Commun. Math. Phys. 9, 1–18 (1983)

Communications in
Mathematical
Physics

Surviving Extrema for the Action on the Twisted SU(∞) One Point Lattice

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Abstract. We give a simplified construction of "twist eating" configurations, based on a theorem due to Frobenius. These configurations are defined through the equation: $U_\mu U_\nu U_\mu^+ U_\nu^+ = \exp(2\pi i n_{\mu\nu}/N)$ with $U_\mu \in SU(N)$, $\mu = 1$ to d and $n_{\mu\nu}$ an antisymmetric matrix with integer entries. In the (Twisted)-Eguchi-Kawai model they yield extrema some of which survive for $N \to \infty$. Comparison is made with the Monte Carlo data of the internal energy in the small coupling region.

1. Introduction

The recently introduced Twisted-Eguchi-Kawai (TEK) model [2] combines four powerful approaches to the strong interaction theories. The $1/N$-expansion [3], the lattice approximation to space time [4], the twisted boundary condition for gauge fields in a box [5] and the loop-equations [6]. For the Eguchi-Kawai model [1] to work, one needs zero vacuum expectation values for all open loops. This is only guaranteed in the strong coupling region. Quenching [7] has been introduced to extend it to all coupling. It however considerably complicates calculations, unlike the twisted version, where the model is defined by a simple modification of the action:

$$S = \sum_{\mu \neq \nu = 1}^{4} \mathrm{Tr}(1 - Z_{\mu\nu} U_\mu U_\nu U_\mu^+ U_\nu^+). \tag{1}$$

Here U_μ are the SU(N) link variables on a one point lattice and $Z_{\mu\nu} = Z_{\nu\mu}^*$, an element of the centre Z_N of SU(N), is the twist. It is labelled by the twist tensor $n_{\mu\nu}$ through:

$$Z_{\nu\mu}(n) = \exp(2\pi i n_{\mu\nu}/N), \tag{2}$$

where $n_{\mu\nu}$ is an antisymmetric 4×4 matrix with integer entries defined modulo N.

The lower bound of the action (1) is zero and is saturated if and only if there exist four elements Ω_μ of SU(N) satisfying:

$$\Omega_\mu \Omega_\nu \Omega_\mu^+ \Omega_\nu^+ = Z_{\nu\mu}(n). \tag{3}$$

These configurations are called twist eating [8] (zero action solutions with nonzero twist) and were first discussed in the context of the continuum [9]. Because of the revived interest in the solutions to Eq. (3) we think it is useful to present a more straightforward construction for these solutions. It is based on a number theoretic theorem due to Frobenius, which brings $n_{\mu\nu}$ in a suitable standard form. This can be found in Sect. 2, along with relevant examples in Sect. 3.

In Sect. 4 we study the stationary configurations of the action (1). Configurations satisfying Eq. (3), but with $n_{\mu\nu}$ replaced by any allowable twist tensor $m_{\mu\nu}$ obviously are stationary, they are similar to the Z_N excitations discussed in [15]. They are stable (to all orders) if m is "close enough" to n and describe flux-like excitations which we prefer to call *fluxons*[1], to be contrasted with the torons of [17] and vortices on an infinite lattice [15, 19]. We show that in the Twisted-Eguchi-Kawai model there are surviving fluxons for $N\to\infty$, because their action is proportional to $1/N$. They can be compared with the continuum solutions in a box with twisted boundary conditions [10]. There are also stationary points which are not of the above simple form. In Sect. 5 we deal with some relevant applications to the TEK-model and compare with existing Monte Carlo data [2], for which there is reasonable agreement.

In Sect. 6 we give conclusions and an outlook for further progress. The motivation for this work is the "standard" belief that fluxlike excitations are responsible for confinement. Work is in progress to use the fluxons for a better understanding of this problem.

An appendix deals with a technical point on the solution space of Eq. (3), which is completely categorized. A formula for its dimension is given in terms of $n_{\mu\nu}$ and N only.

2. Construction of Twist Eating Configurations

We will first concentrate on the 4 dimensional case. Some comments on generalisation to any dimension $d \geqq 2$ are made at the end of Sect. 3. From the continuum we learn that the existence of a solution to Eq. (3) implies [5, 11] "orthogonal" twist: $\kappa(n)=0 \bmod N$, where $\kappa(n)$ is the Pfaffian of n:

$$\kappa(n)=\tfrac{1}{8}\varepsilon_{\mu\nu\alpha\beta}n_{\mu\nu}n_{\alpha\beta}\,. \tag{4}$$

The converse has also been proved [10, 11], that is construction of solutions to Eq. (3) for all [11] N and orthogonal $n_{\mu\nu}$.

We can view $n_{\mu\nu}$ as an alternating form on the lattice spanned by the four basis vectors $(e^{\mu})_\nu=\delta_{\mu\nu}$. A slightly adapted version of the Frobenius theorem [12] tells us that for any nonzero $n_{\mu\nu}$ there is a base $(f^{\mu})_\nu$ such that $n_{\mu\nu}$ is of the form:

$$n'_{\mu\nu}=(f^{\mu})_\lambda n_{\lambda\sigma}(f^{\nu})_\sigma=\begin{pmatrix} \multicolumn{2}{c}{\theta} & e_1 & 0 \\ & & 0 & e_2 \\ -e_1 & 0 & \multicolumn{2}{c}{\theta} \\ 0 & -e_2 & & \end{pmatrix}, \tag{5}$$

1 Sometimes used for "vortex" continuum solutions [18]

with e_1 the greatest common divisor (gcd) of the absolute value of the entries of $n_{\mu\nu}$ $[e_1 = \gcd(|n_{\mu\nu}|)]$. All this implies that

$$X_{\mu\nu} = (f^\mu)_\nu = \langle f^\mu, e^\nu \rangle \tag{6}$$

is an invertible 4×4 matrix with integer entries. Therefore both $\det X$ and $\det(X^{-1}) = (\det X)^{-1}$ are integer, so $\det X = \pm 1$. Now $\kappa(n') = (\det X)\kappa(n) = -e_1 e_2$, and we simply force $\det X = 1$ by choosing $e_2 = -\kappa(n)/e_1$. Note that the greatest common divisor is invariant under $SL(4, \mathbb{Z})$ transformations, so e_1 divides e_2. [If $\kappa(n) = 0$, we are *free* to choose $\det X = 1$.]

We will now use a trick, also employed by Brihaye [14] in a more restrictive case, to split N into the product of N_1 and N_2, which divide, respectively, e_1 and e_2, which is possible since $e_1 e_2 / N$ is an integer. When $e_2 = 0 \bmod N$, we can immediately apply 't Hooft's procedure [10], which is contained in the following by putting $N_2 = N$ and $N_1 = 1$, with $P_1 = Q_1 = 1$. A solution to Eq. (3) is:

$$\Omega_\mu = P_{N_1}^{y_\mu} Q_{N_1}^{z_\mu} \otimes P_{N_2}^{s_\mu} Q_{N_2}^{t_\mu}, \tag{7}$$

with

$$s_\mu = (X^{-1})_{\mu 1}, \quad t_\mu = \frac{e_1}{N_1}(X^{-1})_{\mu 3}, \quad y_\mu = (X^{-1})_{\mu 2},$$
$$z_\mu = \frac{e_2}{N_2}(X^{-1})_{\mu 4}, \tag{8}$$

and P_M, Q_M SU(M) matrices satisfying [10]:

$$P_M Q_M P_M^+ Q_M^+ = \exp(2\pi i/N). \tag{9}$$ [2]

Equation (8) is just one of the possible solutions to:

$$(s_\mu t_\nu - s_\nu t_\mu)/N_2 + (y_\mu z_\nu - y_\nu z_\mu)/N_1 = n_{\mu\nu}/N,$$

which can be easily checked using (5) and (6). With this information it is not hard to verify that (7) yields a solution to Eq. (3). In the appendix it is shown how this generates the most general solution.

Since knowledge of the matrix X is essential in the construction, we will give a recipe (the interested reader can infer from this the proof of the Frobenius theorem as we stated it above). We look for the minimum of the strictly positive values of $E(a,b) = a_\mu n_{\mu\nu} b_\nu$. This minimum can be shown to be e_1, the greatest common divisor of $n_{\mu\nu}$. There exist f^1 and f^3 which saturate this minimum, so: $e_1 = E(f^1, f^3)$. When Λ is the two dimensional sublattice orthogonal to f^1 and f^3 [i.e. points x on the lattice belong to Λ if and only if $E(x, f^1) = E(x, f^3) = 0$] then projection onto Λ is given by:

$$P_\Lambda x = x + \frac{E(x, f^1)}{e_1} f^3 - \frac{E(x, f^3)}{e_1} f^1; \tag{10}$$

2 The solutions to (9) are all gauge equivalent. See the appendix for details

P_Λ is a linear operator. Therefore the lattice can be seen as the direct sum of the kernel of P_Λ (spanned by f^1 and f^3) and the image Λ of P_Λ. Λ is obviously spanned by the set $\{P_\Lambda e^\mu; \mu=1,2,3,4\}$, but its dimension is two. So we can find f^2 and f^4 which span Λ and together with f^1 and f^3 form a base for the lattice. We already showed that this implies $\det X = \pm 1$. By interchanging f^2 and f^4 or multiplying one of them by -1, we can change $\det X = -1$ into $\det X = 1$. It is not hard to see that we can choose for f^2 and f^4 any two independent vectors from the set $\{P_\Lambda e^\mu\}$. This is how the following examples are constructed.

3. Examples

To illustrate the construction we will give three examples:

$$n_{\mu\nu}=\begin{pmatrix} 0 & -L & -L & -L \\ L & 0 & -L & -L \\ L & L & 0 & -L \\ L & L & L & 0 \end{pmatrix}, \quad N=L^2, e_1=L, e_2=L,$$

$$X=\begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 \end{pmatrix}, \quad \Omega_\mu=\{1_L\otimes Q_L, Q_L\otimes Q_L P_L, P_L\otimes Q_L P_L, 1_L\otimes P_L\}. \tag{11a}$$

This is the so-called symmetric twist, used by Gonzalez-Arroyo and Okawa [2b] to construct the TEK model. It guarantees, at least in weak coupling, zero vacuum expectation values for open loops on a $L\times L\times L\times L$-sublattice. For $N=4$ and 16 they checked it by Monte Carlo calculations. A further example is in the class which gave us most difficulties in our original construction [11]:

$$n_{\mu\nu}=\begin{pmatrix} \multicolumn{2}{c}{\theta} & 4 & 0 \\ & & 2 & 4 \\ -4 & -2 & \multicolumn{2}{c}{\theta} \\ 0 & -4 & & \end{pmatrix}, \quad N=2^4=16, e_1=2, e_2=8,$$

$$X=\begin{pmatrix} 0 & 1 & \multicolumn{2}{c}{\theta} \\ -1 & 2 & & \\ \multicolumn{2}{c}{\theta} & 1 & 0 \\ & & -2 & 1 \end{pmatrix}, \quad \Omega_\mu=\{Q_2\otimes P_8^2, 1_2\otimes P_8, 1_2\otimes Q_8, P_2\otimes Q_8^2\}. \tag{11b}$$

This solution is nevertheless the same as we would obtain from [11, Theorem 4.2].

The two methods are not always equivalent as the last example shows:

$$n_{\mu\nu}=\begin{pmatrix} & \theta & 6 & 0\\ & & 0 & 5\\ -6 & 0 & & \\ 0 & -5 & & \theta \end{pmatrix},\quad N=30, e_1=1, e_2=30=0\,(\mathrm{mod}\,30),$$

$$X=\begin{pmatrix} 1 & -1 & 0 & 0\\ -5 & 6 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 5 & 6\end{pmatrix},\quad \Omega_\mu=\{P^6_{30}, P^5_{30}, Q^6_{30}, Q^{-5}_{30}\}. \tag{11c}$$

The method of [11] yields $\Omega_\mu=\{P_5\otimes 1_2\otimes 1_3,\ 1_5\otimes P_2\otimes Q_3,\ Q_5\otimes 1_2\otimes 1_3,\ 1_5\otimes Q_2\otimes P_3\}$. Another solution suggests itself directly from the simple form of $n_{\mu\nu}$: $\Omega_\mu=\{P_5\otimes 1_6, 1_5\otimes P_6, Q_5\otimes 1_6, 1_5\otimes Q_6\}$. These three solutions can be transformed into each other by a similarity transformation.

Generalisation to $d\neq 4$ is obvious. First of all, the 2-dimensional case is trivial to solve with no condition on $n_{\mu\nu}$. The three dimensional case can be seen as a special case for $d=4$, namely by putting $n_{i4}=0$. The case $d=2n$, $n>2$ is a straightforward generalisation of the 4-dimensional case, but now N is split in general in the product of n integers. For this the pfaffian of $n_{\mu\nu}$ [*no* longer given by (4)] needs to be $0 \bmod N$. We do not know if this is a necessary condition for $n>2$, it also is not sufficient, but it is hardly of any physical relevance at present. The case $2n-1$ is again obtained by "dimensional reduction" from $d=2n$.

4. Extrema for the TEK-Action

If $U_\mu\equiv\Omega_\mu$ defines an extremum for the action (1), we can label the fluctuations around this extremum, as in the background field method [13], by:

$$U_\mu=\Omega_\mu\exp(-X_\mu). \tag{12}$$

where X_μ is an element of the algebra of SU(N) in the fundamental representation. Let λ_i for $i=1$ upto N^2-1 be the generators of SU(N). They are antihermitian $N\times N$ matrices satisfying $\mathrm{Tr}(\lambda_i)=0$ and $\mathrm{Tr}(\lambda_i\lambda_j)=-\frac{1}{2}\delta_{ij}$. Any complex $N\times N$ matrix A can be expanded as $A=\alpha_0 I+\sum_{i=1}^{N^2-1}\alpha_i\lambda_i$, with $\alpha_i\in\mathbb{C}$. If furthermore $\mathrm{Tr}(A)=0$ then $\alpha_0=0$. So $\mathrm{Tr}(XA)=0$, for any X in the algebra of SU(N), implies that $A\equiv 0$. We can apply this to find the equations of motion. Expanding the action (1) up to first order in X_μ and demanding stationarity of the action, we find $\mathrm{Tr}(L_\mu X_\mu)=0$ for any X_μ in the algebra and $\mathrm{Tr}(L_\mu)=0$. So the equations of motion are:

$$L_\mu\equiv\sum_{\nu\neq\mu}[\Omega_\nu(P_{\mu\nu}-P_{\nu\mu})\Omega_\nu^+-(P_{\mu\nu}-P_{\nu\mu})]=0, \tag{13}$$

with $P_{\mu\nu}=P^+_{\nu\mu}$ the plaquette variables:

$$P_{\mu\nu}=Z_{\mu\nu}(n)\Omega^+_\mu\Omega^+_\nu\Omega_\mu\Omega_\nu. \tag{14}$$

Finding the most general solution is very hard. In the EK-reduction scheme one eliminates the translational degrees of freedom, yielding the simple one point lattice. The price one pays is that large groups occur.

As a first step we discuss the most obvious solutions. One of these is of course the analogue of the Z_N-fluctuations [15], for which $P_{\mu\nu}$ is a multiple of the identity. However, not all multiples are allowed, even if $U_\mu \in \mathrm{U}(N)$, we always have $P_{\mu\nu} \in \mathrm{SU}(N)$, and thus it must be an element of the centre:

$$\Omega_\mu \Omega_\nu \Omega_\mu^+ \Omega_\nu^+ = Z_{\nu\mu}(m), \tag{15}$$

where $m_{\mu\nu}$ can be *any orthogonal* twist tensor, not necessarily equal to $n_{\mu\nu}$. Expanding the action S around this solution up to second order gives:

$$\begin{aligned} S(n;m) = 2N \sum_{\mu \neq \nu = 1}^{4} \sin^2\left[\frac{\pi}{N}(n_{\mu\nu} - m_{\mu\nu})\right] \\ + \frac{1}{2} \sum_{\mu \neq \nu = 1}^{4} \cos\left[\frac{2\pi}{N}(n_{\mu\nu} - m_{\mu\nu})\right] \mathrm{Tr}(F_{\mu\nu} F_{\mu\nu}^+), \end{aligned} \tag{16}$$

with

$$\begin{aligned} F_{\mu\nu} = D_\mu X_\nu - D_\nu X_\mu = -F_{\mu\nu}^+ = -F_{\nu\mu}, \\ D_\mu X_\nu = \Omega_\mu^+ X_\nu \Omega_\mu - X_\nu . \end{aligned} \tag{17}$$

X_μ is defined through (12), and Ω_μ is a solution of (15).

It is claimed that the fluxon solution (15) is stable if and only if for all μ, ν:

$$\cos\left(\frac{2\pi}{N}(n_{\mu\nu} - m_{\mu\nu})\right) \geqq 0 . \tag{18}$$

This is obvious up to second order. But there are zero modes which are not connected with symmetries of the action, and in these directions higher order contributions can destabilize the solution, at least in principle. (As an example of these zero modes we mention the fluctuations around singular torons [17], with $n = m = 0$, $U_\mu = I$. Here all fluctuations are zero up to third order, but the quartic term is positive.) The proof goes by induction. We write:

$$\begin{aligned} S_{\mu\nu} = Z_{\mu\nu}(m) \mathrm{Tr}(U_\mu U_\nu U_\mu^+ U_\nu^+) = \mathrm{Tr}(\exp(-D_\mu X_\nu - X_\nu) \\ \cdot \exp(D_\nu X_\mu + X_\mu) \exp(X_\nu) \exp(-X_\mu)) . \end{aligned} \tag{19}$$

Using the Campbell-Baker-Hausdorff formula this can be expressed as:

$$S_{\mu\nu} = \mathrm{Tr}\left(\exp\left(\sum_{i=1}^{\infty} A_i\right) \exp\left(-\sum_{i=1}^{\infty} B_i\right)\right), \tag{20}$$

with

$$B_i = \sum (\beta_i)_{\mu_1 \ldots \mu_i} X_{\mu_1} \ldots X_{\mu_i},$$
$$A_i = (-1)^{i+1} \sum (\beta_i)_{\mu_1 \ldots \mu_i} Y_{\mu_1} \ldots Y_{\mu_i},$$
$$\mu_i = \mu, \nu,$$
$$Y_\mu = D_\nu X_\mu + X_\mu, \qquad Y_\nu = D_\mu X_\nu + X_\nu. \tag{21}$$

The specific values of (β_i) are of no interest to us here. In lowest order $S_{\mu\nu} = \mathrm{Tr}(\exp(A_1 - B_1 - \frac{1}{2}[A_1, B_1] + A_2 - B_2 + \mathcal{O}(X^3))) = \mathrm{Tr}(1 + \frac{1}{2}(A_1 - B_1)^2 + \mathcal{O}(X^3))$, so $A_1 - B_1 = F_{\nu\mu}$. A zero mode implies $A_1 = B_1$, and we do not want a decreasing action in this direction. But $A_1 = B_1$ implies

$$S_{\mu\nu} = \mathrm{Tr}(\exp(A_2 - B_2 + \mathcal{O}(X^3))) = \mathrm{Tr}(1 + \tfrac{1}{2}(A_2 - B_2)^2 + \mathcal{O}(X^5)),$$

which indeed gives an increasing action as long as $A_2 \neq B_2$ and condition (18) is satisfied. The induction argument is now to assume that in all directions except those given by $A_j = B_j$ for all j up to i, S is an increasing function, as long as (18) is satisfied. In this exceptional direction however one has

$$S_{\mu\nu} = \mathrm{Tr}(1 + \tfrac{1}{2}(A_{i+1} - B_{i+1})^2 + \mathcal{O}(X^{2i+3})),$$

and thus S is increasing in all directions except those given by $A_j = B_j$ for all j up to $i+1$. So S stays constant along the direction X_μ with $A_j = B_j$ for *all* j, including at least the gauge directions, and increases in all other directions, which proves the claimed stability up to all orders if condition (18) is satisfied.

The weight factor in the functional integral is traditionally given by $\exp(-g^{-2}S) = \exp(-\beta S)$, where g is the bare lattice coupling constant. In the $N \to \infty$ limit $g^2 N$ is kept fixed, so we naively expect only those configurations to survive for which S is proportional to $1/N$ (compare [10]); at least for g^2 small enough. Thus for the fluxon's satisfying Eq. (15) we must have $\sin\left[\frac{\pi}{N}(n_{\mu\nu} - m_{\mu\nu})\right]$ of order $1/N$, which also means that all these surviving fluxons must be minima of the action.

A generalization is obtained by putting $P_{\mu\nu} - P_{\nu\mu} = i\lambda_{\mu\nu} I$. Unitarity implies $|\lambda| \leqq 1$ and up to a gauge transformation:

$$P_{\mu\nu} + P_{\nu\mu} = \begin{pmatrix} \sqrt{1-\lambda_{\mu\nu}^2}\, I_{n_+} & \theta \\ \theta & -\sqrt{1-\lambda_{\mu\nu}^2}\, I_{n_-} \end{pmatrix} \tag{22}$$

with $N = n_+ + n_-$. The *most general* expansion of the action around an extremum up to second order is given by:

$$S(n) = \sum_{\mu \neq \nu = 1}^{4} \mathrm{Tr}\{1 - \tfrac{1}{2}(P_{\mu\nu} + P_{\mu\nu}) - \tfrac{1}{2}([D_\mu X_\nu + X_\mu, D_\nu X_\mu + X_\nu] + \tfrac{1}{2}[X_\mu, X_\nu])(P_{\mu\nu} - P_{\nu\mu}) + \tfrac{1}{2} F_{\mu\nu}(P_{\nu\mu} + P_{\nu\mu}) F_{\mu\nu}^+\}. \tag{23}$$

So apart from the fact that the extremum is unstable if $n_- \neq 0$, we have furthermore that $S(n) \geqq 2$ if there is a (μ, ν) pair with $n_- \neq 0$. In conclusion extrema of the form $P_{\mu\nu} - P_{\nu\mu} = \lambda I$ are stable iff they survive for $N \to \infty$ iff Eqs. (15) and (18) are satisfied.

From (23) we learn that probably any extremum with $P_{\mu\nu}-P_{\nu\mu}$ not a multiple of I is unstable. Furthermore any extremum which survives will stabilize for $N\to\infty$; they will become approximate minima, somewhat analogous to multiinstantons built up from single instantons at a large average separation. For $\mathrm{Tr}(1-\frac{1}{2}(P_{\mu\nu}+P_{\nu\mu}))\sim 1/N$ implies $P_{\mu\nu}-P_{\nu\mu}\sim 1/N$ and $\frac{1}{2}(P_{\mu\nu}+P_{\nu\mu})\sim 1+1/N^2$. These extrema can be important too; one might suspect a large entropy for them. The final class of solutions we managed to construct are of this type, for this we make the ansatz:

$$\Omega_\mu \in \mathrm{U}(N_1)\oplus\mathrm{U}(N_2)\oplus\ldots\oplus U(N_i)/\mathrm{U}(1), \tag{24}$$

where $N=\sum_{j=1}^{i} N_j$ and $\oplus$ is the direct sum. So we choose simultaneously all Ω_μ in a block form. The equations of motion restricted to each block are of the same form as Eq. (13). The phase factors in each block are irrelevant in constructing solutions, so one can choose $\Omega_\mu=\sum_{j=1}^{i}\Omega_\mu^{(i)}$, $\Omega_\mu^{(i)}\in\mathrm{SU}(N_i)$. This problem we already solved, and we can have only surviving extrema if $\Omega_\mu^{(j)}$ are of the Z_{N_j}-type, i.e.:

$$\Omega_\mu^{(j)+}\Omega_\nu^{(j)+}\Omega_\mu^{(j)}\Omega_\nu^{(j)}=\exp(2\pi i m_{\mu\nu}^{(j)}/N_j), \tag{25}$$

with of course orthogonal twist: $\kappa(m^{(j)})=0 \bmod N_j$. We could interpret these solutions as superpositions of fluxons; "separation" is now in the group manifold instead of space-time.

5. Applications

We will discuss situations relevant for the TEK-model and for which Monte Carlo data are available [2]. We will restrict ourselves here to the internal energy in the weak coupling (large β) domain. The internal energy is defined through:

$$E(n,\beta)=\left\langle\frac{1}{12N}\sum_{\mu\neq\nu=1}^{4} Z_{\mu\nu}(n)\,\mathrm{Tr}(U_\mu U_\nu U_\mu^+ U_\nu^+)\right\rangle=1+\frac{1}{12N}\frac{d}{d\beta}\ln Z(n,\beta). \tag{26}$$

Here $Z(n,\beta)$ is the partition function based on the action S of Eq. (1) [dU is the SU(N) invariant Haar measure]:

$$Z(n,\beta)=\int\prod_{\mu=1}^{4} dU_\mu \exp(-\beta S). \tag{27}$$

The expansion around the absolute minimum of the action $S_0(n)$ gives the following weak coupling behavior (see the appendix for a derivation):

$$E(n,\beta)=1-\frac{S_0(n)}{12N}-\frac{(N^2-\frac{1}{3}(i(n)^2+2))}{8N\beta}+\mathcal{O}(g^3). \tag{28}$$

The lower bound for the action is zero, it can be saturated only if the twist $n_{\mu\nu}$ is orthogonal. The minimum action manifold is the solution space to Eq. (3). In the appendix we show that for these solutions:

$$i(n)=\gcd(n_{\mu\nu}, N, \kappa(n)/N). \tag{29}$$

If $i(n)=1$ all solutions are gauge equivalent (see footnote 5), but for $i(n)\neq 1$ there are singular points in the solution manifold, where quartic fluctuations occur, similar to the singular torons of [17]. These torons correspond to $n_{\mu\nu}=0$, $i(n)=N$, and the fluctuations around the most singular (=neutral) torons $U_\mu=I$ dominate the partition function. One finds $E(n=0,\beta)=1-\frac{(N^2-1)}{12N\beta}+\mathcal{O}(g^3)$, using (28) with $S_0(n)=0$. This phenomenon of "ground state metamorphosis" also occurs if there is nonzero twist, with $i(n)\neq 1$, because the fluctuations around singular points will dominate in that case[3]. Details of this highly nontrivial fact can be found in the appendix. For any orthogonal twist $n_{\mu\nu}\neq 0$ we therefore find the result [2]: $E(n,\beta)=1-\frac{1}{8}\frac{N}{\beta}$ for $N\to\infty$, $i(n)/N\to 0$ and $\beta\to\infty$. This corresponds to the expression for the infinitive lattice. There is good agreement with the Monte Carlo data of [2a, b].

For the orthogonal twist in [2a] ($Z_{\mu\nu}=-1$, $N=4k$) we have $i(n)=\frac{1}{2}N$. Here the weak coupling behavior $E(n,\beta)=1-\frac{11}{96}\frac{N}{\beta}$ mildly deviates from the infinite lattice case. This should not surprise us since only the smallest open loops are guaranteed to have zero vacuum expectation value. For the symmetric twist we expect full agreement for $N\to\infty$, and indeed here $i(n)=1$, so at least the weak and strong coupling region are in excellent agreement with the infinite lattice case. We can also understand that fluctuations remain small for β/N not too small and $i(n)=1$[4].

If $n_{\mu\nu}$ is not orthogonal, $S_0(n)>0$, the lower bound $S=0$ can not be saturated. We assume that the minimum action configuration is of the Z_N-type [Eq. (15)] with m orthogonal and closest to n, by which we mean that $\sum_{\mu\neq\nu=1}\cos\left[\frac{2\pi}{N}(n_{\mu\nu}-m_{\mu\nu})\right]$ is maximal. The full stability of the Z_N-fluxons is consistent with the minimum action condition. Note however that the groundstate will be generate, whenever there is more than one twist tensor $m^{(0)}_{\mu\nu}$, which is orthogonal and closest to $n_{\mu\nu}$ in the above sense. The internal energy now behaves as:

$$E(n_1\beta)=1-\frac{1}{6}\sum_{\mu\neq\nu=1}^{4}\sin^2\left[\frac{\pi}{N}(n_{\mu\nu}-m^{(0)}_{\mu\nu})\right]-\frac{(N^2-\frac{1}{3}(i(m^{(0)})^2+2))}{8N\beta}+\mathcal{O}(g^3). \quad (30)$$

From this we deduce that there is an upper bound for the asymptotic value of the internal energy within the class of *nonorthogonal* twist $n_{\mu\nu}$:

$$E(n,\infty)\leqq 1-\frac{1}{3}\sin^2\left(\frac{\pi}{N}\right). \quad (31)$$

It is saturated if $\sum_{\mu>\nu=1}^{4}\left(n_{\mu\nu}-m^{(0)}_{\mu\nu}\right)^2=1$. To compare with Monte Carlo data [2a]

3 Contrary to what was used in [17, Eq. (6.1)]

4 Fluctuations will be of order $(i(n)-2)(i(n)-1)/(24N\beta)$ because of competition between regular and singular fluxons for moderately large β/N [17] (see also the appendix)

we take $N=2(2k+1)$ and $n_{\mu\nu}=\frac{N}{2}$ for all $\mu>\nu$. This yields $\kappa(n)=\frac{N^2}{4}=\frac{N}{2} \bmod N$, and we choose as an example to saturate the bound (31):

$$m_{\mu\nu}^{(0)}=\begin{pmatrix} 0 & -\frac{N}{2} & -\frac{N}{2} & -\frac{N}{2} \\ \frac{N}{2} & 0 & -\frac{N}{2} & -\frac{N}{2} \\ \frac{N}{2} & \frac{N}{2} & 0 & 1-\frac{N}{2} \\ \frac{N}{2} & \frac{N}{2} & \frac{N}{2}-1 & 0 \end{pmatrix}. \tag{32}$$

There are 12 distinct possibilities for $m_{\mu\nu}^{(0)}$. For all of them $i(m^{(0)})=1$, so within each class all configurations are gauge equivalent. The ground state is 12-fold degenerate. This is of course under the assumption that there are no lower fluxon levels. The asymptotic behavior:

$$E(n,\beta)=1-\frac{1}{3}\sin^2\left(\frac{\pi}{N}\right)-\frac{(N^2-1)}{8N\beta}$$

is in good agreement with Monte Carlo data [2a] for SU(6) and SU(10), so the assumption is not contradicted.

Finally we will construct the action for the first few surviving fluxons in the case of symmetric twist [2b] [Eq. (11a)], within the class we considered. Fluxons of the form (25) will only survive if $\frac{m^{(j)}}{N_j}-\frac{n}{N}=\mathcal{O}\left(\frac{1}{N}\right)$. This implies that $\kappa(m^{(j)}) =\kappa\left(\frac{m^{(j)}}{N_j}\right)N_j^2=\left[\kappa\left(\frac{n}{N}\right)+\mathcal{O}\left(\frac{1}{NL}\right)\right]N_j^2=\frac{N_j^2}{N}\cdot\left(1+\mathcal{O}\left(\frac{1}{L}\right)\right)$. Therefore all $m^{(j)}$ cannot be orthogonal simultaneously. The only surviving fluxons in our class are therefore of the Z_N-type [Eq. (15)]. From Eq. (16) we find for $N\to\infty$ the following expression for the action [with $m_{\mu\nu}=n_{\mu\nu}+l_{\mu\nu}$, and $l_{\mu\nu}=\mathcal{O}(1)$]:

$$S(n,m)=\frac{4\pi^2}{N}\sum_{\mu>\nu} l_{\mu\nu}^2\,. \tag{33}$$

The orthogonality of m gives the following constraints on $l_{\mu\nu}$:

$$\kappa(l)=0\,,\qquad l_{\mu\nu}n_{\alpha\beta}\varepsilon_{\mu\nu\alpha\beta}=0\,. \tag{34}$$

From the second constraint we learn that $\sum_{\mu>\nu} l_{\mu\nu}^2$ is always even. We therefore label the Z_N fluxon levels by an integer k, such that $S_k=\frac{8\pi^2}{N}k$. All Z_N-fluxons of the k^{th} level are thus categorized by $l_{\mu\nu}$ satisfying $\frac{1}{2}\sum_{\mu>\nu} l_{\mu\nu}^2=k$ and the constraints (34). Since surviving fluxons have $\kappa(m)=N$, they all are gauge equivalent for a given $l_{\mu\nu}$.

Table 1. The first eight fluxon levels for the symmetric twist: $n_{\mu\nu}=L, \mu>\nu$, and $N=L^2$, along with a representative for $m_{\mu\nu}$ at each level. N is taken large

$$S_1=\frac{8\pi^2}{N},\ m_{\mu\nu}=\begin{pmatrix} 0 & & & \\ L-1 & 0 & * & \\ L-1 & L & 0 & \\ L & L & L & 0 \end{pmatrix};\qquad S_2=\frac{16\pi^2}{N},\ m_{\mu\nu}=\begin{pmatrix} 0 & & & \\ L-1 & 0 & * & \\ L-1 & L & 0 & \\ L & L+1 & L+1 & 0 \end{pmatrix}$$

$$S_3=\frac{24\pi^2}{N},\ m_{\mu\nu}=\begin{pmatrix} 0 & & & \\ L-2 & 0 & * & \\ L-1 & L & 0 & \\ L+1 & L & L & 0 \end{pmatrix};\qquad S_4=\frac{32\pi^2}{N},\ m_{\mu\nu}=\begin{pmatrix} 0 & & & \\ L-2 & 0 & * & \\ L-2 & L & 0 & \\ L & L & L & 0 \end{pmatrix}$$

$$S_5=\frac{40\pi^2}{N},\ m_{\mu\nu}=\begin{pmatrix} 0 & & & \\ L-2 & 0 & * & \\ L-2 & L & 0 & \\ L & L+1 & L+1 & 0 \end{pmatrix};\qquad S_6=\frac{48\pi^2}{N},\ \text{no solution !}$$

$$S_7=\frac{56\pi^2}{N},\ m_{\mu\nu}=\begin{pmatrix} 0 & & & \\ L+3 & 0 & * & \\ L+1 & L & 0 & \\ L-2 & L & L & 0 \end{pmatrix};\qquad S_8=\frac{64\pi^2}{N},\ m_{\mu\nu}=\begin{pmatrix} 0 & & & \\ L+2 & 0 & * & \\ L+2 & L & 0 & \\ L & L+2 & L+2 & 0 \end{pmatrix}$$

There are some values of k (e.g. $k=6, 11$) for which the constraint (34) cannot be satisfied; we call them forbidden fluxon levels. In Table 1 we give the first 8 levels. In the appendix we consider the Z_N-fluxon contribution to $Z(n, \beta)$.

6. Discussion

Even if there are more solutions to the equations of motion not of the simple form we considered, the above shows there is an interesting non-perturbative structure in the TEK-model. Since the perturbative sector of the TEK-model reproduces beautifully planar expansion [2b], it is hard to see how one should extract confinement from this [16]. This is in accordance with the standard belief that Z_N-vortices are thought to be responsible for confinement [19]. We showed that Z_N-excitations in the form of fluxons are also present for the TEK-model and can survive from a naive point of view for $N\to\infty$.

There are nevertheless several problems to be solved. First of all the Eguchi-Kawai reduction [1] process only guarantees that the Wilson loop expectation values satisfy the same set of equations, but do not necessarily yield the same numerical values, although Monte Carlo data [2] suggest that they do. Secondly the $N\to\infty$ limit is taken in a strange way, namely only covering the integers which are squares. The choice of the symmetric twist also seems very special. It is however not hard to see that any m from Table 1 can be taken as the twist tensor n in the action (1), without changing e.g. the fluxon spectrum. The finite N corrections will however become more complicated.

The Monte Carlo data for the χ-ratio support the conjecture that confinement survives for $N\to\infty$ (compare Fig. 11 of [2b], and for SU(4) see [20]). From renormalization group arguments we expect the string tension to behave as $\sigma(\beta) \sim \left(\frac{48\pi^2\beta}{11N}\right)^{\frac{102}{121}} \cdot e^{-\frac{48\pi^2}{11}\frac{\beta}{N}}$. The best one might hope for in weak coupling approximations is a factor $e^{-8\pi^2\frac{\beta}{N}}$ if there are no lower lying fluxons.

It is hoped that the relative simplicity of a one point lattice will bring us closer to a *quantitative* understanding of confinement.

Appendix

A.1. Dimension of Solution Manifold for d=2

We first study the solutions to Eq. (9) ($P_N, Q_N \in \mathrm{SU}(N)$).

$$P_N Q_N P_N^+ Q_N^+ = z \equiv \exp(2\pi i/N). \tag{A.1}$$

We will often drop the suffix N, if no confusion is possible. It is claimed that all solutions are gauge equivalent to those given in [10]:

$$\hat{Q} = z^{(N-1)/2} \begin{pmatrix} 1 & & & \\ & z & & \theta \\ & & \ddots & \\ \theta & & & z^{M-1} \end{pmatrix}, \quad \hat{P} = z^{(N-1)/2} \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \theta \\ & & \ddots & \ddots & \\ 0 & & \theta & 0 & 1 \\ 1 & 0 & & & 0 \end{pmatrix} \tag{A.2}$$

$\hat{P}$ and $\hat{Q}$ are only simultaneously invariant under transformation with the centre Z_N of SU(N), and the solution space is therefore isomorphic to SU(N)/Z_N or the adjoint representation of SU(N). All solutions to (A.1) are given by:

$$P = \Omega \hat{P} \Omega^+, \quad Q = \Omega \hat{Q} \Omega^+, \quad \Omega \in \mathrm{SU}(N). \tag{A.3}$$

Let us prove the claim by bringing Q to a diagonal form; we will show that all eigenvalues of Q are different and fixed apart from trivial rearrangements. It is necessarily of the form $\hat{Q}$. Finally P is then uniquely fixed to be $\hat{P}$ up to a diagonal gauge which leaves $\hat{Q}$ invariant. So we have:

$$Q_{ij} = \lambda_i \delta_{ij}, \quad P_{ij}\lambda_j = z\lambda_i P_{ij}. \tag{A.4}$$

If π is a N-permutation and sg(π) its sign, we have:

$$\det P = \sum_\pi \mathrm{sg}(\pi) \prod_{i=1}^{N} P_{i\pi(i)} = 1 \neq 0. \tag{A.5}$$

So there is a π such that $\prod_{i=1}^{N} P_{i\pi(i)} \neq 0$, implying:

$$\lambda_{\pi(i)} = z\lambda_i. \tag{A.6}$$

We can write any permutation as a string of cyclic one's, $\pi=((i_1\ldots i_s)(j_1\ldots j_t)\ldots)$ in an obvious notation. Suppose now a cycle of length s occurs, then (A.6) forces $z^s=1$; $s=N$ is the smallest possibility, so π is cyclic itself and after some rearrangements $\lambda_i=z^{i-1}\lambda_1$. Then $\det Q=1$ forces λ_1 to be $z^{(N-1)/2}u$, with u an of the centre. However, $u^{-1}\lambda_i$ defines the same *set* of eigenvalues (Z_N is closed under multiplication) and Q is up to a gauge given by $\hat{Q}$. In this gauge all entries of P are zero except for $P_{i,i+1}$, $i=1$ up to $N-1$ and $P_{N,1}$. From $\det P=1$ we must have $P_{12}P_{23}\ldots P_{N-1N}P_{N1}=(-1)^{N-1}$. We can therefore write $P_{i,i+1}=z^{(N-1)/2}\mu_i\mu_{i+1}^{-1}$, with $\mu_{i+1}=z^{(N-1)/2}P_{i,i+1}^{-1}\mu_i$, where we furthermore used the information that $PP^+=I$, implying $P_{i,i+1}\in \mathrm{U}(1)$. We can therefore choose $\mu_i\in \mathrm{U}(1)$ (we define $N+1$ to be equal to 1 as an index). It is a consistent set of equations, because one easily shows that $\mu_{i+1}=z^{i(N-1)/2}(P_{1,2}\ldots P_{i,i+1})^{-1}\mu_1$, which gives an identity for $i=N$, since $z^{N(N-1)/2}=(-1)^{N-1}$.

Finally we choose μ_1 such that $\prod_{i=1}^{N}\mu_i=1$. Therefore $P=\Omega\hat{P}\Omega^+$, with $\Omega=\mathrm{diag}(\mu_1\ldots\mu_N)\in \mathrm{SU}(N)$. This completes the proof of our claim.

The situation is somewhat more complicated if we want to construct the solution space to

$$PQP^+Q^+=z^n. \tag{A.7}$$

For $n=0$ the solution can be found in [17] in the form of torons. They are simply commuting matrices and the dimension is N^2+N-2. However, the solution manifold contains singular points (the singular torons) whenever there is a degeneracy. Its topology is much more intricate. Set theoretically we can label it as $H_N\oplus H_N\oplus \mathrm{SU}(N)/H_N$ with H_N the maximal abelian subgroup of SU(N). For the TEK-model this point is, however, of no practical importance as we will see. If $n\neq 0$, $1 \bmod N$ we will need some simple results on finite abelian groups, which are all of the form Z_N (compare e.g. [21]). z^n generates a subgroup of Z_N given by

$$\langle z^n\rangle=\{z^{in}|i=0,1,\ldots,o(n)-1\}, \tag{A.8}$$

where $o(n)$ is the order of z^n, the smallest nonzero integer such that $z^{o(n)\cdot n}=1$. We can look at the orbits of $\langle z^n\rangle$ in Z_N, which are given through $u\langle z^n\rangle$, for $u\in Z_N$. The number of disjoint orbits is called the index of $\langle z^n\rangle$ in Z_N, which we will denote by $i(n)$. They completely cover Z_N; one now easily deduces [21]

$$o(n)\cdot i(n)=N, \qquad i(n)=\gcd(n,N). \tag{A.9}$$

Repeating the analysis of the case $n=1$ we can use Eq. (A.7), but now π is allowed to have cycles of length $o(n)$ or a multiple thereof. We can have at most $i(n)$ of these cycles each defining a free parameter. We can arange things such that

$$\tilde{Q}=\Lambda\otimes\hat{Q}_{o(n)}=\begin{pmatrix}\mu_1\hat{Q}_{o(n)} & & \theta\\ & \ddots & \\ \theta & & \mu_{(n)}\hat{Q}_{o(n)}\end{pmatrix}, \tag{A.10}$$

with $\Lambda = \text{diag}(\mu_1 \ldots \mu_{i(n)}) \in (U(1)/Z_{o(n)})^{i(n)}$. From $\det\tilde{Q}=1$ we have $\det\Lambda \in Z_{o(n)}$; by absorbing $(\det\Lambda)^{-1}$ in f.e. μ_1 we can fix $\det\Lambda = 1$. As for the case $n=1$, it is easy to verify that in the gauge in which Q is diagonal $(=\tilde{Q})$, P is fixed up to diagonal gauge-transformations. These can be used to bring P in a form which cannot be further reduced:

$$\tilde{P} = \Theta \otimes \hat{P}_{o(n)}^{\frac{n}{i(n)}} = \begin{pmatrix} \nu_1 \hat{P}_{o(n)}^{\frac{n}{i(n)}} & & \\ \theta & \ddots & \theta \\ & & \nu_{i(n)} \hat{P}_{o(n)}^{\frac{n}{i(n)}} \end{pmatrix}, \tag{A.11}$$

with Θ having the same properties as Λ. To count the number of solutions we have to realize that (A.11) is invariant under gauge-transformations of the form:

$$\Omega = \begin{pmatrix} \omega_1 I_{o(n)} & & \\ & \ddots & \theta \\ \theta & & \omega_{i(n)} I_{o(n)} \end{pmatrix} = \omega \otimes I_{o(n)}, \tag{A.12}$$

with ω again of the same type as Λ. The dimension of the solution manifold is therefore:

$$\dim_2(n, N) = N^2 - 1 + \text{gcd.}(n, N) - 1, \tag{A.13}$$

with the obvious convention, $\gcd(o, N) = N$, it also covers the case $n=0$. The manifold has for $\gcd(n, N) \neq 1$ singular points analogous to those for $n=0$, also leading to quartic fluctuations. Note that for n relatively prime to N, the solution is of the form (A.3), and no singular fluxons occur.

A.2. Dimension of Solution Manifold for $d=3$

As in the 2-dimensional case we define:

$$i(n) = \gcd(n_{\mu\nu}, N). \tag{A.14}$$

It is the index for the subgroup generated by $z^{n_{\mu\nu}}$ for all μ, ν, denoted by $\langle z^{n_{\mu\nu}} \rangle$, it is *singly* generated by $z^{i(n)}$ because $z^{n_{\mu\nu}} = \hat{z}^{\hat{n}_{\mu\nu}}$, with

$$\hat{z} = z^{i(n)}, \qquad \hat{n}_{\mu\nu} = \frac{n_{\mu\nu}}{i(n)}. \tag{A.15}$$

Furthermore we define $o(n)$ again to be the order of this subgroup:

$$o(n) = N/i(n) = \hat{N}; \tag{A.16}$$

$\hat{n}_{\mu\nu}$ defines a twist tensor for the SU($\hat{N}$) case for which we know that there exist solutions $\hat{\Omega}_\mu$ satisfying:

$$\hat{\Omega}_\mu \hat{\Omega}_\nu \hat{\Omega}_\mu^+ \hat{\Omega}_\nu^+ = \exp\left(\frac{2\pi i \hat{n}_{\mu\nu}}{\hat{N}}\right). \tag{A.17}$$

We leave it to the reader to verify that all solutions of (A.17) are gauge equivalent[5]. One uses that $\gcd(\hat{n}_{\mu\nu}, \hat{N}) = 1$ and the result for $d=2$ for pairs of $\hat{\Omega}_\mu$[6]. The full solution is now simply given up to a gauge by:

$$\Omega_\mu = \Lambda \otimes \hat{\Omega}_\mu , \tag{A.18}$$

with Λ, $\Lambda_\mu \in \mathrm{SU}(i(n))$ and diagonal, where $\Lambda \otimes I_{\hat{N}}$ again defines the invariance group of (A.18). So the parameters are easy to count, yielding:

$$\dim_3(n_{\mu\nu}, N) = N^2 - 1 + 2(\gcd(n_{\mu\nu}, N) - 1). \tag{A.19}$$

A.3. Dimension for $d=4$

If we repeat the analysis of the last paragraph we could take $\hat{n}_{\mu\nu}$ as a twist tensor for SU($\hat{N}$), however $\kappa(\hat{n})$ is orthogonal if and only if $\kappa(n)/N$ contains $i(n)$ as a factor, since then $\kappa(\hat{n}) = \kappa(n)/i(n)^2$ is a multiple of $\hat{N}$. Therefore we define a *new* value for $i(n)$:

$$i(n) = \gcd(n_{\mu\nu}, \kappa(n)/N, N). \tag{A.20}$$

$\hat{n}$ and $\hat{N}$ are now defined as in (A.15); of course $\kappa(n)$ is supposed to be $0 \bmod N$ in (A.20). We showed in Sect. 2 and [11] that there exist solutions $\hat{\Omega}_\mu$ satisfying (A.17), and again the reader is invited to show that all these solutions are SU($\hat{N}$) gauge equivalent. The SU(N) solution is now obviously of the form of (A.18) with the same invariance group. So we finally arrive at the most important result:

$$\dim_4(n_{\mu\nu}, N) = N^2 - 1 + 3\left(\gcd\left(n_{\mu\nu} \frac{\kappa(n)}{N}, N\right) - 1\right). \tag{A.21}$$

As a very important application we mention that $|\kappa(n)| = N$ implies that there are only gauge modes in the zero-mode spectrum. Thus for the TEK model with the symmetric twist (11a) there are *no physical* zero-modes, for all surviving extrema we considered [surviving fluxons cannot change $\kappa(n)$]. This means that the partition function in the weak coupling region is of the form:

$$Z(n, \beta) = \beta^{-\frac{3}{2}(N^2-1)} \sum_{k=0}^{\infty} C_k e^{-8\pi^2 k\beta/N} \quad (1 + \mathcal{O}(g^3)) \tag{A.22}$$

if no other fluxons are present. C_k is purely determined by the number of fluxons and the gaussian integration. Note that some C_k will be zero, such as C_6 or C_{11}. We leave a further investigation for a future publication.

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5 Throughout the article gauge equivalence for $d \neq 2$ means that each connected component is a gauge orbit. These components are related by multiplication with centre elements

6 This is most easily seen by realizing that if $PQP^+Q^+ = z^l$, then $P^N = \tilde{\Lambda} \otimes I_{o(l)}$, $Q^N = \tilde{\Theta} \otimes I_{o(l)}$. In the gauge in which f.e. Ω_1 is diagonal, we can compare the different conditions on Ω_μ^N

Let us finally study the case of singular fluxons, which occur whenever $i(n) \neq 1$. The fluxon of the form $\Omega_\mu = \Lambda_\mu \otimes \hat{\Omega}_\mu$ (see above for notations) is obviously most singular if $\Omega_\mu = I_{i(n)} \otimes \hat{\Omega}_\mu$ for all μ. It has the largest symmetry group, and gives the largest number of zero-modes. This number can be found by counting the number of independent solutions to $F_{\mu\nu}=0$ [see Eqs. (16) and (17)]. We write X_μ in a suitable block form:

$$X_\mu = \begin{pmatrix} \lambda_\mu^{(11)} I_{o(n)} + Y_\mu^{(11)} \dots \lambda_\mu^{(1i)} I_{o(n)} + Y_\mu^{(1i)} \\ \vdots \qquad\qquad \vdots \\ \lambda_\mu^{(i1)} I_{o(n)} + Y_\mu^{(i1)} \dots \lambda_\mu^{(ii)} I_{o(n)} + Y_\mu^{(ii)} \end{pmatrix} \tag{A.23}$$

with $i \equiv i(n)$, Y_μ^{kl} a complex traceless $o(n) \times o(n)$ matrix satisfying $(Y_\mu^{(kl)})^\dagger = -(Y_\mu^{(lk)})$, $(\bar{\lambda}_\mu^{(kl)}) = -\lambda_\mu^{(kl)}$ and $\mathrm{Tr}(\lambda_\mu)=0$. $F_{\mu\nu}=0$ reduces to $\hat{F}_{\mu\nu}^{(kl)} = \hat{D}_\mu Y_\mu^{(kl)} = 0$. We know the number of independent solutions to this equation to be $o(n)^2 - 1 + 3(i(\hat{n})-1) = o(n)^2 - 1$ if $Y_\mu^{(kl)}$ is traceless and antihermitian. From this one easily deduces that there are $4(N^2-1) - i(n)^2(o(n)^2-1) - 4(i(n)^2-1) = 3(N^2 - i(n)^2)$ quadratic modes and $4(i(n)^2-1)$ quartic modes[7]. As in [17] one finds a contribution to $Z(n,\beta)$ proportional to:

$$\beta^{-\frac{3}{2}(N^2 - i(n)^2) - (i(n)^2 - 1)} \tag{A.24}$$

for the maximally singular fluxon, and

$$\beta^{-\frac{3}{2}(N^2 - i(n))} \tag{A.25}$$

for a regular fluxon (with all eigenvalues of Λ_μ different). All other fluxons give contributions in between (A.24) and (A.25). Therefore a singular fluxon will dominate for large β only if:

$$\tfrac{3}{2}(N^2 - i(n)) > \tfrac{3}{2}(N^2 - i(n)^2) + (i(n)^2 - 1)\,. \tag{A.26}$$

This is equivalent to $(i(n)-2)(i(n)-1) > 0$.

For $i(n)=1$ no singular fluxons occur. For $i(n)=2$ there is competition between singular and regular fluxons (compare for torons in SU(2) [17]). For $N \to \infty$ one expects Eq. (28) to remain valid, for finite N there can be $\ln\beta$ corrections [17]. For $i(n) > 2$ singular fluxons dominate. Our analysis avoids gauge *noninvariant* calculations in the spirit of the usefulness of lattice gauge theories. Note for example that for torons which are singular $D_\mu X_\mu = 0$ for all X_μ. The standard gauge fixing term [7] $D_\mu X_\mu$ does not fix any gauge parameter here.

After the completion of this manuscript I was informed through private communication that Y. Brihaye obtained some results similar to those in the appendix, concerning the *general* solution to Eq. (3).

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7 Note: There are $N^2 - 1 + 3(i(n)^2 - 1)$ zero-modes, $N^2 - i(n)^2$ of them are pure gauges [remember that there are $i(n)^2 - 1$ gauges which leave Ω_μ invariant]

Note added. These results can be found in the following two papers: Brihaye, Y., Rossi, P.: The Twisted Eguchi-Kawai model fails to reproduce the weak coupling of the Wilson model. Phys. Lett. **125**B, 415 (1983), and Brihaye, Y., Maiella, G., Rossi, P.: Twisted Eguchi-Kawai models: an analysis of the saddle points. Nucl. Phys. B**222**, 309 (1983). The overlap with the present work for the first paper is the observation that there is a slight discrepancy between the $Z_{\mu\nu} = -1$ twist and the infinite lattice. They use the method of [7]. We therefore find a somewhat different expression for the weak coupling internal energy. This difference is already present between the results of [7, 17]. The second paper essentially finds the same result as we find in Sect. A.3 along a different route. Their number J corresponds to our $i(n)$.

Acknowledgements. We thank A. Gonzalez-Arroyo for explaining his ideas and stimulation, Sander Bais, J. Groeneveld, and J. Smit for discussions, Bert van Geemen for explaining the Frobenius theorem, and Th. Ruygrok for reading the manuscript. Furthermore we acknowledge Gerard 't Hooft for criticism, discussions and pointing out an error in an earlier draft of the manuscript.

This work is part of the research program of the "Stichting voor Fundamenteel Onderzoek der Materie (F.O.M.)," which is financially supported by the "Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek (ZWO)."

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Communicated by R. Stora

Received February 25, 1983; in revised form April 15, 1983