that $2 \in A^*$ if $\Phi = C_2$. Then (R2) implies the existence of $\beta, \gamma \in \Phi$ such that

$$\hat{U}(\alpha, \mathfrak{p}^2) \subset [\hat{U}(\beta, \mathfrak{p}), \hat{U}(\gamma, \mathfrak{p})] \cdot \hat{U}(S, \mathfrak{p}^2)$$

where $S \subset \Phi$ and $\alpha \notin S$. Therefore

$$[\hat{U}(\alpha, \mathfrak{p}^2), \hat{U}(-\alpha, \mathfrak{q})] \subset [[\hat{U}(\beta, \mathfrak{p}), \hat{U}(\gamma, \mathfrak{p})] \cdot \hat{U}(S, \mathfrak{p}^2), \hat{U}(-\alpha, \mathfrak{q})] \subset \mathrm{St}(\mathfrak{p}\mathfrak{q}). \tag{6}$$

(The last inclusion follows from [14, (2.1)] and (5).)

Finally, $\hat{K}(\Phi, \mathfrak{q})$ is generated by elements of the form $\{u, v\}_\beta \hat{h}_\beta(v)$, $u \in A^*$, $v \in (1 + \mathfrak{q})^*$. Therefore since $\{u, v\}_\beta$ is central, relation (R6) implies

$$[x_\alpha(p), \{u, v\}_\beta \hat{h}_\beta(v)] = [x_\alpha(p), \hat{h}_\beta(v)] = x_\alpha(p'q')$$

for some $p' \in \mathfrak{p}$, $q' \in \mathfrak{q}$, which implies that

$$[\hat{U}(\alpha, \mathfrak{p}), \hat{K}(\mathfrak{q})] \subset \mathrm{St}(\mathfrak{p}\mathfrak{q}). \tag{7}$$

Clearly (b$''$) is a consequence of (5), (6), (7); this is true under either hypothesis of (b) since (b$'$) is equivalent to

$$\mathrm{St}(\mathfrak{q}) \subset (\mathrm{St}(\mathfrak{p}\mathfrak{q}): \mathrm{St}(\mathfrak{p}^2)).$$

From (5) and (7) we also conclude that

$$[\hat{U}(\alpha,\ \mathfrak{p}), \mathrm{St}(\mathfrak{q})] = \mathrm{St}(\mathfrak{p}\mathfrak{q}) \cdot [\hat{U}(\alpha,\ \mathfrak{p}),\ \hat{U}(-\alpha,\ \mathfrak{q})].$$

It is easily checked, moreover, that in SL(2, A)

$$[U(\alpha,\ \mathfrak{p}),\ U(-\alpha,\ \mathfrak{q})] \subset E(\mathfrak{p}\mathfrak{q})$$

and therefore

$$[\hat{U}(\alpha,\ \mathfrak{p}), \mathrm{St}(\mathfrak{q})] \subset \mathrm{St}(\mathfrak{p}\mathfrak{q}) \cdot (L(\Phi,\ \mathfrak{q}) \cap \mathrm{St}_\alpha(A)).$$

Since $L(\Phi, A) \cap \mathrm{St}_\alpha(A)$ is central in $\mathrm{St}(\Phi, A)$ (by [14, (5.1)] if $\mathrm{rk}\, \Phi > 1$ and by hypothesis if $\mathrm{rk}\, \Phi = 1$), (a) is proved.

(2.16) **Corollary.** *Let* $(A,\ \mathfrak{q})$ *be a radical pair and assume* $A = \mathbf{Z}[A^*]$. *If* $\mathfrak{p} \subset A$ *is an ideal such that* $\mathfrak{p}\,\mathfrak{q} = 0$, *then* $[\mathrm{St}(\Phi,\ \mathfrak{p}), \mathrm{St}(\Phi,\ \mathfrak{q})]$ *is central in* $\mathrm{St}(\Phi, A)$.
Moreover if $\mathrm{rk}\, \Phi > 1$ *and* $2 \in A^*$ *if* $\Phi = C_2$, *then for all* $i \geq 2$,

$$[\mathrm{St}(\Phi,\ \mathfrak{p}^i), \mathrm{St}(\Phi,\ \mathfrak{q})] = [\mathrm{St}(\Phi,\ \mathfrak{p}), \mathrm{St}(\Phi,\ \mathfrak{q}^i)] = \{1\}.$$

(2.17) **Corollary.** *Let* $(A,\ \mathfrak{q})$ *be as in* (2.15) *and suppose further that* $\mathfrak{q}^{n+1} = 0$. *Then* $\Gamma = [\mathrm{St}(\Phi,\ \mathfrak{q}^i), \mathrm{St}(\Phi,\ \mathfrak{q}^j)]$ *is central in* $\mathrm{St}(\Phi, A)$ *if* $i + j \geq n + 1$.
If $\mathrm{rk}\, \Phi > 1$ *and if* $2 \in A^*$ *if* $\Phi = C_2$, Γ *is trivial when* $i + j \geq n + 2$.

3. Some computations for local rings.

(3.1) **Proposition.** *For any pair* $(A,\ \mathfrak{q})$, *the sequence*

$$1 \longrightarrow L(\Phi,\ \mathfrak{q}) \longrightarrow L(\Phi, A) \longrightarrow L(\Phi, A/\mathfrak{q})$$

is exact.

Except for the "1" on the left, this is just [16, (3.2)]. Exactness at the left holds because the group $L(\Phi,\ \mathfrak{q})$ used here is the image under the natural homomorphism of the group $L(\Phi,\ \mathfrak{q})$ of [16], and is therefore a subgroup of $L(\Phi, A)$.

(3.2) **Proposition** [17, 3.3]. *If* k *is an algebraic extension of a finite field*, $L(\Phi, k) = 1$.

(3.3) **Proposition.** (a) *For every positive integer* m *not divisible by* 4, $L(\Phi, \mathbf{Z}/m\mathbf{Z}) = 1$, *provided* $\mathrm{rk}\, \Phi \geq 2$.

(b) *For every integer* $n \geq 2$, *the groups* $L(\Phi, \mathbf{Z}/2^{n+1}\mathbf{Z})$ *and* $L(\Phi, \mathbf{Z}/2^n\mathbf{Z})$ *are isomorphic and are generated by the symbol* $\{-1, -1\}$, *which has order at most* 2 *if* Φ *is nonsymplectic.*

Proof. (a) Since $L(\Phi, \ )$ commutes with finite products, the Chinese remainder theorem implies we may assume $m = p^n$, p a prime; we may further assume $n > 1$ and $p \neq 2$ by (3.2). Since $\mathbf{Z}/p^n\mathbf{Z}$ satisfies the hypotheses of (2.13), it follows from (3.2) and from (3.1) with $\mathfrak{q} = \mathrm{rad}(\mathbf{Z}/p^n\mathbf{Z}) = p\mathbf{Z}/p^n\mathbf{Z}$ that $L(\Phi, \mathbf{Z}/p^n\mathbf{Z})$ is isomorphic to $L(\Phi, p\mathbf{Z}/p^n\mathbf{Z})$ which, according to (2.5), is generated by all $\{u, v\}$, $u \in (\mathbf{Z}/p^n\mathbf{Z})^*$, $v \in (1 + p\mathbf{Z}/p^n\mathbf{Z})$.

Now $(\mathbf{Z}/p^n\mathbf{Z})^*$ is a cyclic group of order $(p-1)p^{n-1}$, isomorphic to the direct product $(\mathbf{Z}/p\mathbf{Z})^* \times (1 + p\mathbf{Z}/p^n\mathbf{Z})$. Hence (1.1)(S7), (S8) imply $\{u, v^2\} = 1$ (u, v as above). Since p is odd, every element of $1 + p\mathbf{Z}/p^n\mathbf{Z}$ is a square, which proves (a).

(b) Again the hypotheses of (2.13) are satisfied. It follows from (1.1)(S1) that $\{-1, -1\}$ is the only possibly nontrivial symbol in $L(\Phi, \mathbf{Z}/4\mathbf{Z})$, and if Φ is non-symplectic, (1.1)(S°2) implies that the order of this symbol is at most 2. Since $(\mathbf{Z}/2^{n+1}\mathbf{Z})^* \to (\mathbf{Z}/2^n\mathbf{Z})^*$ is surjective, we have, by (2.13) and (3.1), an exact sequence

$$1 \to L(\Phi, 2^n\mathbf{Z}/2^{n+1}\mathbf{Z}) \to L(\Phi, \mathbf{Z}/2^{n+1}\mathbf{Z}) \to L(\Phi, \mathbf{Z}/2^n\mathbf{Z}) \to 1$$

for all $n \geq 1$ and all Φ. Thus to complete the proof of (b) it suffices to show

$$L(\Phi, 2^n\mathbf{Z}/2^{n+1}\mathbf{Z}) = 1 \quad \text{for } n \geq 2.$$

Let $n \geq 2$. According to (2.5), $L(\Phi, 2^n\mathbf{Z}/2^{n+1}\mathbf{Z})$ is generated by the symbols $\{1 + 2^n, u\}$, $u \in (\mathbf{Z}/2^{n+1}\mathbf{Z})^*$. Now $(\mathbf{Z}/2^{n+1}\mathbf{Z})^*$ is the direct product of the group $\{\pm 1\}$ with the cyclic group of order 2^{n-1} generated by the residue class of 5 modulo 2^{n+1}. Moreover, an easy induction argument shows that for all $n \geq 2$,

$$(1) \qquad 1 + 2^n \equiv 5^s \mod 2^{n+1}, \qquad s = 2^{n-2}.$$

Now assume $n \geq 3$. Then $1 + 2^n$ is a square and (1.1)(S6) implies that $L(\Phi, 2^n\mathbf{Z}/2^{n+1}\mathbf{Z})$ is generated by the two symbols $\{1 + 2^n, -1\}$, $\{1 + 2^n, 5\}$; since $\{1 + 2^n, -1\} = \{1 + 2^n, 1 + 2^n\} = \{1 + 2^n, 5\}^s$ by (1), this group is generated by the single symbol $\{1 + 2^n, 5\}$. Again applying (1) and computing in $L(\Phi, \mathbf{Z}/2^{n+1}\mathbf{Z})$, we have $\{1 + 2^n, 5\} = \{5^s, 5\} = 1$ by (1.1)(S8).

Now suppose $n = 2$. Then it follows from (2.5) and (1.1)(S1) and (S4) that $L(\Phi, 4\mathbf{Z}/8\mathbf{Z})$ is also generated by $\{5, -1\}$. Take $q = 2$, $u = v' = -1$, $u' = v = 5$ in (2.8) to conclude that, in $L(\Phi, \mathbf{Z}/8\mathbf{Z})$, $1 = \{5, -1\}$.

Note. For the functor $K_2 = \lim_{l\to\infty} L(A_l, \;)$, this proposition was proved by Milnor [9] using his computation of $K_2(\mathbf{Z})$ (cf. [11], [19]) and results of Mennicke, Bass, Lazard and Serre [1] on the congruence subgroup problem.

(3.4) **Proposition.** *Let A be an artinian ring such that A^* is cyclic, and suppose* $\operatorname{rk} \Phi \geq 2$. *Then* $L(\Phi, A) = 1$, *except possibly when A has a direct factor isomorphic to* $\mathbf{Z}/4\mathbf{Z}$.

Eldridge and Fischer [4] have shown that if A is artinian and A^* is finitely generated, then A is finite. Moreover, a finite ring is a finite product of primary rings $A_1, \cdots, A_n$ (rings with a unique prime ideal); if A^* is cyclic, A_i^* must also be cyclic for $i = 1, \cdots, n$ with $|A_i^*|$ and $|A_j^*|$ relatively prime for $i \neq j$. Gilmer

[5] has determined all primary rings with cyclic groups of units; they are

(a) $\mathbf{F}_q$, q a prime power,
(b) $\mathbf{Z}/p^m\mathbf{Z}$, p an odd prime, $m > 1$,
(c) $\mathbf{Z}/4\mathbf{Z}$,
(d) $\mathbf{F}_p[X]/(X^2)$, p prime,
(e) $\mathbf{F}_2[X]/(X^3)$,
(f) $\mathbf{Z}[X]/(4, 2X, X^2 - 2)$.

Since $L(\Phi, \ )$ commutes with finite products, it suffices to compute $L(\Phi, A)$ when A is one of the rings in (a)–(f) and we may apply (2.13). Propositions 3.2 and 3.3 above settle cases (a)–(c). In (d), (e), (f) we let x denote the residue class of X in A.

In (d) we use (3.1), with $\mathfrak{q} = \operatorname{rad} A = 1 + Ax$, and (3.2) to conclude that $L(\Phi, A) \approx L(\Phi, 1 + Ax)$. If ζ is a generator of $\mathbf{F}_p^*$, A^* is the product of the cyclic group $\langle \zeta \rangle$ of order $p - 1$ with the cyclic group $\langle 1 + x \rangle = 1 + Ax$ of order p. If p is odd, $1 + x$ is a square, and $L(\Phi, 1 + Ax)$ is generated by $\{\zeta, 1 + x\}$ and $\{1 + x, 1 + x\}$ according to (2.5) and (1.1)(S6). That these symbols are trivial follows from (1.1)(S6), (S8).

If $p = 2$ in (d), $\zeta = 1$ and $L(\Phi, 1 + Ax)$ is generated by

$$\{1 + x, 1 + x\} = \{1 + x, -(1 + x)\} = 1$$

by (S4) of (1.1).

In (e) and (f), A^* is cyclic of order 4, generated by $1 + x$, and $L(\Phi, A)$ is generated by $\{1 + x, 1 + x\}$. In (e) we have

$$\{1 + x, 1 + x\} = \{1 + x, -(1 + x)\} = 1,$$

and in (f)

$$\{1 + x, 1 + x\} = \{1 + x, (1 + x)^{-1}\} = \{1 + x, -(1 + x)\} = 1,$$

which completes the proof of (3.4).

Our next objective is to generalize Proposition 3.3. *Throughout the rest of this section we will assume A is a local ring whose maximal ideal $\mathfrak{p}$ is principal and generated by μ. We further assume that $A/\mathfrak{p}$ is a finite field containing $q = p^s$ elements.*

For $n \geq 0$, the group of units $(A/\mathfrak{p}^{n+1})^*$ is the direct product $\langle \zeta \rangle \times (1 + \mathfrak{p}/\mathfrak{p}^{n+1})$, where $\zeta \in (A/\mathfrak{p}^{n+1})^*$ is of order $q - 1$ and maps to a generator of $(A/\mathfrak{p})^* \approx (\mathbf{F}_q)^*$. Since A and $A/\mathfrak{p}^{n+1}$ are local, they are generated by their units.

(3.5) **Lemma.** *For all $n \geq 0$ and $1 \leq i \leq n + 1$, the additive group $\mathfrak{p}^i/\mathfrak{p}^{n+1}$ and the multiplicative group $1 + \mathfrak{p}^i/\mathfrak{p}^{n+1}$ have exponent p^{n-i+1}. Hence if p is odd, every element of $1 + \mathfrak{p}/\mathfrak{p}^{n+1}$ is a square.*

The map $a \mapsto \bar{a}\bar{\mu}^n$ induces, for all $n \geq 0$, an isomorphism of additive groups

$$A/\mathfrak{p} \approx \mathfrak{p}^n/\mathfrak{p}^{n+1}$$

where we write $\bar{a}$ for the residue class of $a \in A$ modulo $\mathfrak{p}^{n+1}$. Since $(\mathfrak{p}^n/\mathfrak{p}^{n+1})^2 = 0$, $1 + \mathfrak{p}^n/\mathfrak{p}^{n+1} \approx \mathfrak{p}^n/\mathfrak{p}^{n+1}$ and both, therefore, have exponent p. The lemma follows by descending induction on i and the exact sequences

$$0 \to \mathfrak{p}^{i+1}/\mathfrak{p}^{n+1} \to \mathfrak{p}^i/\mathfrak{p}^{n+1} \to \mathfrak{p}^i/\mathfrak{p}^{i+1} \to 0,$$

$$1 \to (1 + \mathfrak{p}^{i+1}/\mathfrak{p}^{n+1}) \to (1 + \mathfrak{p}^i/\mathfrak{p}^{n+1}) \to (1 + \mathfrak{p}^i/\mathfrak{p}^{i+1}) \to 1.$$

(3.6) **Lemma.** *Let k be a finite field. Every element of k is a sum of squares. Every element of k is a sum of fourth powers if and only if $k \neq \mathbf{F}_9$.*

Let $k = \mathbf{F}_q$, $q = p^n$, and let d be a positive nonzero integer. The subset S of k consisting of sums of dth powers is closed under addition, multiplication and subtraction, since $-1 = p - 1 = 1^d + \cdots + 1^d$. Hence S, being a subdomain of a finite field, is a subfield of k, and $S = \mathbf{F}_r$, $r = p^m$ for some m dividing n. In particular, $p^m - 1$ divides $p^n - 1$ with quotient c.

Choose an $x \in k^*$ of order $p^n - 1$. Then $x^d \in S$ and thus $x^{d(p^m-1)} = 1$, which implies $p^n - 1 \mid d(p^m - 1)$. Hence $c(p^m - 1) \mid d(p^m - 1)$ and $c \mid d$. If $d = 2$, then $c = 1$ or 2. If $c = 2$, then

$$2p^m - 2 = p^n - 1, \qquad p^m(2 - p^{n-m}) = 1, \qquad p = 1.$$

Thus $c = 1$ and $n = m$.

If $d = 4$ we must have $c = 1$, 2 or 4, and we have seen above that $c = 2$ leads to a contradiction. If $c = 4$, then

$$p^m(4 - p^{n-m}) = 3, \qquad p = 3,\ m = 1,\ n = 2,$$

and it is easily checked that $(\mathbf{F}_9)^4 = \mathbf{F}_3$.

Note. I would like to thank Armand Brumer who supplied the neat proof of this lemma.

(3.7) **Corollary.** *The symbols $\{1 + s, 1 + t\}$, $s \in \mathfrak{p}/\mathfrak{p}^{n+1}$, $t \in \mathfrak{p}^n/\mathfrak{p}^{n+1}$ generate $D(\Phi, \mathfrak{p}^n/\mathfrak{p}^{n+1})$.*

Recall from (1.3) that $D(\Phi, \mathfrak{p}^n/\mathfrak{p}^{n+1})$ is the subgroup of $L(\Phi, \mathfrak{p}^n/\mathfrak{p}^{n+1})$ generated by all $\{u, 1 + t\}$, $u \in (A/\mathfrak{p}^{n+1})^*$, $t \in \mathfrak{p}^n/\mathfrak{p}^{n+1}$. Write $u = \zeta^i(1 + s)$, $s \in \mathfrak{p}/\mathfrak{p}^{n+1}$, where ζ is of order $q - 1$. Then if p is odd, $1 + s$ is a square by (3.5), and if $p = 2$, ζ^i is a square. In either case (1.1)(S6) implies

$$\{u, 1 + t\} = \{\zeta^i, 1 + t\}\{1 + s, 1 + t\}$$

and we must show $\{\zeta^i, 1 + t\} = 1$. Suppose $1 + t$ is a square and let $v \in 1 + \mathfrak{p}^n/\mathfrak{p}^{n+1}$, $v^2 = 1 + t$. Then v has exponent p by (3.5) and ζ^i has order prime to p. Hence ζ^i and v generate a cyclic subgroup of $(A/\mathfrak{p}^{n+1})^*$ and $\{\zeta^i, 1 + t\} = 1$ by (1.1)(S7) and (S8). If $1 + t$ is not a square, we must have $p = 2$ and ζ^i is a square; a similar argument applied to $(\zeta^i)^{1/2}$ and $1 + t$ again yields $\{\zeta^i, 1 + t\} = 1$.

(3.8) **Lemma.** *If* rk $\Phi = 1$, *assume* $A/\mathfrak{p} \neq \mathbf{F}_9$. *Then* $L(\Phi, \mathfrak{p}^n/\mathfrak{p}^{n+1})$ *is generated by all*

$$\{1 + u\bar{\mu}^i, 1 + u\bar{\mu}^n\}, \qquad 1 \leq i \leq n,$$

where u *is a power of* ζ *and* $\bar{\mu}$ *denotes the image of* μ *in* $A/\mathfrak{p}^{n+1}$.

Moreover if $\Phi \neq A_1, C_2$, *or if* $\Phi = C_2$ *and* p *is odd, then these symbols are trivial except possibly when* $i = 1$.

We begin by proving that the additive group $\mathfrak{p}^m/\mathfrak{p}^{n+1}$ is generated by all $\xi\bar{\mu}^k$, $m \leq k \leq n$, where ξ is an *even* power of ζ (resp. ξ is a *fourth* power of ζ if $A/\mathfrak{p} \neq \mathbf{F}_9$). By (3.6) this is true if $m = n$, for $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ is isomorphic to $A/\mathfrak{p}$. By definition of ζ, $\mathfrak{p}^{m-1}/\mathfrak{p}^{n+1}$ is generated by all $v\bar{\mu}^k$, $m - 1 \leq k \leq n$, where v is a power of ζ. According to (3.6), $v \equiv a_1 + \cdots + a_r$ modulo $\mathfrak{p}/\mathfrak{p}^{n+1}$ where the a_i are even (resp. fourth) powers of ζ. Therefore $v\bar{\mu}^k = a_1\bar{\mu}^k + \cdots + a_r\bar{\mu}^k + b$ for some $b \in \mathfrak{p}^m/\mathfrak{p}^{n+1}$; by descending induction on m, b is of the desired form.

Our hypothesis on p assures us, by (2.5), that $L(\Phi, \mathfrak{p}^n/\mathfrak{p}^{n+1}) = D(\Phi, \mathfrak{p}^n/\mathfrak{p}^{n+1})$ and is generated, according to (3.7), by all

$$\{1 + s, 1 + \xi\bar{\mu}^n\}_\alpha, \qquad s \in \mathfrak{p}/\mathfrak{p}^{n+1}, \tag{2}$$

where $\xi = b_1 + \cdots + b_r$ is a sum of even (resp. fourth) powers of ζ, and α is any fixed long root. (The "resp." statements hold under the hypothesis $A/\mathfrak{p} \neq \mathbf{F}_9$.)

Now if Φ is nonsymplectic, there is a $\beta \in \Phi$ with $\langle\alpha, \beta\rangle = 1$, where α is the root occuring in (2). We now show that the same is true if $\Phi = C_l$, $l \geq 2$, and p is odd. In that case $1 + s = (1 + s')^2$ for some $s' \in \mathfrak{p}/\mathfrak{p}^{n+1}$ by (3.5), and we have, by (4) of §1 and (1.1)(S°3),

$$\begin{aligned}\{1 + s, 1 + t\}_\alpha &= \{(1 + s')^2, 1 + t\}_\alpha \\ &= \{1 + t, 1 + s'\}_\gamma^{-1} = \{1 + s', 1 + t\}_\gamma\end{aligned}$$

where $\gamma \in \Phi$ is a short root such that $\langle\alpha, \gamma\rangle = 2$, $\langle\gamma, \alpha\rangle = 1$. Replacing α by γ in (2), we are done.

Because $(\mathfrak{p}/\mathfrak{p}^{n+1})(\mathfrak{p}^n/\mathfrak{p}^{n+1}) = 0$, we may apply (2.9), (2.17), and the commutator identities of [14, (2.1)] to conclude

$$\begin{aligned}\{1 + s, 1 + \xi\bar{\mu}^n\}_\alpha &= [x_{-\alpha}(s), x_\alpha(\xi\bar{\mu}^n)] \\ &= [x_{-\alpha}(s), x_\alpha(b_1\bar{\mu}^n)\cdot \ldots \cdot x_\alpha(b_r\bar{\mu}^n)] \\ &= [x_{-\alpha}(s), x_\alpha(b_1\bar{\mu}^n)]\cdot \ldots \cdot [x_{-\alpha}(s), x_\alpha(b_r\bar{\mu}^n)] \\ &= \{1 + s, 1 + b_1\bar{\mu}^n\}_\alpha\cdot \ldots \cdot\{1 + s, 1 + b_r\bar{\mu}^n\}_\alpha\end{aligned} \tag{3}$$

which shows we may assume in (2) that ξ itself is an even (resp. fourth) power of ζ (and not just a sum of such powers).

Conjugating

$$\{1+s,\ 1+\xi\bar{\mu}^n\}_\alpha = [x_{-\alpha}(s),\ x_\alpha(\xi\bar{\mu}^n)]$$

by $\hat{h}_{-\alpha}(\xi^{1/2})$ yields

$$\{1+s,\ 1+\xi\bar{\mu}^n\}_\alpha = [x_{-\alpha}(\xi s),\ x_\alpha(\bar{\mu}^n)] = \{1+\xi s,\ 1+\bar{\mu}^n\}_\alpha,$$

and $L(\Phi, \mathfrak{p}^n/\mathfrak{p}^{n+1})$ is thus generated by all

$$\{1+s,\ 1+\bar{\mu}^n\}_\alpha, \qquad s \in \mathfrak{p}/\mathfrak{p}^{n+1}. \tag{4}$$

Now we may write $s = a_1\bar{\mu} + \cdots + a_n\bar{\mu}^n$, where each a_i is a sum of even (resp. fourth) powers of ζ. Arguing as for (3) above, we have

$$\begin{aligned}\{1+s,\ 1+\bar{\mu}^n\}_\alpha &= [x_{-\alpha}(s),\ x_\alpha(\bar{\mu}^n)]\\ &= [x_{-\alpha}(a_1\bar{\mu}),\ x_\alpha(\bar{\mu}^n)]\cdot\ldots\cdot[x_\alpha(a_n\bar{\mu}^n),\ x_{-\alpha}(\bar{\mu}^n)]\\ &= \{1+a_1\bar{\mu},\ 1+\bar{\mu}^n\}_\alpha\cdot\ldots\cdot\{1+a_n\bar{\mu}^n,\ 1+\bar{\mu}^n\}_\alpha,\end{aligned} \tag{5}$$

and a further argument of this type shows we may assume each a_i is itself an even (resp. fourth) power of ζ. We conclude, therefore, from (4) and (5) that $L(\Phi, \mathfrak{p}^n/\mathfrak{p}^{n+1})$ is generated by the symbols

$$\{1+a\bar{\mu}^i,\ 1+\bar{\mu}^n\}_\alpha = [x_{-\alpha}(a\bar{\mu}^i),\ x_\alpha(\bar{\mu}^n)], \qquad 1 \le i \le n, \tag{6}$$

where a is an even (resp. fourth) power of ζ.

Now if Φ is nonsymplectic, or if p is odd and $\Phi = C_l$, $l \geq 2$, take β so that $\langle\alpha, \beta\rangle = 1$ and let v be a power of ζ such that $v^2 = a$. If $\Phi = A_1$, or if $p = 2$ and $\Phi = C_l$, $l \geq 2$, take $\beta = \alpha$ and let v be a power of ζ such that $v^4 = a$ (these choices are possible by our hypotheses and the previous discussion). Conjugating (6) by $\hat{h}_\beta(v)$ yields

$$\begin{aligned}\{1+a\bar{\mu}^i,\ 1+\bar{\mu}^n\}_\alpha &= {}^{\hat{h}_\beta(v)}[x_{-\alpha}(a\bar{\mu}^i),\ x_\alpha(\bar{\mu}^n)]\\ &= [x_{-\alpha}(u\bar{\mu}^i),\ x_\alpha(u\bar{\mu}^n)] = \{1+u\bar{\mu}^i,\ 1+u\bar{\mu}^n\}_\alpha\end{aligned}$$

where $u = v^{\langle\alpha,\beta\rangle}$ is a power of ζ as desired.

Finally if $\Phi \neq A_1$, C_2, or if $\Phi = C_2$ and p is odd, it follows from (2.9) and (2.17) that for $i > 1$,

$$\{1+u\bar{\mu}^i,\ 1+u\bar{\mu}^n\} = [x_{-\alpha}(u\bar{\mu}^i),\ x_\alpha(u\bar{\mu}^n)] = 1.$$

(3.9) **Lemma.** *For every $u \in A^*$ and all $n \geq 1$,*

$$(1+u\mu^k)^{p^{n-k}} \equiv 1 + up^{n-k}\mu^k \mod \mathfrak{p}^{n+1}, \qquad 2 \le k \le n.$$

If $p \neq 2$, this congruence holds for $k = 1$ as well.

If $k = n$ the congruence is clearly true, and we will prove the remaining cases by induction on $(n-k, n+1)$ (lexicographically ordered).

Our induction hypothesis implies

$$(1 + u\mu^k)^{p^{n-k-1}} \equiv 1 + up^{n-k-1}\mu^k \mod \mathfrak{p}^n$$

and, therefore, for some $s \in \mathfrak{p}^n/\mathfrak{p}^{n+1}$,

$$\begin{aligned}(1 + u\mu^k)^{p^{n-k-1}} &\equiv 1 + up^{n-k-1}\mu^k + s \\ &\equiv (1 + up^{n-k-1}\mu^k)(1 + s) \mod \mathfrak{p}^{n+1}\end{aligned}$$

since $s\mu^k = 0$.

Thus modulo $\mathfrak{p}^{n+1}$ we have

$$\begin{aligned}(1 + u\mu^k)^{p^{n-k}} &\equiv ((1 + u\mu^k)^{p^{n-k-1}})^p \\ &\equiv (1 + up^{n-k-1}\mu^k)^p(1 + s)^p \\ &\equiv (1 + up^{n-k-1}\mu^k)^p \\ &\equiv 1 + up^{n-k}\mu^k + \sum_{i=2}^{p} \binom{p}{i} (up^{n-k-1}\mu^k)^i\end{aligned}$$

since $1 + \mathfrak{p}^n/\mathfrak{p}^{n+1}$ has exponent p by (3.5), and it suffices to show

$$\binom{p}{i} p^{ni-ki-i}\mu^{ki} \equiv 0 \mod \mathfrak{p}^{n+1}$$

for $2 \leq i \leq p$.

According to (3.5), $\mathfrak{p}^{ki}/\mathfrak{p}^{n+1}$ has additive exponent p^{n-ki+1}. Since $\binom{p}{i}$ is divisible by p if $2 \leq i \leq p - 1$, we must have

$$\begin{aligned}ni - ki - i + 1 &\geq n - ki + 1, \qquad 2 \leq i \leq p - 1, \\ np - kp - p &\geq n - kp + 1.\end{aligned}$$

That is, we must have

$$\begin{aligned}i &\geq n/(n - 1), \qquad 2 \leq i \leq p - 1, \\ p &\geq (n + 1)/(n - 1).\end{aligned}$$

These identities are satisfied *except* when $n = 1$ (in which case the lemma is trivial) and when $p = 2$, $n = 2$.

This completes the proof when p is odd. If $p = 2$, the lemma holds for $n = 2$, $k = 2$ and hence by induction for all (n, k) with $n \geq 2$, $k \geq 2$. The cases $(n, 1)$, $n \geq 1$ are true exceptions.

(3.10) **Theorem.** *Let A be a local ring whose residue field is a finite field with $q = p^s$ elements and whose maximal ideal $\mathfrak{p}$ is principal, generated by $\overline{p} = \mu$, the image of p in A. If* $\operatorname{rk} \Phi = 1$, *assume that $A/\mathfrak{p} \neq \mathbf{F}_9$. Then for all $n \geq 0$ and all odd primes p, $L(\Phi, A/\mathfrak{p}^{n+1}) = 1$. Moreover, if $p = 2$, the groups $L(\Phi, A/\mathfrak{p}^{n+1})$ and $L(\Phi, A/\mathfrak{p}^n)$ are isomorphic for all $n \geq 2$ and are generated by the $2^s - 1$ symbols $\{1 + \zeta^i\overline{\mu}, 1 + \zeta^i\overline{\mu}\}$, $1 \leq i \leq 2^s - 1$, where $\zeta \in (A/\mathfrak{p}^{n+1})^*$ has* *order* $2^s - 1$ *and maps to a generator of* $A/\mathfrak{p}$. *Each of these symbols has order at most* 2.

Since $\overline{p} = \overline{\mu}$ generates $\mathfrak{p}/\mathfrak{p}^{n+1}$ (we identify $\overline{p} \in A$ with its image in $A/\mathfrak{p}^{n+1}$), (3.9) implies, for p odd, that

$$1 + u\overline{\mu}^n = 1 + u\overline{p}^{n-i}\overline{\mu}^i = (1 + u\overline{\mu}^i)^{p^{n-i}}$$

and it follows from (3.8) that $L(\Phi, \mathfrak{p}^n/\mathfrak{p}^{n+1})$ is generated by all

$$\{1 + u\overline{\mu}^i, (1 + u\overline{\mu}^i)^{p^{n-i}}\}, \qquad 1 \leq i \leq n, \tag{7}$$

where u is a power of ζ. Since p is odd, (3.5) implies that $1 + u\overline{\mu}^i$ is a square, and

$$\{1 + u\overline{\mu}^i, (1 + u\overline{\mu}^i)^{p^{n-i}}\} = \{1 + u\overline{\mu}^i, 1 + u\overline{\mu}^i\}^{p^{n-i}} = 1$$

by (1.1)(S6), (S7) and (S8). The first part of the theorem now follows by induction on n from (3.2) and the exact sequence

$$1 \to L(\Phi, \mathfrak{p}^n/\mathfrak{p}^{n+1}) \to L(A/\mathfrak{p}^{n+1}) \to L(A/\mathfrak{p}^n) \to 1. \tag{8}$$

Suppose, then, that $p = 2$. The above argument still applies if $2 \leq i \leq n$, and we conclude that

$$\{1 + u\overline{\mu}^i, 1 + u\overline{\mu}^n\} = \{1 + u\overline{\mu}^i, (1 + u\overline{\mu}^i)^{2^{n-i}}\} = 1$$

so long as $2 \leq i \leq n$ and $(1 + u\overline{\mu}^i)^{2^{n-i}}$ is a square; that is, when $n - i \geq 1$. Thus these symbols are trivial whenever $n \geq i + 1 \geq 3$ and $i \geq 2$.

If $i = 1$, it follows from the argument of (3.8) that we may assume $u = \zeta^{2k}$ is an even power of ζ. Then, since we may take $\overline{\mu} = 2$, we have

$$\begin{aligned} \{1 + \zeta^{2k}\overline{\mu}, 1 + \zeta^{2k}\overline{\mu}^n\} &= [x_{-\alpha}(\zeta^{2k}\overline{\mu}), x_\alpha(\zeta^{2k}\overline{\mu}^n)] \\ &= {}^{x_{-\alpha}(-\overline{\mu})\hat{h}_\alpha(\zeta^k)}[x_{-\alpha}(\zeta^{2k}\overline{\mu}), x_\alpha(\zeta^{2k}\overline{\mu}^n)] \\ &= {}^{x_{-\alpha}(-\overline{\mu})}[x_{-\alpha}(\overline{\mu}), x_\alpha(\zeta^{4k}\overline{\mu}^n)] = [x_\alpha(\zeta^{4k}\overline{\mu}^n), x_{-\alpha}(-\overline{\mu})] \\ &= \{1 + \zeta^{4k}\overline{\mu}^n, 1 - \overline{\mu}\}^{-1} = \{(1 + \zeta^{4k}\overline{\mu}^2)^{2^{n-2}}, -1\}^{-1} = 1 \end{aligned} \tag{9}$$

if $n - 2 \geq 1$; that is if $n \geq 3$. Thus we have shown that $L(\Phi, \mathfrak{p}^n/\mathfrak{p}^{n+1}) = 1$ for all $n \geq 3$.

Finally suppose $n = 2$, and continue to take $\overline{\mu} = 2$. Then the characteristic of $A/\mathfrak{p}^3$ is 8, and for any $u \in A^*$,

$$\begin{aligned} \{1 + 4u, 1 + 4u\} &= [x_{-\alpha}(4u), x_\alpha(4u)] \\ &= {}^{\hat{w}_\alpha(1)}[x_{-\alpha}(4u), x_\alpha(4u)] = [x_\alpha(4u), x_{-\alpha}(4u)] = \{1 + 4u, 1 + 4u\}^{-1}. \end{aligned} \tag{10}$$

Thus $\{1+4u, 1+4u\}^2 = 1$ for any $u \in A^*$. Now $L(\Phi, \mathfrak{p}^2/\mathfrak{p}^3)$ is generated by the symbols $\{1+4u, 1+4u\}$, $\{1+2u, 1+4u\}$. But

$$\begin{aligned}\{1+4u, 1+4u\} &= [x_{-\alpha}(4u), x_\alpha(4u)] \\ &= [x_{-\alpha}(2u), x_\alpha(4u)]^2 = \{1+2u, 1+4u\}^2\end{aligned} \tag{11}$$

and we may take the symbols $\{1+2u, 1+4u\}$, $u = \zeta^{2k}$, as generators. But (9), (10), (11) then imply

$$\begin{aligned}\{1+2\zeta^{2k}, 1+4\zeta^{2k}\} &= \{1+4\zeta^{4k}, -1\}^{-1} \\ &= \{1+4\zeta^{4k}, 1+4\zeta^{4k}\}^{-1} = \{1+4\zeta^{4k}, 1+4\zeta^{4k}\} \\ &= [x_{-\alpha}(4\zeta^{4k}), x_\alpha(4\zeta^{4k})] = [x_{-\alpha}(2\zeta^{4k}), x_\alpha(4\zeta^{4k})]^2 \\ &= \{1+2\zeta^{4k}, 1+4\zeta^{4k}\}^2 = \{1+4\zeta^{8k}, -1\}^{-2} = 1.\end{aligned}$$

(Note that the last 3 lines of this computation follow from (9) by substituting $2k$ for k.)

Thus by (8), $L(\Phi, A/\mathfrak{p}^{n+1}) \approx L(\Phi, A/\mathfrak{p}^n)$ for all $n \geq 2$ as stated. If $n = 1$, then (8) and (3.2) imply $L(\Phi, A/\mathfrak{p}^2) \approx L(\Phi, \mathfrak{p}/\mathfrak{p}^2)$ is generated by the symbols $\{1+u\overline{\mu}, 1+u\overline{\mu}\}$ where $u = \zeta^i$, $1 \leq i \leq 2^s - 1$. Since the characteristic of $A/\mathfrak{p}^2$ is 4, an argument similar to (10) shows that each of these symbols has order at most 2.

(3.11) **Corollary.** *Under the hypothesis of* (3.10) *assume further that* $\mathfrak{p}$ *is nilpotent. Then if* p *is odd*, $L(\Phi, A) = 1$, *and if* $p = 2$, $L(\Phi, A)$ *is generated by the* $2^s - 1$ *symbols* $\{1+\zeta^i\overline{\mu}, 1+\zeta^i\overline{\mu}\}$, $1 \leq i \leq 2^s - 1$, *which have order at most* 2.

The corollary follows from the theorem, since if $\mathfrak{p}^{n+1} = 0$, $A/\mathfrak{p}^{n+1} = A$.

(3.12) **Corollary.** *Let* $\mathfrak{O}$ *be the ring of integers in an algebraic number field and let* $0 \neq \mathfrak{p} \subset \mathfrak{O}$ *be a prime ideal which is unramified over* $p\mathbf{Z} = \mathfrak{p} \cap \mathbf{Z}$. *If* $\operatorname{rk}\Phi = 1$, *assume that* $\mathfrak{O}/\mathfrak{p} \neq \mathbf{F}_9$. *Then if* p *is odd*, $L(\Phi, \mathfrak{O}/\mathfrak{p}^{n+1}) = 1$ *for all* $n \geq 0$. *Moreover, if* $p = 2$, *the groups* $L(\Phi, \mathfrak{O}/\mathfrak{p}^{n+1})$ *are isomorphic for all* $n \geq 1$ *and are generated by the* $2^s - 1$ *symbols* $\{1+2\zeta^i, 1+2\zeta^i\}$, $1 \leq i \leq 2^s - 1$, *where* $|\mathfrak{O}/\mathfrak{p}| = 2^s$ *and* $\zeta \in (\mathfrak{O}/\mathfrak{p}^{n+1})^*$ *has order* $2^s - 1$ *and maps to a generator of* $(\mathfrak{O}/\mathfrak{p})^*$. *These symbols have order at most* 2.

This follows from (3.11) with $A = \mathfrak{O}/\mathfrak{p}^{n+1}$.

Note. For the groups of type A_l, $l \geq 2$, this corollary is due to Christofides [2].

4. **Stability for $H_2(E(\Phi, A), \mathbf{Z})$.** Throughout this section, A denotes a local ring with maximal ideal $\mathfrak{p}$. We set $k = A/\mathfrak{p}$, but do *not* assume that k is finite or that $\mathfrak{p}$ is principal, as in §3.

We fix an $l > 1$ (depending on Φ and A) such that $L(\Phi_l, A) \approx H_2(E(\Phi_l, A), \mathbf{Z})$

and write $\Phi = \Phi_l$. It follows from [14, Theorem 5.3] that for a given A and Φ there is an $l_0 \geq 1$ such that every $l \geq l_0$ satisfies this condition, and it is clear that l_0 depends only on Φ and $A/\mathrm{rad}\, A = k$.

We abbreviate the functors $\mathrm{St}(A_1, \ )$ and $L(A_1, \ )$ by $\mathrm{St}_1(\ )$ and $L_1(\ )$ and we write $H_i(G)$ for the homology groups $H_i(G, \mathbf{Z})$ of the group G, $i = 1, 2$. Recall that the functor $E(A_1, \ )$ is $\mathrm{SL}_2(\ )$.

(4.1) **Theorem.** $H_2(\mathrm{SL}_2(A)) \to H_2(E(\Phi, A))$ *is surjective whenever* $|k| \geq 4$.

Apply the homology spectral sequence [6] to the diagram of group extensions

$$\begin{array}{ccccccccc} 1 & \to & L_1(A) & \to & \mathrm{St}_1(A) & \to & \mathrm{SL}_2(A) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & L(\Phi, A) & \to & \mathrm{St}(\Phi, A) & \to & E(\Phi, A) & \to & 1 \end{array}$$

to obtain the following commutative diagram with exact rows:

$$(1) \qquad \begin{array}{ccccccc} H_2(\mathrm{SL}_2(A)) & \xrightarrow{\phi} & L_1(A) & \to & \mathrm{St}_1(A)^{\mathrm{ab}} & \to & \mathrm{SL}_2(A)^{\mathrm{ab}} \to 0 \\ \downarrow & & \downarrow & & & & \\ H_2(E(\Phi, A)) & \xrightarrow{\approx} & L(\Phi, A) & & & & \\ & & \downarrow & & & & \\ & & 1 & & & & \end{array}$$

The surjectivity of $L_1(A) \to L(\Phi, A)$ is a consequence of (2.13). If $|k| \geq 4$, there exists $u \in A^*$ with $u^2 - 1 \in A^*$ and by [14, (4.4)], $\mathrm{St}_1(A)^{\mathrm{ab}} = 0$. Thus the theorem follows from (1).

We shall require the following unpublished result of Bass.

(4.2) **Lemma.** *Let* $\mathfrak{q} \subset A$ *be the ideal generated by all* $u^2 - 1$, $u \in A^*$. *If* $k = \mathbf{F}_2$, *assume that* $\mathfrak{p}$ *is principal, generated by* μ. *Then* $\mathrm{St}_1(A)^{\mathrm{ab}} \approx \mathrm{St}_1(A/\mathfrak{q})^{\mathrm{ab}}$ *and both groups are quotients of* $A/\mathfrak{q}$. *Moreover,* $\mathfrak{q} = A$ *except in the following cases*:

$$\begin{array}{llll} k = \mathbf{F}_3, & & \mathfrak{q} = \mathfrak{p}, & A/\mathfrak{q} = \mathbf{F}_3, \\ k = \mathbf{F}_2, & \mu A = 2A, & \mathfrak{q} = 8A, & A/\mathfrak{q} = \mathbf{Z}/2^n\mathbf{Z}, \quad n = 1, 2 \text{ or } 3, \\ k = \mathbf{F}_2, & 2 \in \mu^2 A, & \mathfrak{q} = \mu^2 A, & A/\mathfrak{q} \approx \mathbf{F}_2[X]/(X^2). \end{array}$$

Denote the image in $\mathrm{St}_1(A)^{\mathrm{ab}}$ of $g \in \mathrm{St}_1(A)$ by $[g]$, and set $\langle t \rangle = [x_\alpha(t)]$ for $t \in A$. It follows from relation (R1) that $t \mapsto \langle t \rangle$ is a homomorphism $A^+ \to \mathrm{St}_1(A)^{\mathrm{ab}}$. By relation (R3)

$$\hat{w}_\alpha(u) x_{-\alpha}(t) \hat{w}_\alpha(-u) = x_\alpha(-u^2 t), \qquad u \in A^*,$$

we have $[x_{-\alpha}(t)] = \langle -u^2 t \rangle$; hence $t \mapsto \langle t \rangle$ is surjective. Moreover by (R6)

$$[\hat{h}_\alpha(u), x_\alpha(t)] = x_\alpha((u^2 - 1)t)$$

and therefore $\langle t\rangle = 0$ for $t \in \mathfrak{q}$. This proves that $\mathrm{St}_1(A)^{ab}$ is a quotient of $A/\mathfrak{q}$ and that $\mathrm{St}_1(\mathfrak{q}) \subset [\mathrm{St}_1(A), \mathrm{St}_1(A)]$. Hence there is a surjective homomorphism $\mathrm{St}_1(A/\mathfrak{q}) \to \mathrm{St}_1(A)^{ab}$ which factors through $\mathrm{St}_1(A/\mathfrak{q})^{ab}$; the projection $\mathrm{St}_1(A)^{ab} \to \mathrm{St}_1(A/\mathfrak{q})^{ab}$ is an inverse to this induced homomorphism.

Now let us determine the ideal $\mathfrak{q}$. Since A is local, $\mathfrak{q} = A$ if and only if $|k| \geq 4$. If $k = \mathbf{F}_3$, we have $A^* = \{1 + x, x - 1, x \in \mathfrak{p}\}$. Hence if $u \in A^*$, $u^2 - 1 = x(2 + x)$ or $x(x - 2)$ for some $x \in \mathfrak{p}$; since $2 + x, 2 - x \in A^*$, this proves $\mathfrak{q} = \mathfrak{p}$.

If $k = \mathbf{F}_2$, write $2A = \mu^e A$ with $e = \infty$ if $2A = 0$. If $e = 1$ we may assume $\mu = 2$, and $(1 + 2x)^2 - 1 = 4x + 4x^2 \equiv 0 \bmod 8A$. Taking $x = 1$, we see that $\mathfrak{q} = 8A$ and, therefore, that $A/\mathfrak{q} \approx \mathbf{Z}/2^n\mathbf{Z}$, $n = 1, 2$ or 3. If $e > 1$, write $2 = \mu^e v$, $v \in A^*$. Then

$$(1 + \mu)^2 - 1 = 2\mu + \mu^2 = v\mu^{e+1} + \mu^2 = \mu^2(1 + v\mu^{e-1}).$$

Since $1 + v\mu^{e-1} \in A^*$, $\mathfrak{q} = \mu^2 A$ and $A/\mathfrak{q} \approx \mathbf{F}_2[X]/(X^2)$ as desired.

(4.3) **Theorem.** *The map*

$$H_2(\mathrm{SL}_2(A)) \to H_2(E(\Phi, A))$$

is surjective if $k \approx \mathbf{F}_3$.

It suffices, by (1), to show that $L_1(A) \to \mathrm{St}_1(A)^{ab}$ is 0, and this map factors, by (4.2), as

$$\begin{array}{ccc} L_1(A) & \to & \mathrm{St}_1(A)^{ab} \\ \downarrow & & \downarrow \wr \\ L_1(A/\mathfrak{q}) & \to & \mathrm{St}_1(A/\mathfrak{q})^{ab} \end{array}$$

But $L_1(A/\mathfrak{q}) = L_1(\mathbf{F}_3) = 1$ by (3.2).

(4.4) **Lemma.** *Let* $\{u, v\} \in L_1(A)$. *Then* $[\{u, v\}] = \langle 3(u - 1)(v - 1)\rangle$ *in* $\mathrm{St}_1(A)^{ab}$. *Moreover,* $\{u, v\}$ *lies in the image of* $H_2(\mathrm{SL}_2(A))$ *if and only if* $[\{u, v\}] = 1$.

Since $[x_{-\alpha}(t)] = \langle -u^2 t\rangle$ (cf. the proof of (4.2)), taking $t = -u^{-1}$, we have $[x_{-\alpha}(-u^{-1})] = \langle u\rangle$. Hence $[\hat{w}_\alpha(u)] = [x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u)] = \langle 3u\rangle$ and $[\hat{h}_\alpha(u)] = [\hat{w}_\alpha(u)\hat{w}_\alpha(-1)] = \langle 3(u - 1)\rangle$. Finally,

$$\begin{aligned}[\{u, v\}] &= [\hat{h}_\alpha(uv)\hat{h}_\alpha(u)^{-1}\hat{h}_\alpha(v)^{-1}] \\ &= \langle 3(uv - 1) - 3(u - 1) - 3(v - 1)\rangle = \langle 3(uv - 1 - u + 1 - v + 1)\rangle = \langle 3(u - 1)(v - 1)\rangle.\end{aligned}$$

Now consider the commutative diagram

$$\text{(2)} \qquad \begin{array}{ccccccccc} 1 & \to & L_1(A) & \longrightarrow & \mathrm{St}_1(A) & \longrightarrow & \mathrm{SL}_2(A) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & L_1(A)/\phi(H_2(\mathrm{SL}_2(A))) & \to & \mathrm{St}_1(A)^{ab} & \to & \mathrm{SL}_2(A)^{ab} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & 1 & & \end{array}$$

Its columns and top row are clearly exact. Since the bottom row is obtained by factoring out the image of $H_2(\mathrm{SL}_2(A))$ from the top row of (1), it too is exact. The second part of the lemma follows easily from (2).

(4.5) **Proposition.** *The map* $H_2(\mathrm{SL}_2(\mathbf{Z}/2^n\mathbf{Z})) \longrightarrow L_1(\mathbf{Z}/2^n\mathbf{Z})$ *is surjective for* $n = 1, 2$ *but not for* $n \geq 3$. *Therefore the map*

$$H_2(\mathrm{SL}_2(\mathbf{Z}/4\mathbf{Z})) \to H_2(E(\Phi, \mathbf{Z}/4\mathbf{Z}))$$

is surjective.

It is clear from (1) that the second statement is implied by the first. For $n = 1$, the first assertion is trivial since $L_1(\mathbf{Z}/2\mathbf{Z}) = 1$ by (3.2). Now $L_1(\mathbf{Z}/4\mathbf{Z})$ is generated by the symbol $\{-1, -1\}$ whose image in $\mathrm{St}_1(\mathbf{Z}/4\mathbf{Z})^{\mathrm{ab}}$ is $\langle 3(-1-1)(-1-1)\rangle = 1$. This completes the proof for $n = 2$ by (4.4).

Now suppose $n \geq 3$. According to (4.2), $\mathrm{St}_1(\mathbf{Z}/2^n\mathbf{Z})^{\mathrm{ab}} \approx \mathrm{St}_1(\mathbf{Z}/8\mathbf{Z})^{\mathrm{ab}}$ for all $n \geq 3$; thus (1) implies that

$$\phi\colon\ H_2(\mathrm{SL}_2(\mathbf{Z}/2^n\mathbf{Z})) \to L_1(\mathbf{Z}/2^n\mathbf{Z})$$

is surjective for $n = 3$ if and only if ϕ is surjective for all $n \geq 3$.

Suppose that this is the case. Then from (1) we have

$$\mathrm{St}_1(\mathbf{Z}/2^n\mathbf{Z})^{\mathrm{ab}} \approx \mathrm{SL}_2(\mathbf{Z}/2^n\mathbf{Z})^{\mathrm{ab}}$$

for all $n \geq 3$, and the same must be true for the 2-adic integers

$$\mathrm{St}_1(\hat{\mathbf{Z}}_2)^{\mathrm{ab}} \approx \mathrm{SL}_2(\hat{\mathbf{Z}}_2)^{\mathrm{ab}}.$$

Hence $H_2(\mathrm{SL}_2(\hat{\mathbf{Z}}_2)) \longrightarrow L_1(\hat{\mathbf{Z}}_2) \longrightarrow L_\infty(\hat{\mathbf{Z}}_2) = K_2(\hat{\mathbf{Z}}_2)$ is surjective by (1) and (2.13).

Dualizing, we have

$$\mathrm{Hom}(H_2(\mathrm{SL}_2(\hat{\mathbf{Z}}_2)), \mathbf{Q}/\mathbf{Z}) \approx H^2(\mathrm{SL}_2(\hat{\mathbf{Z}}_2), \mathbf{Q}/\mathbf{Z})$$

by the universal coefficient theorem [7, p. 77]. But $H^2(\mathrm{SL}_2(\hat{\mathbf{Z}}_2), \mathbf{Q}/\mathbf{Z}) = 0$ [1, Proposition 2]. Therefore if ϕ is surjective, we conclude that $K_2(\hat{\mathbf{Z}}_2) = 0$; in particular $\{-1, -1\} = 0$ in $K_2(\hat{\mathbf{Q}}_2)$. But it follows from results of Moore [10] and Matsumoto [8] that $\{-1, -1\} \neq 0$ in $K_2(\hat{\mathbf{Q}}_2)$, whence the proposition.

(4.6) **Corollary.** *The symbol* $\{-1, -1\}$ *is nontrivial in* $L_1(\mathbf{Z}/4\mathbf{Z})$.

Since $\{-1, -1\}$ generates $L_1(\mathbf{Z}/4\mathbf{Z})$, if it is 1 we conclude from (3.1) that $L_1(\mathbf{Z}/8\mathbf{Z}) \approx L_1(4\mathbf{Z}/8\mathbf{Z})$ is generated by the symbols $\{1 + 4a, 1 + 2b\}$, $a, b \in \mathbf{Z}$. But in $\mathrm{St}_1(\mathbf{Z}/8\mathbf{Z})^{\mathrm{ab}}$, $[\{1 + 4a, 1 + 2b\}] = \langle 3(4a)(2b)\rangle = 0$, which implies that $H_2(\mathrm{SL}_2(\mathbf{Z}/8\mathbf{Z})) \longrightarrow L_1(\mathbf{Z}/8\mathbf{Z})$ is surjective by (4.4). This contradicts (4.5).

Note. Despite (4.6), we cannot conclude that $\{-1, -1\} \neq 0$ in $K_2(\mathbf{Z}/4\mathbf{Z})$ since $K_2(\mathbf{Z}/4\mathbf{Z})$ is a quotient of $L_1(\mathbf{Z}/4\mathbf{Z})$ by (2.13).

Added in proof. Much more extensive information on the functor $K_2 = \lim_{l\to\infty} L(A_l,)$ has been obtained since this paper was written. Dennis ([20], [21]) has proved the conjecture of the Introduction, showing that when Φ is of type A_l, the maps $\theta(l, m)$ are surjective for all $m \geq l \geq d + 3$, where d is the dimension of the maximal ideal space of A.

The results concerning K_2 of a semilocal ring (Theorem 2.13) have been completed by Stein and Dennis [24]. They have also proved ([22], [23]) that for nonsymplectic Φ, the maps $\theta(l, m)$ are injective (and hence isomorphisms) when A is a discrete valuation ring or a quotient thereof, and they have given a presentation of the K_2 of such a ring. These papers also compute K_2 of a ring of algebraic integers modulo any nonzero ideal, generalizing the results of §3. Among the consequences of this computation is the nontriviality of the symbol $\{-1, -1\} \in K_2(\mathbf{Z}/4\mathbf{Z})$ (see the Note at the end of §4).

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