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Surjectivity of convolution operators on spaces of ultradifferentiable functions of Roumieu type

by

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Abstract. Let $\mathcal{E}_{\{\omega\}}(I)$ denote the space of all ω-ultradifferentiable functions of Roumieu type on an open interval I in $\mathbb{R}$. In the special case $\omega(t) = t$ we get the real-analytic functions on I. For $\mu \in \mathcal{E}_{\{\omega\}}(I)'$ with $\operatorname{supp}(\mu) = \{0\}$ one can define the convolution operator $T_\mu : \mathcal{E}_{\{\omega\}}(I) \to \mathcal{E}_{\{\omega\}}(I)$, $T_\mu(f)(x) := \langle \mu, f(x - \cdot)\rangle$. We give a characterization of the surjectivity of T_μ for quasianalytic classes $\mathcal{E}_{\{\omega\}}(I)$, where $I = \mathbb{R}$ or I is an open, bounded interval in $\mathbb{R}$. This characterization is given in terms of the distribution of zeros of the Fourier Laplace transform $\widehat{\mu}$ of μ.

Let $\omega : [0, \infty[\to [0, \infty[$ be a continuous increasing function which satisfies some technical conditions. By $\mathcal{E}_{\{\omega\}}(I)$ we denote the space of ω-ultradifferentiable functions of Roumieu type on an open interval $I \subset \mathbb{R}$. This notion is an extension of the classical Gevrey classes $\Gamma^{\{d\}}(\mathbb{R})$, $d > 1$. In the special case $\omega(t) = t$ we get the real-analytic functions on I.

For $\mu \in \mathcal{E}_{\{\omega\}}(\mathbb{R})'$ the convolution operator $T_\mu : \mathcal{E}_{\{\omega\}}(\mathbb{R}) \to \mathcal{E}_{\{\omega\}}(\mathbb{R})$ is defined by

$$T_\mu(f)(x) := \langle \mu, f(x - \cdot)\rangle, \quad f \in \mathcal{E}_{\{\omega\}}(\mathbb{R}),\ x \in I.$$

Many authors have investigated the surjectivity of convolution operators on various classes of infinitely differentiable functions. For non-quasianalytic classes $\mathcal{E}_{\{\omega\}}(\mathbb{R})$ (i.e. when ω satisfies $\int_1^\infty \omega(t)t^{-2}\,dt < \infty$) Braun, Meise and Vogt [7, 3.8] have shown that a convolution operator T_μ on $\mathcal{E}_{\{\omega\}}(\mathbb{R})$ is surjective if and only if the following two conditions are satisfied: (i) T_μ admits a fundamental solution, (ii) there exists a decomposition $J_1 \cup J_2$ of the zero set of the Fourier Laplace transform $\widehat{\mu}$ of μ with

$$\lim_{|z|\to\infty,\, z\in J_1} |\operatorname{Im} z|/\omega(z) = 0 \quad \text{and} \quad \liminf_{|z|\to\infty,\, z\in J_2} |\operatorname{Im} z|/\omega(z) > 0.$$

Because of the topological structure of $\mathcal{E}_{\{\omega\}}(I)$, which is a projective limit of (DFS)-spaces, this theorem is more difficult to prove then for example for the classes $C^\infty(I)$, where the existence of a fundamental solution is already

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equivalent to surjectivity. In the real-analytic case ($\omega(t) = t$) an analogous result for $\mu \in \mathcal{E}_{\{\omega\}}(\mathbb{R})'$ with $\operatorname{supp}(\mu) = \{0\}$ and open intervals I in $\mathbb{R}$ has been shown by Korobeĭnik [9] and independently by Napalkov and Rudakov [21].

In the present article we extend these results for $\mu \in \mathcal{E}_{\{\omega\}}(\mathbb{R})'$ with $\operatorname{supp}(\mu) = \{0\}$ to quasianalytic classes $\mathcal{E}_{\{\omega\}}(I)$ where $I = \mathbb{R}$ or I is an open, bounded interval in $\mathbb{R}$. For this purpose we choose a projective spectrum $(E_n, \pi^n_{n+1})_{n\in\mathbb{N}}$ of (DFS)-spaces with limit $\mathcal{E}_{\{\omega\}}(I)$ such that T_μ also operates on each E_n. We call T_μ locally surjective if T_μ is surjective on all E_n. The main result of the present article is the following theorem.

THEOREM. *A convolution operator T_μ on $\mathcal{E}_{\{\omega\}}(I)$ is surjective if and only if the following two conditions are satisfied*:

(i) *T_μ is locally surjective*,

(ii) *there exists a decomposition $J_1 \cup J_2$ of the zero set of the Fourier Laplace transform $\widehat{\mu}$ of μ with*

$$\lim_{|z|\to\infty,\, z\in J_1} |\operatorname{Im} z|/\omega(z) = 0 \quad \textit{and} \quad \liminf_{|z|\to\infty,\, z\in J_2} |\operatorname{Im} z|/\omega(z) > 0.$$

In the non-quasianalytic case the local surjectivity is equivalent to the existence of a fundamental solution for T_μ.

As in Braun, Meise and Vogt [7] the proof is based on the essential properties of the projective limit functor of Palamodov [22] and on Vogt's [25] elementary approach to it. To apply these methods, we first show $\operatorname{proj}^1 \mathcal{E}^I_{\{\omega\}} = 0$ for a projective spectrum $\mathcal{E}^I_{\{\omega\}}$ with limit $\mathcal{E}_{\{\omega\}}(I)$. In the non-quasianalytic case this follows immediately from the existence of cut-off functions (see [8, 1.8]). In the quasianalytic case we use a result of Braun [5, 2.3.5], who introduces two conditions which are sufficient for $\operatorname{proj}^1 \mathcal{E}^I_{\{\omega\}} = 0$. The proof of these two conditions is the main part of Section 3.

Then the surjectivity of a convolution operator T_μ which is locally surjective is equivalent to $\operatorname{proj}^1 \mathcal{K}^I(\omega,\mu) = 0$, where $\mathcal{K}^I(\omega,\mu)$ is a particular projective spectrum of (DFS)-spaces with limit $\ker T_\mu$. To evaluate the condition $\operatorname{proj}^1 \mathcal{K}^I(\omega,\mu) = 0$ as in [7] we need a sequence space representation of $\ker T_\mu$. For this purpose, and to characterize the local surjectivity of T_μ by means of growth conditions on the Fourier Laplace transform $\widehat{\mu}$ of μ, we show the equivalence of local surjectivity for T_μ and a new slowly decreasing condition (SD) for $\widehat{\mu}$. Using (SD) we construct a sequence space representation of $\ker T_\mu$ similar to Meise [12]. Together with a result of Vogt [25] characterizing $\operatorname{proj}^1 \Lambda^I(\gamma,\delta) = 0$ for certain projective spectra of (DF)-sequence spaces, this yields the equivalence of $\operatorname{proj}^1 \mathcal{K}^I(\omega,\mu) = 0$ and condition (ii) of the theorem.

Another problem which does not appear in the non-quasianalytic case is that surjectivity of a convolution operator implies local surjectivity. This is shown in 3.12 using de Wilde's open mapping theorem, a theorem of Grothendieck on topological homomorphisms and some results of Vogt on projective spectra of (DF)-spaces.

The article has three sections. In the first section we introduce the classes of ultradifferentiable functions of Roumieu type and fix some notation. The second part deals with the characterization of the local surjectivity of a convolution operator T_μ with different slowly decreasing conditions, and the sequence space representation of $\ker T_\mu$. In the last section we show $\operatorname{proj}^1 \mathcal{E}^I_{\{\omega\}} = 0$ and the theorem stated above.

1. Preliminaries.

In this section we introduce the spaces $\mathcal{E}_{\{\omega\}}$ of ultradifferentiable functions, which we will use in the sequel. We begin with the definition of weight functions in the sense of Braun, Meise and Taylor [6].

1.1. *Weight functions.* Let $\omega : [0,\infty[\to [0,\infty[$ be a continuous increasing function. We consider the following properties:

$$(\alpha) \quad \omega(2t) = O(\omega(t)),\ t\to\infty, \qquad (\beta) \quad \int_0^\infty \frac{\omega(t)}{1+t^2}\,dt < \infty,$$

$$(\gamma) \quad \log t = o(\omega(t)),\ t\to\infty, \qquad (\delta) \quad \varphi : t \mapsto \omega(e^t) \text{ is convex.}$$

$$(\varepsilon) \quad \omega(t) = O(t),\ t\to\infty,$$

ω is called a *weight* function if it satisfies conditions (α), (γ), (δ) and (ε). If in addition (β) holds, then ω is called a *non-quasianalytic weight function*; otherwise it is *quasianalytic*. In both cases we denote by φ^* the *Young conjugate* of φ, i.e.

$$\varphi^* : [0,\infty[\to [0,\infty[, \qquad \varphi^*(x) = \sup_{y\ge 0}(xy - \varphi(y)).$$

We extend ω to the complex plane by setting $\omega(z) := \omega(|z|)$ for $z \in \mathbb{C}$.

1.2. REMARK. (a) If we replace (α) by the stronger condition (α'): $\forall s,t > 0 : \omega(s+t) \le \omega(s) + \omega(t)$ and (γ) by the weaker condition (γ') : $\log t = O(\omega(t))$, then (α'), (β) and (γ') are the conditions which were used by Beurling [3] and Björck [4] to develop a theory of ultradifferentiable functions and ultradistributions.

(b) There exists a weight function $\sigma \le \omega$ satisfying $\sigma|_{[0,1]} = 0$ and $\sigma(t) = \omega(t)$ for all large $t > 0$. Since the subsequent definitions do not change if ω is replaced by σ, we will assume that $\omega|_{[0,1]} = 0$. Then φ^* has only non-negative values and $\varphi^{**} = \varphi$.

1.3. *Ultradifferentiable functions.* Let ω be a weight function and $I \subset \mathbb{R}$ an open interval. We define

$$\mathcal{E}_{\{\omega\}}(I) := \Big\{ f \in C^\infty(I) \,\Big|\, \text{for each } K \Subset I \text{ there exists } m \in \mathbb{N} \text{ with}$$
$$q_{K,m}(f) := \sup_{j \in \mathbb{N}_0} \sup_{x \in K} |f^{(j)}(x)| \exp\Big(-\frac{1}{m}\varphi^*(jm)\Big) < \infty \Big\}$$

and endow $\mathcal{E}_{\{\omega\}}(I)$ with the topology given by the projective limit over $K \Subset I$ of the inductive limits over $m \in \mathbb{N}$.

As in Braun, Meise and Taylor [6, 4.4] one can show that $\mathcal{E}_{\{\omega\}}(I)$ is a nuclear locally convex algebra with continuous multiplication (for the quasi-analytic case see [16]).

1.4. Definition and Remark. Let ω be a weight function. We will see that it suffices to consider the case $I =]-1,1[$ instead of an arbitrary bounded, open interval. Therefore we fix $I = \mathbb{R}$ or $I =]-1,1[$. Then we choose a strictly increasing sequence $(a_n)_{n\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ with $\lim_{n\to\infty} a_n = 1$ (for $I =]-1,1[$), resp. $\lim_{n\to\infty} a_n = \infty$ (for $I = \mathbb{R}$), and define for $n, m \in \mathbb{N}$ the compact sets $K_n := [-a_n, a_n]$, $K_{n,m} := [-a_n - 1/m, a_n + 1/m]$ and the normed spaces

$$\mathcal{E}_{\{\omega\},n,m} := \{ f \in C^\infty(\mathring{K}_{n,m}) \mid q_{\mathring{K}_{n,m},m}(f) < \infty \}.$$

We write

$$q_{n,m}(f) := q_{K_{n,m},m}(f) = q_{\mathring{K}_{n,m},m}(f)$$

and remark that $(\mathcal{E}_{\{\omega\},n,m})_{m\in\mathbb{N}}$ together with the restriction mappings is an inductive spectrum for each $n \in \mathbb{N}$. We set

$$\mathcal{E}_{\{\omega\},n} := \operatorname*{ind}_{m\to} \mathcal{E}_{\{\omega\},n,m}, \qquad \pi_n : \mathcal{E}_{\{\omega\}}(I) \to \mathcal{E}_{\{\omega\},n}, \qquad \pi_n(f) = f|_{K_n}$$

and

$$\pi^n_{n+1} : \mathcal{E}_{\{\omega\},n+1} \to \mathcal{E}_{\{\omega\},n}, \quad \pi^n_{n+1}(f) = f\big|_{K_n}$$

to get a projective spectrum $(\mathcal{E}_{\{\omega\},n}, \pi^n_{n+1})_{n\in\mathbb{N}}$ with $\mathcal{E}_{\{\omega\}}(I) = \operatorname{proj}_{\leftarrow n} \mathcal{E}_{\{\omega\},n}$. In addition we define

$$\mathcal{E}_{\{\omega\},0} := \operatorname*{ind}_{m\to} E_m$$

where

$$E_m := \{ f \in C^\infty(\mathring{L}_m) \mid q_{L_m,m}(f) < \infty \}, \quad L_m = [-1/m, 1/m],$$

and remark that $\mu \in (\mathcal{E}_{\{\omega\},0})'$ if and only if $\mu \in \mathcal{E}_{\{\omega\}}(\mathbb{R})'$ and the following condition holds:

$$\forall m \in \mathbb{N}\ \exists C_m > 0\ \forall f \in \mathcal{E}_{\{\omega\}}(\mathbb{R}) : \quad |\langle \mu, f\rangle| \le C_m q_{L_m,m}(f).$$

1.5. *Convolution operators.* Let ω be a weight function and $\mu \in (\mathcal{E}_{\{\omega\},0})'$. Then

$$T_\mu : \mathcal{E}_{\{\omega\}}(I) \to \mathcal{E}_{\{\omega\}}(I), \qquad T_\mu(f) = \mu * f \quad \text{and}$$
$$T_{\mu,n} : \mathcal{E}_{\{\omega\},n} \to \mathcal{E}_{\{\omega\},n}, \qquad T_{\mu,n}(f) = \mu * f$$

with $\mu * f(x) = \langle \mu, f(x - \cdot)\rangle$ are linear and continuous mappings. T_μ and $T_{\mu,n}$ are called *convolution operators.* Taking an open, bounded interval $]a,b[$ and the transformations

$$T(t) = \frac{b-a}{2}t + \frac{b+a}{2}, \qquad t \in]-1,1[,$$
$$\widetilde{T} : \mathcal{E}_{\{\omega\}}(]a,b[) \to \mathcal{E}_{\{\omega\}}(]-1,1[), \qquad \widetilde{T}(f) = f \circ T,$$

we see that $\mathcal{E}_{\{\omega\}}(]a,b[)$ and $\mathcal{E}_{\{\omega\}}(]-1,1[)$ are linear topologically isomorphic. So we can reduce our considerations for open, bounded intervals to the case $I =]-1,1[$.

1.6. *Fourier Laplace transformation.* Let ω be a weight function, $(a_n)_{n\in\mathbb{N}}$ the sequence from 1.4 and $p(z) = |\operatorname{Im} z| + \omega(z)$. Then we define the following spaces of entire functions ($A(U)$ for open $U \subset \mathbb{C}$ denotes the space of holomorphic functions on U):

$$A_{\{\omega\},n,m} := \Big\{ f \in A(\mathbb{C}) \,\Big|\, |f|_{n,m} := \sup_{z\in\mathbb{C}} |f(z)| \exp\Big(-a_n|\operatorname{Im} z| - \frac{1}{m}p(z)\Big) < \infty \Big\},$$
$$A^0_{\{\omega\},n} := \operatorname*{proj}_{\leftarrow m} A_{\{\omega\},n,m}, \qquad A^I_{\{\omega\}} := \operatorname*{ind}_{n\to} A^0_{\{\omega\},n}$$

and

$$A^0_p := \Big\{ f \in A(\mathbb{C}) \,\Big|\, \forall m \in \mathbb{N} : |f|_m := \sup_{z\in\mathbb{C}} |f(z)| \exp\Big(-\frac{1}{m}p(z)\Big) < \infty \Big\}.$$

$A^0_{\{\omega\},n}$ and A^0_p endowed with the locally convex topology induced by the seminorm systems $(|\ |_{n,m})_{m\in\mathbb{N}}$, respectively $(|\ |_m)_{m\in\mathbb{N}}$, are nuclear Fréchet spaces. As an inductive limit of nuclear (F)-spaces, $A^I_{\{\omega\}}$ is a nuclear (LF)-space. Moreover, A^0_p is an algebra with the continuous multiplication $(f \cdot g)(z) = f(z) \cdot g(z)$. Setting $f_z(x) = \exp(-ixz)$ for $x \in \mathbb{R}$ and $z \in \mathbb{C}$ we define the *Fourier Laplace transformations*

$$\mathcal{F} : \mathcal{E}_{\{\omega\}}(I)'_b \to A^I_{\{\omega\}}, \qquad \mathcal{F}(\mu)[z] = \langle \mu, f_z\rangle,$$
$$\mathcal{F}_n : (\mathcal{E}_{\{\omega\},n})'_b \to A^0_{\{\omega\},n}, \qquad \mathcal{F}_n(\mu)[z] = \langle \mu, f_z\rangle,$$
$$\mathcal{F}_0 : (\mathcal{E}_{\{\omega\},0})'_b \to A^0_p, \qquad \mathcal{F}_0(\mu)[z] = \langle \mu, f_z\rangle.$$

The index b denotes the strong topology in the dual space. Using a result of Taylor [24] one can show (see [16, Chapter 5]) that the Fourier Laplace transformations $\mathcal{F}$, $\mathcal{F}_n$ and $\mathcal{F}_0$ are linear topological isomorphisms. We also use the notation $\widehat{\mu}$ instead of $\mathcal{F}(\mu)$.

1.7. *Multiplication operators.* As in [6] each functional $\mu \in (\mathcal{E}_{\{\omega\},0})'$ also defines a convolution operator on the dual spaces of $\mathcal{E}_{\{\omega\}}(I)$ and $\mathcal{E}_{\{\omega\},n}$ in the following way:

$$\mathcal{E}_{\{\omega\}}(I)' \to \mathcal{E}_{\{\omega\}}(I)', \quad \nu \mapsto \mu * \nu,$$

with $\mu * \nu(f) := \langle \nu, \check{\mu} * f\rangle = \nu \circ T_{\check{\mu}}(f)$, $\langle \check{\mu}, f\rangle := \langle \mu, \check{f}\rangle$ and $\check{f}(x) := f(-x)$. The definition for $\mathcal{E}_{\{\omega\},n}$ is analogous. For $\nu_1 \in \mathcal{E}_{\{\omega\}}(I)'$ and $\nu_2 \in (\mathcal{E}_{\{\omega\},n})'$ we get immediately

$$\mathcal{F}(\mu * \nu_1) = \mathcal{F}_0(\mu)\mathcal{F}(\nu_1), \quad \mathcal{F}_n(\mu * \nu_2) = \mathcal{F}_0(\mu)\mathcal{F}_n(\nu_2).$$

Let $g \in A_p^0$. Then we can define the multiplication operators

$$M_g : A^I_{\{\omega\}} \to A^I_{\{\omega\}}, \qquad M_g(f) = gf \quad \text{and}$$
$$M_{g,n} : A^0_{\{\omega\},n} \to A^0_{\{\omega\},n}, \quad M_{g,n}(f) = gf.$$

If we denote by T^t_μ (resp. $T^t_{\mu,n}$) the transposed map of T_μ (resp. $T_{\mu,n}$), i.e. $T^t_\mu(\nu) = \nu \circ T_\mu$, $\nu \in \mathcal{E}_{\{\omega\}}(I)'$ (resp. $T^t_{\mu,n}(\nu) = \nu \circ T_{\mu,n}$, $\nu \in (\mathcal{E}_{\{\omega\},n})'$), then we get

$$T^t_\mu = \mathcal{F}^{-1} \circ M_{\mathcal{F}_0(\check{\mu})} \circ \mathcal{F} \quad \text{and} \quad T^t_{\mu,n} = \mathcal{F}_n^{-1} \circ M_{\mathcal{F}_0(\check{\mu})} \circ \mathcal{F}_n.$$

2. Local surjectivity of convolution operators and a sequence space representation of $\ker T_\mu$. In order to characterize the surjectivity of a convolution operator T_μ on $\mathcal{E}_{\{\omega\}}(I)$ we need some preparations. In the first part of this section we characterize the local surjectivity, i.e. the surjectivity of $T_{\mu,n}$ on $\mathcal{E}_{\{\omega\},n}$, by a new slowly decreasing condition for $\widehat{\mu}$. This new condition is equivalent to a condition of Ehrenpreis (2.1(E)), which was already used by Braun, Meise and Vogt [7] in the non-quasianalytic case. In the second part we use the slowly decreasing condition to get a sequence space representation of the kernel of T_μ. This construction uses methods of Meise [12] and modifications of those by Momm [19].

2.1. Proposition. *Let ω be a weight function, $p(z) = |\mathrm{Im}\, z| + \omega(z)$ and $F \neq 0$ a function in A_p^0. Then the following conditions are equivalent*:

(i) *For all $k \in \mathbb{N}$ there exists $x_k > 0$ such that for all $x \in \mathbb{R}$ with $|x| \geq x_k$ there exists $w \in \mathbb{C}$ such that $|w - x| \leq \omega(x)/k$ and $|F(w)| \geq \exp(-\omega(w)/k)$.*

(E) *For all $k \in \mathbb{N}$ there exists $x_k > 0$ such that for all $x \in \mathbb{R}$ with $|x| \geq x_k$ there exists $t \in \mathbb{R}$ such that $|t - x| \leq \omega(x)/k$ and $|F(t)| \geq \exp(-\omega(t)/k)$.*

(ii) *For all $k \in \mathbb{N}$ there exists $x_k > 0$ such that for all $z \in \mathbb{C}$ with $|z| \geq x_k$ there exists $w \in \mathbb{C}$ such that $|w - z| \leq (\omega(\mathrm{Re}\, z) + |\mathrm{Im}\, z|)/k$ and $|F(w)| \geq \exp(-(\omega(w) + |\mathrm{Im}\, w|)/k)$.*

(iii) *For all $k \in \mathbb{N}$ there exists $x_k > 0$ such that for all $z \in \mathbb{C}$ with $|z| \geq x_k$ there exists a circle T around z with* $\mathrm{diam}\, T \leq (\omega(\mathrm{Re}\, z) + |\mathrm{Im}\, z|)/k$ *such that for all $\zeta \in T$ we have $|F(\zeta)| \geq \exp(-(\omega(\zeta) + |\mathrm{Im}\, \zeta|)/k)$.*

(SD) *For all $k \in \mathbb{N}$ there exists $C_k > 0$ such that for all components S of $S_{\omega,k}(F) := \{z \in \mathbb{C} \mid |F(z)| < \exp(-(\omega(z) + |\mathrm{Im}\, z|)/k)\}$ we have* $\mathrm{diam}\, S \leq \inf_{z \in S}((\omega(\mathrm{Re}\, z) + |\mathrm{Im}\, z|)/k) + C_k$.

(iv) *For all $n, m \in \mathbb{N}$ there exist $k \in \mathbb{N}$ and $C_m > 0$ such that for all $f \in A^0_{\{\omega\},n}$ we have $|f|_{n,m} \leq C_m |Ff|_{n,k}$.*

By $\mathrm{diam}\, S = \sup_{z,w \in S} |z - w|$ we have denoted the diameter of S.

Proof. The proof follows the proof of Momm [18, Proposition 1], which follows the proof of Meise, Taylor and Vogt [14, 2.3] for the steps (E)⇒(ii)⇒(iii)⇒(SD). The differences here are the weight system $(k^{-1}\omega)_{k\in\mathbb{N}}$ instead of $(k\omega)_{k\in\mathbb{N}}$, different quantifiers and the need of appropriate estimates for the diameters of the components of the set $S_{\omega,k}(F)$. For technical details see [17].

2.2. Remark. In the proof of 2.1(iv)⇒(i) it is shown (see [17, 2.2], resp. Momm [18, Prop. 1]) that if the Ehrenpreis condition (E) does not hold for a function $F \in A_p^0$ (and therefore 2.1(i) does not hold) then we have the following: For each $l \in \mathbb{N}$ there exists a sequence $(f_j)_{j\in\mathbb{N}}$ in $A^0_{\{\omega\},l}$ such that:

(1) $(Ff_j)_{j\in\mathbb{N}}$ is bounded in $A^0_{\{\omega\},l}$ and therefore in $A^I_{\{\omega\}}$.

(2) $(f_j)_{j\in\mathbb{N}}$ is unbounded in $A^0_{\{\omega\},\lambda}$ for all $\lambda \in \mathbb{N}$, $\lambda \geq l$.

To use Proposition 2.1 for the investigation of convolution operators, we now give our definition of local surjectivity.

2.3. Definition. Let ω be a weight function and $\mu \in (\mathcal{E}_{\{\omega\},0})'$. Then the convolution operator T_μ is called *locally surjective* if for all $n \in \mathbb{N}$ the operator $T_{\mu,n} : \mathcal{E}_{\{\omega\},n} \to \mathcal{E}_{\{\omega\},n}$ is surjective.

2.4. Theorem. *Let ω be a weight function. Take $0 \neq \mu \in (\mathcal{E}_{\{\omega\},0})'$ and set $F = \widehat{\mu}$. Then the following conditions are equivalent*:

(i) *T_μ is locally surjective.*

(ii) *For all $n \in \mathbb{N}$, $M_{F,n} : A^0_{\{\omega\},n} \to A^0_{\{\omega\},n}$ is an injective topological homomorphism.*

(iii) *For all $n, m \in \mathbb{N}$ there exist $k \in \mathbb{N}$ and $C_m > 0$ such that for all $f \in A^0_{\{\omega\},n}$ we have $|f|_{n,m} \leq C_m |Ff|_{n,k}$.*

(E) *For all $k \in \mathbb{N}$ there exists $x_k > 0$ such that for all $x \in \mathbb{R}$ with $|x| \geq x_k$ there exists $t \in \mathbb{R}$ such that $|t - x| \leq \omega(x)/k$ and $|F(t)| \geq \exp(-\omega(t)/k)$.*

(SD) *For all $k \in \mathbb{N}$ there exists $C_k > 0$ such that for all components S of $S_{\omega,k}(F) := \{z \in \mathbb{C} \mid |F(z)| < \exp(-(\omega(z) + |\mathrm{Im}\, z|)/k)\}$ we have* $\mathrm{diam}\, S \leq \inf_{z\in S}((\omega(\mathrm{Re}\, z) + |\mathrm{Im}\, z|)/k) + C_k$.

Proof. By definition T_μ is locally surjective if and only if $T_{\mu,n}$ is surjective for all $n \in \mathbb{N}$. Consider the short exact sequence for $n \in \mathbb{N}$:

$$0 \to \ker T_{\mu,n} \to \mathcal{E}_{\{\omega\},n} \xrightarrow{T_{\mu,n}} \mathcal{E}_{\{\omega\},n} \to 0.$$

Since $\mathcal{E}_{\{\omega\},n}$ is a (DFS)-space, hence reflexive, we can apply Meise and Vogt [15, 26.4, 26.22] to the dual sequence

$$0 \to (\mathcal{E}_{\{\omega\},n})' \xrightarrow{T^t_{\mu,n}} (\mathcal{E}_{\{\omega\},n})' \to (\ker T_{\mu,n})' \to 0$$

and find that $T_{\mu,n}$ is surjective if and only if $T^t_{\mu,n}$ is an injective topological homomorphism. Fourier Laplace transformation yields the equivalence of (i) and (ii).

(ii)⇒(iii). If (ii) is satisfied then the map $M^{-1}_{F,n} : M_{F,n}(A^0_{\{\omega\},n}) \to A^0_{\{\omega\},n}$ is continuous with respect to the topology induced by $(|\ |_{n,m})_{m\in\mathbb{N}}$. This gives the estimate.

(iii)⇒(E)⇒(SD) is a part of 2.1.

(SD)⇒(ii). First we want to show that if (SD) holds, then

(∗) for all $n \in \mathbb{N}$ and $g \in A^0_{\{\omega\},n}$ with $g/F \in A(\mathbb{C})$ we have $g/F \in A^0_{\{\omega\},n}$.

This can be done as in Proposition 3 of Berenstein and Taylor [2]. The estimate for $g/F \in A^0_{\{\omega\},n}$ outside a set $S_{\omega,m}(F)$ is immediate. Inside $S_{\omega,m}(F)$ we use the maximum principle, $g/F \in A(\mathbb{C})$ and the estimates of the diameter of the components of $S_{\omega,m}(F)$ to derive $g/F \in A^0_{\{\omega\},n}$. Because $A^0_{\{\omega\},n}$ is continuously embedded in $A(\mathbb{C})$, from (∗) we deduce that $FA^0_{\{\omega\},n}$ is a closed subspace of $A^0_{\{\omega\},n}$ (for a sequence $Ff_j \in FA^0_{\{\omega\},n}$ we can take $F \cdot \lim_{j\to\infty} f_j$ (the limit of f_j in $A(\mathbb{C})$)). Since $FA^0_{\{\omega\},n}$ is an (F)-space we get the assertion by the open mapping theorem.

2.5. Definition and Remark. In the sequel we fix a weight function ω and $p : \mathbb{C} \to [0,\infty[$, $p(z) = |\mathrm{Im}\, z| + \omega(z)$. For $F \in A^0_p \subset A^I_{\{\omega\}}$ we set

$$I_{\mathrm{loc}}(F) := \{f \in A^I_{\{\omega\}} \mid f \text{ has at least the zeros of } F \text{ with multiplicities}\}.$$

Then $I_{\mathrm{loc}}(F)$ is a closed subspace of $A^I_{\{\omega\}}$ and $FA^I_{\{\omega\}} \subset I_{\mathrm{loc}}(F)$. If $F \in A^0_p$ is slowly decreasing (i.e. F satisfies 2.1(SD)), then $FA^I_{\{\omega\}}$ is a closed subspace of $A^I_{\{\omega\}}$ and $I_{\mathrm{loc}}(F) = FA^I_{\{\omega\}}$. This can be seen as in 2.4(SD)⇒(ii).

We restrict the following considerations to the case $I =]-1,1[$. First we give a variation of a theorem of Berenstein and Taylor [2].

2.6. Theorem. *Let $0 \neq F \in A^0_p$ and $q : \mathbb{C} \to \mathbb{R}^+$ be an upper semicontinuous function which satisfies the following conditions*:

(1) $q(z) = o(p(z))$, *i.e.* $\forall \varepsilon > 0\ \exists D_\varepsilon > 0\ \forall z \in \mathbb{C} : q(z) \le \varepsilon p(z) + D_\varepsilon$,

(2) $\forall z \in \mathbb{C} : \quad |F(z)| \le \exp(q(z))$.

For all $n \in \mathbb{N}_0$ define

$$q_n(z) := (n+1) \sup_{|z-w|\le n} q(w) + n \log(1+|z|^2)$$

and

$$S_{q_n}(F,C) := \left\{ z \in \mathbb{C} \,\middle|\, |F(z)| < \frac{1}{C}\exp(-q_n(z)) \right\}.$$

Then for every function $\widetilde{f} \in A(S_{q_0}(F,1))$ with

(3) $$\exists 0 < \eta < 1\ \forall m \in \mathbb{N}\ \exists D_m > 0\ \forall z \in S_{q_0}(F,1) : \quad |\widetilde{f}(z)| \le D_m \exp\left(\eta|\mathrm{Im}\, z| + \frac{1}{m}\omega(z)\right)$$

there exist numbers $k_1 \in \mathbb{N}$, $C_1 \ge 1$ and functions $f \in A^I_{\{\omega\}}$ and $\alpha \in A(S_{q_{k_1}}(F,C_1))$ such that

(4) $$\forall z \in S_{q_{k_1}}(F,C_1) : \quad f(z) = \widetilde{f}(z) + \alpha(z)F(z).$$

Proof. Since p satisfies (α) and (γ), from (1) we get $q_k = o(p)$ for all $k \in \mathbb{N}$. Now let $\widetilde{f} \in A(S_{q_0}(F,1))$ satisfy (3). As in Momm [19, Proposition 6] (see also Momm [20, 2.4]) there exist numbers $k_1 \in \mathbb{N}$ and $C_1 \ge 1$ such that

(5) $$\forall z \in S_{q_{k_1}}(F,C_1) : \quad \mathrm{dist}(z, \mathbb{C} \setminus S_{q_0}(F,1)) \ge \frac{1}{C_1}\exp(-q_{k_1}(z)).$$

Following Berenstein und Taylor [2, page 120] we get the existence of a function $\chi \in C^\infty(\mathbb{C})$ with $0 \le \chi \le 1$, $\mathrm{supp}(\chi) \subset S_{q_0}(F,1)$, $\chi \equiv 1$ on $S_{q_{k_1}}(F,C_1)$ as well as numbers $k_2 \in \mathbb{N}$ and $C_2 \ge 1$ with

(6) $$|\overline{\partial}\chi| \le C_2 \exp(q_{k_2}).$$

Since $\overline{\partial}(\chi\widetilde{f}) = \widetilde{f}\,\overline{\partial}\chi$ on $S_{q_{k_1}}(F,C_1)$ equals zero, the function

$$v(z) := -\frac{1}{F(z)}\overline{\partial}(\chi\widetilde{f})(z)$$

lies in $C^\infty(\mathbb{C})$ and $\mathrm{supp}(v) \subset S_{q_0}(F,1) \setminus S_{q_{k_1}}(F,C_1)$. By definition we get

(7) $$\frac{1}{|F(z)|} \le C_1 \exp(q_{k_1}(z)) \quad \text{for } z \in \mathbb{C} \setminus S_{q_{k_1}}(F,C_1).$$

Now from (3), (6) and (7) it follows that for all $m \in \mathbb{N}$ and $z \in \mathbb{C}$,

$$|v(z)| \le C_1 D_m C_2 \exp\left(q_{k_1}(z) + \eta|\mathrm{Im}\, z| + \frac{1}{m}\omega(z) + q_{k_2}(z)\right)$$
$$\le D'_m \exp\left(\frac{1}{m}p(z) + \eta|\mathrm{Im}\, z| + \frac{1}{m}\omega(z)\right)$$

for suitably chosen $D'_m > 0$ according to $q_{k_1}, q_{k_2} = o(p)$. Hence

$$\exists \widetilde{\eta}, \eta < \widetilde{\eta} < 1 \ \forall m \in \mathbb{N} \ \exists D''_m > 0 \ \forall z \in \mathbb{C}:$$

$$|v(z)| \le D''_m \exp\Big(\widetilde{\eta}|\operatorname{Im} z| + \frac{2}{m}\omega(z)\Big).$$

Since $\log(1+|t|) = o(\omega(t))$ for all $m \in \mathbb{N}$ there exists $E_m > 0$ with

$$\int_{\mathbb{C}} \Big(|v(z)| \exp\Big(-\widetilde{\eta}|\operatorname{Im} z| - \frac{1}{m}\omega(z)\Big)\Big)^2 \, d\lambda(z) \le E_m,$$

where $d\lambda$ denotes the Lebesgue measure on $\mathbb{C}$. Now we can apply Meise [12, 2.4] and the hypoellipticity of $\overline{\partial}$ to get a function $u \in C^\infty(\mathbb{C})$ with $\overline{\partial} u = v$ such that for all $m \in \mathbb{N}$ there exists $E'_m > 0$ with

$$\text{(8)} \qquad \int_{\mathbb{C}} \Big(|u(z)| \exp\Big(-\widetilde{\eta}|\operatorname{Im} z| - \frac{1}{m}\omega(z)\Big)\Big)^2 \, d\lambda(z) \le E'_m.$$

If we define $f = \chi\widetilde{f} + uF$, then we have $\overline{\partial} f = 0$, i.e. $f \in A(\mathbb{C})$. Taking $\alpha(z) = u(z)$ for $z \in S_{q_{k_1}}(F, C_1)$, we get $\overline{\partial}\alpha(z) = v(z) = 0$ for $z \in S_{q_{k_1}}(F, C_1)$, hence $\alpha \in A(S_{q_{k_1}}(F, C_1))$ and the desired representation $f = \widetilde{f} + \alpha F$ on $S_{q_{k_1}}(F, C_1)$ holds. From (8), (3) and $F \in A_p^0$ we conclude that $f = \chi\widetilde{f} + uF \in A^I_{\{\omega\}}$.

Now, for every slowly decreasing function $F \in A_p^0$, we have to construct an upper semicontinuous function $q : \mathbb{C} \to \mathbb{R}_+$ satisfying the assumptions of Theorem 2.6. This is done in the following lemma.

2.7. Lemma. *Let $0 \ne F \in A_p^0$ be slowly decreasing. Then there exists an upper semicontinuous function $q : \mathbb{C} \to \mathbb{R}_+$ with the following properties*:

(i) *for all $z \in \mathbb{C}$, $|F(z)| \le \exp(q(z))$,*
(ii) $q = o(p)$,
(iii) *for all $k \in \mathbb{N}$ there exists $C_k > 0$ such that for all components $\widetilde{S}$ of $S_q(F, 1) = \{z \in \mathbb{C} \mid |F(z)| < \exp(-q(z))\}$ we have*

$$\operatorname{diam} \widetilde{S} \le \frac{1}{k} \inf_{z \in \widetilde{S}} (|\operatorname{Im} z| + \omega(\operatorname{Re} z)) + C_k.$$

Proof. By assumption for all $k \in \mathbb{N}$ there exists $C_k > 0$ such that for all components $\widetilde{S}$ of

$$S_k := S_{p/k}(F, 1) = \Big\{ z \in \mathbb{C} \,\Big|\, |F(z)| < \exp\Big(-\frac{1}{k}p(z)\Big)\Big\}$$

we have

$$\operatorname{diam} \widetilde{S} \le \frac{1}{k} \inf_{z \in \widetilde{S}} (|\operatorname{Im} z| + \omega(\operatorname{Re} z)) + C_k.$$

Since $S_k \subset S_l$ for $l \ge k$, there exists a sequence $(C_k)_{k\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ such that for all $k \in \mathbb{N}$ and all components $\widetilde{S}$ of S_k and all $l \ge k$ we have

$$\operatorname{diam} \widetilde{S} \le \frac{1}{l} \inf_{z \in \widetilde{S}} (|\operatorname{Im} z| + \omega(\operatorname{Re} z)) + C_l.$$

Applying this to $k \in \mathbb{N}$ and $l = 2k$, we see that there exists $R_k > 0$ such that for all components $\widetilde{S}$ of S_k lying outside $B_{R_k}(0) = \{z \in \mathbb{C} \mid |z| \le R_k\}$ we have

$$\text{(1)} \qquad \operatorname{diam} \widetilde{S} \le \frac{1}{k} \inf_{z \in \widetilde{S}} (|\operatorname{Im} z| + \omega(\operatorname{Re} z)).$$

Without loss of generality we choose R_k strictly increasing with $\lim_{k\to\infty} R_k = \infty$. Now we label the components of S_k outside $B_{R_k}(0)$ as $(S_{k,j})_{j\in\mathbb{N}}$ in such a way that $\sup\{|z| \mid z \in S_{k,j}\}$ is increasing with $j \in \mathbb{N}$. Then we define

$$S := \bigcup_{j,k\in\mathbb{N}} S_{k,j} \cup B_{R_0}(0)$$

where $R_0 = R_1 + C_1 + \sup_{z\in B_{R_1}(0)}(|\operatorname{Im} z| + \omega(\operatorname{Re} z))$. This assures that the zero set $V(F)$ of F lies in S, because the components $\widetilde{S}$ of S_1 not lying in $\bigcup_{j\in\mathbb{N}} S_{1,j}$ must lie in $B_R(0)$. We get immediately: For all $k \in \mathbb{N}$ there exists $R_k > 0$ such that for all components $\widetilde{S}$ of S outside $B_{R_k}(0)$ we have $\operatorname{diam} \widetilde{S} \le k^{-1} \inf_{z\in\widetilde{S}}(|\operatorname{Im} z| + \omega(\operatorname{Re} z))$, or equivalently: For all $k \in \mathbb{N}$ there exists $C_k > 0$ such that for all components $\widetilde{S}$ of S we have

$$\text{(2)} \qquad \operatorname{diam} \widetilde{S} \le \frac{1}{k} \inf_{z \in \widetilde{S}} (|\operatorname{Im} z| + \omega(\operatorname{Re} z)) + C_k.$$

Since $S_k \setminus B_{R_k}(0) \subset S \setminus B_{R_k}(0)$ it follows that if $z \in \mathbb{C} \setminus (B_{R_k}(0) \cup S)$ then $z \in \mathbb{C} \setminus (B_{R_k}(0) \cup S_k)$ and $|F(z)| \ge \exp(-p(z)/k)$, i.e.

$$\text{(3)} \qquad \forall k \in \mathbb{N} \ \forall z \in \mathbb{C} \setminus (B_{R_k}(0) \cup S): \quad |F(z)| \ge \exp\Big(-\frac{1}{k}p(z)\Big).$$

Now we can define $q : \mathbb{C} \to \mathbb{R}_+$ by

$$q(z) := \max(\ln|F(z)|, \chi_{\mathbb{C}\setminus S}(z)\ln(|F(z)|^{-1}), 0),$$

where $\chi_{\mathbb{C}\setminus S}$ denotes the characteristic function of $\mathbb{C} \setminus S$. As $V(F) \subset S$ the function $\chi_{\mathbb{C}\setminus S}(z)\ln(|F(z)|^{-1})$ admits only finite values. Moreover, $\chi_{\mathbb{C}\setminus S}$ and therefore q are upper semicontinuous, since $\mathbb{C} \setminus S$ is a closed set. Now condition (i) follows by definition. Condition (ii) follows from $F \in A_p^0$, i.e. $\ln|F(z)| = o(p)$, and (3), i.e. $\chi_{\mathbb{C}\setminus S} \ln|F(z)|^{-1} = o(p)$. To get (iii) it suffices to show $S_q(F, 1) \subset S$. To do this, let $z \notin S$ be given. Then $q(z) \ge \ln(|F(z)|^{-1})$, i.e. $|F(z)| \ge \exp(-q(z))$, hence $z \notin S_q(F, 1)$.

Now we are able to give a sequence space representation of $A^I_{\{\omega\}}$ ($I =]-1,1[$) modulo a closed subspace $FA^I_{\{\omega\}}$. For this purpose let $0 \neq F \in A^0_p$ be slowly decreasing, $q : \mathbb{C} \to \mathbb{R}_+$ as in 2.7 and $\operatorname{card} V(F) = \infty$. Let $(S_j)_{j\in\mathbb{N}}$ be the components S of $S_q(F,1)$ with $S \cap V(F) \neq \emptyset$ arranged that so $\beta := (\sup_{z\in S_j} \omega(z))_{j\in\mathbb{N}}$ is an increasing sequence. We define $\alpha := (\sup_{z\in S_j} |\operatorname{Im} z|)_{j\in\mathbb{N}}$ and denote by m_a the multiplicity of a zero a of F. The finite-dimensional $\mathbb{C}$-vector spaces E_j are defined by

$$E_j := \prod_{a\in S_j\cap V(F)} \mathbb{C}^{m_a}$$

for every $j \in \mathbb{N}$. We consider the linear surjective mappings

$$\varrho_j : A^\infty(S_j) \to E_j, \quad g \mapsto ((g^{(j)}(a))_{j=0}^{m_a-1})_{a\in S_j\cap V(F)},$$
$$A^\infty(S_j) := \{f \in A(S_j) \mid \|f\|_{A^\infty(S_j)} = \sup_{z\in S_j} |f(z)| < \infty\},$$

and define the quotient norm $\|\ \|_j : E_j \to \mathbb{R}_+$ by $\|x\|_j := \inf\{\|g\|_{A^\infty(S_j)} \mid g \in A^\infty(S_j),\ \varrho_j(g) = x\}$. We set $\mathbb{E} = (E_j, \|\ \|_j)_{j\in\mathbb{N}}$. Finally, for a sequence $(a_n)_{n\in\mathbb{N}}$ as in 1.4 we define

$$K^I(\alpha,\beta,\mathbb{E}) := \Big\{x = (x_j)_{j\in\mathbb{N}} \in \prod_{j\in\mathbb{N}} E_j \ \Big|\ \exists n \in \mathbb{N}\ \forall m \in \mathbb{N} :$$
$$|||x|||_{n,m} := \sup_{j\in\mathbb{N}} \|x_j\|_j \exp\Big(-a_n\alpha_j - \frac{1}{m}\beta_j\Big) < \infty\Big\},$$

and endow this space with the natural (LF)-space topology by taking the inductive limit over $n \in \mathbb{N}$ of the projective limits over $m \in \mathbb{N}$. With $K_n(\alpha,\beta,\mathbb{E}) := \{x \in \prod_{j\in\mathbb{N}} E_j \mid \forall m \in \mathbb{N} : |||x|||_{n,m} < \infty\}$ we get $K^I(\alpha,\beta,\mathbb{E}) = \operatorname{ind}_{n\to} K_n(\alpha,\beta,\mathbb{E})$.

2.8. THEOREM. *Let $0 \neq F \in A^0_p$ be slowly decreasing, $q : \mathbb{C} \to \mathbb{R}_+$ be as in 2.7 and $\operatorname{card} V(F) = \infty$. Then, with the notations above, the map $\varrho : A^I_{\{\omega\}} \to K^I(\alpha,\beta,\mathbb{E})$, $f \mapsto (\varrho_j(f|_{S_j}))_{j\in\mathbb{N}}$, induces a linear topological isomorphism $\widetilde{\varrho}$ between*

$$A^I_{\{\omega\}}/I_{\mathrm{loc}}(F) \quad \text{and} \quad K^I(\alpha,\beta,\mathbb{E}).$$

Proof. For $f \in A^I_{\{\omega\}}$ there is $0 < \eta < 1$ such that for all $m \in \mathbb{N}$ there exists $C_m > 0$ with

$$|f(z)| \le C_m \exp\Big(\eta|\operatorname{Im} z| + \frac{1}{m}\omega(z)\Big) \quad \text{for } z \in \mathbb{C}.$$

Hence for all $j \in \mathbb{N}$,

$$\|\varrho_j(f|_{S_j})\|_j \le \|f|_{S_j}\|_{A^\infty(S_j)} \le C_m \exp\Big(\eta \sup_{z\in S_j} |\operatorname{Im} z| + \frac{1}{m} \sup_{z\in S_j} \omega(z)\Big)$$
$$= C_m \exp\Big(\eta\alpha_j + \frac{1}{m}\beta_j\Big).$$

From this we get the existence and continuity of the linear map ϱ.

To show the surjectivity of ϱ, let $x = (x_j)_{j\in\mathbb{N}} \in K^I(\alpha,\beta,\mathbb{E})$ be given. Since ϱ_j is surjective there exists a function $f_j \in A^\infty(S_j)$ with $\varrho_j(f_j) = x_j$ and $\|f_j\|_{A^\infty(S_j)} \le 2\|x_j\|_j$. For $x \in K^I(\alpha,\beta,\mathbb{E})$ there is $0 < \eta < 1$ such that for all $m \in \mathbb{N}$ there exists $D_m > 0$ with

$$\sup_{j\in\mathbb{N}} \|x_j\|_j \exp\Big(-\eta\alpha_j - \frac{1}{m}\beta_j\Big) \le D_m. \tag{4}$$

If we define

$$\widetilde{f} : S_q(F,1) \to \mathbb{C} \quad \text{by} \quad \widetilde{f}(z) := \begin{cases} f_j(z) & \text{for } z \in S_j \\ 0 & \text{for } z \in S_q(F,1) \setminus \bigcup_{j\in\mathbb{N}} S_j, \end{cases}$$

then $\widetilde{f} \in A(S_q(F,1))$. For $\eta < \widetilde{\eta} < 1$ and $m' \in \mathbb{N}$ we choose $l \in \mathbb{N}$ with $\eta(1+1/l) < \widetilde{\eta}$ and $\eta/l < 1/(2m')$. Taking $C_l > 0$ from condition 2.7(iii), for all $j \in \mathbb{N}$ and $z \in S_j$ we get

$$\sup_{w\in S_j} |\operatorname{Im} w| \le \inf_{w\in S_j} |\operatorname{Im} w| + \operatorname{diam} S_j$$
$$\le |\operatorname{Im} z| + \frac{1}{l}|\operatorname{Im} z| + \frac{1}{l}\omega(z) + C_l$$
$$\le \frac{\widetilde{\eta}}{\eta}|\operatorname{Im} z| + \frac{1}{2m'\eta}\omega(z) + C_l,$$

and

$$\sup_{w\in S_j} \omega(w) \le \inf_{w\in S_j} \omega(|w| + \operatorname{diam} S_j) \le \omega\Big(|z| + \frac{1}{l}(\omega(z) + |\operatorname{Im} z|) + C_l\Big)$$
$$\le K\omega(z) + K,$$

for some $K \ge 1$ depending on $l \in \mathbb{N}$. If we choose $m \in \mathbb{N}$ with $K/m < 1/(2m')$ then for all $j \in \mathbb{N}$ and $z \in S_j$ we get

$$|\widetilde{f}(z)| = |f_j(z)| \le \|f_j\|_{A^\infty(S_j)} \le 2\|x_j\|_j \le 2D_m \exp\Big(\eta\alpha_j + \frac{1}{m}\beta_j\Big)$$
$$= 2D_m \exp\Big(\eta \sup_{w\in S_j} |\operatorname{Im} w| + \frac{1}{m} \sup_{w\in S_j} \omega(w)\Big)$$
$$\le 2D_m \exp\Big(\eta\Big(\frac{\widetilde{\eta}}{\eta}|\operatorname{Im} z| + \frac{1}{2m'\eta}\omega(z) + C_l\Big) + \frac{1}{m}(K\omega(z) + K)\Big)$$
$$= 2D_m \exp\Big(\eta C_l + \frac{1}{2m'}\Big) \exp\Big(\widetilde{\eta}|\operatorname{Im} z| + \frac{1}{m'}\omega(z)\Big).$$

Since $\widetilde{f}$ satisfies the assumption of 2.6, there exist numbers $n \in \mathbb{N}$, $K_1 \geq 1$ and functions $f \in A^I_{\{\omega\}}$ and $\alpha \in A(S_{q_n}(F, K_1))$ with

$$f(z) = \widetilde{f}(z) + \alpha F \quad \text{for all } z \in S_{q_n}(F, K_1).$$

Obviously, $\varrho(f|_{S_q(F,1)}) = \varrho(\widetilde{f}) = x$, i.e. ϱ is surjective and by the open mapping theorem for (LF)-spaces even topological. Since $\ker \varrho = I_{\text{loc}}(F)$, we get the assertion.

2.9. REMARK. For further considerations in Section 3, we need the following result for the spaces $K_n(\alpha, \beta, \mathbb{E})$ and $A^0_{\{\omega\},n}$. If $0 \neq F \in A^0_p$ is slowly decreasing, $q : \mathbb{C} \to \mathbb{R}_+$ as in 2.7 and $\operatorname{card} V(F) = \infty$, then for all $n \in \mathbb{N}$,

$$\sigma_n : A^0_{\{\omega\},n} \to K_n(\alpha, \beta, \mathbb{E}), \quad \sigma_n(f) = (\varrho_j(f|_{S_j}))_{j\in\mathbb{N}},$$

is a continuous linear mapping with $\ker \sigma_n = I_{\text{loc}}(F) = FA^0_{\{\omega\},n}$ ($I_{\text{loc}}(F)$ in $A^0_{\{\omega\},n}$). In particular, we have the linear, continuous and injective mapping

$$\widetilde{\sigma}_n : A^0_{\{\omega\},n}/FA^0_{\{\omega\},n} \to K_n(\alpha, \beta, \mathbb{E}), \quad \widetilde{\sigma}_n(f + FA^0_{\{\omega\},n}) = \sigma_n(f).$$

This follows directly from the proof of 2.8.

Now we transfer the sequence space representation via Fourier Laplace transformation and duality theory to the kernel of a convolution operator T_μ with slowly decreasing $\widehat{\mu}$.

2.10. DEFINITION. Let $(\alpha_j)_{j\in\mathbb{N}}, (\beta_j)_{j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}_+$ and $\mathbb{E} = (E_j, \|\ \|_j)_{j\in\mathbb{N}}$ be a sequence of Banach spaces. For a sequence $(a_n)_{n\in\mathbb{N}}$ as in 1.4 we define

$$\lambda^I(\alpha, \beta, \mathbb{E}) := \Big\{ x = (x_j)_{j\in\mathbb{N}} \in \prod_{j\in\mathbb{N}} E_j \ \Big|\ \forall n \in \mathbb{N}\ \exists m \in \mathbb{N} : \|x\|_{n,m} = \sum_{j=1}^{\infty} \|x_j\|_j \exp\Big(a_n \alpha_j + \frac{1}{m}\beta_j\Big) < \infty \Big\}.$$

If $\mathbb{E} = (\mathbb{C}, |\ |)_{j\in\mathbb{N}}$ we will write $\lambda^I(\alpha, \beta)$ instead of $\lambda^I(\alpha, \beta, \mathbb{E})$. If in addition $\lim_{j\to\infty} \beta_j = \infty$ and $\dim E_j < \infty$ for all $j \in \mathbb{N}$, then we see, as in Meise [12, Proposition 1.6], that $\lambda^I(\alpha, \beta, \mathbb{E})$ is a complete Schwartz space and that $\lambda^I(\alpha, \beta, \mathbb{E})'_{\text{b}}$ is linear topologically isomorphic to $K^I(\alpha, \beta, \mathbb{E}')$ ($\mathbb{E}' = (E'_j, \|\ \|'_j)_{j\in\mathbb{N}}$), via the canonical bilinear form

$$\langle x, y\rangle := \sum_{j\in\mathbb{N}} \langle x_j, y_j\rangle_j, \quad x = (x_j)_{j\in\mathbb{N}} \in \prod_{j\in\mathbb{N}} E_j,\ y = (y_j)_{j\in\mathbb{N}} \in \prod_{j\in\mathbb{N}} E'_j.$$

2.11. THEOREM. *Let $\mu \in (\mathcal{E}_{\{\omega\},0})'$, $\widehat{\mu} \in A^0_p$ be slowly decreasing and let $T_\mu : \mathcal{E}_{\{\omega\}}(I) \to \mathcal{E}_{\{\omega\}}(I)$ ($I = \mathbb{R}$ or $I =]-1,1[$) with $\dim(\ker T_\mu) = \infty$. If $\gamma = (\gamma_j)_{j\in\mathbb{N}}$, $\gamma_j = |\operatorname{Im} a_j|$, and $\delta = (\delta_j)_{j\in\mathbb{N}}$, $\delta_j = \omega(a_j)$, where $(a_j)_{j\in\mathbb{N}}$ counts the zeros of $\widehat{\mu}$ with multiplicities, then $\ker T_\mu$ is linear topologically isomorphic to $\lambda^I(\gamma, \delta)$.*

Proof. *Case $I =]-1,1[$.* Since $\widehat{\mu}$ is slowly decreasing, so is $\widehat{\check{\mu}}$ and we have

$$\overline{\widehat{\check{\mu}} A^I_{\{\omega\}}} = \widehat{\check{\mu}} A^I_{\{\omega\}} = I_{\text{loc}}(\widehat{\check{\mu}}).$$

If we consider the transposed map $j^t : \mathcal{E}_{\{\omega\}}(I)'_{\text{b}} \to (\ker T_\mu)'_{\text{b}}$ of the injection $j : \ker T_\mu \to \mathcal{E}_{\{\omega\}}(I)$ and apply the Fourier Laplace transformation, duality theory and 1.7, we get a linear topological isomorphism between $(\ker T_\mu)'_{\text{b}}$ and $A^I_{\{\omega\}}/\overline{\widehat{\check{\mu}} A^I_{\{\omega\}}}$. With 2.8, 2.10 and the notations after 2.7 we get

$$(\ker T_\mu)'_{\text{b}} \cong A^I_{\{\omega\}}/\overline{\widehat{\check{\mu}} A^I_{\{\omega\}}} \cong A^I_{\{\omega\}}/I_{\text{loc}}(\widehat{\check{\mu}}) \cong K^I(\alpha, \beta, \mathbb{E}) \cong \lambda^I(\alpha, \beta, \mathbb{E}')'_{\text{b}}.$$

$K^I(\alpha, \beta, \mathbb{E})$ does not change if we switch from $\widehat{\mu}$ to $\widehat{\check{\mu}}$.

Now $\ker T_\mu$ and $\lambda^I(\alpha, \beta, \mathbb{E}')$ are complete Schwartz spaces, hence semireflexive. Therefore the transposed map of the above isomorphism is a linear bijection between $\ker T_\mu$ and $\lambda^I(\alpha, \beta, \mathbb{E}')$. As in Meise [12, 2.7, 3.4] one can show that the equicontinuous sets in $\lambda^I(\alpha, \beta, \mathbb{E}')'_{\text{b}}$ and $(\ker T_\mu)'_{\text{b}}$ coincide. Hence the bijection is a linear topological isomorphism. Since $\ker T_\mu$ is a closed subspace of a nuclear space, $\ker T_\mu$ and $\lambda^I(\alpha, \beta; \mathbb{E}')$ are also nuclear. As in the proof of Meise [13, 1.3(1)⇒(3)] the nuclearity implies

$$\forall \varepsilon > 0 : \quad \sum_{j=1}^{\infty} \dim E_j \exp(-\varepsilon(\alpha_j + \beta_j)) < \infty.$$

A slight modification of Meise [12, 1.7] yields an isomorphism

$$\lambda^I(\alpha, \beta, \mathbb{E}') \cong \lambda^I(\widetilde{\gamma}, \widetilde{\delta}),$$

where the sequence $\widetilde{\gamma}$ (resp. $\widetilde{\delta}$) is obtained from α (resp. β) by repeating α_j (resp. β_j) $\dim E'_j = \dim E_j$ times. Since the dimension of E_j equals the number of zeros of $\widehat{\mu}$ in S_j with multiplicity, and because of 2.7(iii), we may replace $\widetilde{\gamma}$ and $\widetilde{\delta}$ by γ and δ. For an index k with $\widetilde{\gamma}_j = \alpha_k$ we have $a_j \in S_k$. Therefore

$$\alpha_j = \sup_{z\in S_j} |\operatorname{Im} z|, \quad \beta_j = \sup_{z\in S_j} \omega(z) \quad (S_j \text{ as in } 2.8)$$

yields $\gamma_j \leq \widetilde{\gamma}_j$ und $\delta_j \leq \widetilde{\delta}_j$ for all $j \in \mathbb{N}$. From the proof of 2.8 we get the existence of $B > 0$ such that for all $j \in \mathbb{N}$,

$$(1) \qquad \sup_{z\in S_j} \omega(z) \leq B(1 + \inf_{z\in S_j} \omega(z)), \quad \text{thus} \quad \widetilde{\delta}_j \leq B(1 + \delta_j).$$

Now, 2.7(iii) implies

$$\forall \varepsilon > 0\ \exists C_\varepsilon > 0\ \forall k \in \mathbb{N} : \quad \operatorname{diam} S_k \leq \varepsilon(\inf_{z\in S_k} |\operatorname{Im} z| + \omega(\operatorname{Re} z)) + C_\varepsilon,$$

and

(2) $\forall \varepsilon > 0\ \exists C_\varepsilon > 0\ \forall j \in \mathbb{N}\ \exists k \in \mathbb{N}:$

$$\widetilde{\gamma}_j = \sup_{z \in S_k} |\operatorname{Im} z| \le \inf_{z \in S_k} |\operatorname{Im} z| + \operatorname{diam} S_k$$
$$\le \gamma_j + \varepsilon(\inf_{z \in S_k} |\operatorname{Im} z| + \omega(\operatorname{Re} z)) + C_\varepsilon \le \gamma_j + \varepsilon\gamma_j + \varepsilon\delta_j + C_\varepsilon.$$

If we denote by $\| \ \|'_{k,m}$ a norm in $\lambda^I(\widetilde{\gamma}, \widetilde{\delta})$, then (1) and (2) implies

$$\forall k_2 \in \mathbb{N}\ \exists k_1 \in \mathbb{N}\ \forall m_1 \in \mathbb{N}\ \exists m_2 \in \mathbb{N}\ \forall x \in \lambda^I(\gamma, \delta):$$
$$\|x\|'_{k_2,m_2} \le \exp\Big(a_{k_2} C_\varepsilon + \frac{1}{2m_1}\Big)\|x\|_{k_1,m_1}.$$

Since the reverse estimate is obvious, we get the isomorphism

$$\ker T_\mu \cong \lambda^I(\widetilde{\gamma}, \widetilde{\delta}) \cong \lambda^I(\gamma, \delta).$$

Case $I = \mathbb{R}$. The proof is analogous to the case $I =]-1, 1[$, because the only difference is the absence of the restriction $0 < \eta < 1$ in 2.6 and 2.8. In the non-quasianalytic case this has already been proved by Meise [12, Theorem 3.6].

3. Surjectivity of convolution operators. In the previous section we characterized the local surjectivity of a convolution operator T_μ by a slowly decreasing condition for the Fourier Laplace transformation $\widehat{\mu}$ of μ. To turn to global surjectivity one can use the following idea. Take any $f \in \mathcal{E}_{\{\omega\}}(I)$ and consider functions $g_n \in \mathcal{E}_{\{\omega\},n}$ with $T_{\mu,n}(g_n) = f|_{K_n}$, which exist if T_μ is locally surjective. Then change g_n by adding a function from $\ker T_{\mu,n}$ such that the new sequence $\widetilde{g}_n \in g_n + \ker T_{\mu,n}$ converges to $g \in \mathcal{E}_{\{\omega\}}(I)$ with $T_\mu(g) = f$. As a requirement for this construction the projective spectrum of the kernels of $T_{\mu,n}$ has to satisfy a certain condition.

The above construction will be used in an abstract setup, based on the works of Palamodov [22] and Vogt [26], which was already used by Braun, Meise and Vogt [7] to characterize surjectivity of convolution operators on non-quasianalytic classes $\mathcal{E}_{\{\omega\}}(\mathbb{R})$. To apply these methods we need some preparations on projective spectra of linear spaces. We follow the approach of Vogt [26].

3.1. *Projective spectra.* (i) A sequence $\mathcal{X} = (X_n, \iota^n_{n+1})_{n\in\mathbb{N}}$ of vector spaces X_n and linear mappings $\iota^n_{n+1} : X_{n+1} \to X_n$ is called a *projective spectrum*. We set $\iota^n_n = \mathrm{id}_{X_n}$ and $\iota^n_m = \iota^n_{n+1} \circ \ldots \circ \iota^{m-1}_m$ for $m > n+1$.

(ii) For a projective spectrum $\mathcal{X} = (X_n, \iota^n_{n+1})_{n\in\mathbb{N}}$ we define the vector spaces

$$\operatorname{proj}^0 \mathcal{X} := \Big\{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n \,\Big|\, \forall n \in \mathbb{N} : \iota^n_{n+1}(x_{n+1}) = x_n\Big\},$$

$$B(\mathcal{X}) := \Big\{(a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n \,\Big|\, \exists (b_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n : \forall n \in \mathbb{N} : a_n = \iota^n_{n+1}(b_{n+1}) - b_n\Big\}$$

and

$$\operatorname{proj}^1 \mathcal{X} := \Big(\prod_{n\in\mathbb{N}} X_n\Big)/B(\mathcal{X}).$$

(iii) A *map* $\Phi : \mathcal{X} \to \mathcal{Y}$ between projective spectra $\mathcal{X} = (X_n, \iota^n_{n+1})_{n\in\mathbb{N}}$ and $\mathcal{Y} = (Y_n, j^n_{n+1})_{n\in\mathbb{N}}$ is a sequence $\varphi^n_{k(n)} : X_{k(n)} \to Y_n$ of linear maps with $k(n) \le k(n+1)$ and $\varphi^n_{k(n)} \circ \iota^{k(n)}_{k(n+1)} = j^n_{n+1} \circ \varphi^{n+1}_{k(n+1)}$ for all $n \in \mathbb{N}$. For $m \ge k(n)$ we set $\varphi^n_m = \varphi^n_{k(n)} \circ \iota^{k(n)}_m$.

(iv) Let $\Phi : \mathcal{X} \to \mathcal{Y}$, $\Phi = (\varphi^n_{k(n)})_{n\in\mathbb{N}}$, and $\Psi : \mathcal{Y} \to \mathcal{Z}$, $\Psi = (\psi^n_{l(n)})_{n\in\mathbb{N}}$, be two maps between projective spectra $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$. Then we define the composition $\Psi \circ \Phi : \mathcal{X} \to \mathcal{Z}$ by

$$\Psi \circ \Phi = (\chi^n_{k(l(n))})_{n\in\mathbb{N}}, \quad \chi^n_{k(l(n))} = \psi^n_{l(n)} \circ \varphi^{l(n)}_{k(l(n))}.$$

(v) Two maps $\Phi : \mathcal{X} \to \mathcal{Y}$, $\Phi = (\varphi^n_{k(n)})_{n\in\mathbb{N}}$, and $\Psi : \mathcal{X} \to \mathcal{Y}$, $\Psi = (\psi^n_{l(n)})_{n\in\mathbb{N}}$, are called *equivalent* if

$$\forall n \in \mathbb{N}\ \exists m(n) \ge \max(k(n), l(n)) : \quad \varphi^n_{m(n)} = \psi^n_{m(n)}.$$

(vi) Two projective spectra $\mathcal{X}$ and $\mathcal{Y}$ are called *equivalent* if there exist maps $\Phi : \mathcal{X} \to \mathcal{Y}$ and $\Psi : \mathcal{Y} \to \mathcal{X}$ such that $\Phi \circ \Psi$ and $\mathrm{id}_\mathcal{Y}$, and respectively $\Psi \circ \Phi$ and $\mathrm{id}_\mathcal{X}$, are equivalent. By $\mathrm{id}_\mathcal{X}$ and $\mathrm{id}_\mathcal{Y}$ we denote the identity maps $(\iota^n_n)_{n\in\mathbb{N}}$ and $(j^n_n)_{n\in\mathbb{N}}$.

(vii) Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \Phi$ and Ψ be as in (iv). The sequence

$$\mathcal{X} \xrightarrow{\Phi} \mathcal{Y} \xrightarrow{\Psi} \mathcal{Z}$$

is called *exact* in $\mathcal{Y}$ if the following two conditions are satisfied:

(1) $\Psi \circ \Phi$ is equivalent to the zero map.

(2) $\forall n \in \mathbb{N}, N \ge k(n)\ \exists \mu \in \mathbb{N}, m \ge \max(n, l(\mu))$, with $\operatorname{im} \varphi^n_N \supset j^n_m \ker \psi^\mu_m$.

(viii) In the special case $\Phi : \mathcal{X} \to \mathcal{Y}$, $\Phi = (\varphi^n_n)_{n\in\mathbb{N}}$, and $\Psi : \mathcal{Y} \to \mathcal{Z}$, $\Psi = (\psi^n_n)_{n\in\mathbb{N}}$, where all the sequences

$$0 \to X_n \xrightarrow{\varphi^n_n} Y_n \xrightarrow{\psi^n_n} Z_n \to 0$$

are exact, we get a *short exact sequence of projective spectra*

$$0 \to \mathcal{X} \xrightarrow{\Phi} \mathcal{Y} \xrightarrow{\Psi} \mathcal{Z} \to 0.$$

3.2. Definitions and Notations. (i) The spaces $\mathcal{E}_{\{\omega\},n}$ together with the maps

$$\pi_{n+1}^n : \mathcal{E}_{\{\omega\},n+1} \to \mathcal{E}_{\{\omega\},n}, \quad \pi_{n+1}^n(f) = f|_{K_n}, \quad K_n = [-a_n, a_n],$$

form a projective spectrum, denoted by $\mathcal{E}_{\{\omega\}}^I$ (with $(a_n)_{n\in\mathbb{N}}$ as in 1.4). The map

$$\Pi : \mathcal{E}_{\{\omega\}}(I) \to \prod_{n\in\mathbb{N}} \mathcal{E}_{\{\omega\},n}, \quad \Pi(f) = (\pi_n(f))_{n\in\mathbb{N}} = (f|_{K_n})_{n\in\mathbb{N}},$$

induces a natural isomorphism between $\mathcal{E}_{\{\omega\}}(I)$ and $\operatorname{proj}^0 \mathcal{E}_{\{\omega\}}^I$.

(ii) Let $\mu \in (\mathcal{E}_{\{\omega\},0})'$ and consider the sequence $T_{\mu,n} : \mathcal{E}_{\{\omega\},n} \to \mathcal{E}_{\{\omega\},n}$, $n \in \mathbb{N}$. Then the map

$$\mathcal{T}_\mu : \mathcal{E}_{\{\omega\}}^I \to \mathcal{E}_{\{\omega\}}^I, \quad \mathcal{T}_\mu = (T_{\mu,n})_{n\in\mathbb{N}},$$

defines a map on the projective spectrum $\mathcal{E}_{\{\omega\}}^I$. For $n \in \mathbb{N}$ we set

$$k_n = \ker T_{\mu,n}, \quad q_{n+1}^n : k_{n+1} \to k_n, \quad q_{n+1}^n(f) = \pi_{n+1}^n|_{k_{n+1}}(f) = f|_{K_n},$$

to get the projective spectrum $\mathcal{K}^I(\omega,\mu) = (k_n, q_{n+1}^n)_{n\in\mathbb{N}}$.

With $q_n : \ker T_\mu \to k_n$, $q_n(f) = f|_{K_n}$, we have

$$\operatorname{proj}_{\leftarrow n} k_n = \ker T_\mu.$$

In addition, for all $n \in \mathbb{N}$ we define

$$j_n : k_n \to \mathcal{E}_{\{\omega\},n}, \quad j_n(f) = f, \quad \text{and} \quad J : \mathcal{K}^I(\omega,\mu) \to \mathcal{E}_{\{\omega\}}^I, \quad J = (j_n)_{n\in\mathbb{N}}.$$

(iii) For $\ker T_\mu$ with slowly decreasing $\widehat{\mu} \in A_p^0$ we have the sequence space representation $\lambda^I(\gamma,\delta)$ from 2.11. Let

$$\Lambda^I(\gamma,\delta) := (\lambda_n(\gamma,\delta), i_{n+1}^n)_{n\in\mathbb{N}}, \quad i_{n+1}^n(x) = x,$$

with

$$\lambda_n(\gamma,\delta) := \left\{ x \in \mathbb{C}^\mathbb{N} \;\middle|\; \exists m \in \mathbb{N} : \sum_{j\in\mathbb{N}} |x_j| \exp\left(a_n\gamma_j + \frac{1}{m}\delta_j\right) < \infty \right\}$$

be the corresponding projective spectrum.

A main point in using the methods of Palamodov and Vogt is the vanishing of $\operatorname{proj}^1 \mathcal{E}_{\{\omega\}}^I$. For the non-quasianalytic case this is more or less obvious (see Braun, Meise and Vogt [7]), because of the existence of cut-off functions. In the quasianalytic case we use a result of Braun [5], who gives sufficient conditions for $\operatorname{proj}^1 \mathcal{E}_{\{\omega\}}^I = 0$. We begin by proving a density condition with methods from [16]. There an idea of Taylor [24] was used to show the injectivity of the Fourier Laplace transform on $\mathcal{E}_{\{\omega\}}(\mathbb{R})'$. To simplify the calculations, we introduce a new projective spectrum with limit $\mathcal{E}_{\{\omega\}}(I)$ which is equivalent to $\mathcal{E}_{\{\omega\}}^I$.

3.3. Definition. Let ω be a weight function and $(a_n)_{n\in\mathbb{N}}$, $(K_n)_{n\in\mathbb{N}}$ be as in 1.4. For $n, m \in \mathbb{N}$ and $f \in C^\infty(\mathring{K}_n)$ we define

$$Q_{n,m}(f) := q_{\mathring{K}_n,m}(f) = \sup_{j\in\mathbb{N}_0} \sup_{x\in\mathring{K}_n} |f^{(j)}(x)| \exp\left(-\frac{1}{m}\varphi^*(jm)\right),$$

we introduce the Banach space

$$E_{n,m} := \{ f \in C^\infty(\mathring{K}_n) \mid Q_{n,m}(f) < \infty \}$$

and denote by $B_{n,m} = \{ f \in E_{n,m} \mid Q_{n,m}(f) \le 1 \}$ the closed unit ball in $E_{n,m}$. We also define

$$E_n := \operatorname{ind}_{m\to} E_{n,m}$$

and the restriction maps $\pi_{n+1}^n : E_{n+1} \to E_n$, $\pi_{n+1}^n(f) = f|_{K_n}$.

The corresponding representation of $A_{\{\omega\}}^I$ is given by

$$A_{\{\omega\},n,m}^* := \left\{ f \in A(\mathbb{C}) \;\middle|\; |f|_{n,m}^* := \sup_{z\in\mathbb{C}} |f(z)| \exp\left(-a_n|\operatorname{Im} z| - \frac{1}{m}\omega(z)\right) < \infty \right\},$$

$$A_{\{\omega\},n}^* := \operatorname{proj}_{\leftarrow m} A_{\{\omega\},n,m}^*, \quad A_{\{\omega\}}^I = \operatorname{ind}_{n\to} A_{\{\omega\},n}^*.$$

$A_{\{\omega\},n}^*$ is not equal to $A_{\{\omega\},n}^0$, but the inductive limits coincide. As in 1.6 the Fourier Laplace transformation is a linear topological isomorphism between $(E_n)_b'$ and $A_{\{\omega\},n}^*$ (see [16]).

3.4. Lemma (density condition). *Let ω be a weight function. Then*

(D) $\quad \forall n \in \mathbb{N}\ \exists k \ge n\ \forall l \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall K \ge k\ \exists M \in \mathbb{N}:$

$$\pi_k^n E_{k,l} \subset \pi_K^n E_{K,M} + B_{n,m}.$$

Proof. For all $k, m \in \mathbb{N}$ the exponential functions $f_z(x) = \exp(-ixz)$, $z \in \mathbb{C}$, and therefore the linear span $E = \operatorname{span}\{f_z \mid z \in \mathbb{C}\}$, lie in $E_{k,m}$, as a short calculation shows. To prove the assertion we show

(1) $$\forall k, l \in \mathbb{N}\ \exists m \in \mathbb{N}: \quad E_{k,l} \subset \overline{E}^{E_{k,m}}.$$

To deduce (D) from (1) let $n \in \mathbb{N}$, choose $k = n$ and for $l \in \mathbb{N}$ choose $m \in \mathbb{N}$ according to (1). For given $K \ge k$, $M \in \mathbb{N}$ and $f \in E_{k,l}$ there exists $g \in E \subset \pi_K^k E_{K,M}$ with $Q_{k,m}(f-g) < 1$, i.e. $f - g \in B_{k,m}$. With

$$f = g + (f-g) \in \pi_K^k E_{K,M} + B_{k,m}$$

we get the assertion.

Proof of (1): For $k, m \in \mathbb{N}$ we define

$$b_{k,m} := \exp\left(\frac{1}{m}\varphi^*(km)\right), \quad b_m := (b_{k,m})_{k\in\mathbb{N}_0}, \quad b_m^- := (b_{k,m}^{-1})_{k\in\mathbb{N}_0},$$

$$F_k := C(K_k, \sup_{x\in K_k} |f(x)|),$$

and we introduce the Banach spaces

$$l_k^1(b_m) := \Big\{\mu = (\mu_j)_{j\in\mathbb{N}_0} \in \prod_{j\in\mathbb{N}_0} F_k' \,\Big|\, \|\mu\|_{k,m}^1 := \sum_{j\in\mathbb{N}_0} \|\mu_j\|_{F_k'} b_{j,m} < \infty\Big\}$$

and

$$c_k^0(b_m^-) := \Big\{f = (f_j)_{j\in\mathbb{N}_0} \in \prod_{j\in\mathbb{N}_0} F_k \,\Big|\, \lim_{j\to\infty} \|f_j\|_{F_k} b_{j,m}^{-1} = 0\Big\},$$

where we endow $c_k^0(b_m^-)$ with the norm $\|f\|_{k,m}^\infty := \sup_{j\in\mathbb{N}_0} \|f_j\|_{F_k} b_{j,m}^{-1}$. An easy calculation shows the isomorphy $c_k^0(b_m^-)' \cong l_k^1(b_m)$ via the canonical bilinear form

$$\langle(\mu_j)_{j\in\mathbb{N}_0}, (f_j)_{j\in\mathbb{N}_0}\rangle = \sum_{j\in\mathbb{N}_0} \langle\mu_j, f_j\rangle.$$

By elementary estimates of the Young conjugate φ^* (see [16, 2.5]) one can prove the following Schwartz condition for $(b_{k,m})_{k\in\mathbb{N}_0, m\in\mathbb{N}}$:

(S) $\qquad \forall l \in \mathbb{N}\ \exists m \in \mathbb{N}, m > l: \quad \lim_{j\to\infty} b_{j,l}/b_{j,m} = 0,$

hence for all $l \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that the map

$$\phi : E_{k,l} \to c_k^0(b_m^-), \quad \phi(f) = (f^{(j)})_{j\in\mathbb{N}_0},$$

is linear, continuous and injective. Let $\mu \in (E_{k,m})'$ and define $\widetilde{\mu} = \mu \circ \iota_{k,l}^m$ (where $\iota_{k,l}^m : E_{k,l} \to E_{k,m}$, $f \mapsto f$). Then the map $\widetilde{\mu} \circ \phi^{-1}$ is well defined on $\operatorname{im}\phi$ and continuous because

$$\begin{aligned}|\langle\widetilde{\mu} \circ \phi^{-1}, (f^{(j)})_{j\in\mathbb{N}_0}\rangle| &= |\langle\widetilde{\mu}, f\rangle| = |\langle\mu, f\rangle| \\ &\le CQ_{k,m}(f) = C\|f\|_{k,m}^\infty, \qquad f \in E_{k,l}.\end{aligned}$$

By the Hahn–Banach theorem we find $\widetilde{\widetilde{\mu}} \in c_k^0(b_m^-)' = l_k^1(b_m)$ with $\widetilde{\mu} = \widetilde{\widetilde{\mu}} \circ \phi$. Setting $\widetilde{\widetilde{\mu}} = (\mu_j)_{j\in\mathbb{N}_0} \in l_k^1(b_m)$ we get for all $f \in E_{k,l}$,

$$\langle\widetilde{\mu}, f\rangle = \langle\widetilde{\widetilde{\mu}} \circ \phi, f\rangle = \langle\widetilde{\widetilde{\mu}}, (f^{(j)})_{j\in\mathbb{N}_0}\rangle = \sum_{j\in\mathbb{N}_0} \langle\mu_j, f^{(j)}\rangle.$$

As in the proof of Taylor [24, Theorem 2.8] (see also [16, 5.3] or [17, 4.5]), from (2) we get

(2) $\quad \forall l \in \mathbb{N}\ \exists m \in \mathbb{N}, m > l\ \forall \mu \in (E_{k,m})':$

$$\forall z \in \mathbb{C} : \langle\mu, f_z\rangle = 0 \Rightarrow \forall f \in E_{k,l} : \langle\mu, f\rangle = 0.$$

To show (1) we take $f \in E_{k,l}$ and suppose $f \notin \overline{E}^{E_{k,m}} \subset E_{k,m}$. By Hahn–Banach there is a $\mu \in (E_{k,m})'$ with $\mu|_{\overline{E}^{E_{k,m}}} = 0$ and $\langle\mu, f\rangle = 1$, which contradicts (2) and shows the assertion.

3.5. LEMMA. *Using the notations above, we introduce the following two conditions*:

(P) $\quad \forall \nu \in \mathbb{N}\ \exists n \in \mathbb{N}, k \ge \nu\ \forall m \in \mathbb{N}\ \exists N \in \mathbb{N}\ \forall K \ge k\ \exists S > 0:$

$$\pi_k^\nu B_{k,m} \subset S(\pi_K^\nu B_{K,N} + B_{\nu,n}),$$

(P*) $\quad \forall \nu \in \mathbb{N}\ \exists n \in \mathbb{N}, k \ge \nu\ \forall m \in \mathbb{N}\ \exists N \in \mathbb{N}\ \forall K \ge k\ \exists S > 0\ \forall f \in A_{\{\omega\},\nu}^*:$

$$|f|_{k,m}^* \le S(|f|_{K,N}^* + |f|_{\nu,n}^*).$$

Then (P*) *implies* (P).

Proof. First we transfer (P*) via Fourier Laplace transformation from $(A_{\{\omega\},n}^*)_{n\in\mathbb{N}}$ to $((E_n)_b')_{n\in\mathbb{N}}$. For $k, m \in \mathbb{N}$ and $\mu \in E_k'$ we define the dual norms

$$\|\mu\|_{k,m}' := \sup\{|\langle\mu, f\rangle| \mid Q_{k,m}(f) \le 1\}.$$

Then a short calculation, using the property $\varphi^{**} = \varphi$ of the function $\varphi : x \mapsto \omega(e^x)$, shows that for each $k, m \in \mathbb{N}$, $\mu \in E_k'$ and $z \in \mathbb{C}$ we have

$$|\widehat{\mu}(z)| \le \|\mu\|_{k,m}' \exp\left(k|\operatorname{Im} z| + \frac{1}{m}\omega(z)\right),$$

and therefore $\|\widehat{\mu}\|_{k,m}^* \le \|\mu\|_{k,m}'$. An important point for further calculations is that the index m does not change. Since the Fourier Laplace transformation is a linear topological isomorphism, we get a converse estimate directly:

(1) $\qquad \forall k, m'\ \exists m, C_{m'} > 0\ \forall \mu \in E_k': \quad \|\mu\|_{k,m'}' \le C_{m'}\|\widehat{\mu}\|_{k,m}^*.$

An obvious choice of indices shows that (P*) implies (P′) where

(P′) $\quad \forall \nu' \in \mathbb{N}\ \exists n' \in \mathbb{N}, k' \ge \nu'\ \forall m' \in \mathbb{N}\ \exists N' \in \mathbb{N}\ \forall K' \ge k'\ \exists S' > 0\ \forall \mu \in E_k':$

$$\|\mu\|_{k',m'}' \le S'(\|\mu\|_{K',N'}' + \|\mu\|_{\nu',n'}').$$

To dualize (P′) we need for $k, \nu, n \in \mathbb{N}$ with $k \ge \nu$ the sets

$$A_{k,m}^\nu := \{\mu \in E_\nu' \mid \|\mu\|_{k,m}' \le 1\}.$$

In the dual system $\langle E_\nu, E_\nu'\rangle$ we have

$$A_{k,m}^\nu = (\pi_k^\nu B_{k,m})^\circ.$$

As polars the sets $A_{k,m}^\nu$ are absolutely convex and weakly closed. In addition, $A_{\nu,m}^\nu$ is a zero neighborhood in E_ν', hence by Alaoglu–Bourbaki $(A_{\nu,m}^\nu)^\circ$ is weakly compact and $(A_{k,m}^\nu)^\circ + (A_{\nu,m}^\nu)^\circ$ is weakly closed.

Condition (S) (see proof of 3.4) implies

$$\forall n\ \exists m\ \forall \nu: \quad B_{\nu,n} \text{ is relatively compact in } E_{\nu,m}.$$

Since the closure of $B_{\nu,n}$ in $E_{\nu,m}$, respectively E_ν, is compact, we get

(2) $\qquad \forall n\ \exists n'\ \forall \nu\ \exists C_{\nu,n,n'}: \quad \overline{B_{\nu,n}}^{E_\nu} = \overline{B_{\nu,n}}^{E_{\nu,n'}} \subset C_{\nu,n,n'} B_{\nu,n'}.$

In terms of the sets $A_{k,m}^\nu$, (P′) can be written in the form

$$\forall \nu \in \mathbb{N}\ \exists n \in \mathbb{N}, k \ge \nu\ \forall m \in \mathbb{N}\ \exists N \in \mathbb{N}\ \forall K \ge k\ \exists S > 0:$$

$$SA_{k,m}^\nu \supset A_{K,N}^\nu \cap A_{\nu,n}^\nu.$$

To show (P) let $\nu \in \mathbb{N}$. Choose $k \geq \nu$ and $n \in \mathbb{N}$ according to (P′). (From now on we omit the upper index ν). For n we take n' by (2). For $m \in \mathbb{N}$ there exist N by (P′) and N' by (2) (not depending on K) such that for all $K \geq k$ there exists a suitable $S > 0$. We set $S' = \max(C_{K,N,N'}, C_{\nu,n,n'})S$. Using the bipolar theorem and well-known computation rules for polars we get

$$\begin{aligned}\frac{1}{S}B_{k,m} &\subset \frac{1}{S}B_{k,m}^{\circ\circ} = \frac{1}{S}A_{k,m}^{\circ} = (SA_{k,m})^{\circ} \subset (A_{K,N} \cap A_{\nu,n})^{\circ}\\ &= (A_{K,N}^{\circ} \cup A_{\nu,n}^{\circ})^{\circ\circ} \subset (A_{K,N}^{\circ} + A_{\nu,n}^{\circ})^{\circ\circ} = A_{K,N}^{\circ} + A_{\nu,n}^{\circ}\\ &= (\pi_K^{\nu}B_{K,N})^{\circ\circ} + (B_{\nu,n})^{\circ\circ} = \overline{(\pi_K^{\nu}B_{K,N})}^{E_\nu} + \overline{B_{\nu,n}}^{E_\nu}\\ &\subset \pi_K^{\nu}(\overline{B_{K,N}}^{E_K}) + C_{\nu,n,n'}B_{\nu,n'}\\ &\subset \pi_K^{\nu}(C_{K,N,N'}B_{K,N'}) + C_{\nu,n,n'}B_{\nu,n'}\\ &\subset \max(C_{K,N,N'}, C_{\nu,n,n'})(\pi_K^{\nu}B_{K,N'} + B_{\nu,n'})\end{aligned}$$

and hence the assertion.

3.6. LEMMA. *Let $x \in \mathbb{R}$, $R, D > 0$ and $B = \{z \in \mathbb{C} \mid \operatorname{Im} z \geq 0, |z - x| \leq R\}$. Let u be a continuous function on B which is subharmonic in the interior of B. If u satisfies* (i) *and* (ii) *then also* (iii), *where*

(i) $\forall \xi \in \mathbb{R} \cap B: \quad u(\xi) \leq 0,$

(ii) $\forall \zeta \in \partial B, \operatorname{Im}\zeta > 0: \quad u(\zeta) \leq D,$

(iii) $\forall 0 < y \leq R: \quad u(x + iy) \leq \frac{4}{\pi} \cdot \frac{y}{R} D.$

Proof. This can be seen by a scaling argument as in Ahlfors [1, 3.4], where the harmonic measure is estimated on the unit half circle.

3.7. THEOREM. *Let ω be a weight function. Then* $\operatorname{proj}^1 \mathcal{E}_{\{\omega\}}^I = 0$.

Proof. Braun [5] has shown that the conditions (P) and (D) are sufficient for $\operatorname{proj}^1 \mathcal{E}_{\{\omega\}}^I = 0$. By 3.4 and 3.5 it suffices to show that (P*) holds. We first take $I =]-1, 1[$ and remark that (P*) is equivalent to the following condition:

$\forall 0 < \nu < 1\ \exists n \in \mathbb{N}, \nu \leq k < 1\ \forall m \in \mathbb{N}\ \exists N \in \mathbb{N}\ \forall k \leq K < 1\ \exists S > 0$ $\forall f \in A_{\{\omega\}}^I$: (i) & (ii) $\Rightarrow$ (iii), where

(i) $\forall z \in \mathbb{C}: \quad \ln|f(z)| \leq \nu|\operatorname{Im} z| + \frac{1}{n}\omega(z),$

(ii) $\forall z \in \mathbb{C}: \quad \ln|f(z)| \leq K|\operatorname{Im} z| + \frac{1}{N}\omega(z),$

(iii) $\forall z \in \mathbb{C}: \quad \ln|f(z)| \leq k|\operatorname{Im} z| + \frac{1}{m}\omega(z) + \ln S.$

We will show this for subharmonic functions on $\mathbb{C}$ instead of $\ln|f|$. To do this let $c_2 > 1$ be such that $\omega(z) \leq c_2|z| + c_2$ for all $z \in \mathbb{C}$ according to 1.1(ε) and choose $C > 1$ such that for all $z \in \mathbb{C}$,

$$\omega((c_2 + 1)|z| + c_2) \leq C\omega(z) + C \quad \text{and} \quad \omega(3|z|) \leq C\omega(z) + C.$$

Let $0 < \nu < 1$. Choose $n \in \mathbb{N}$ such that $k := \nu + 16C/(\pi^2 n) < 1$ and for given $m \in \mathbb{N}$ take $N \in \mathbb{N}$ with $N \geq Cm$. Finally, we take an arbitrary K with $k \leq K < 1$ and set $S = \exp(C + C/N)$. Now we have the following two cases:

(A) $z \in \mathbb{C}$ with $|\operatorname{Im} z| < \pi\omega(z)/8$. We only treat the case $\operatorname{Im} z > 0$ since $\operatorname{Im} z < 0$ can be handled in the same way.

We define $R = \omega(z)$ and $B = \{\zeta \in \mathbb{C} \mid \operatorname{Im}\zeta \geq 0, |\zeta - \operatorname{Re} z| \leq R\}$. Then for a subharmonic function u on $\mathbb{C}$ the conditions (i) and (ii) yield:

(i)′ $\forall \zeta \in B: \quad u(\zeta) \leq \nu|\operatorname{Im}\zeta| + \frac{1}{n}\omega(\zeta),$

(ii)′ $\forall \xi \in \mathbb{R} \cap B: \quad u(\xi) \leq \frac{1}{N}\omega(\xi).$

For $\zeta \in B$ we have

$$\omega(\zeta) \leq \omega(|\operatorname{Re} z| + R) \leq \omega(|z| + \omega(z)) \leq \omega(|z| + c_2|z| + c_2) \leq C\omega(z) + C.$$

Hence by (i)′ and (ii)′ we get

$$\forall \zeta \in B: \quad u(\zeta) \leq \nu|\operatorname{Im}\zeta| + \frac{C}{n}\omega(z) + \frac{C}{n},$$
$$\forall \xi \in \mathbb{R} \cap B: \quad u(\xi) \leq \frac{1}{N}(C\omega(z) + C).$$

Defining $\widetilde{u}(\zeta) := u(\zeta) - \nu|\operatorname{Im}\zeta| - N^{-1}(C\omega(z) + C)$ gives

$$\forall \zeta \in B: \quad \widetilde{u}(\zeta) \leq \frac{C}{n}\omega(z) + \frac{C}{n},$$
$$\forall \xi \in \mathbb{R} \cap B: \quad \widetilde{u}(\xi) \leq 0.$$

Now we can apply 3.6 to the function $\widetilde{u}$ to get the following estimates for $\widetilde{u}$ and u:

$$\widetilde{u}(z) \leq \frac{4}{\pi}\frac{|\operatorname{Im} z|}{R}\left(\frac{C}{n}\omega(z) + \frac{C}{n}\right),$$

hence

$$\begin{aligned}u(z) &\leq \nu|\operatorname{Im} z| + \frac{1}{N}(C\omega(z) + C) + \frac{4}{\pi}\frac{|\operatorname{Im} z|}{\omega(z)}\left(\frac{C}{n}\omega(z) + \frac{C}{n}\right)\\ &\leq \left(\nu + \frac{4C}{n\pi}\right)|\operatorname{Im} z| + \frac{C}{N}\omega(z) + \frac{C}{N} + \frac{C}{2n}\\ &\leq k|\operatorname{Im} z| + \frac{1}{m}\omega(z) + \ln S.\end{aligned}$$

(B) $z \in \mathbb{C}$ with $|\operatorname{Im} z| \ge \pi\omega(z)/8$. As in (A) we only need to treat the case $\operatorname{Im} z > 0$. We define $R = 2|\operatorname{Im} z|$ and $B = \{\zeta \in \mathbb{C} \mid \operatorname{Im}\zeta \ge 0, |\zeta - \operatorname{Re} z| \le R\}$. Now $\zeta \in B$ satisfies

$$\omega(\zeta) \le \omega(|\operatorname{Re} z| + 2|\operatorname{Im} z|) \le \omega(3|z|) \le C\omega(z) + C \le \frac{8C}{\pi}|\operatorname{Im} z| + C.$$

For a subharmonic function u on $\mathbb{C}$, from (i) and (ii) we get

$$\forall \zeta \in B: \quad u(\zeta) \le \nu|\operatorname{Im}\zeta| + \frac{8C}{n\pi}|\operatorname{Im} z| + \frac{C}{n},$$
$$\forall \xi \in \mathbb{R} \cap B: \quad u(\xi) \le \frac{1}{N}(C\omega(z) + C).$$

Defining $\widetilde{u}(\zeta) := u(\zeta) - \nu|\operatorname{Im}\zeta| - N^{-1}(C\omega(z) + C)$ we get

$$\forall \zeta \in B: \quad \widetilde{u}(\zeta) \le \frac{8C}{n\pi}|\operatorname{Im} z| + \frac{C}{n},$$
$$\forall \xi \in \mathbb{R} \cap B: \quad \widetilde{u}(\xi) \le 0.$$

As in case (A) it follows by 3.6 that

$$\widetilde{u}(z) \le \frac{4}{\pi}\frac{|\operatorname{Im} z|}{R}\left(\frac{8C}{n\pi}|\operatorname{Im} z| + \frac{C}{n}\right),$$

hence

$$\begin{aligned} u(z) &\le \nu|\operatorname{Im} z| + \frac{1}{N}(C\omega(z) + C) + \frac{4}{2\pi}\left(\frac{8C}{n\pi}|\operatorname{Im} z| + \frac{C}{n}\right) \\ &\le \left(\nu + \frac{16C}{n\pi^2}\right)|\operatorname{Im} z| + \frac{C}{N}\omega(z) + \frac{C}{N} + \frac{2C}{n\pi} \\ &\le k|\operatorname{Im} z| + \frac{1}{m}\omega(z) + \ln S, \end{aligned}$$

which shows the assertion in the case $I =]-1,1[$. The proof in the case $I = \mathbb{R}$ is an easy consequence of the above proof.

3.8. PROPOSITION. *Let ω be a weight function and $\mu \in (\mathcal{E}_{\{\omega\},0})'$. For every locally surjective convolution operator $T_\mu : \mathcal{E}_{\{\omega\}}(I) \to \mathcal{E}_{\{\omega\}}(I)$ the following conditions are equivalent*:

(i) $T_\mu : \mathcal{E}_{\{\omega\}}(I) \to \mathcal{E}_{\{\omega\}}(I)$ *is surjective*,
(ii) $\operatorname{proj}^1 \mathcal{K}^I(\omega,\mu) = 0$.

Proof. The proof follows Braun, Meise and Vogt [7, 3.3]. Since T_μ is locally surjective, the operators $T_{\mu,n}$ are surjective for every $n \in \mathbb{N}$ so we obtain the short exact sequences

$$0 \to k_n \xrightarrow{j_n} \mathcal{E}_{\{\omega\},n} \xrightarrow{T_{\mu,n}} \mathcal{E}_{\{\omega\},n} \to 0.$$

By 3.1,

$$0 \to \mathcal{K}^I(\omega,\mu) \xrightarrow{J} \mathcal{E}^I_{\{\omega\}} \xrightarrow{T_\mu} \mathcal{E}^I_{\{\omega\}} \to 0$$

is a short exact sequence of projective spectra. By Vogt [25, 1.11] there exists the following exact sequence of six spaces:

$$0 \to \operatorname{proj}^0 \mathcal{K}^I(\omega,\mu) \xrightarrow{J^0} \operatorname{proj}^0 \mathcal{E}^I_{\{\omega\}} \xrightarrow{T^0_\mu} \operatorname{proj}^0 \mathcal{E}^I_{\{\omega\}}$$
$$\xrightarrow{\delta^*} \operatorname{proj}^1 \mathcal{K}^I(\omega,\mu) \xrightarrow{J^1} \operatorname{proj}^1 \mathcal{E}^I_{\{\omega\}} \xrightarrow{T^1_\mu} \operatorname{proj}^1 \mathcal{E}^I_{\{\omega\}} \to 0.$$

By 3.2 we can identify $\operatorname{proj}^0 \mathcal{E}^I_{\{\omega\}}$ with $\mathcal{E}_{\{\omega\}}(I)$. Then T^0_μ corresponds to the operator T_μ. If $\operatorname{proj}^1 \mathcal{K}^I(\omega,\mu)$ vanishes, the exactness at the third place of the sequence gives the surjectivity of T_μ. Otherwise, if T_μ is surjective, we need $\operatorname{proj}^1 \mathcal{E}^I_{\{\omega\}} = 0$ (3.7) to get $\operatorname{proj}^1 \mathcal{K}^I(\omega,\mu) = 0$ from the exactness at the fourth place.

To evaluate the condition $\operatorname{proj}^1 \mathcal{K}^I(\omega,\mu) = 0$ using the sequence space representation 2.11, we need three lemmas.

3.9. LEMMA. *Let ω be a weight function and $\mu \in (\mathcal{E}_{\{\omega\},0})'$ with $\widehat{\mu} \in A^0_p$ be slowly decreasing. Consider the convolution operator T_μ on $\mathcal{E}_{\{\omega\}}(]-1,1[)$ and the following diagram*:

$$\begin{array}{ccccccc} (\ker T_\mu)'_b & \xrightarrow{A} & \mathcal{E}_{\{\omega\}}(I)'_b/(\ker T_\mu)^\perp & \xrightarrow{B} & A^I_{\{\omega\}}/\widehat{\mu}A^I_{\{\omega\}} & \xrightarrow{\widetilde{\varrho}} & K^I(\alpha,\beta,\mathbb{E}) \\ \uparrow{\scriptstyle (q_n)^t} & & & & & & \uparrow{\scriptstyle \iota_n} \\ (k_n)'_b & \xrightarrow{A_n} & (\mathcal{E}_{\{\omega\},n})'_b/k_n^\perp & \xrightarrow{B_n} & A^0_{\{\omega\},n}/\widehat{\mu}A^0_{\{\omega\},n} & \xrightarrow{\widetilde{\sigma}_n} & K_n(\alpha,\beta,\mathbb{E}) \end{array}$$

where $\widetilde{\varrho}$ is the isomorphism from 2.8, and $\widetilde{\sigma}_n$ is as in 2.9. The canonical maps A, B, A_n and B_n will be defined in the proof. Under these assumptions the diagram is commutative and $(q_n)^t$ is an injective map. For $I = \mathbb{R}$ we get the same result if we replace $\widetilde{\varrho}$ by the corresponding isomorphism of Meise [12, 2.6].

Proof. Using the Hahn–Banach theorem we get for every $\nu_0 \in (k_n)'_b$ a functional $\nu_1 \in (\mathcal{E}_{\{\omega\},n})'_b$ with $\nu_1 \circ j_n = \nu_0$ and a linear topological isomorphism

$$A_n : (k_n)'_b \to (\mathcal{E}_{\{\omega\},n})'_b/k_n^\perp, \quad A_n(\nu_0) = \nu_1 + k_n^\perp,$$

because $\mathcal{E}_{\{\omega\},n}$ is a (DFS)-space. The map A is defined in the same way. We have $k_n^\perp = \overline{\operatorname{im} T^t_{\mu,n}}$ and $\operatorname{im} M_{\check{\mu}} = \widehat{\mu}A^0_{\{\omega\},n}$ by [16, 7.5] and Fourier Laplace transformation, so we can define an isomorphism

$$B_n : (\mathcal{E}_{\{\omega\},n})'_b/k_n^\perp \to A^0_{\{\omega\},n}/\widehat{\mu}A^0_{\{\omega\},n}, \quad B_n(\nu_1 + k_n^\perp) = \widehat{\nu}_1 + \widehat{\mu}A^0_{\{\omega\},n}$$

(and B analogously). Using the definition of $\widetilde{\sigma}_n$ from 2.9, an easy calculation shows $\iota_n \circ \widetilde{\sigma}_n \circ B_n \circ A_n = \widetilde{\varrho} \circ B \circ A \circ (q_n)^t$. Since A_n and B_n are isomorphisms and ι_n and $\widetilde{\sigma}_n$ are injective, $\widetilde{\varrho} \circ B \circ A \circ (q_n)^t$ is also injective. Hence the injectivity of $(q_n)^t$ follows, because $\widetilde{\varrho} \circ B \circ A$ is an isomorphism.

3.10. Lemma. *Let ω be a weight function, $\mu \in (\mathcal{E}_{\{\omega\},0})'$ slowly decreasing and γ and δ as in* 2.11. *Then*

$$\operatorname{proj}^1 \Lambda^I(\gamma,\delta) = 0 \Leftrightarrow \operatorname{proj}^1 \mathcal{K}^I(\omega,\mu) = 0.$$

Proof. We have

$$\operatorname*{proj}_{\leftarrow n} \lambda_n^I(\gamma,\delta) = \lambda^I(\gamma,\delta) \cong \ker T_\mu = \operatorname*{proj}_{\leftarrow n} k_n$$

By Vogt [26, 5.3] two reduced projective spectra of complete (LB)-spaces generating the same space are equivalent. Since $\lambda_n^I(\gamma,\delta)$ and k_n are (DFS)-spaces, hence complete (LB)-spaces, and $\Lambda^I(\gamma,\delta)$ is a reduced spectrum, it suffices to show that $\mathcal{K}^I(\omega,\mu)$ is reduced. By 3.9, $(q_n)^t$ is injective for all $n \in \mathbb{N}$, which means the image of q_n is dense. So we get $\operatorname{proj}^1 \Lambda^I(\gamma,\delta) \cong \operatorname{proj}^1 \mathcal{K}^I(\omega,\mu)$ from Vogt [26, 1.2].

3.11. Lemma. *Let $(\alpha_j)_{j\in\mathbb{N}}$ and $(\beta_j)_{j\in\mathbb{N}}$ be sequences in $\mathbb{R}_+$ with $\lim_{j\to\infty}\beta_j = \infty$ and let ω be a weight function. Then the following conditions are equivalent*:

(i) $\operatorname{proj}^1 \Lambda^I(\alpha,\beta) = 0$.

(ii) *There exists a decomposition $J_1 \cup J_2$ of $\mathbb{N}$ such that*

$$\lim_{j\to\infty,\, j\in J_1} \alpha_j/\beta_j = 0 \quad \textit{and} \quad \liminf_{j\to\infty,\, j\in J_2} \alpha_j/\beta_j > 0.$$

Proof. For increasing sequences $(r_k)_{k\in\mathbb{N}}$ and $(\varrho_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} r_k = r$ and $\lim_{k\to\infty}\varrho_k = \varrho$ we define $a_{j;k,m} = \exp(r_k\alpha_j - \varrho_m\beta_j)$. Setting $\varrho = 0$ and $r = 1$ for $I =]-1,1[$ (resp. $r = \infty$ for $I = \mathbb{R}$) we get the assertion directly from Vogt [26, 4.3].

In the non-quasianalytic case it is obvious that surjectivity implies local surjectivity (see Braun, Meise and Vogt [7, 2.5, 2.7]), because of the existence of cut-off functions. In the quasianalytic case we need duality theory and Remark 2.2.

3.12. Lemma. *Let ω be a weight function. Then every surjective convolution operator $T_\mu : \mathcal{E}_{\{\omega\}}(I) \to \mathcal{E}_{\{\omega\}}(I)$ is locally surjective.*

Proof. Since $\operatorname{proj}^1 \mathcal{E}^I_{\{\omega\}} = 0$ we deduce from Vogt [25, 5.7] that $\mathcal{E}_{\{\omega\}}(I)$ is ultrabornological and barreled. Being a projective limit of inductive limits of (B)-spaces, $\mathcal{E}_{\{\omega\}}(I)$ is a webbed space, so we can apply de Wilde's open mapping theorem (see Köthe [10, §35]) to the surjective operator $T_\mu : \mathcal{E}_{\{\omega\}}(I) \to \mathcal{E}_{\{\omega\}}(I)$. This yields that T_μ is a topological homomorphism. We consider the transposed map $T^t_\mu : \mathcal{E}_{\{\omega\}}(I)'_{\mathrm{b}} \to \mathcal{E}_{\{\omega\}}(I)'_{\mathrm{b}}$ of T_μ. Since T_μ is surjective, T^t_μ is injective and a theorem of Grothendieck on topological homomorphisms (see [10, 32.4(3)]) gives: For every equicontinuous set M in $T^t_\mu(\mathcal{E}_{\{\omega\}}(I)')$ the set $T^{-1}_\mu(M)$ is equicontinuous in $\mathcal{E}_{\{\omega\}}(I)'$. Since $\mathcal{E}_{\{\omega\}}(I)$ is barreled, the equicontinuous sets in $\mathcal{E}_{\{\omega\}}(I)'$ coincide with the bounded sets. Via Fourier Laplace transform we get for the multiplication operator

$$M_F : A^I_{\{\omega\}} \to A^I_{\{\omega\}}, \quad F := \widehat{\mu},$$

the following:

(1) For every bounded set $B \subset M_F(A^I_{\{\omega\}})$ the set $M_F^{-1}(B)$ is bounded in $A^I_{\{\omega\}}$.

Now we want to show the Ehrenpreis condition (E) for F, which is equivalent to the local surjectivity of T_μ (2.4). Suppose (E) is not true. By 2.2 we get for every $l \in \mathbb{N}$ a sequence $(f_j)_{j\in\mathbb{N}} \in (A^0_{\{\omega\},l})^{\mathbb{N}}$ with $(Ff_j)_{j\in\mathbb{N}}$ bounded in $A^I_{\{\omega\}}$, but $(f_j)_{j\in\mathbb{N}}$ unbounded in $A^0_{\{\omega\},\lambda}$ for $\lambda \ge l$. Since $\operatorname{proj}^1 \mathcal{E}^I_{\{\omega\}} = 0$ and the spaces $A^0_{\{\omega\},\lambda}$, $\lambda \in \mathbb{N}$, are reflexive it follows by Vogt [27, 4.1, 4.4], that $A^I_{\{\omega\}}$ is a regular space, i.e. every bounded set $B \subset A^I_{\{\omega\}}$ lies in a space $A^0_{\{\omega\},\lambda}$ for some λ and is bounded there. This is not true for $(f_j)_{j\in\mathbb{N}}$, so this sequence is unbounded in $A^I_{\{\omega\}}$ and we have a contradiction with (1).

We are now able to give the final result.

3.13. Theorem. *Let ω be a weight function, $\mu \in (\mathcal{E}_{\{\omega\},0})'$ and I an open bounded interval in $\mathbb{R}$ or $I = \mathbb{R}$. Then the convolution operator*

$$T_\mu : \mathcal{E}_{\{\omega\}}(I) \to \mathcal{E}_{\{\omega\}}(I), \quad T_\mu(f) = \mu * f,$$

is surjective if and only if the following two conditions are satisfied:

(i) *T_μ is locally surjective.*

(ii) *There exists a disjoint partition $V(\widehat{\mu}) = J_1 \cup J_2$ with*

$$\lim_{|z|\to\infty,\, z\in J_1} |\operatorname{Im} z|/\omega(z) = 0 \quad \textit{and} \quad \liminf_{|z|\to\infty,\, z\in J_2} |\operatorname{Im} z|/\omega(z) > 0.$$

Proof. We consider only the cases $I =]-1,1[$ and $I = \mathbb{R}$. The case $I =]a,b[$ can be easily reduced to $]-1,1[$ by the transformation from 1.5. Defining $\gamma = (\gamma_j)_{j\in\mathbb{N}} = (|\operatorname{Im} z_j|)_{j\in\mathbb{N}}$ and $\delta = (\delta_j)_{j\in\mathbb{N}} = (\omega(z_j))_{j\in\mathbb{N}}$, where $(z_j)_{j\in\mathbb{N}}$ counts the zeros of $\widehat{\mu}$, we can write (ii) in the form:

There exists a decomposition $J_3 \cup J_4$ of $\mathbb{N}$ with

$$\lim_{j\to\infty,\, j\in J_3} \gamma_j/\delta_j = 0 \quad \text{and} \quad \liminf_{j\to\infty,\, j\in J_4} \gamma_j/\delta_j > 0.$$

Now 3.10 and 3.11 imply $\operatorname{proj}^1 \mathcal{K}^I(\omega,\mu) = 0$. By 3.8 we get the surjectivity of T_μ.

Conversely, if T_μ is surjective, then 3.12 yields the local surjectivity of T_μ. Applying 3.8, 3.10 and 3.11 we get (ii) and hence the assertion.

Addendum. (a) In the meanwhile, Wengenroth [28] proved that condition (P_2) of Vogt [25], [26], which is slightly weaker than (P^*), is sufficient for a reduced spectrum $\mathcal{X}$ of (DFM)-spaces to satisfy $\text{proj}^1 \mathcal{X} = 0$. The necessity of (P_2) is known by Vogt [26, 2.7]. The main difference for the proof of Theorem 3.7 is that condition (D) is no longer needed.

(b) For an extension of the result of Korobeĭnik, Napalkov and Rudakov (which has been mentioned in the introduction) to the case $\text{supp}(\mu) \neq \{0\}$ but $I = \mathbb{R}$, we refer to a recent publication of Langenbruch [11].

References

[1] L. V. Ahlfors, *Conformal Invariants. Topics in Geometric Function Theory*, McGraw-Hill, 1973.

[2] C. A. Berenstein and B. A. Taylor, *A new look at interpolation theory for entire functions of one variable*, Adv. in Math. 33 (1979), 109–143.

[3] A. Beurling, *Quasianalyticity and general distributions*, Lectures 4 and 5, Amer. Math. Soc. Summer Institute (Stanford, 1961).

[4] G. Björck, *Linear partial differential operators and generalized distributions*, Ark. Mat. 6 (1965), 351–407.

[5] R. W. Braun, *A sufficient criterion for the vanishing of* proj^1 *for* (DFS)*-spectra*, preprint.

[6] R. W. Braun, R. Meise and B. A. Taylor, *Ultradifferentiable functions and Fourier analysis*, Results Math. 17 (1990), 206–237.

[7] R. W. Braun, R. Meise and D. Vogt, *Existence of fundamental solutions and surjectivity of convolution operators on classes of ultra-differentiable functions*, Proc. London Math. Soc. (3) 61 (1990), 344–370.

[8] —, —, —, *Characterization of the linear partial differential operators with constant coefficients which are surjective on non-quasianalytic classes of Roumieu type*, Math. Nachr. 168 (1994), 19–54.

[9] Yu. F. Korobeĭnik, *Solvability of a convolution equation in some classes of analytic functions*, Mat. Zametki 49 (2) (1991), 76–83 (in Russian); English transl.: Math. Notes 49 (1991), 165–172.

[10] G. Köthe, *Topological Vector Spaces II*, Grundlehren Math. Wiss. 237, Springer, 1979.

[11] M. Langenbruch, *Hyperfunction fundamental solutions of surjective convolution operators on real analytic functions*, J. Funct. Anal. 131 (1995), 78–93.

[12] R. Meise, *Sequence space representations for zero-solutions of convolution equations on ultradifferentiable functions of Roumieu type*, Studia Math. 92 (1989), 211–230.

[13] —, *Sequence space representations for* (DFN)*-algebras of entire functions modulo closed ideals*, J. Reine Angew. Math. 363 (1985), 59–95.

[14] R. Meise, B. A. Taylor and D. Vogt, *Equivalence of slowly decreasing conditions and local Fourier expansions*, Indiana Univ. Math. J. 36 (1987), 729–756.

[15] R. Meise und D. Vogt, *Einführung in die Funktionalanalysis*, Vieweg, Braunschweig, 1992.

[16] T. Meyer, *Die Fourier–Laplace-Transformation quasianalytischer Funktionale und ihre Anwendung auf Faltungsoperatoren*, Diplomarbeit, Düsseldorf, 1989.

[17] —, *Surjektivität von Faltungsoperatoren auf Räumen ultradifferenzierbarer Funktionen vom Roumieu Typ*, Thesis, Düsseldorf, 1992.

[18] S. Momm, *Closed principal ideals in nonradial Hörmander algebras*, Arch. Math. (Basel) 58 (1992), 47–55.

[19] —, *Convolution equations on the analytic functions on convex domains in the plane*, Bull. Sci. Math. 118 (1994), 259–270.

[20] —, *Ideale in gewichteten Algebren holomorpher Funktionen auf dem Einheitskreis*, Dissertation, Düsseldorf, 1988.

[21] V. V. Napalkov and I. A. Rudakov, *Convolution operator in the space of real analytic functions*, Mat. Zametki 49 (3) (1991), 57–65 (in Russian); English transl.: Math. Notes 49 (1991), 266–271.

[22] V. P. Palamodov, *The projective limit functor in the category of linear topological spaces*, Math. USSR-Sb. 4 (1968), 529–559.

[23] C. Roumieu, *Sur quelques extensions de la notion de distribution*, Ann. Sci. École Norm. Sup. (3) 77 (1960), 41–121.

[24] B. A. Taylor, *Analytically uniform spaces of infinitely differentiable functions*, Comm. Pure Appl. Math. 24 (1971), 39–51.

[25] D. Vogt, *Lectures on projective spectra of* (DF)*-spaces*, Universität Wuppertal, 1987.

[26] —, *Topics on projective spectra of* (LB)*-spaces*, in: Advances in the Theory of Fréchet Spaces, T. Terzioğlu (ed.), NATO Adv. Sci. Inst. Ser. C 287, Kluwer, Dordrecht, 1989, 11–27.

[27] —, *Regularity properties of* (LF)*-spaces*, in: Progress in Functional Analysis, K. D. Bierstedt, J. Bonet, J. Horváth and M. Maestre (eds.), North-Holland Math. Stud. 170, North-Holland, 1992, 57–84.

[28] J. Wengenroth, *Acyclic inductive spectra of Fréchet spaces*, Studia Math. 120 (1996), 247–258.

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