Statistical Science
1995, Vol. 10, No. 1, 86–103

Survival in Dynamic Environments

Nozer D. Singpurwalla

Abstract. This expository paper is an overview of a relatively new class of failure models, both univariate and multivariate, that are suitable for describing the lifelength of items that operate in dynamic environments. Many of the currently used failure models are developed under the premise that the operating environment is static: these models turn out to be special cases of the new models that are overviewed here. These new models are derived by describing the underlying failure-causing mechanisms, such as degradation and wear, using suitable stochastic processes: this is the underlying mathematical theme that drives their development. Because of their generality, the new models should provide improved descriptions of failure data and assessments of item survivability. Furthermore, they may signal a new philosophy of life-testing experiments wherein one also monitors the environmental factors that govern the tests. This overview categorizes the models by the various strategies used for their development, outlines the salient assumptions underlying them and provides a convenient road map to the pertinent references, which are scattered among the applied probability, engineering, reliability and survival analysis literatures. It is our hope that this paper sets a tone for the direction of future work in the development of models for survival wherein the physics of failure and the characteristics of the operating environment play a key role.

Key words and phrases: Biometry, biostatistics, Cameron–Martin formula, diffusion, dynamic linear models, extended gamma process, gamma process, hazard rate, Lévy processes, Markov additive processes, multivariate distributions, reliability, shot noise, stochastic processes, survival analysis, Wiener process.

1. INTRODUCTION AND OVERVIEW

The last few years have witnessed much criticism about the failure of reliability theory to have a tangible impact on the problems of modern science and technology. The lack of meaningful assessments of the integrity of composite materials and the survivability of structural elements have been cited as examples. Such criticism is justified because much of the literature in reliability tends to focus on old themes such as a characterization of classes of survival distributions or inference for parameters of failure models chosen from a limited inventory of such models. Parametric families of distributions, such as the exponential and the Weibull, have been used as failure models for almost 30 years. Besides tradition and convenience, a typical reason for selecting these models has been a subjective assessment about the aging characteristics of an item and/or the model's goodness-of-fit to failure data. Often the failure data used is obtained from life-testing experiments conducted under controlled laboratory environments that are static. To many engineers and scientists, this black-box approach to model selection is unsatisfactory. A more appealing approach would be to choose a model based on the physics of failure and the characteristics of the operating environment. Some efforts in this direction are already underway, and an aim of this overview is to highlight this trend.

Over the last few years, some literature devoted to the evolution of a relatively new class of failure models, both univariate and multivariate, for applications in reliability and biometry has begun to appear. A distinguishing feature of these new models is the fundamental theme that drives their development: they have been derived by considering stochastic processes that are presumed to describe the failure-generating mechanisms. Describing fail-

Nozer D. Singpurwalla is Professor, Department of Operations Research, School of Engineering, George Washington University, Washington, DC 20052.

ure-generating mechanisms by stochastic processes is particularly germane when the environment under which the items operate is dynamic, that is, when the induced stresses (or covariates) vary over time. This is because dynamic environments induce internal stresses in an item that change the rates and the modes by which the item degrades to failure. The purpose of this paper is to survey this new literature and to present an overview of the salient developments. The focus of this overview is on probabilistic modelling. Inferential issues are beyond the scope of what is discussed here and, indeed, require further research.

Because dynamic environments induce changes in the physics of failure, a stochastic-process approach to failure modelling provides flexibility with respect to describing the failure-generating mechanisms. This flexibility results in a better description of the failure data and an improved assessment of item survivability. This is perhaps the main reason why reliability engineers should be interested in the approach of this paper. Furthermore, the approach also raises the general level of the state of the art in reliability theory and survival analysis, especially in the ability to describe the survivability of multistate items. A disadvantage of the stochastic-process-based approach to failure modelling is that the resulting expressions for the survival function take unmanageable forms and can only be expressed via their Laplace transforms. However, with the rapid advances in the use of computer and simulation-based technologies in the statistical sciences, together with a widespread use of numerical techniques such as saddle-point approximations, the disadvantage of not having closed-form expressions will gradually disappear. Consequently, future research in reliability will also have to focus on computation and computability. Another aim of this paper is to emphasize the importance of this direction and to enable reliability theory to regain its lost appeal.

In developing an approach to failure modelling based on a stochastic process, it appears that four strategies have evolved; the two predominant ones are emphasized here. With the first strategy, the *item state* (or, equivalently, its *wear*) has been described by a diffusion process: typically a Wiener process, a gamma process or a deterministic diffusion. It can be shown that deterministic diffusions give rise to some of the well-known failure models that are in use today. Diffusion processes are stochastic processes with continuous sample paths. With the second strategy, it is the *failure rate* (also known as the *hazard rate*) of the item that is described by a stochastic process: typically a gamma process; a shot-noise process; functionals of a Wiener process; or, in general, a Lévy process. Lévy processes have stationary independent increments, and their marginal distributions are infinitely divisible. A more recent trend has been the development of failure models based on a consideration of two processes, one for the item state or wear and the other for a covariate that drives it. Covariate processes that drive the wear are referred to in the engineering literature as *excitation processes*. A study of models derived from excitation processes may signal a new philosophy of life-testing experiments wherein one must also monitor the conditions of the test. The third strategy for developing failure models focuses on describing the damage-causing environment by a stochastic process, typically a shock-inflicting Poisson process; the resulting failure models are known as *shock models*. The fourth and final strategy, and one for which there has been little development, is that in which a *response variable* that is strongly correlated with the lifelength, such as temperature, is described by a stochastic process, typically a stationary, continuous-time Gaussian process.

The text of this paper is based on a synopsis of material from the following chronologically ordered references which focus on one or the other of the first two strategies mentioned above: Mercer (1961); Gaver (1963); Antelman and Savage (1965); Çinlar (1977); Arjas (1981); Myers (1981); Dykstra and Laud (1981); Yashin (1985, 1993); Lemoine and Wenocur (1985, 1986); Meinhold and Singpurwalla (1987); Blackwell and Singpurwalla (1988); Wenocur (1989); Çinlar and Ozekici (1987); Gamerman (1991); Kebir (1991); and Singpurwalla and Youngren (1991, 1993). The flowchart depicted in Figure 1 is a convenient road map that shows the disposition of the material in some of these references. The above list is not to be viewed as exhaustive, and it is very likely that some key references, particularly those from the biostatistical literature, may have been overlooked. The work on models for failure based on the occurrence of shocks stems from the seminal paper of Esary, Marshall and Proschan (1973) and essentially culminates with the papers by A-Hameed and Proschan (1973, 1975). Desmond (1985) outlines an overall scheme for developing failure models under the fourth strategy, namely, that involving a response variable; Schäbe (1990) applies it when the response variable is a crack due to fatigue; also see Sobczyk (1987).

An historical comment may be of interest. It appears that Mercer (1961) may have been the first to consider the idea of describing item state by a stochastic process, and Gaver (1963) the first to propose modelling the item failure rate as such. Failure models based on covariate processes also

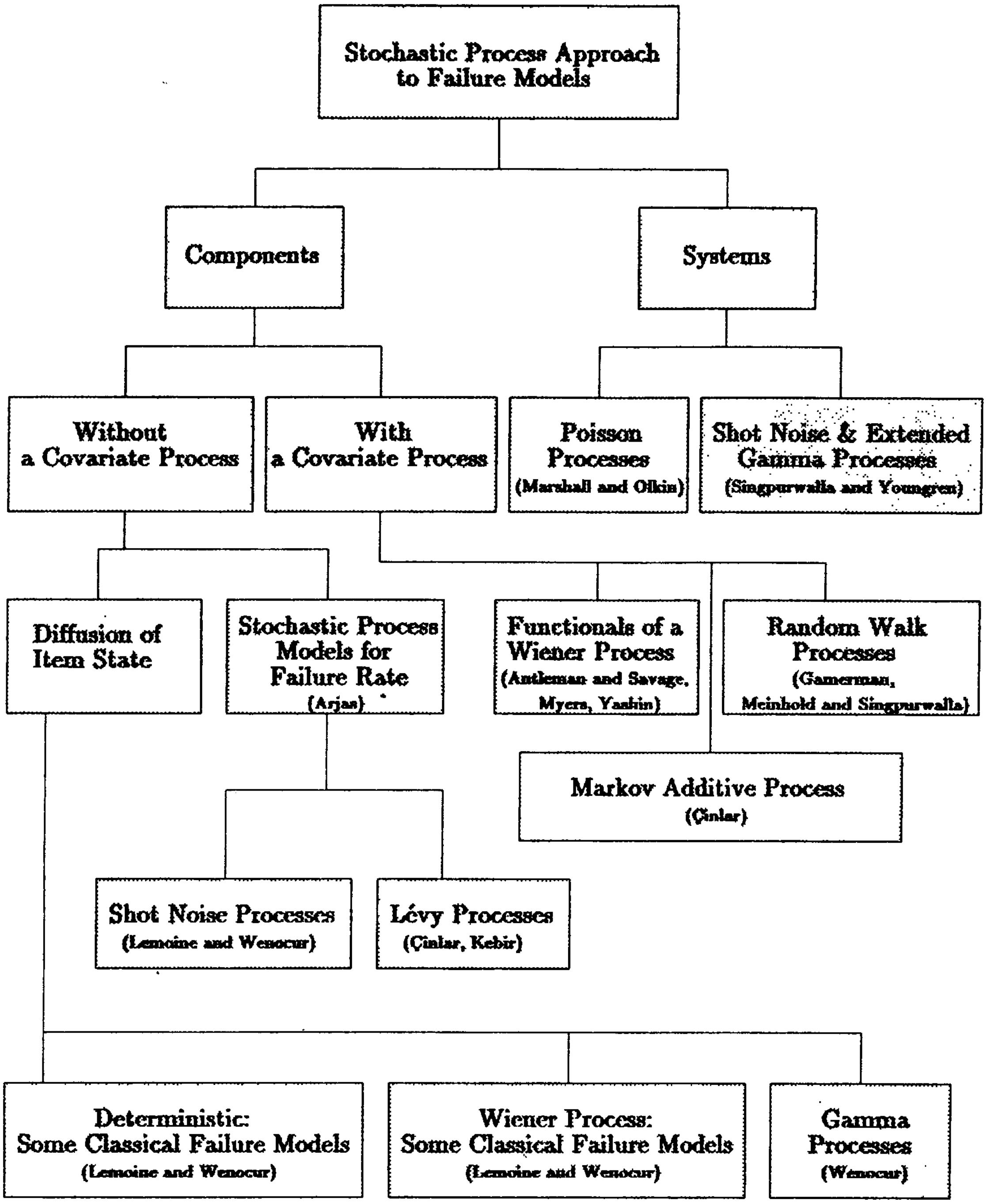

FIG. 1. *Modelling strategies and a road map to the literature.*

appear to have been first envisioned by Mercer (1961) in his remarkably imaginative and far-sighted paper. However, it is Çinlar (1972) who, via his notion of Markov additive processes, puts this idea on a firm mathematical basis and in so doing provides the needed boost for its formal development.

2. ORGANIZATION

The organization of the rest of this paper is as follows. Section 3, which constitutes the bulk of what is written here, pertains to models for the failure of components, whereas Section 4 pertains to failure models for multicomponent systems and networks. Work in the arena of the reliability of multicomponent systems, though currently sparse, is very important; it shows signs of extensive further development. Section 5 concludes the paper with an indication of directions for continued research in the interesting and potentially useful ideas that are described in Sections 3 and 4.

Section 3 is broken down into four subsections, each devoted to a specific modelling strategy. Thus, Section 3.1 pertains to the premise that the state of a component changes over time, with the various state transitions being described by a stochastic process. The processes considered are a deterministic diffusion, a Wiener process and a gamma process. In Section 3.2, it is the failure rate of the item that is stochastic. This is in contrast to the traditional approach wherein the failure rate is taken to be a deterministic function of time with unknown parameters. Processes that are considered for describing the stochastic nature of the failure rate function are the shot-noise process and the Lévy process, with the gamma and the compound Poisson processes as special cases. Sections 3.3 and 3.4 pertain to failure models derived via consideration of the behavior of covariates that influence failure. Thus, for example, in Section 3.3.1 the covariates are assumed to be time-varying and are described by a Brownian motion, whereas in Section 3.3.2 they are described by a random walk. Section 3.3.1 leads us to the famous Cameron–Martin formula. Section 3.4 is different in spirit from Section 3.3; it involves a consideration of two stochastic processes, one for the item state and the other for the covariate that drives the state. This potentially fertile idea, though proposed by Çinlar as early as 1972, offers much promise regarding a sense of the future, but remains to be fully exploited.

Section 4 is based on the notion that dependencies between component lifelengths are due to a common but dynamic environment in which the components function as a system, and that the effects of the environment are described by a stochastic process. Two processes are considered: the first is an extension of the gamma process and the second is a shot-noise process. The former generalizes the shock model framework of Esary, Marshall and Proschan (1973) and yields the famous multivariate exponential distribution of Marshall and Olkin (1967) as a special case. The latter results in a new family of multivariate distributions with exponential marginals as a special case.

To summarize, much of the material of Sections 3 and 4 provides concrete probabilistic results pertaining to models for failure and survival. The implication of these models for practical applications and the associated inferential matters are topics that need to be researched.

3. MODELS FOR THE FAILURE OF COMPONENTS

The simplest way to start is via the setup of Section 3.1, which considers the failure of an item due to the competing causes of deterioration and trauma. The deterioration is described by some stochastic process, and the rate of occurrence of trauma by a Poisson process with a state-dependent intensity function. The proposed setup is quite natural and intuitively appealing; it leads to a general expression for the survival function of the item [see equation (3.5)]. A consideration of special cases for the stochastic process (Sections 3.1.1–3.1.3) leads us to either an alternative perspective on some of the commonly used failure models or to some new failure models. The alternative perspectives facilitate a critique of some of the commonly used models. By expanding the inventory of plausible failure distributions, the new models provide added flexibility for dealing with failure data.

3.1 Modelling the Item State by a Stochastic Process

Suppose that the time axis is divided into contiguous and equispaced intervals of length h, and that the successive endpoints of the intervals are denoted by $h, 2h, 3h, \dots, nh, \dots$. Let $X(n)$ denote the state of an item at time nh, $n = 1, 2, \dots$, where $X(n)$ is some physically observable entity, such as the size of a crack.

The *item wear* is defined as the increment $X(n+1) - X(n)$, and the following discrete approximation to the diffusion of $X(n)$ has been considered by Lemoine and Wenocur (1985) and by Wenocur (1989):

$$X(n+1) - X(n) = \sigma(X(n))\varepsilon_n + \mu(X(n))h, \tag{3.1}$$

where the $\{\varepsilon_n\}$ are independent and identically distributed (i.i.d.) variables, with expectations $E(\varepsilon_n)$ and variances $\mathrm{Var}(\varepsilon_n)$; σ and μ are functions of their arguments. Thus if the item at the start is x, then over h units of time its wear will increase on the average by $\sigma(x)E(\varepsilon_n) + \mu(x)h$, and the standard deviation of the average increase in wear is $\sigma(x)[\mathrm{Var}(\varepsilon_n)]^{1/2}$. Clearly, if $\sigma(\cdot)$, $\mu(\cdot)$ and ε_n are nonnegative, then the wear is monotonically increasing. It should be noted, however, that there are many situations in which the wear need not be increasing. For instance, cracks due to fatigue loading sometimes reflect a tendency to heal.

The item is said to have failed when its state reaches a level (say, r) or when a traumatic event such as a shock of large magnitude annihilates it. Suppose that traumatic events occur as a Poisson process with a rate (called *killing rate*) which depends on the item's state. The latter is a meaningful assumption, since the weaker the item, the more susceptible it is to failure due to trauma. Then, per the postulates of a Poisson process,

$$\begin{aligned} &P\{\text{item fails due to trauma by } h \text{ time} \\ &\quad \text{units} \mid \text{item state remains at } x,\ x \neq r\} \\ &\qquad = 1 - \exp\{-hk(x)\}, \end{aligned} \tag{3.2}$$

where $k(x)$ is the killing rate when item state is x. Thus given $\mathbf{X}(j) = [X(0), X(1), \ldots, X(j)]$, a history of the item states, and assuming that $X(i) \neq r$, $i = 0, \ldots, j$,

$$P\{\text{item is surviving at time } jh \mid \mathbf{X}(j)\} = \exp\left\{-h \sum_{i=0}^{j-1} k(X(i))\right\} I_{[0,r)}(X(j)), \tag{3.3}$$

where $I_{[0,r)}(X(j))$ is 1 if $X(j) < r$, and 0 otherwise; recall that jh represents the endpoint of the jth interval. Conditioning only on $X(0) = x$, and letting $jh \to t$,

$$P\{\text{item is surviving at } t \mid X(0) = x\} = E^x\left[\exp\left\{\int_0^t - k(X(s))\, ds\right\} I_{[0,r)}(X(t))\right], \tag{3.4}$$

where $E^x[\cdot]$ denotes expectation of the argument within the square brackets, with $X(0)$ fixed at x. A scenario in which $X(0)$ is random is considered by Crowder, Kimber, Smith and Sweeting (1991, page 74). However, their development incorporates results from fracture mechanics for developing a suitable failure model.

Thus if T is the time to failure of the item and $\mathscr{T}$ is the time to first passage of the $X(\cdot)$ process to the level r, then a general expression for the survivability of the item for a *mission time* (say, t) is of the form

$$\begin{aligned} P(T > t \mid X(0) = x) &\overset{\text{def}}{=} P^x(T > t) \\ &= E^x\left[\exp\left\{-\int_0^t k(X(s))\, ds\right\} I_{\{\mathscr{T} > t\}}\right]. \end{aligned} \tag{3.5}$$

The next step would be to evaluate (3.5) under specific assumptions about the i.i.d. sequence $\{\varepsilon_n\}$. For this, it is helpful to go back to (3.1) and to consider its continuous-time analogue. The discrete version (3.1) is conceptually easier to motivate and is therefore chosen as a starting point. Accordingly, we allow h to decrease to 0, and we focus on the resulting stochastic differential equation

$$dX(t) = \sigma X(t)\, d\gamma(t) + \mu[X(t)]\, dt, \tag{3.6}$$

where $\{\gamma(t)\}$ will be a *gamma process* if the $\{\varepsilon_n\}$ are assumed to be i.i.d. gamma, and it will be a *Wiener process* if the $\{\varepsilon_n\}$ are assumed to be i.i.d. Gaussian. The essential features of these two processes will be summarized in Sections 3.1.2 and 3.1.3, respectively; however, we note now that in integral terms (3.6) becomes

$$\begin{aligned} X(t) = X(0) &+ \int_0^t \sigma[X(s)]\, d\gamma(s) \\ &+ \int_0^t \mu[X(s)]\, ds, \end{aligned} \tag{3.7}$$

a form that will be subsequently used.

3.1.1 *A deterministic diffusion for item state.* Suppose that in (3.7), $\sigma(\cdot) = 0$ and that $X(0) = x$. Then the process $\{X(t), t \geq 0\}$ satisfies the integral equation

$$X(t) = \int_0^t \mu(X(s))\, ds + x,$$

or, alternatively, the following differential equation with initial condition $X(0) = x$:

$$dX(t) = \mu[X(t)]\, dt.$$

The assumption $\sigma(\cdot) = 0$ implies that the deterioration of the item is deterministic, and hence the expectation operator of (3.5) is not needed. Consequently,

$$P^x(T > t) = \exp\left\{-\int_0^t k(X(s))\, ds\right\} I_{\{\mathscr{T} > t\}}. \tag{3.8}$$

Equation (3.8) is noteworthy because it relates the notion of the failure rate, familiar to those working in reliability and in survival analysis, to that of the killing rate. Specifically, (3.8) suggests that under a deterministic diffusion the failure rate at time u equals the state-dependent killing rate at u. For example, if the survival function of T were judged to be a Weibull with scale parameter α and shape parameter β, that is, if $P(T \geq t \mid \alpha, \beta) = \exp(-\alpha t^{\beta})$, then this judgment is equivalent to stating that an item's failure is due to the competing causes of a deterministic deterioration of the item state and the occurrence of trauma with a killing rate at time t of $\alpha\beta t^{\beta-1}$. Note that $\alpha\beta t^{\beta-1}$ is also the failure rate of a Weibull distribution at time t. Furthermore, if $\beta \neq 1$ and $k(x) = x^{\beta-1}$, then $X(t) = t(\alpha\beta)^{1/(\beta-1)} = tw$, say, so that $\mu(x)$, the rate of deterioration of the item, is a constant w. If $\beta = 1$, that is, if the survival function is judged to be an exponential, then $k(X(t)) = \alpha$, and this implies that either the killing rate is a constant or that the item wear is 0, or both. Lemoine and Wenocur (1985) draw similar analogies when the survival function is a Gumbel or a Makeham distribution. This is the promised alternative perspective on some well-known failure models such as the exponential, the Weibull and the Gumbel. But has anything noteworthy been gained by this alternative perspective? The answer is yes; the deterioration of an item is generally a function of usage and/or the operating environment, and since the latter are typically time-varying and random the assumption of a deterministic deterioration is a gross simplification. Consequently, the routine use of the exponential and Weibull distributions in reliability and survival analysis, even when defended by formal goodness-of-fit tests, needs to be reevaluated. The physics of failure may not support such a choice. The failure models derived in Sections 3.1.2 and 3.1.3 appear to have a more realistic motivation and thus are worthy of serious consideration. Indeed, they may provide a better goodness-of-fit to failure data than the exponential and the Weibull distributions.

3.1.2 *A Wiener process diffusion for item state.* Suppose that the diffusion of the item state is described by a Wiener process, also known as *Brownian motion*. Specifically, this means that the process $\{\gamma(s),\ s \geq 0\}$ of (3.7) is such that $\gamma(0) = 0$, $\gamma(s)$ has stationary independent increments and, for every $s > 0$, $\gamma(s) \sim \mathcal{N}(0, c^2 s)$ for some $c > 0$. If $\gamma(s) \sim \mathcal{N}(\mu s, c^2 s)$ for some $\mu > 0$, then the process is called a *Brownian motion with drift*. With either case, $\gamma(s)$ is a continuous function of s. The independent increments property of the Wiener process implies that, for any $t > 0$,

$$P\{\gamma(t+s) \leq a \mid \gamma(s) = x, \gamma(u), 0 \leq u \leq s\} = P\{\gamma(t+s) \leq a \mid \gamma(s) = x\},$$

and so the Wiener process is a *Markov process*.

A disadvantage of assuming a Wiener process for the diffusion of item state is that the wear is no longer monotonically increasing. In many applications the assumption of decreasing wear may not be meaningful; an exception is the healing of cracks caused by fatigue. An advantage is that the mathematics of Wiener processes has been studied extensively. More important, as will be described below, some commonly used distributions in reliability and survival analysis can be motivated as special cases for diffusion under such processes. For example, consider the simplifying scenario in which trauma occurs at a rate that is independent of item state, that is, $k(x) = \lambda$, the case of *constant killing*. Then the expectation operator of (3.5) is again not necessary, and

$$P^x(T > t) = \exp(-\lambda t) P(\mathcal{T} > t). \tag{3.9}$$

Thus it suffices to derive the distribution of the first passage time $\mathcal{T}$.

Calculating the first passage time distribution for an arbitrary diffusion is generally quite difficult. However, there is a significant class of diffusions that gives rise to mathematically tractable first passage time distributions. For example, it is well known (see Barndorff-Nielsen, Blæsild and Halgreen, 1978), that for certain diffusion processes with drift, such as the Wiener process, the *generalized inverse Gaussian distribution* with a nonpositive power parameter is the first passage time distribution. The inverse Gaussian, the inverted gamma, the hyperbolic and, as a limiting case, the gamma are special case of the generalized inverse Gaussian distribution. Distributions such as the gamma and the inverse Gaussian have in the past been used as failure models; thus the above development can be viewed as an alternative motivation for these distributions. However, since the assumption that the rate of occurrence of trauma is independent of the item state is not always realistic, one must refrain from the routine use of the gamma and the inverse Gaussian distributions, even when a goodness-of-fit test supports their choice.

3.1.3 *A gamma process diffusion for item state.* The use of a gamma process for diffusion of item state can be justified on the basis of two considerations, the main one being that under a gamma process the item wear is always nondecreasing. The other consideration stems from the fact that the gamma process is the limit of a *shot-noise process with exponential decay*. Shot-noise processes are discussed by Cox and Isham (1980) and are natural in many practical situations where the environment is comprised of shocks and jolts; more will be said on these in Section 3.2.1.

A gamma process for item state results if the $\{\gamma(s);\ s \geq 0\}$ of (3.7) is described by a gamma process. Specifically, this means that $\gamma(0) = 0$, that $\{\gamma(s)\}$ has stationary independent increments and that, for $t > s$, $(\gamma(t) - \gamma(s))$ has density g_{t-s}, which is a gamma density with scale 1 and shape $(t - s)$ [i.e., $g_h(u) = e^{-u} \cdot u^{h-1}/\Gamma(h)$]. Like the Wiener process, the gamma process also possesses the Markov property.

To proceed further, it is helpful to generalize the last term of (3.4) by writing $I_{[0,r)}(X(t))$ as $f(X(t))$, where $f(\cdot)$ is some function. The ensuing expression,

$$P^x(T > t) = E^x\left[\exp\left\{-\int_0^t k(X(s))\,ds\right\} f(X(t))\right], \tag{3.10}$$

known as the *Kac functional equation*, facilitates a use of known results. In particular, under certain conditions on the functions $\sigma(\cdot)$, $\mu(\cdot)$, $f(\cdot)$ and $k(\cdot)$, equation (3.10), the reliability of the item, can be evaluated. This has been done by Wenocur (1989), from which the following examples are taken.

If, for all $s \geq 0$, $\sigma(s) = 1$, $\mu(s) = 0$, $f(s) = 1$ and $k(s) = s$, that is, the item has a constant killing rate, the critical state $r = +\infty$ and the diffusion of item state is described by a gamma process without an underlying drift [i.e., $\mu(\cdot) \equiv 0$], then, using some limiting arguments to show that $P^x(T > t)$ satisfies the Feynman–Kac equation, Wenocur (1989) argues that

$$(3.11)\quad P^x(T > t) = \exp(-t(x-1) - (1+t)\ln(1+t)).$$

Observe that, when $x = 1$,

$$Y = (1+T)\ln(1+T) \sim \text{exponential}(1).$$

If the gamma process has a drift, say, $\mu(s) = 1$, for all $s \geq 0$, then

$$(3.12)\quad P^x(T > t) = \exp\left\{-\ln(1+t)(1+t) + t(1-x) - \frac{t^2}{2}\right\}.$$

If, for all $s \geq 0$, $\sigma(s) = \mu(s) = 1$, $f(s) = I_{[0,r)}(s)$ and $k(s) = 0$, that is, the failure of the item is due to wear alone, then

$$(3.13)\quad P^x(T > t) = \int_0^{(r-x)^+} \frac{y^{t-1}e^{-y}}{\Gamma(t)}\, dy,$$

the cumulative of a gamma density with shape parameter t, the mission time.

Thus consideration of a gamma process diffusion for the item state results, via (3.11)–(3.13), in a new family of failure models whose properties remain to be explored and whose utility needs to be established. Of the three new failure models given by (3.11)–(3.13), the first two appear to be idealized because of the assumption that the item state never reaches a critical value. The model given by (3.13) is quite intriguing because the mission time t has also become the shape parameter of an underlying distribution. For a fixed value of $(r-x)^+$, $P^x(T > t)$ decreases in t. This decrease implies that the underlying distribution shifts toward the right as the mission time t increases, which is a true for a gamma distribution with shape parameter t. The model also has an intuitive appeal because failure of many items is more likely to be due to wear than due to trauma.

3.2 Modelling Item Failure Rate by a Stochastic Process

Dynamic environments exert random stresses on the item. Since the notion of stress is a conceptual one (i.e., stress is not a directly observable entity), one way to capture the effect of stress on the item's lifelength is by its failure rate. An approach for modelling the lifelength of items in a dynamic environment is to describe the item's failure rate by a stochastic process. Arjas (1981) has referred to such processes as *hazard rate processes*. To develop failure models based on an assumed form for the hazard rate process, two candidate processes are considered: the shot-noise process, to be discussed in Section 3.2.1; and the *Lévy process*, to be discussed in Section 3.2.2. Despite the fact that these processes lead to survival functions that have interesting functional forms, they lack a stronger motivation. In contrast, the material of Section 3.3 is more appealing because it motivates, via consideration of the stochastic behavior of covariates, a functional of a Wiener process and a random walk process for the failure rate function. However, before proceeding further, we wish to point out an interesting relationship between a hazard rate process and a *doubly stochastic Poisson process* (also known as a *Cox process*).

Let T denote the lifelength of the item, and let $\{\lambda(s),\ s \geq 0\}$ be a nonnegative, real-valued, right-continuous process. Then $\{\lambda(s),\ s \geq 0\}$ is said to be the *hazard rate process of* T if

$$P(T \geq t \mid \lambda(s), 0 \leq s < t) = \exp\left[-\int_0^t \lambda(s)\, ds\right].$$

Consequently,

$$P(T \geq t) = E\left\{\exp\left[-\int_0^t \lambda(s)\, ds\right]\right\}, \quad t \geq 0.$$

Next, suppose that $\{N(s),\ s \geq 0\}$ is a doubly stochastic Poisson process with a stochastic intensity function $\{\lambda(s),\ s \geq 0\}$, and let S denote the time until the arrival of the first event. Then, for $t \geq 0$,

$$P(S \geq t) = E\left\{\exp\left[-\int_0^t \lambda(s)\, ds\right]\right\} = P(T \geq t).$$

Thus the lifelength of an item in a dynamic environment whose effect is described by a hazard rate process $\{\lambda(s),\ s \geq 0\}$ can also be viewed as the time to first arrival in a doubly stochastic Poisson process with intensity function given by the process $\{\lambda(s),\ s \geq 0\}$. A rigorous proof of the above relationship is in Grandell (1976).

3.2.1 *A shot noise for the hazard rate process.* The shot-noise process provides a natural setting for describing the time-dependent effects of damage due to nontraumatic events.

Suppose that an item operates in a dynamic environment whose net effect is to inflict shocks of varying magnitude. Suppose that shocks occur over time u, according to a Poisson process with rate

$m(u)$, $u \geq 0$, and that the consequence of each shock is a stress on the item contributing to its failure rate. Specifically, if a shock of magnitude D occurs at an epoch S, then at time $S + t$ the contribution of this shock to the item's failure rate is $Dh(t)$, where the *attenuation function* $h(t)$ is a nonincreasing function of t. One example wherein the assumption of a decreasing attenuation function is meaningful is the human heart muscle's tendency to heal after a heart attack. Another example is from materials science and pertains to cracks due to fatigue which tend to close up after the material has borne a load which has caused the cracks to grow. In many applications, however, $h(t)$ would be a constant function of t.

If $\{S_n, n \geq 1\}$ are the epochs at which the shocks occur and $\{D_n, n \geq 1\}$ are the respective shock magnitudes, then the item's failure rate at time t is

$$\lambda(t) = \sum_{n=1}^{\infty} D_n h(t - S_n),$$

where $h(t) = 0$, when $t < 0$.

Suppose that $\lambda(0) \equiv 0$ and that the D_n's are the realizations of a random variable D. Also, suppose that the sequences $\{D_n\}$ and $\{S_n\}$ are both serially and contemporaneously independent.

With stress comes susceptibility to failure, and if for some parameter $k \geq 0$ the rate of occurrence of fatal trauma, given that the failure rate is x, is kx, then T, the time to failure of the item, would be such that

$$P\{T < \delta + u \mid T > u, \lambda(u) = x\} = \delta kx + o(\delta).$$

It now follows that $R(t)$, the survival function of T, is given by

$$(3.14)\quad P\{T > t\} = E\left\{\exp\left(-\int_0^t k\,\lambda(u)\,du\right)\right\}.$$

If $\mathscr{L}^*$ is the Laplace transform of the distribution of D, $H(t) = \int_0^t h(u)\,du$ and $M(t) = \int_0^t m(u)\,du$, then it can be shown (see Lemoine and Wenocur, 1986, or Singpurwalla and Youngren, 1993) that

$$(3.15)\quad \begin{aligned} R(t) &= \exp[-M(t)] \\ &\cdot \exp\left[\int_0^t \mathscr{L}^*(kH(u))m(t-u)\,du\right]. \end{aligned}$$

Special cases of the above result provide us with concrete probabilistic results for some new families of failure distributions, distributions which have not previously been considered as failure models. For example, suppose the following: that all the shocks are of a constant magnitude, so that $D = d$; that $h(u) = (1 + u)^{-1}$, implying that the shock-induced stress decays slowly; that $\int_0^t m(u)\,du = mt$, so that the shock-generating process is a homogeneous Poisson; and that $kd = 1$. Then

$$(3.16)\quad R(t) = e^{-mt}(1 + t)^m.$$

Observe that this form of $R(t)$ is the survival function of a Pareto distribution of the third kind (cf. Johnson and Kotz, 1970, page 234). Thus we have here an alternative motivation for the Pareto as a failure model.

If, on the contrary, D has an exponential distribution with parameter b, then, for $a > 0$, $h(u) = \exp[-au]$, $\int_0^t m(u)\,du = mt$ and $k = 1$,

$$(3.17)\quad \begin{aligned} R(t) &= \exp\left(-\frac{mabt}{1+ab}\right) \\ &\cdot \left\{\frac{1 + ab - \exp(-at)}{ab}\right\}^{mb/(1+ab)}. \end{aligned}$$

The above form of the survival function is unfamiliar, but interesting. It is interesting because, for $a = b = 1$ and $m = 2$, equation (3.17) implies that $X \overset{\text{def}}{=} \exp(-T)$ has a beta distribution on (0, 1), with parameters 1 and 2. Since inference procedures for the beta distribution based on Bernoulli trials are well established, a scenario which transforms lifelengths to follow a beta distribution appears to have an attractive advantage.

3.2.2 *A Lévy process for the hazard rate*. Suppose that the hazard rate process $\{\lambda(s), s \geq 0\}$ is right-continuous and increasing with independent increments. Candidates for such processes, to name a few, are the Poisson and the compound Poisson processes; the gamma process; and stable processes (Breiman, 1968, page 316). The *Itô decomposition* (Itô, 1969) of $\{\lambda(s), s \geq 0\}$ facilitates an evaluation of the survival function generated by such processes. Specifically, $\lambda(s)$ can be represented as

$$(3.18)\quad \lambda(s) = r(s) + \sum_{s_i \leq s} W_i + \int_0^s \int_{\mathbb{R}^+} zN(du, dz),$$

where the following hold:

- $r(s)$ is a deterministic, increasing and continuous real-valued function of $s \geq 0$;
- the s_i's are some fixed points on the positive real line $\mathbb{R}^+$;
- the W_i's are mutually independent, strictly positive random variables which may or may not depend on the s_i's;
- N is a Poisson measure on the space $(\mathbb{R}^+ \times \mathbb{R}^+)$, with mean $n(du, dz)$, independent of the W_i's; that is, for every Borel subset B of $(\mathbb{R}^+ \times \mathbb{R}^+)$, $P(N(B) = j) = \exp[-n(B)](n(B))^j/j!$.

If the increments of the process described above have a stationary distribution, then the resulting process is known as a *Lévy process*. The Itô decomposition for Lévy processes has the simplifying features that $r(s)$ is linear in s, say, $r(s) = \alpha s$, where the constant $\alpha \geq 0$ is known as the *drift rate* of the process, and that there are no random variables W_i. Furthermore, the mean $n(du, dz) = du\, \nu(dz)$, where $\nu(dz)$ does not depend on u; $\nu(dz)$ is known as the *Lévy measure*.

Besides its generality, a motivation for describing the hazard rate by a Lévy process is due to the following striking result of Kebir (1991), who shows that the survival function of T is given by

$$R(t) = \exp\left\{-\frac{\alpha t^2}{2}\right\} \cdot \exp\left\{-\int_0^t ds \cdot \int_0^\infty [1 - \exp(-(t-s)x)]\nu(dx)\right\}. \tag{3.19}$$

The result is striking because its implication is that the reliability of an item with a Lévy process for its hazard rate is identical to the reliability of an item with a *deterministic* hazard rate $h(t)$, where

$$h(t) = \alpha t + \int_0^\infty [1 - \exp(-tx)]\nu(dx).$$

Special cases of (3.19) yield some models of failure that appear to be new. For example, suppose that the Lévy measure of the hazard rate process is of the form

$$\nu(dy) = by^{-1}\exp(-cy)\,dy, \quad y \geq 0.$$

Then, for $t \geq 0$,

$$R(t) = \exp\left[-\int_0^t b \ln\left(\frac{c+s}{c}\right) ds\right], \tag{3.20}$$

implying that the item has a deterministic hazard rate function of the form $b \ln((c+s)/c)$, which for $s \geq 0$ is increasing in s. A further simplification of (3.20) results in the survival function

$$R(t) = e^{bt}\left(\frac{c}{c+t}\right)^{b(c+t)}, \quad t \geq 0,$$

which to the best of our knowledge is not of any recognizable form.

As a second illustration of the use of (3.19) for generating some meaningful failure models, suppose that $\{\lambda(s), s \geq 0\}$ is a compound Poisson process with a constant intensity ω and with increments having distribution ϕ. Then, for $n(ds, dy) = \omega\, ds\, \phi(dy)$, it can be seen (cf. Kebir, 1991) that

$$R(t) = \exp\left[-\int_0^t \omega\left(1 - \mathscr{L}_\phi(s)\right) ds\right], \tag{3.21}$$

where $\mathscr{L}_\phi$ is the Laplace transform of ϕ.

The very general result above is of value because it provides us with a computable expression for the survival function, especially because hazard rate processes which are compound Poisson are easy to conceive. For example, consider an item operating in an environment comprised of shocks with each shock inflicting a random amount of damage to the item. If the shock-generating process is assumed to be Poisson and the successive amounts of damage assumed to have a distribution ϕ, then the assumption of a compound Poisson process for the hazard rate is meaningful. It is perhaps useful to distinguish between the shock model scenario of Esary, Marshall and Proschan (1973) and the scenario described above. In the former, the amount of some physically observable entity, such as the amount of wear or the size of a crack, is modelled by a compound Poisson process, whereas here the hazard rate (which is an unobservable entity) is so modelled. With the former, the item fails when the observable entity reaches a threshold, whereas here there is no parallel notion of a threshold for the hazard rate.

As a special case of (3.21), suppose that ϕ is a gamma distribution with shape (scale) parameter $b(c)$. Then $\mathscr{L}_\phi(s) = (c/(c+s))^b$, and so, for $t \geq 0$, the survival function of the item is

$$R(t) = \begin{cases} e^{-wt}\left(\dfrac{c+t}{c}\right)^{wc}, & \text{if } b = 1, \\ e^{-w}\left(t + \dfrac{c}{b-1}\left(\dfrac{c}{c+t}\right)^{b-1} - \dfrac{c}{b-1}\right), & \text{if } b \neq 1. \end{cases}$$

Observe that, for the case $b = c = 1$, $R(t)$ would be identical to (3.16).

3.3 Covariate Induced Hazard Rate Processes

The new classes of survival models in Section 3.2 were based on assuming two particular forms for the hazard rate process: a shot-noise process and a Lévy process. The assumed forms were intuitively satisfactory, but because one cannot directly observe the failure rate these forms lack a definitive justification. One way of overcoming this reservation is to model directly the stochastic behavior of the covariates believed to influence the hazard rate, and, by so doing, to induce a hazard rate process. We emphasize that inducing a hazard rate process via a covariate process does not in itself make the former realistic. Rather, the premise here is that it is conceptually more direct to propose a particular form of a stochastic process for an observable entity such as a covariate than for an unobservable entity such as the failure rate.

With the above in mind, we consider two approaches for specifying particular forms of the covariate process. In Section 3.3.1 the covariates are assumed to evolve over time according to a Brownian motion, whereas in Section 3.3.2 the coefficients associated with each covariate follow a random walk. A special case of the former results in the hazard rate process being a functional of the Wiener process, and this in turn leads to the famous formula of Cameron and Martin (1944) and its multidimensional version (Liptser and Shiryayev, 1977). The Cameron–Martin formula enables us to compute the characteristic function of the definite integral of a squared Brownian motion process. The approach of Section 3.3.2 was designed to facilitate a statistical analysis of lifetime data using the dynamic linear models technology of West and Harrison (1989); however, the setup is attractive enough to be regarded as a general strategy for failure modelling.

3.3.1 *The hazard rate as a functional of Wiener processes—the Cameron–Martin formula.* The use of Brownian motion (Wiener process) to describe hazard rate processes has appeared in the biostatistical and demographic literature; Yashin (1985) provides an overview. The motivation in biostatistics stems from the following scenario. In a clinical trial, $\mathbf{Z}$, the vector of physiologic, laboratory and hematologic variables, together with treatment indicators, is measured at the time of entry into a study. The survival time T, viewed as a dependent variable, is regressed on $\mathbf{Z}$ using techniques such as those introduced by Cox (1972). A linear diffusion with time-dependent, continuous and nonrandom coefficients is used to describe the changes in $\mathbf{Z}$ over time. This framework is used to induce a hazard rate process that satisfies a linear stochastic differential equation driven by a Wiener process. Some highly technical machinery involving the solution of a matrix Ricatti equation (cf. Myers, 1981) is then invoked to obtain the survival function of T. The special case in which the diffusion process is one-dimensional with constant coefficients results in the Cameron–Martin formula.

Specifically, following Myers (1981), suppose that the m-dimensional covariate vector $\mathbf{Z}_s$ represents the values at time s of the physiologic, environmental or treatment variables relevant to the viability of an individual. If the instantaneous hazard at s is $\lambda(s, \mathbf{Z}_s, \beta(s))$, where $\beta(s)$ is a parameter function (a vector or a matrix), then the survival function of T, given that the unit is surviving at t_0 and follows the covariate path $\mathbf{Z}_s = \mathbf{z}_s$, is, for $t \geq t_0$, of the form

$$R(t \mid t_0, \mathbf{z}_s, \beta(s)) = \exp\left(-\int_{t_0}^{t} \lambda(s, \mathbf{z}_s, \beta(s))\, ds \right).$$

It is important to bear in mind that the above exponential formula is valid only if the elements of the covariate vector $\mathbf{Z}_s$ are what Kalbfleisch and Prentice (1980, page 123) call "external covariates." With "internal covariates," difficulties of the type mentioned by Yashin and Arjas (1988) arise, and the exponential formula is no longer valid; also see Singpurwalla and Wilson (1995).

In clinical trials, often covariates cannot be tracked continuously over time so that it is not possible to obtain $\mathbf{z}_s$, $s \geq 0$. Rather, all that can be observed is $\mathbf{z}_0$, the value of $\mathbf{Z}_0$ at t_0, the time of entry into the trial. In this case, the relevant survival function is the conditional expected survival obtained by averaging the above expression over all covariate paths $\mathbf{z}_s$ which pass through $(t_0, \mathbf{z}_0)$, that is,

$$\begin{aligned} &R(t \mid t_0, \mathbf{z}_0, \beta(s)) \\ &\quad = E\left\{ \exp\left(-\int_{t_0}^{t} \lambda(s, \mathbf{Z}_s, \beta(s))\, ds \right) \Big| \mathbf{z}_0 \right\}. \end{aligned} \tag{3.22}$$

With $t_0 = 0$, the similarities among (3.22), (3.10) and (3.14) are quite apparent.

As it stands, (3.22) is too general for most purposes, but the choice of a diffusion for $\mathbf{Z}_s$ with the instantaneous hazard being a time-dependent quadratic function of $\mathbf{Z}_s$ makes the needed computations feasible. Recall that with (3.10) and (3.14) the assumptions of a gamma process for item state and a shot noise for the hazard rate process facilitated the computations. The choice of a diffusion for $\mathbf{Z}_s$ appears to be more realistic for applications to reliability than for biostatistics and demography. This is because in many engineering scenarios the applied stresses and loads, such as fatigue and temperature, tend to fluctuate rapidly around a central value and are better described as diffusions. Sobczyk and Spencer (1992, page 117) discuss examples in which fatigue cracks are described by diffusion processes.

Suppose that, for some matrix B, vector C and constant d, the instantaneous hazard $\lambda(s)$ is the quadratic function $\lambda(s) = \mathbf{Z}_s^* B \mathbf{Z}_s + 2C^* \mathbf{Z}_s + d$, where $\mathbf{Z}^*$ denotes the transpose of a vector $\mathbf{Z}$. If B is symmetric, then $\lambda(s)$ can be written as $\lambda(s) = \mathbf{Y}_s^* B \mathbf{Y}_s + \nu$, where $\mathbf{Y}_s = \mathbf{Z}_s - \mu$, and ν and μ are constants. If B is positive definite, then μ can be interpreted as the healthiest point in the covariate space. The incorporation of aging effects in the model can be achieved by letting $B = B(s)$, a continuous deterministic function of time. Substituting the parameterization for $\lambda(s)$ in (3.22) leads us to

$$E\left\{\exp\left(-\int_0^t \mathbf{Y}_s^* B(s)\mathbf{Y}_s\, ds\right)\right\},$$

whose general solution has been obtained by Myers (1981) when $\mathbf{Y}_s$ is of the form $d\mathbf{Y}_s = a(s)\mathbf{Y}_s\, ds + b(s)\, d\mathbf{W}_s$, with $a(s)$ and $b(s)$ continuous, and $\mathbf{W}_s$ is a standardized m-dimensional Brownian motion with independent coordinates. Myers' proof involves an exponential martingale in $\mathbf{Z}_s$ with Itô's lemma applied indirectly to the exponent. His result, though explicit, is not in closed form. Specifically, his theorem states that, under the conditions given above,

$$\begin{aligned} R(t \mid 0, \mathbf{Z}_0, B, \mu, v) &= \exp\Big(-rt + (\mathbf{z}_0 - \mu)^* U_t(0)(\mathbf{z}_0 - \mu) \\ &\qquad + \operatorname{tr}\int_0^T b(s)b^*(s)U_t(s)\, ds\Big), \end{aligned}$$

where $U(s) = U_t(s)$ is a solution of the matrix Riccati equation

$$\begin{aligned} B(s) &= \dot{U}(s) + (U(s) + U^*(s))a(s) \\ &\quad + \tfrac{1}{2}(U(s) + U^*(s))b^*(s)(U(s) + U^*(s)), \end{aligned}$$

with terminal condition $U(t) = 0$. Effective algorithmic methods for solving the general Riccati equation are given by Bellman and Kalaba (1965).

The survival function $R(t \mid 0, \mathbf{Z}_0, B, \mu, r)$ can be evaluated in closed form if $\mathbf{Y}_s$ is one-dimensional and the coefficients $a(s)$ and $b(s)$ constant, say, $a \geq 0$ and $b \geq 0$. In particular, if $\lambda(s) = BZ_s^2$ and if $Z_s = Z_0 + bW_s$ is a one-dimensional Brownian motion with scale b and which starts at Z_0, then the logarithm of the survival function $R(t \mid 0, Z_0, B)$ is such that

$$\text{(3.23)} \qquad \begin{aligned} -\frac{d}{dt}\ln R(t \mid 0, Z_0, B) &= BZ_0^2 \operatorname{sech}^2(Gt) + G\tanh(Gt), \end{aligned}$$

where $G = (2b^2B)^{1/2}$.

Because the right-hand side of (3.23) is the failure rate of an item at time t, this result implies that the reliability of an item having the square of a one-dimensional Brownian motion with constant coefficients for its hazard rate is identical to the reliability of an item with a *deterministic* failure rate $BZ_0^2 \operatorname{sech}^2(Gt) - G\tanh(Gt)$. The foregoing result parallels that of Section 3.2.2, which considered a Lévy process for the hazard rate and which arrived at an analogous conclusion. Note that when $Z_0 = 0$ and $B = b = 1$, the failure rate at t is $\sqrt{2}\tanh(\sqrt{2}\,t)$; it is monotonically increasing but bounded like the failure rate of a gamma distribution. This observation and the text of Section 3.1.2 provide support for a naive use of the gamma distribution as a failure model. Formula (3.23) was first derived by Cameron and Martin (1944), after whom it is now named. Antleman and Savage (1965) appear to have been the first to mention its use in reliability (and in survival analysis). More recently, Yashin (1993) has extended the development for hazard rate processes that are functionals of piecewise-continuous Gaussian martingales, such as the Orstein–Uhlenbeck processes and the pure jump process.

3.3.2 *A random walk model for the evolution of covariate effects.* A recent development in reliability and in survival analysis has been the use of random walk processes to describe the stochastic evolution of the *effects* of the covariates on the survival time T. This activity has been spawned by the success of *dynamic linear models*, also known as *Kalman filter models* or *state-space models*, with problems of time series analysis and forecasting. Even though the initial motivation for considering dynamic linear models in reliability (cf. Meinhold and Singpurwalla, 1987, and Blackwell and Singpurwalla, 1988) and in survival analysis (cf. Gamerman, 1991) centered around failure data and inference, the underlying structure of such models is broad enough to provide a general strategy for model development. The setup described below is prompted by the work of Gamerman (1991); it has some commonality with the material of Section 3.3.1 and uses parallel notation.

Suppose that $\mathbf{Z} = (Z_1, \ldots, Z_p)$ is a row vector denoting the p covariate values believed to influence the instantaneous hazard of T, at time s, via the relationship

$$\text{(3.24)} \qquad \lambda(s, \mathbf{Z}, \beta(s)) = \exp\{\mathbf{Z}^*\beta(s)\},$$

where $\mathbf{Z}^* = (1, \mathbf{Z})$, $\beta(s)$ is a $(p+1) \times 1$ vector of regression parameters of the form $\beta'(s) = (\beta_0(s), \beta_1(s), \ldots, \beta_p(s))$ and $\beta_0(s) = \ln \lambda_0$; λ_0 is known as the *baseline hazard* (cf. Cox, 1972).

In (3.24), the effect of the covariates on the instantaneous hazard changes over time, while the actual covariate values are assumed fixed. Such a strategy is in contrast to that of Section 3.3.1, where the covariate values themselves changed over time so that $\mathbf{Z}$ was indexed by s and a diffusion assumed for $\mathbf{Z}_s$. In biomedicine a motivation for model (3.24) is the scenario in which a single exposure to a drug or a hazardous substance has a progressively deleterious effect on survival. In engineering a motivation is the growth of a crack caused by a single fatigue-causing shock that progressively damages an item and decreases its probability of survival. A dynamic model results when one specifies the manner in which $\beta(s)$ changes with s. However, before we do this, the next step in the model-development process is specifying a particular form for $\lambda(\mathbf{s}, \mathbf{Z}, \beta(s))$. A simplifying choice is the piecewise-constant hazard; specifically, for some integer $k \geq 2$,

$$\lambda(s, \mathbf{Z}, \beta(s)) = \begin{cases} \lambda_1(\mathbf{Z}, \beta(s)), & s \in I_1 = [s_0, s_1], \\ \lambda_i(\mathbf{Z}, \beta(s)), & s \in I_i = (s_{i-1}, s_i], \\ & 2 \leq i < k-1, \\ \lambda_k(\mathbf{Z}, \beta(s)), & s \in I_k = (s_{k-1}, \infty), \end{cases}$$

where $\tau = \{s_1, \ldots, s_{k-1}\}$ is a partition of the time axis with $0 \equiv s_0 \leq s_1 \leq \cdots \leq s_{k-1}$. Thus given $\lambda = (\lambda_1, \ldots, \lambda_k)$ and τ, T is said to have a *piecewise-exponential distribution* with *guide relation*

$$\ln \lambda_i = \ln \lambda_0 + \sum_{j=1}^{p} Z_j \beta_j(i), \tag{3.25}$$

where for $j = 1, \ldots, p$, $\beta_j(i) = \beta_j(s)$, for $s \in I_i$, $i = 1, \ldots, k$.

To specify the evolution of each $\beta_j(s)$ over time s, that is, to specify the *system equation*, a common but simple approach is the random walk model,

$$\beta_j(i) = \beta_j(i-1) + \omega_j(i), \tag{3.26}$$

where ω_j, the vector of innovation terms, equals $\{\omega_j(1), \ldots, \omega_j(k)\}$ and has mean (vector) $\mathbf{0}$ and a specified covariance matrix $\mathbf{W}_j$. With a judicious choice of the $\mathbf{W}_j$'s, relationship (3.26) can be used to provide a suitable degree of smoothness for the function λ_i, $i = 1, \ldots, k$.

With (3.25), (3.26) and an assumed distributional form for the $\omega_j(i)$'s, the specification of a failure model for T is, in principle, complete. For example, if for each j the $\omega_j(i)$'s are assumed to be independently and normally distributed with variances that are functions of I_i, the width of the ith interval, and if the intervals are of equal widths, then a limiting argument would suggest that the diffusion of the λ_i's, $i = 1, \ldots, k$, would result in approximating the process $\{\lambda(s, \mathbf{Z}, \beta(s)); z \geq 0\}$ by a Brownian motion with drift, and the development of Section 3.3.1 would apply. If on the contrary the $\omega_j(i)$'s are assumed to be independently and identically distributed as a gamma and if relationship (3.24) were of the form $\lambda(s, \mathbf{Z}, \beta(s)) = \mathbf{Z}^*\beta(s)$, then a diffusion of the λ_i's, $i = 1, \ldots, k$, would suggest that $\{\lambda(s, \mathbf{Z}, \beta(s); s \geq 0\}$ could be approximated by a Lévy process, and the development of Section 3.2.2 would apply. Thus many scenarios involving a random walk model for the evolution of covariate effects result in the failure rate being described by processes whose treatments have been previously considered.

3.4 Markov Additive Processes for Describing the Item State and a Covariate

Markov additive processes, abbreviated MAP, were introduced by Çinlar (1972). They provide a generalized approach for modelling the lifelengths of items given the stochastic behavior of a covariate. The idea here, using the notation of the previous sections, is that at any time t one looks at two stochastic processes: $X(t) \in \mathbb{R}^+$, representing the state of the item; and $Z(t) \in \mathscr{E}$, representing a covariate which is assumed to excite (influence) $X(t)$. For example, $Z(t)$ could be binary with $Z(t) = 1$ denoting the item in use and $Z(t) = 0$ denoting the item at rest. In general, $Z(t)$ could represent the state of the environment at time t, with $\mathscr{E}$ being either a countable set or a subset of the real line $\mathbb{R}$. Both $X(t)$ and $Z(t)$ are assumed to be right-continuous and have left-hand limits everywhere, and $X(t)$ is nondecreasing. The probabilistic structure of a MAP has been investigated by Çinlar (1972); its key features are summarized below.

For each $z \in \mathscr{E}$ there is a probability measure p^z on a suitably defined probability space such that $p^z(X(0) = 0,\ Z(0) = z) = 1$. Let $Q_t(z, A, B) = p^z(X(t) \in A, Z(t) \in B)$. Then $\{X(t), Z(t), t \geq 0\}$ is a *Markov additive process* if $p^z\{(X(s+t) - X(s)) \in A, Z(s+t) \in B \mid \mathscr{H}(s)\} = Q_t(Z(s), A, B)$, where $\mathscr{H}(s)$ is the history of the process until time s. Let $P_t(z, A) = Q_t(z, A, \mathbb{R}^+)$. Then it can be shown that $\{Z(t), t \geq 0\}$ is a Markov process with a state space $\mathscr{E}$ and transition function $P_t(\cdot,\cdot)$ and that, given $Z(t)$, the conditional law of $X(t)$ is that of an increasing process with independent increments. Furthermore, *locally*, that is, during a small interval of time $(t, t + dt)$, the probability law of $X(t)$ is that of an increasing Lévy process whose parameters depend on the state of the excitation process $Z(\cdot)$. This latter property is useful for describing situations involving the occurrence of many shocks in very small intervals of time (such as fatigue due to vibrations) where no individual shock causes any measurable damage to an item. Item states that are described by increasing Lévy processes such as the compound Poisson, the gamma (see Section 3.1.3) and the increasing stable processes can be viewed as special cases of MAP's with $\mathscr{E}$ having only one element.

Examples of items operating in scenarios meaningfully described by a MAP are given by Çinlar (1977). A few of these are outlined below; they demonstrate the versatility of this failure modelling methodology.

3.4.1 *Compound Poisson shocks in random environments.* Suppose that $\mathscr{E}$ consists of only two states $\mathscr{E} = \{r, w\}$ and that $\{Z(t); t \geq 0\}$ is a two-state Markov process with $Z(t) = r$ denoting the item at rest or in repair at time t and $Z(t) = w$ denoting the item working at t. Suppose that $Z(\cdot)$ stays in w for an exponentially distributed length of time, then jumps to r and stays there for an exponentially distributed length of time and then jumps back to w and so on. Suppose that, when $Z(\cdot)$ is in state w, shocks occur according to a Poisson process with a rate c and that each shock inflicts a random amount of damage to the item. Suppose that the amount of damage is independent of everything else and has a distribution ϕ. When $Z(\cdot)$ is in state r, there are no shocks and therefore no damage. Suppose that damage is cumulative, and let $X(t)$ denote the cumulative damage at time t. Clearly, $\{X(t), Z(t), t \geq 0\}$ is a MAP; its sample path is shown in Figure 2. The conditional probability law of $X(\cdot)$ given $Z(\cdot)$ is described (see Çinlar, 1972) by its Laplace transform at fixed time t. Specifically, for any $\lambda \geq 0$,

$$\begin{aligned} &E^z[\exp(-\lambda X(t)) \mid Z(t), t \geq 0] \\ (3.27) \qquad &= \exp\left[-A_t \int_0^\infty c\phi(dx)(1 - \exp(\lambda x))\right], \end{aligned}$$

where A_t is the amount of time spent by $Z(\cdot)$ in state w during $(0, t]$; thus A_t can be regarded as a parameter of the excitation process. The superscript z on the expectation operator E denotes the fact that expectations are with respect to the measure p^z. The distribution of the time to item failure is the distribution of the hitting time of the $X(\cdot)$ process to some threshold, say, X_0. An analytical determination of this distribution appears to be an onerous task; simulation offers an alternative. An attractive feature of results given by (3.27) is our ability to incorporate the stochastic behavior of the covariate (environment) process as a parameter of the process of interest. A disadvantage is the absence of a closed-form probabilistic expression for the survival function. A possible strategy for easing this difficulty is to use the saddlepoint approximations technique of Daniels (1954) to obtain the marginal distribution of $X(t)$ and then to use the marginal distributions to simulate the distribution of the hitting times. Thus the need for developing computational methodologies for reliability and survival analysis is a pressing one, for only then will the effectiveness of modern stochastic modelling technologies come to fruition.

3.4.2 *General Markov additive process with a finite* $\mathscr{E}$. Suppose that $\{Z(t); t \geq 0\}$ is a Markov process with a finite state space $\mathscr{E}$; one may view $Z(t)$ as the state of the environment at time t. Let $X(t)$ denote the cumulative damage at time t, and suppose that, when $Z(t) = i$, $X(t)$ increases as a Lévy process with a drift rate $\alpha(i)$ and Lévy measure $\nu(i, dx)$ (see Section 3.2.2). Increases in damage over the different environments are assumed to be linearly additive. In addition, suppose that every change of state from i to j is accompanied by a shock which causes an additional random amount of damage whose distribution is $F(i, j, \cdot)$. The amount of damage accompanying a shock is assumed to be independent of the cumulative damage at the time the shock occurs. The process $\{X(t), Z(t); t \geq 0\}$ is a MAP. Analogously to (3.27),

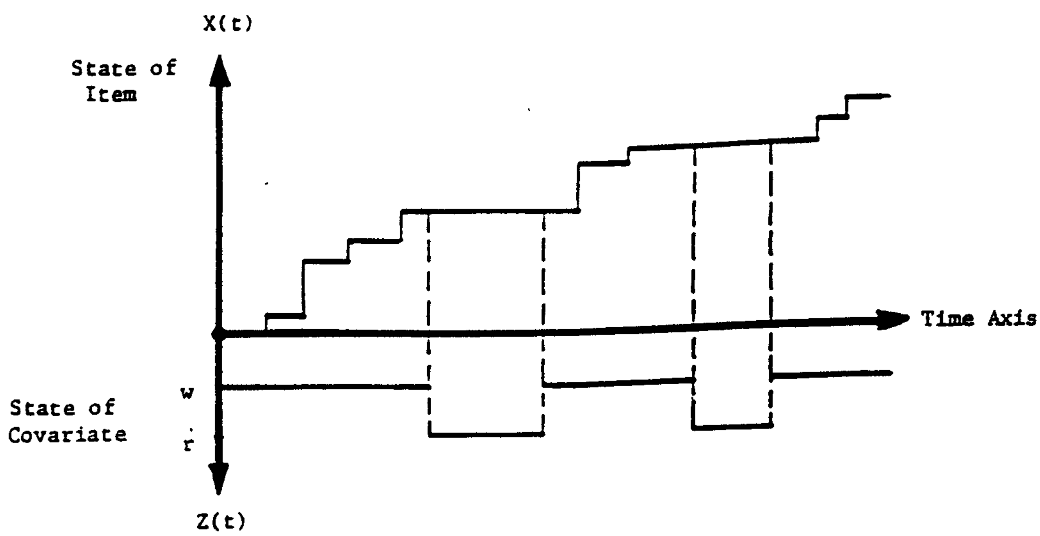

FIG. 2. *Sample path of a two-state Markov additive process with Poisson shocks (from Çinlar, 1977).*

given $Z(\cdot)$, the conditional law of $X(\cdot)$ is, for some constant $\lambda \geq 0$, described by

$$
\begin{aligned}
&E^z[\exp(-\lambda X(t)) \mid Z(t), t \geq 0] \\
&\quad = \exp\left[-\lambda \int_0^t \alpha(Z(s))\,ds - \int_0^t ds \right. \\
(3.28) &\qquad \left. \cdot \int_0^\infty \nu(Z(s), dx)(1 - \exp(-\lambda x))\right] \\
&\quad \cdot \left[\prod_{s \leq t} F^\lambda(Z(s^-), Z(s))\right],
\end{aligned}
$$

where $F^\lambda(i, j) = \int_0^\infty F(i, j, dx)\exp(-\lambda x)$ if $i \neq j$ and is 1 when $i = j$ (see Çinlar, 1977).

Figure 3 shows the sample path for the above process when $\mathscr{E}$ consists of three states: i, j, and k. The distribution of the time to item failure is the distribution of the hitting time of the $X(\cdot)$ process to a threshold X_0. The comment at the end of Section 3.4.1 now takes on a stronger meaning; we have here a model that is quite realistic and general, but in a form that resists application. As of this writing we are unable to offer any suggestions for the practical implementation of expressions given by (3.28).

3.4.3 *Gamma process in random environments.* As a final illustration of the use of MAP's in failure modelling, we consider the scenario of failures induced by metal creep and fatigue. Let $X(t)$ be the creep level at time t, and let $Z(t)$ be the state of the covariate at t. We suppose that $\{Z(t);\ t \geq 0\}$ is a Markov process and that $\{X(t);\ t \geq 0\}$ is locally a gamma process with a shape $\beta(X(t))$ and scale $\alpha(X(t))$; it has been claimed that gamma processes offer attractive possibilities for modelling creep. Then, analogously to (3.27) and (3.28), it can be shown that

$$
\begin{aligned}
&E^z[\exp(-\lambda X(t)) \mid Z(t), t \geq 0] \\
(3.29) &\quad = \exp\left[-\int_0^t \beta(Z(t)) \ln\left(1 + \frac{1}{\alpha(Z(t))}\right) dt\right].
\end{aligned}
$$

Here again the distribution of the time to failure must be obtained via numerical techniques and simulation though the computational burden appears to be less than that of (3.27) and (3.28).

4. MODELS FOR THE FAILURE OF SYSTEMS

The development of failure models for the reliability of systems and networks of components is an active area of research in reliability and survival analysis. The impetus comes because many available models have been based on the simplifying assumption of component independence; this assumption has resulted in inadequate assessments of system reliability. Failure models for system sur-

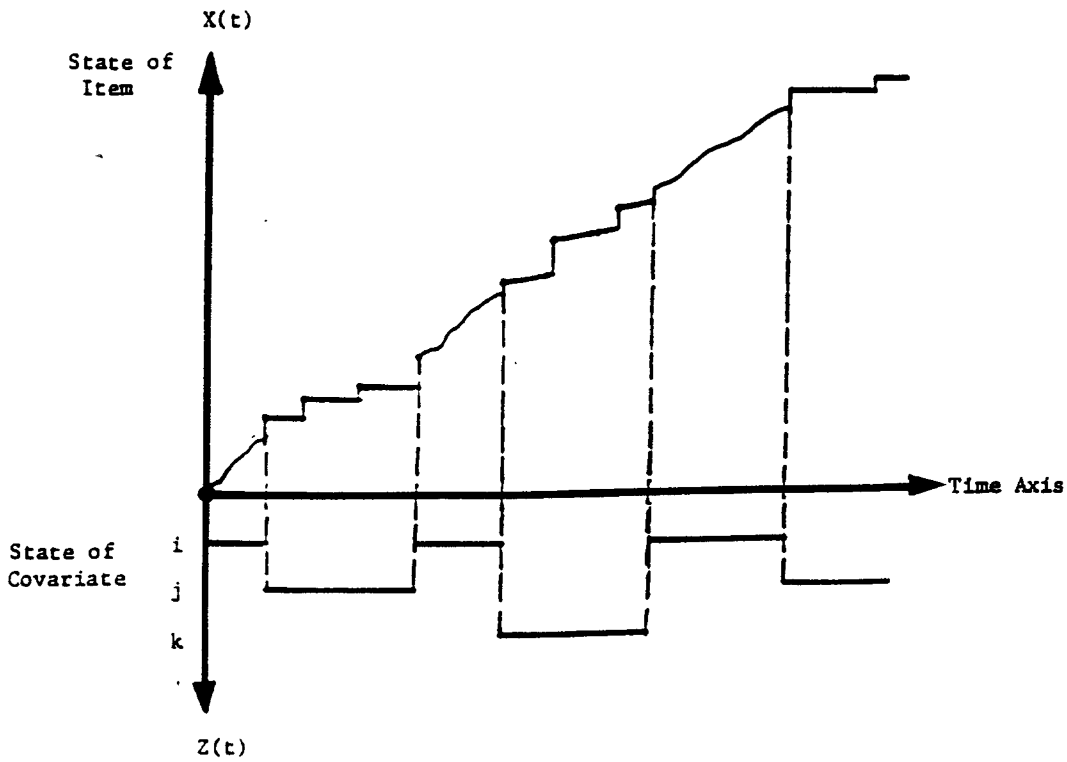

FIG. 3. *Sample path of a three-state Markov additive process with a Lévy process wear and accompanying shocks with change of state (from Çinlar, 1977).*

vival that incorporate component dependencies were proposed as early as 1961 (Freund, 1961), but subsequent progress has been limited. Furthermore, almost all of the existing models assume a static operating environment; exceptions are the work of Marshall and Olkin (1967) and Esary and Marshall (1974). The latter, however, is mainly qualitative and leads to results that are not statistically tractable. The objective of this section is to review a class of models for system reliability when the environment is dynamic; the distribution of Marshall and Olkin will turn out to be a special case. The theme underlying the development of this class of models was echoed by Marshall (1975) and exploited by Lindley and Singpurwalla (1986), who argue that component dependencies are due to a lack of knowledge about the effect of the common operating environment on the lifelengths. The same notion has been used by Çinlar and Ozekici (1987), but their aim has not been the development of parametric classes of failure models.

For the sake of completeness, we point out another strand of research in multivariate failure models for reliability conducted by Shaked and his colleagues and summarized in Marshall and Shaked (1986) and Shaked and Shanthikumar (1993). However, the emphasis of such work has been on the characterization of aging and dependence and their preservation properties as opposed to the development of explicit probabilistic expressions for the failure of multiple components. Related point-process models used in reliability are discussed by Arjas (1981) and by Shaked and Shanthikumar (1993).

4.1 The Setup for Model Development

In what follows attention will be restricted to the case of two-component systems; extension to multicomponent systems is straightforward. Suppose that T_1 and T_2, the lifelengths of the two components when they operate in the laboratory (design) environment, have specified failure rates $\lambda_i(u)$, $u \geq 0$, $i = 1, 2$, respectively. Suppose that the system is installed in an operating (use) environment comprising several covariates (stresses) whose presence and intensities change over time. Assume that the net effect of the operating environment is to modulate $\lambda_i(u)$ to $\lambda_i(u)\eta(u)$, where $\eta(u)$ is assumed to be an unknown function of time, referred to by Singpurwalla and Youngren (1991) as the *environmental factor function* (EFF). This notion of modulating the failure rate is analogous to, though not exactly the same as, that proposed by Cox (1972) for proportional hazards. If at any time u the operating environment is harsher than the design environment, then $\eta(u) > 1$; otherwise, $\eta(u) < 1$. Finally, suppose that, conditional on $\lambda_i(u)$ and $\eta(u)$, the lifelengths of the components are judged to be independent. Since $\eta(u)$ is unknown, its uncertainty is described by a suitable stochastic process $\{\eta(u);\ u \geq 0\}$. Two candidate processes have been considered in the literature; these are discussed in Section 4.2. The lack of knowledge about $\eta(u)$ results in a judgment of dependence between the T_i's. The case $\eta(u) = \eta$, for all $u \geq 0$, with the uncertainty about η described by various distributions, has been considered by several authors; Singpurwalla and Youngren (1993) and Oakes (1989) give reviews. Of particular note, especially for biostatistical applications, is the work of Hougaard (1987), who assumes a positive stable distribution for η and generalizes most of the known results.

4.2 Stochastic Processes for the Environmental Factor Function

The two processes that have been considered for describing the effects of uncertainty about the environmental factor function $\eta(u)$ are the "extended gamma process" of Dykstra and Laud (1981) (also see Çinlar, 1980) and the shot-noise process mentioned in Section 3.2.1. A motivation for the former is given by Cornfield and Detre (1977), Kalbfleisch and Prentice (1980, page 203) and Çinlar (1980); also see Singpurwalla and Youngren (1993), who motivate via the decomposition of the gamma process. A motivation for the shot-noise process was given in Section 3.2.1.

4.2.1 *An extended Gamma process for the environmental factor function.* Dykstra and Laud (1981), in proposing a nonparametric Bayes approach for inference in survival analysis, defined an *extended gamma process*. Specifically, if $\alpha(s)$ is a nondecreasing left-continuous real-valued function on $[0, \infty)$ with $\alpha(0) = 0$ and if $\beta \in (0, \infty)$, then $\{Y(s);\ s \geq 0\}$ is said to be a *gamma process* with parameters $\alpha(s)$ and β. It is denoted $Y(s) \in \mathscr{G}(\alpha(s), \beta)$ if $Y(0) = 0$, if $Y(s)$ has independent increments and if, for $0 \leq s \leq t$, $(Y(t) - Y(s))$ has a gamma distribution with shape $(\alpha(t) - \alpha(s))$ and scale $1/\beta$. The definition of the gamma process given in Section 3.1.3 is a special case of the above when $\alpha(u) = u$, for all $u \geq 0$, and $\beta = 1$. The independent increments property of the gamma process makes it unsuitable for describing many environmental phenomena; thus it does not make sense to describe $\eta(u)$ by a gamma process. However, the fact that a gamma process can be represented as the sum of a countable number of jumps of random heights occurring at random points (Ferguson and Klass, 1972) motivates representing the cumulative effect of the environment on the survival function by a gamma process. Therefore the following generalization of the gamma process has been proposed. Suppose that $\gamma(s)$, $s \geq 0$, is a positive right-continuous real-valued function bounded away from 0 with left-hand limits and that $Z(t) = \int_0^t \gamma(s)\,dY(s)$, where $Y(s)$ is a gamma process with $\beta = 1$. Then $\{Z(t);\ t \geq 0\}$ is called an *extended gamma process* with parameters $\alpha(t)$ and $\gamma(t)$, denoted $\mathscr{G}_E(\alpha(t), \gamma(t))$. This process has independent increments and like the gamma process is a pure jump process. Its Laplace–Stieltjes transform, obtained by Dykstra and Laud (1981) (also Çinlar, 1980), is of the form

$$\mathscr{L}^*_{Z(t)}(\omega) = \exp\left[-\int_0^t \ln(1 + \omega\gamma(u))\,d\alpha(u)\right]. \tag{4.1}$$

Reverting to the two-component system, let $Y(t) = \int_0^t dY(u)$, where $dY(u) = \eta(u)\,du$ whenever $\eta(u)$ exists, and suppose that $Y(t) \in \mathscr{G}(\alpha(t), 1/b)$, where $\alpha(t)$ is differentiable for all t with $\alpha'(t) = a(t)$. Then the cumulative hazard of component i, $\Lambda_i(t) = \int_0^t \lambda_i(u)\,dY(u)$, is such that $\Lambda_i(t) \in \mathscr{G}_E(\alpha(t), \lambda_i(t)/b)$. Using (4.1), it can be shown that, for $0 \leq \tau_1 \leq \tau_2$, the bivariate survival function $\overline{F}(\tau_1, \tau_2) \overset{\text{def}}{=} P(T_1 > \tau_1, T_2 > \tau_2)$ is of the form

$$\begin{aligned} &\exp\left[-\int_0^{\tau_1} \ln\left(1 + \frac{\lambda_1(u) + \lambda_2(u)}{b}\right) a(u)\,du\right] \\ &\cdot \exp\left[-\int_{\tau_1}^{\tau_2} \ln\left(1 + \frac{\lambda_2(u)}{b}\right) a(u)\,du\right], \end{aligned} \tag{4.2}$$

from which it follows that the marginal survival function of component i, $P(T_i > \tau)$, $i = 1, 2$, is

$$\exp\left[-\int_0^{\tau} \ln\left(1 + \frac{\lambda_1(u)}{b}\right) a(u)\,du\right]; \tag{4.3}$$

for details see Singpurwalla and Youngren (1993). The case of $\lambda_i(u) = \lambda_i$, $i = 1, 2$, yields a failure rate for L_i that is a constant times the derivative of $\alpha(\tau)$, the shape parameter of the gamma process. Thus different choices of $\alpha(\tau)$ result in different marginal distributions. One choice is $\alpha(\tau) = \alpha\tau$ for some constant $\alpha > 0$. For this choice the marginal distributions are exponential, and the bivariate survival function (4.2) can be shown to reduce to the bivariate exponential distribution of Marshall and Olkin. If $\alpha(\tau) = \alpha\tau^{\delta}$ for some constants α and $\delta > 0$, then the resulting survival function is the bivariate Weibull of Marshall and Olkin (1967).

These results are of interest because they provide an alternative, stochastic-process-based motivation for one of the most widely discussed multivariate life distributions in reliability theory; furthermore, they provide a vehicle for extending it in new directions. Finally, the fact that the extended gamma process results in a familiar family of multivariate exponential distributions is in itself a good motivation for considering such processes.

4.2.2 *A shot-noise process for the environmental factor function.* Following the notation of the previous section, suppose again that $\lambda_i(u) = \lambda_i$, $i = 1, 2$, but that $\eta(u)$ is now described by a shot-noise process. Then it follows that $\{\lambda_i\eta(u);\ u \geq 0\}$ is also a shot-noise process, where $\lambda_i\eta(u)$ is the failure rate of component i under the operating environment. Given λ_1, λ_2, $M(t)$, $H(t)$ and $\mathscr{L}^*$ (see Section 3.2.1 for their definitions), it has been shown by Singpurwalla and Youngren (1993) that, for $0 \leq \tau_1 \leq \tau_2$, $\overline{F}(\tau_1, \tau_2)$ is given by

$$\begin{aligned} &\mathscr{L}^*[\lambda_1 H(\tau_1) + \lambda_2 H(\tau_2)] \\ &\cdot \exp\left[\int_0^{\tau_1} \mathscr{L}^*\{\tau_1 H(\tau_1 - u_1) + \tau_2 H(\tau_2 - u_1)\} m(u_1)\,du_1\right] \\ &\cdot \exp\left[\int_{\tau_1}^{\tau_2} \mathscr{L}^*\{\tau_2 H(\tau_2 - u_2)\} \cdot m(u_2)\,du_2 - M(\tau_2)\right]. \end{aligned} \tag{4.4}$$

The marginal survival function of T_i at some $\tau \geq 0$ follows and is easily seen to be of the form

$$\mathscr{L}^*[\lambda H(\tau)] \exp \times \left[-M(\tau) + \int_0^{\tau} \mathscr{L}^*\{\lambda H(u)\} m(\tau - u)\,du\right]. \tag{4.5}$$

Observe the similarity between the above result and that given by (3.15), which did not involve the modulation of a base failure rate λ but in which there was included a scale factor k.

Special cases of (4.4) result in some interesting distributions, one of which is a new family of bivariate distributions with exponential marginals. Specifically, if $\lambda_1 = \lambda_2 = \lambda$, $h(u) = 1$ and $m(u) = \lambda/b = m$, where the distribution of the shot magnitudes D is a gamma with shape 1 and scale b, then for $\tau_1, \tau_2 > 0$ the joint survival function is

$$\begin{aligned} &\overline{F}(\tau_1, \tau_2) \\ &= \sqrt{\frac{1 - m \cdot \min(\tau_1, \tau_2) + m \cdot \max(\tau_1, \tau_2)}{1 + m(\tau_1 + \tau_2)}} \\ &\cdot \exp[-m \cdot \max(\tau_1, \tau_2)]. \end{aligned} \tag{4.6}$$

The marginal survival function of T_i at some $\tau \geq 0$ follows from (4.6) and is of the form

$$\overline{F}(\tau) = e^{-m\tau}. \tag{4.7}$$

Thus a shot-noise process environment will preserve the nonaging property of a component if the stress-inducing jolts are Poisson with rate λ/b, if the jolts induce a cumulative damage on the component and if the magnitudes of the induced stresses are exponential with scale parameter b.

The joint survival function (4.6) has been developed via considerations that appear to be realistic and natural. However, as of this writing, its properties (such as the joint moments, decomposition into an absolutely continuous and a singular part etc.) have not been computed. Thus it is hard to assess the value of (4.6) for practical applications. This is a potential arena for additional research.

5. CONCLUDING COMMENTS

It appears that the current practice in reliability assessment and survival analysis focuses on the use of a few parametric families of distributions such as the exponential, the gamma, the lognormal, the Pareto, the Weibull and their mixtures and multivariate analogues, or on nonparametric procedures. Subsequent to this, much effort (both Bayesian and frequentist) has been devoted to inferential issues under complete or censored information. A recent book by Crowder, Kimber, Smith and Sweeting (1991) captures the spirit of the current state of the art in reliability. The parametric families mentioned above have been in use for almost 30 years, and most have been motivated by the behavior of the failure rate function whose general form is subjectively specified using assumptions about aging. Sections 3.1.1 and 3.1.2 provide an alternative, more microscopic, approach for motivating some of these distributions. Our conclusion is that the existing models may be meaningful in static operating conditions, but this situation is rare. The stochastic-process-based approach for developing failure models is relatively new and offers a rich environment in which dynamic operating conditions can be meaningfully captured. In many cases it yields concrete probabilistic results pertaining to some new families of failure models, a few of which are given in this paper. Equations (3.11)–(3.13), (3.16), (3.17), (3.20), (3.21) and (4.6) provide specific examples.

The material reviewed in this paper indicates that a stochastic-process-based approach to failure modelling is a promising area of development, offering possibilities which enhance current practice. The approach better exploits the physics of the failure process and offers potential for improved assessments of item survivability. It also facilitates the modelling of dependencies, an important topic that has not been vigorously explored. Major drawbacks of this more microscopic approach are the complexity of the resulting distributional forms and the lack of available inferential procedures. Thus future work is needed before the applications gap can be filled. The author conjectures that modern numerical and simulation-based techniques such as saddlepoint approximations will provide the tools to bridge the gap between theory and practice. Because of the inherent complexity of the proposed models, inferential procedures will tend to be involved and also call for simulation-based techniques. The Gibbs sampling approach of Gelfand and Smith (1990) and Tierney (1994) offers hope.

Finally, one may ask why we need new models for failure, especially more complicated ones. An obvious response is to enhance the state-of-the-art by expanding the inventory of models and providing added flexibility for fitting failure data. However, such an answer may not be a sufficiently strong argument. Perhaps a more appealing answer is that the proposed models reflect a new paradigm for life-testing experiments in which one also monitors environmental factors under which failure data are obtained; the models of Section 3.4 serve as illustrative examples.

ACKNOWLEDGMENTS

This paper is based on a presentation at the NATO Advanced Research Workshop in Survival Analysis and Related Topics at the Ohio State University, June 24–26, 1991. The author is grateful to Professor A. F. M. Smith for suggesting its formal write-up and for stimulating its development. He is also grateful to Professor Anatoly Yashin for drawing attention to several references and for copies of his preprints. Professors Elja Arjas and Erhan Çinlar, Ms. Sylvia Campodonico, Mr. Jingxian Chen and Ms. Celeste Landry provided valuable comments and other assistance. The several comments by the referees and Professor Robert Kass have helped better define the focus of this paper. This work is supported by Office of Naval Research Contract N00014-94-1-0406; Army Research Office Grant DAAH04-93-G-0020; and Air Force Office of Scientific Research Grant AFOSR-F-49620-95-I-0107.

REFERENCES

A-Hameed, M. and Proschan F. (1973). Nonstationary shock models. *Stochastic Process. Appl.* **1** 383–404.

A-Hameed, M. and Proschan, F. (1975). Shock models with underlying birth process. *J. Appl. Probab.* **12** 18–28.

Antelman, G. and Savage, I. R. (1965). Characteristic functions of stochastic integrals and reliability theory. *Naval Res. Logist. Quart.* **12** 199–222.

Arjas, E. (1981). The failure and hazard processes in multivariate reliability systems. *Math. Oper. Res.* **6** 551–562.

Barndorff-Nielsen, O., Blæsid, P. and Halgreen, C. (1978). First hitting time models for the generalized inverse Gaussian distribution. *Stochastic Process. Appl.* **7** 49–54.

BELLMAN, R. and KALABA, R. (1965). *Quasilinearization and Nonlinear Boundary Value Problems*. North-Holland, Amsterdam.

BLACKWELL, L. M. and SINGPURWALLA, N. D. (1988). Inference from accelerated life tests using filtering in coloured noise. *J. Roy. Statist. Soc. Ser. B* **50** 281–292.

BREIMAN, L. (1968). *Probability*. Addison Wesley, Reading, MA.

CAMERON, R. and MARTIN W. (1944). The Wiener measure of Hilbert neighborhoods in the space of real continuous functions. *J. Math. Phys.* **23** 195–209.

ÇINLAR, E. (1972). Markov additive processes. II. *Z. Wahrsch. Verw. Gebiete* **24** 94–121.

ÇINLAR, E. (1977). Shock and wear models and Markov additive process. In *The Theory and Applications of Reliability* (I. N. Shimi and C. P. Tsokos, eds.) 193–214. Academic, New York.

ÇINLAR, E. (1980). On a generalization of gamma processes. *J. Appl. Probab.* **17** 467–480.

ÇINLAR, E. and OZEKICI, S. (1987). Reliability of complex devices in random environments. *Probability in the Engineering and Informational Sciences* **1** 97–115.

CORNFIELD, J. and DETRE, K. (1977). Bayesian analysis of life tables. *J. Roy. Statist. Soc. Ser. B* **39** 86–94.

COX, D. R. (1972). Regression models and life tables. *J. Roy. Statist. Soc. Ser. B* **34** 187–220.

COX, D. R. and ISHAM, V. (1980). *Point Processes*. Chapman and Hall, London.

CROWDER, M. J., KIMBER, A. C., SMITH, R. L. and SWEETING, T. J. (1991). *Statistical Analysis of Reliability Data*. Chapman and Hall, London.

DANIELS, H. E. (1954). Saddlepoint approximations in statistics. *Ann. Math. Statist.* **25** 631–650.

DESMOND, A. (1985). Stochastic models of failure in random environments. *Canad. J. Statist.* **13** 171–183.

DYKSTRA, R. L. and LAUD, P. (1981). A Bayesian nonparametric approach to reliability. *Ann. Statist.* **9** 356–367.

ESARY, J. D. and MARSHALL, A. W. (1974). Families of components and systems exposed to a compound Poisson damage process. In *Reliability and Biometry* (F. Proschan and R. J. Serfling, eds.) 31–46. SIAM, Philadelphia.

ESARY, J. D., MARSHALL, A. W. and PROSCHAN, F. (1973). Shock models and wear processes. *Ann. Probab.* **1** 627–649.

FERGUSON, T. S. and KLASS, M. J. (1972). A representation of independent increment processes without Gaussian components. *Ann. Math. Statist.* **43** 1634–1643.

FREUND, J. E. (1961). A bivariate extension of the exponential distribution. *J. Amer. Statist. Assoc.* **56** 971–977.

GAMERMAN, D. (1991). Dynamic Bayesian models for survival data. *J. Roy. Statist. Soc. Ser. C* **40** 63–79.

GAVER, D. P. (1963). Random hazard in reliability problems. *Technometrics* **5** 211–226.

GELFAND, A. E. and SMITH, A. F. M. (1990). Sampling-based approaches to calculating marginal densities. *J. Amer. Statist. Assoc.* **85** 398–409.

GRANDELL, J. (1976). *Double Stochastic Poisson Processes. Lecture Notes in Math.* **529**. Springer, New York.

HOUGAARD, P. (1987). Modelling multivariate survival. *Scand. J. Statist.* **14** 291–304.

ITÔ, K. (1969). *Stochastic Processes*. Matematisk Institut, Aarhus Univ.

JOHNSON, N. L. and KOTZ, S. (1970). *Continuous Univariate Distributions. Distributions in Statistics* **1**. Houghton Mifflin, New York.

KALBFLEISCH, J. D. and PRENTICE, R. L. (1980). *The Statistical Analysis of Failure Time Data*. Wiley, New York.

KEBIR, Y. (1991). On hazard rate processes. *Naval Res. Logist.* **38** 865–876.

LEMOINE, A. J. and WENOCUR, M. L. (1985). On failure modeling. *Naval Res. Logist. Quart.* **32** 479–508.

LEMOINE, A. J. and WENOCUR, M. L. (1986). A note on shot-noise and reliability modeling. *Oper. Res.* **34** 320–323.

LINDLEY, D. V. and SINGPURWALLA, N. D. (1986). Multivariate distributions for the life lengths of components of a system sharing a common environment. *J. Appl. Probab.* **23** 418–431.

LIPTSER, R. S. and SHIRYAYEV, A. N. (1977). *Statistics of Random Processes*. Springer, New York.

MARSHALL, A. W. (1975). Multivariate distributions with monotone hazard rate. In *Reliability and Fault Tree Analysis* (R. E. Barlow, J. B. Fussell and N. D. Singpurwalla, eds.) 259–284. SIAM, Philadelphia.

MARSHALL, A. W. and OLKIN, I. (1967). A multivariate exponential distribution. *J. Amer. Statist. Assoc.* **62** 30–44.

MARSHALL, A. W. and SHAKED, M. (1986). Multivariate new better than used distributions: a survey. *Scand. J. Statist.* **13** 277–290.

MEINHOLD, R. J. and SINGPURWALLA, N. D. (1987). A Kalman-filter smoothing approach for extrapolations in certain dose–response, damage-assessment, and accelerated-life-testing studies. *Amer. Statist.* **41** 101–106.

MERCER, A. (1961). Some simple wear-dependent renewal processes. *J. Roy. Statist. Soc. Ser. B* **23** 368–376.

MYERS, L. E. (1981). Survival functions induced by stochastic covariate processes. *J. Appl. Probab.* **18** 523–529.

OAKES, D. (1989). Bivariate survival models induced by frailties. *J. Amer. Statist. Assoc.* **84** 487–493.

SCHÄBE, H. (1990). A new stochastic model for crack propagation. *Quality and Reliability Engineering International* **6** 341–344.

SHAKED, M. and SHANTHIKUMAR, J. G. (1993). Dynamic conditional marginal distributions in reliability theory. *J. Appl. Probab.* **30** 421–428.

SINGPURWALLA, N. D. and WILSON, S. P. (1995). The exponentiation formula of reliability and survival: Does it always hold? *Lifetime Data Analysis,* **1**. To appear.

SINGPURWALLA, N. D. and YOUNGREN, M. A. (1991). Models for dependent life lengths induced by common environments. In *Topics in Statistical Dependence* (H. W. Block, A. R. Sampson and T. H. Savits, eds.) 435–441. IMS, Hayward, CA.

SINGPURWALLA, N. D. and YOUNGREN, M. A. (1993). Multivariate distributions induced by dynamic environments. *Scand. J. Statist.* **20** 251–261.

SOBCZYK, K. (1987). Stochastic models for fatigue damage of materials. *Adv. in Appl. Probab.* **19** 652–673.

SOBCZYK, K. and SPENCER, B. F., JR. (1992). *Random Fatigue*. Academic, New York.

TIERNEY, L. (1994). Markov chains for exploring posterior distributions (with discussion). *Ann. Statist.* **22** 1701–1762.

WENOCUR, M. L. (1989). A reliability model based on the gamma process and its analytic theory. *Adv. in Appl. Probab.* **21** 899–918.

WEST, M. and HARRISON, P. J. (1989). *Bayesian Forecasting and Dynamic Models*. Springer, New York.

YASHIN, A. I. (1985). Dynamics of survival analysis: Conditional Gaussian property versus Cameron–Martin formula. In *Statistics and Control of Stochastic Processes* (N. V. Krylov, R. S. Lipster and A. A. Novikov, eds.) 466–485. Springer, New York.

YASHIN, A. I. (1993). An extension of the Cameron–Martin result. *J. Appl. Probab.* **30** 247–251.

YASHIN, A. I. and ARJAS, E. (1988). A note on random intensities and conditional survival functions. *J. Appl. Probab.* **25** 630–635.