Swarm dynamics and equilibria for a nonlocal aggregation model

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Self-organizing animal aggregations

- Animal groups with a high structural order

- The behaviour of individuals is so coordinated, that the group moves as a single coherent entity

- Examples of self-organizing biological groups

  – schooling fish

  – herds of ungulates

  – swarming insects

  – zigzaging flocks of birds

Mathematical models

- The existing models fall into 2 categories: **Lagrangian** and **Eulerian**

- Lagrangian models: trajectories of all individuals of a species are tracked according to a set of interaction and decision rules
  - a large set of coupled ODE's
  - a large set of coupled difference equations (discrete time)

- Eulerian models: the problem is cast as an evolution equation for the population density field
  - parabolic
  - hyperbolic

A nonlocal Eulerian PDE swarming model

- We study the PDE aggregation model in $\mathbb{R}^n$:
  - continuity equation for the density ρ:

$$\rho_t + \nabla \cdot (\rho v) = 0$$

  - the velocity v is assumed to have a functional dependence on the density

$$v = -\nabla K * \rho$$

  - the potential K incorporates social interactions: attraction and repulsion

- The model was first suggested by Mogilner and Keshet, *J. Math. Biol.* [1999]

- Literature on this model has been very rich in recent years

Lagrangian description

N individuals

$X_i(t) =$ spatial location of the i-th individual at time t

$$\frac{dX_i}{dt} = -\frac{1}{N} \sum_{\substack{j=1\ldots N \\ j \neq i}} \nabla_i K(X_i - X_j), \qquad i = 1 \ldots N$$

PDE: continuum approximation, as $N \to \infty$

Assumption: social interactions depend only on the relative distance between the individuals

- radially symmetric potentials

$$K(x) = K(|x|)$$

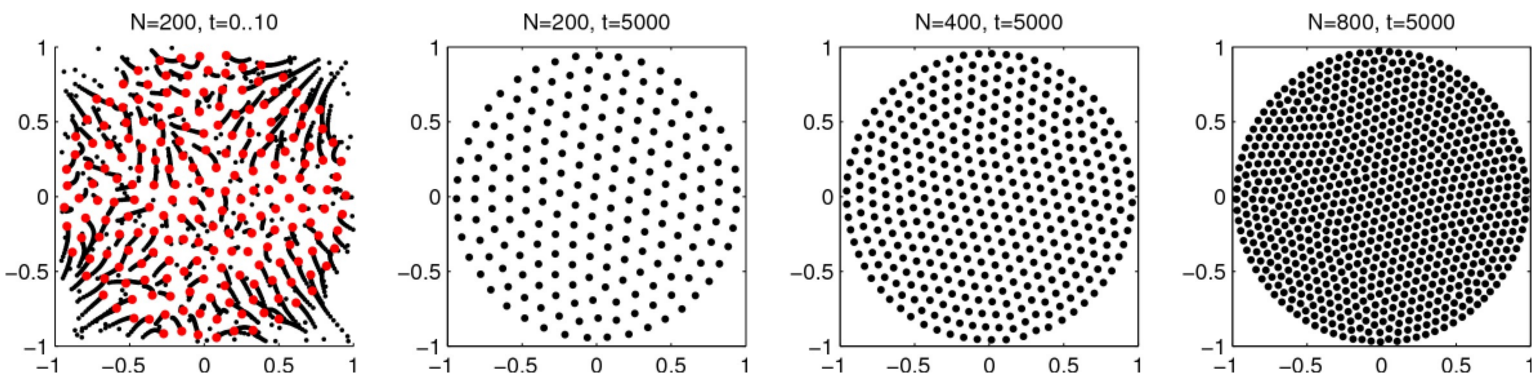
Notation: $F(r) = -K^{'}(r)$

$$\frac{dX_i}{dt} = \frac{1}{N} \sum_{\substack{j=1\dots N \\ j\neq i}} F(|X_i - X_j|) \frac{X_i - X_j}{|X_i - X_j|}, \qquad i = 1 \dots N$$

$F(|X_i - X_j|)$ = magnitude of the force that the individual X_j exerts on the individual X_i, along $X_i - X_j$

Repulsion ($F(r) > 0$) acts at short ranges, attraction ($F(r) < 0$) at long ranges.

Example: $n = 2$, $F(r) = 1/r - r$; random initial conditions inside the unit square. The solution approaches a constant density in the unit disk.



Motivation for this work

- Equilibria of the model should have biologically relevant features:
  - finite densities
  - sharp boundaries
  - relatively constant internal population

- The main motivation for this work is to
  - design interaction potentials K which lead to such equilibria
  - investigate analytically and numerically the well-posedness and long time behaviour of solutions

Interaction potential K

$$\begin{aligned} K(x) &= K_r + K_a \\ &= \phi(x) + \frac{1}{q}|x|^q, \qquad q \geq 2 \end{aligned}$$

$\phi(x) =$ the free-space Green's function for $-\Delta$:

$$\phi(x) = \begin{cases} -\frac{1}{2\pi} \ln |x|, & n = 2 \\ \frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

Continuity equation: $\rho_t + v \cdot \nabla \rho = -\rho \operatorname{div} v$

Calculate $\operatorname{div} v$:

$$\begin{aligned} \operatorname{div} v &= \operatorname{div}(-\nabla K * \rho) \\ &= -\Delta K * \rho \\ &= \rho - \Delta \Big(\frac{1}{q}|x|^q\Big) * \rho \end{aligned}$$

The repulsion term has become local!

Lagrangian approach

Characteristic curves: $\frac{d}{dt}X(\alpha,t) = v(X(\alpha,t),t), \qquad X(\alpha,0) = \alpha$

Evolution equation for $\rho(X(\alpha,t),t)$:

$$\frac{D\rho}{Dt} = -\rho^2 + \rho\,\Delta\Big(\frac{1}{q}|x|^q\Big) * \rho$$

Special case $q = 2$: explicit calculations

$$\Delta\Big(\frac{1}{2}|x|^2\Big) = n, \quad \Delta\Big(\frac{1}{2}|x|^2\Big) * \rho = n \underbrace{\int_{\mathbb{R}^n} \rho(y)dy}_{=M}$$

ODE along characteristics: $\frac{D\rho}{Dt} = -\rho(\rho - nM)$

Exact solution: $\rho(X(\alpha,t),t) = \frac{nM}{1+\left(\frac{nM}{\rho_0(\alpha)}-1\right)e^{-nMt}}$

Asymptotic behaviour as $t \to \infty$?

Asymptotic behaviour

Density: $\rho(X,t) \to nM$, as $t \to \infty$, along particle paths with $\rho_0(\alpha) \neq 0$

Asymptotic behaviour of trajectories: $R_\alpha = \lim_{t\to\infty} |X(\alpha,t)|$

For radial solutions, it can be proved that trajectories are mapped into the ball of $\mathbb{R}^n$ of radius $R_\alpha = \frac{1}{(n\omega_n)^{\frac{1}{n}}}$.

Numerics suggest that *all* solutions have this asymptotic behaviour.

Global attractor: constant, compactly supported density:

$$\bar{\rho}(x) = \begin{cases} nM & \text{if } \ |x| < \frac{1}{(n\omega_n)^{\frac{1}{n}}} \\ 0 & \text{otherwise} \end{cases}$$

Global existence of particle paths

$$v(x) = \int_{\mathbb{R}^n} k(x-y)\rho(y)\,dy - Mx, \qquad (1)$$

where

$$k(x) = \frac{1}{n\omega_n}\frac{x}{|x|^n}$$

The convolution kernel k is singular, homogeneous of degree $1-n$.

Equation (1) is analogous to Biot-Savart law, where vorticity ω is now replaced by density ρ.

Existence and uniqueness of particle paths follow similarly to that for incompressible Euler equations.

Extension to global existence: Beale-Kato-Majda criterion

$$\int_0^t \|\rho(\cdot, s)\|_{L^\infty} ds < \infty, \text{ for all finite times } t$$

Case $q > 2$: Non-constant steady states

Numerics suggests that attractors are radially symmetric.

Assume the model admits a radial steady state supported on a ball $B(0, R)$.

Recall formula for div v: $\quad \text{div}\, v = \rho - \Delta\Big(\frac{1}{q}|x|^q\Big) * \rho$

Equilibria supported on $B(0, R)$:

$$v = 0, \text{ hence } \text{div}\, v = 0 \text{ in } B(0, R)$$

A steady state $\bar{\rho}$ satisfies:

$$\boxed{\bar{\rho} - (n + q - 2) \int_{\mathbb{R}^n} |x - y|^{q-2} \bar{\rho}(y) dy = 0 \qquad \text{in } B(0, R)}$$

Use radial symmetry $\bar{\rho}(x) = \bar{\rho}(r)$.

Radial steady states

The density $\bar{\rho}$ satisfies the homogeneous Fredholm integral equation

$$\bar{\rho}(r) = c(q,n) \int_0^R (r')^{n-1} \bar{\rho}(r') I(r,r') dr', \qquad 0 \le r < R,$$

$$I(r,r') = \int_0^\pi (r^2 + (r')^2 - 2rr' \cos\theta)^{q/2-1} \sin^{n-2}\theta d\theta.$$

In other words, $\bar{\rho}$ is an eigenfunction of the linear operator T_R:

$$T_R\bar{\rho}(r) = c(q,n) \int_0^R (r')^{n-1} \bar{\rho}(r') I(r,r') dr',$$

that corresponds to eigenvalue one: $\boxed{T_R\bar{\rho}(r) = \bar{\rho}(r), \quad r < R}$

The eigenvalue problem: find $\bar{\rho}$ **and** the radius R of the support

Krein-Rutman theorem

Consider case $R = 1$ first.

The kernel $c(q,n)(r')^{n-1}I(r,r')$ is nonnegative, continuous and bounded.

T_1 is a linear, strongly positive, compact operator that maps the space of continuous functions $C([0,1],\mathbb{R})$ into itself.

Krein-Rutman theorem: there exists a *positive* eigenfunction $\bar{\rho}_1$ such that

$$T_1 \bar{\rho}_1 = \lambda \bar{\rho}_1 \tag{2}$$

$\lambda(q,n)$ is the spectral radius of T_1; it is a simple eigenvalue and there is no other eigenvalue with a positive eigenvector.

Define, by rescaling: $\bar{\rho}(r) = \bar{\rho}_1(r/R)$.

Existence and uniqueness of equilibria

Introduce $\bar{\rho}(r) = \bar{\rho}_1(r/R)$ in (2):

$$T_R\bar{\rho}(r) = R^{n+q-2}\lambda\,\bar{\rho}(r)$$

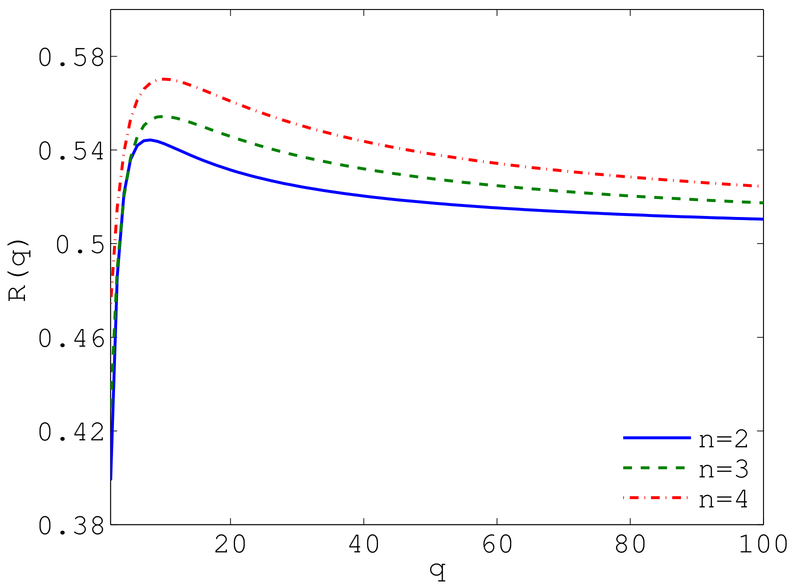
Ask that $\bar{\rho}$ is an eigenfunction of T_R corresponding to e-value 1:

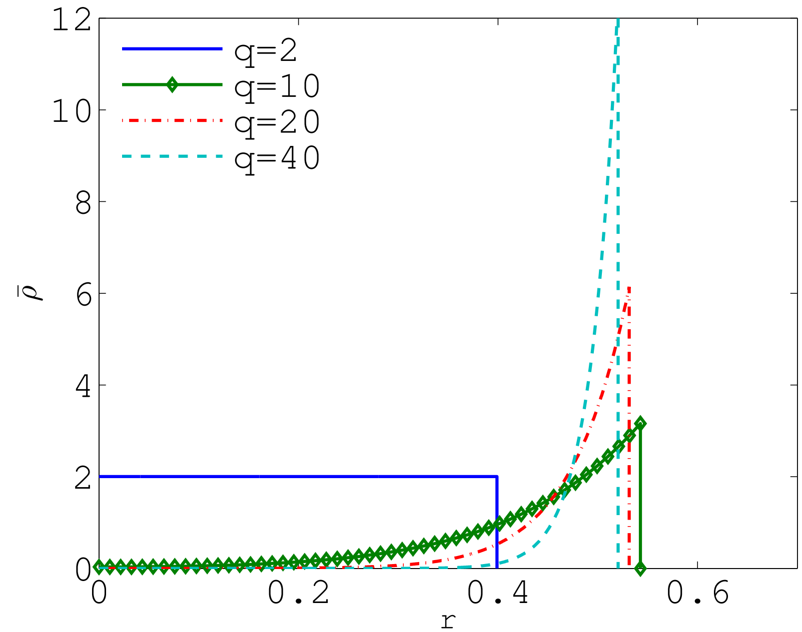
$$R = \lambda^{-\frac{1}{n+q-2}}$$

This gives the radius of the support as a function of q and n.

Once a mass M for $\bar{\rho}$ is set, uniqueness can be inferred from the uniqueness properties of the spectral radius of T_1 and its associated eigenfunction $\bar{\rho}_1$.

Equilibria: numerical results





Left: Plot of the radius of the support R of the steady states as a function of the exponent q, for various space dimensions n.

The plot suggests that the radius R approaches a constant, as $q \to \infty$.

Right: Normalized radially symmetric steady states $\bar{\rho}(r)$ in two dimensions for various values of the exponent q.

For $q = 2$ the steady state is the constant solution in a disk. As q increases, mass aggregates toward the edge of the swarm, creating an increasingly void region in the centre.

Even q: polynomial steady states

Kernel $I(r,r') = \int_0^\pi (r^2 + (r')^2 - 2rr'\cos\theta)^{q/2-1} \sin^{n-2}\theta d\theta$ is separable when q is even.

Define the i-th order moments of the density ($m_0 = M$):

$$m_i = n\omega_n \int_0^R r^{n+i-1} \bar{\rho}(r) dr. \tag{3}$$

Example: $q = 4$

$$I(r,r') = (r^2 + (r')^2) \int_0^\pi \sin^{n-2}\theta d\theta$$

and

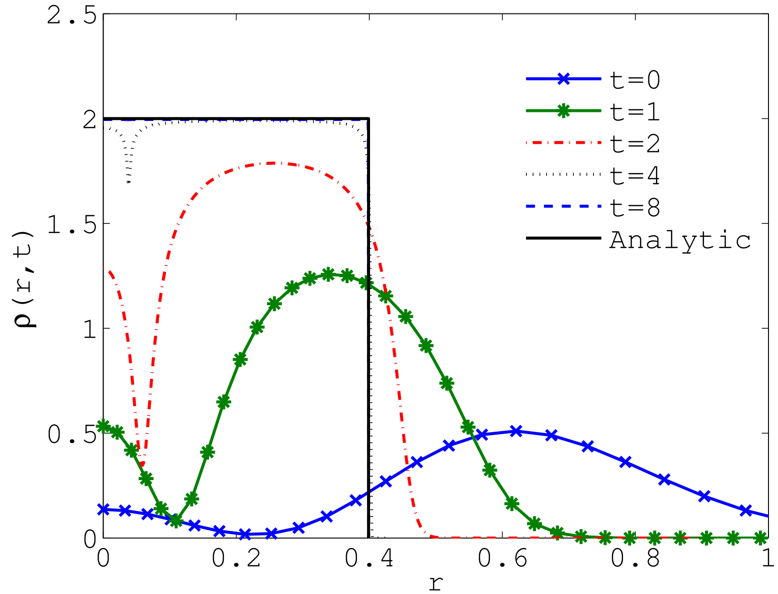
$$\begin{aligned} \bar{\rho}(r) &= n(n+2)\omega_n \int_0^R (r')^{n-1}(r^2 + (r')^2)\bar{\rho}(r')dr' \\ &= (n+2)m_0 r^2 + (n+2)m_2 \end{aligned} \tag{4}$$

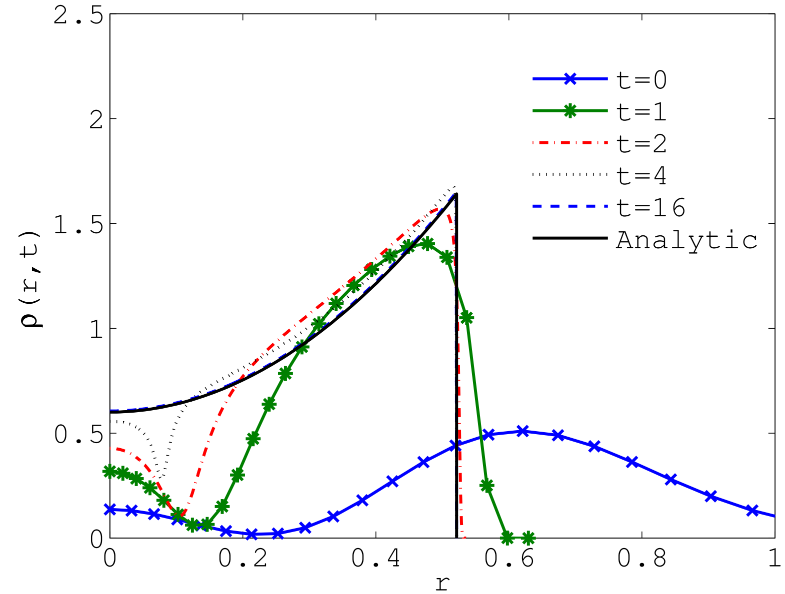
Plug (4) into (3): linear system to find R and m_2

$$\begin{pmatrix} m_0 \\ m_2 \end{pmatrix} = \begin{pmatrix} n\omega_n R^{n+2} & (n+2)\omega_n R^n \\ \frac{n(n+2)}{n+4}\omega_n R^{n+4} & n\omega_n R^{n+2} \end{pmatrix} \begin{pmatrix} m_0 \\ m_2 \end{pmatrix} \tag{5}$$

General q even: $\bar{\rho}(r)$ is a polynomial of even powers of r, of degree $q-2$.

Dynamic evolution: numerical results





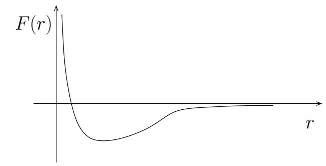
Time evolution of a radially symmetric solution to the aggregation model with $q = 2$ (left) and $q = 4$ (right) in two dimensions

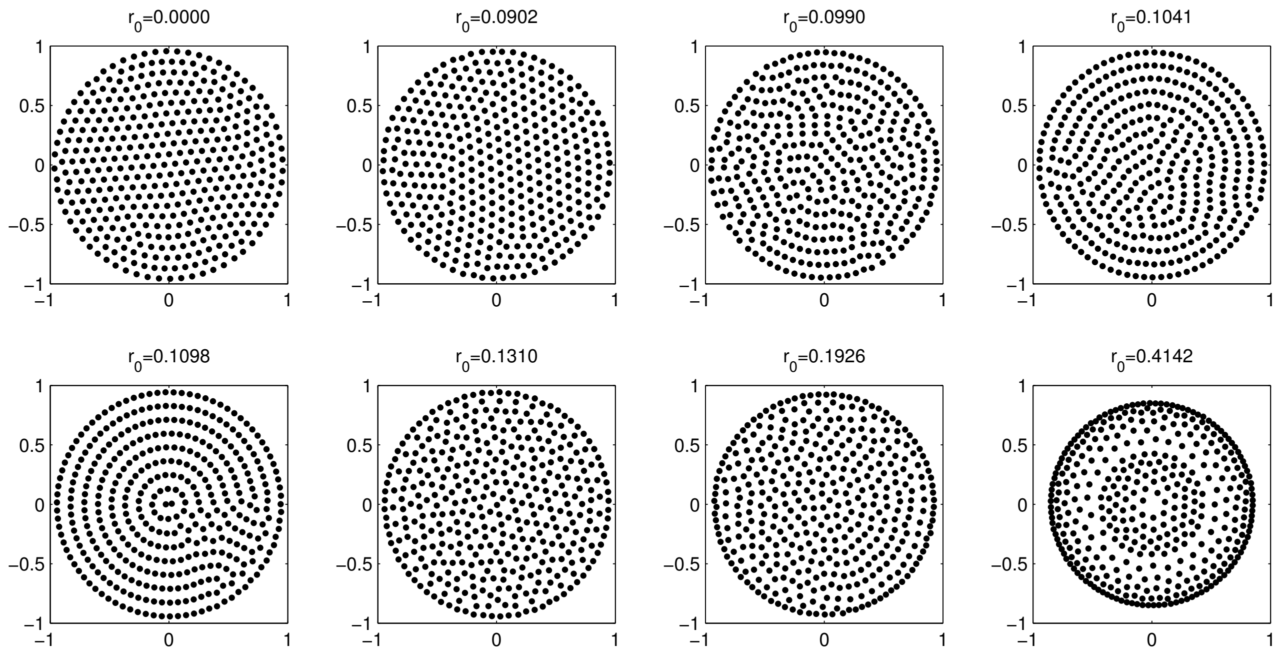
Left: As predicted by the analytical results, the solution approaches asymptotically a constant, compactly supported steady state.

Right: The solution approaches asymptotically the steady state computed analytically

Regularized potentials

$$F(r) = \frac{1}{r} - r \quad \xrightarrow{\text{regularize}} \quad F(r) = \begin{cases} C_1, & 0 \le r < r_0 \\ \frac{1}{r} - r, & r_0 \le r \le 2 \\ -C_2 \exp(-r), & 2 < r \end{cases} \qquad (6)$$





Equilibrium states for the regularized interaction force (6). Initial conditions were chosen at random in the unit square. For $r_0 < 0.09$, the steady state is the same as for $r_0 = 0$ (uniform density in the unit circle).

Inverse problem: custom designed potentials

Inverse problem: given a density $\bar{\rho}(x)$, can we find a potential K for which $\bar{\rho}(x)$ is a steady state of the model?

Answer: Yes, provided $\bar{\rho}(x)$ is radial and is a polynomial in $|x|$.

Theorem: In one dimension, consider an even density $\bar{\rho}$ of the form

$$\bar{\rho}(x) = \begin{cases} b_0 + b_2 x^2 + b_4 x^4 + \ldots + b_{2d} x^{2d} & |x| < R \\ 0 & \text{otherwise.} \end{cases}$$

Define the moments m_i as in (3). Then $\bar{\rho}(x)$ is the steady state corresponding to the force F:

$$F(x) = \frac{1}{2} - \sum_{i=0}^{d} \frac{a_{2i}}{2i+1} x^{2i+1}$$

where the constants $a_0, a_2, \ldots, a_{2d},$ are computed from $b_0,\ b_2, \ldots, b_{2d}$ by solving the following linear system:

$$b_{2k} = \sum_{j=k}^{d} a_{2j} \begin{pmatrix} 2j \\ 2k \end{pmatrix} m_{2(j-k)}, \qquad k = 0 \ldots d. \tag{7}$$

Moreover, system (7) has a unique solution.

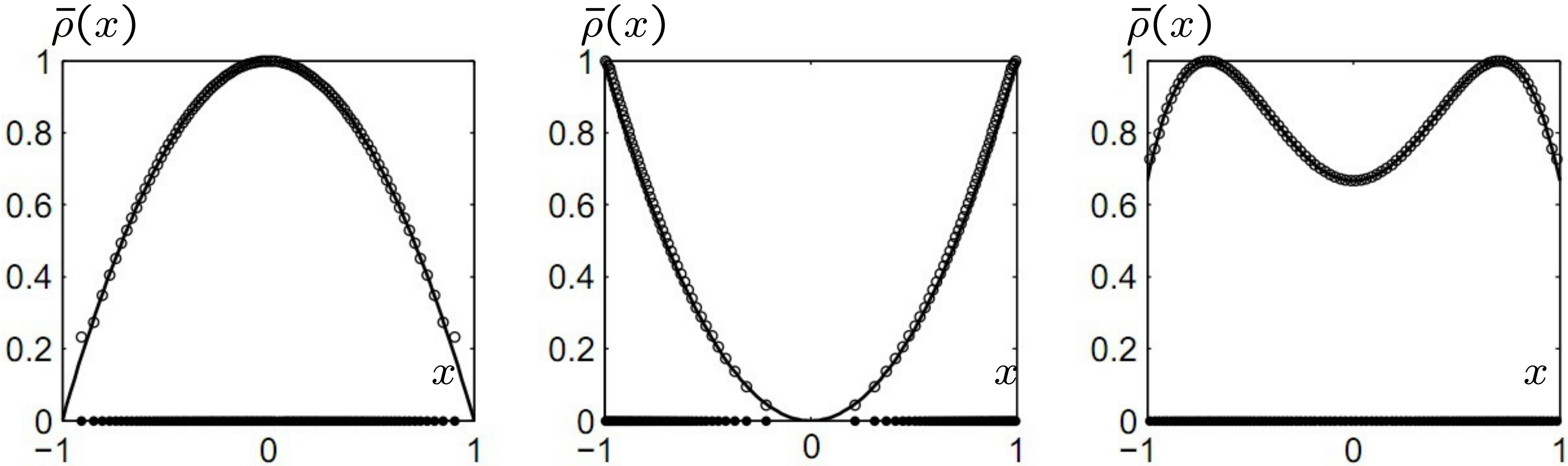
Inverse problem: numerical results

Examples: $R = 1$ and:

$$(a)\quad \bar{\rho}(x) = 1 - x^2; \qquad (b)\quad \bar{\rho}(x) = x^2; \qquad (c)\quad \bar{\rho}(x) = \frac{1}{2} + x^2 - x^4$$

The corresponding forces given by the Theorem are:

$$(a)\quad F(x) = \frac{1}{2} - \frac{9}{10}x + \frac{1}{4}x^3; \quad (b)\quad F(x) = \frac{1}{2} + \frac{9}{10}x - \frac{1}{2}x^3;$$
$$(c)\quad F(x) = \frac{1}{2} + \frac{209425}{672182}x - \frac{2075}{2527}x^3 + \frac{3}{19}x^5.$$



(a) **(b)** **(c)**

Filled circles along the x-axis: the steady states reached by numerical time evolution. Empty circles: density function as computed from the filled circles. Solid line: analytical expression for $\bar{\rho}$.

Bibliography

1. R.C. Fetecau, Y. Huang and T. Kolokolnikov [2011]. Swarm dynamics and equilibria for a nonlocal aggregation model, *Nonlinearity*, Vol. 24, No. 10, pp. 2681-2716 (featured article)

Other recent work / Future Directions

- Studied $q < 2$, in particular the case $q \to 2-n$, when attraction becomes as singular as repulsion (Newtonian potential)

- Investigated properties of the steady states: monotonicity, asymptotic behaviour ($q \to \infty$, $q \to 2-n$)

- Energy considerations: local/ global minima

$$E[\rho] = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y)\rho(x)\rho(y)dydx$$

  The model is a gradient flow with respect to this energy:

$$\frac{d}{dt}E[\rho] = -\int_{\mathbb{R}^n} \rho(x)|\nabla K * \rho(x)|^2 dx \le 0$$