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SWEEPING BY A TAME PROCESS

by Aris DANIILIDIS & Dmitriy DRUSVYATSKIY (*)

Abstract. — We show that any semi-algebraic sweeping process admits piecewise absolutely continuous solutions (trajectories), and any such bounded trajectory must have finite length. Analogous results hold more generally for sweeping processes definable in o-minimal structures. This extends previous work on (sub)gradient dynamical systems beyond monotone sweeping sets.

Résumé. — Nous montrons l'existence des solutions (orbites) absolument continues par morceaux pour le processus de rafle défini par un opérateur multivoque semi-algébrique (ou plus généralement, o-minimal). Nous établissons que de telles orbites bornées sont de longueur finie. Cette contribution, dans le cas particulier où le processus de rafle correspond aux sous-niveaux d'une fonction (non nécessairement régulière), généralise les résultats connus pour les orbites des systèmes dynamiques de type sous-gradient.

1. Introduction

A classical result of Łojasiewicz shows that any bounded trajectory of the gradient system

$$\dot{x} = -\nabla f(x)\,,$$

with a real-analytic potential function f on $\mathbb{R}^n$, necessarily has finite length and hence converges to a critical point of f. This conclusion can fail if the analyticity is weakened to infinite differentiability; see [30] for example. The main ingredient of the argument in the analytic case is the celebrated Łojasiewicz inequality, which has been subsequently generalized by Kurdyka [24] to smooth functions definable in an o-minimal structure;

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see [14] for the relevant definitions. The authors of [2, 19] further eliminated the smoothness assumptions, showing that any bounded solution $x\colon [0,\eta) \to \mathbb{R}^n$ of the subgradient system

$$\dot{x} \in -\partial f(x) \qquad \text{a.e. on } [0,\eta)$$

induced by a semi-algebraic function f (or more generally, by a definable function in an o-minimal structure) has finite length and converges to a generalized critical point of f. Here the subdifferential ∂f is meant in any reasonable sense, such as the limiting subdifferential or the generalized gradient; see e.g. [31]. With the publications of [4, 20], the close relationship between such results and the so-called desingularizing function (traced back to [24] for the C^1 case) became clear.

A salient point is that in the subgradient dynamical system, the function f decreases along the trajectory x. In particular, after a reparametrization, the trajectory $x\colon [a,b) \to \mathbb{R}^n$ satisfies the inclusion

$$\dot{x}(r) \in -N_{[f\leqslant r]}(x(r)) \qquad \text{for a.e. } r \in [a,b)\,.$$

Here $N_{[f\leqslant r]}$ denotes the normal cone to the sublevel set $[f \leqslant r] := \{x \in \mathbb{R}^n : f(x) \leqslant r\}$. See for example [10, 11, 12, 25, 26] for this point of view. Thus the subgradient system is inherently related to a "monotonically evolving sweeping set" $r \mapsto [f \leqslant r]$. This observation then naturally motivates investigation of trajectory length of the more general system

$$\dot{x}(r) \in -N_{S(r)}(x(r)) \qquad \text{for a.e. } r \in [a,b)\,, \tag{1.1}$$

where $S(r)$ is a subset of $\mathbb{R}^n$ varying in time. This dynamical system is precisely the *sweeping process* of Moreau, well-known in mathematical mechanics (see e.g. [23, 28]), and which has recently received much attention [7, 8, 9, 29, 32]. In this paper, much akin to the results of Kurdyka and Łojasiewicz, we prove that bounded absolutely continuous trajectories of the sweeping process, with a semi-algebraic set-valued mapping $S\colon \mathbb{R} \rightrightarrows \mathbb{R}^n$, have finite length and therefore must converge to an equilibrium point. We discuss extensions to the degenerate sweeping process [22], and limitations when the sweeping process is state-dependent in the sense of [21].

As a byproduct, we prove a convenient set-valued extension of the projection formula [3, Prop. 4], and establish a "desingularization" result for semi-algebraic set-valued mappings $S\colon \mathbb{R} \rightrightarrows \mathbb{R}^n$, generalizing the Kurdyka-Łojasiewicz inequality for the sublevel set mapping $r \mapsto [f \leqslant r]$ of a semi-algebraic function f. The desingularization philosophy, combined with [1, 6], allows us to quickly prove that any locally bounded, semi-algebraic

sweeping process always admits piecewise absolutely continuous solutions. The overall trend of the arguments follows along the lines of [2, 24], with some important deviations. Nevertheless, we believe that the striking connection of semi-algebraic and o-minimal geometry to the sweeping process, and in particular to nonmonotone evolution equations, outlined in this paper, will pave the way for new applications and settings to be explored.

The outline of the manuscript is as follows. In Section 2, we record some notation and preliminary results of variational analysis needed in the rest of the paper. In Section 3, we discuss basic elements of semi-algebraic geometry and their interactions with variational constructions. Section 4 contains our main results on the lengths of trajectories generated by the sweeping process. Section 5 discusses the role of desingularization, while Section 6 applies desingularization ideas to deduce existence of piecewise absolutely continuous solutions of the sweeping process.

2. Notation

In this section, we summarize some basic tools we will use. We follow closely the terminology and notation of [31]. Throughout, we consider a Euclidean space which we denote by $\mathbb{R}^n$, along with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. The closed unit ball will be denoted by $\mathcal{B}$. For any set Q in $\mathbb{R}^n$, we let $\operatorname{cl} Q$ and $\operatorname{int} Q$ denote the closure and the interior of Q respectively. The symbol $\operatorname{conv} Q$ will stand for the convex hull of Q, while $\operatorname{par} Q$ will denote the smallest affine space containing Q, translated to the origin, that is, the linear span of the set $Q - Q$. Given two sets Q and L, we say that the orthogonality relation $Q \perp L$ holds, if any pair of points $x \in Q$ and $y \in L$ are orthogonal. The *distance* of a point x to a set Q is defined by

$$\operatorname{dist}(x; Q) := \inf_{y \in Q} \|x - y\|.$$

A set-valued mapping F from $\mathbb{R}^n$ to $\mathbb{R}^m$, denoted $F \colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, is a mapping from $\mathbb{R}^n$ to the powerset of $\mathbb{R}^m$. The *domain* and *graph* of such a mapping F are defined by

$$\operatorname{dom} F := \{x \in \mathbb{R}^n : F(x) \neq \emptyset\},$$
$$\operatorname{gph} F := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in F(x)\},$$

respectively. The inverse of a set-valued mapping F is another set-valued mapping defined by $F^{-1}(y) := \{x \in \mathbb{R}^n : y \in F(x)\}$. A set-valued mapping $L \colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *positively homogeneous* whenever $\operatorname{gph} L$ is a cone, or

equivalently whenever we have

$$0 \in L(0) \qquad \text{and} \qquad L(\lambda x) = \lambda L(x) \quad \text{for all } \lambda > 0 \text{ and } x \in \mathbb{R}^n.$$

In this case, the *outer norm* of L is defined by

$$|L|^+ := \sup_{x \in \mathcal{B}} \sup_{y \in L(x)} \|y\|.$$

Due to positive homogeneity of L, the outer norm coincides with

$$\inf\{\kappa > 0 : \|y\| \leqslant \kappa \|x\| \text{ whenever } y \in L(x)\}.$$

One can now easily deduce that the norm of the inverse admits the representation

$$|L^{-1}|^+ = \frac{1}{\inf_{\|x\|=1} \operatorname{dist}(0; L(x))}. \tag{2.1}$$

Next we pass to certain geometric constructions associated to sets in $\mathbb{R}^n$. In what follows, the symbol "$o(\|x-\bar{x}\|)$ for $x \in Q$" will stand for any function satisfying $\frac{o(\|x-\bar{x}\|)}{\|x-\bar{x}\|} \to 0$ as x tends to $\bar{x}$ in Q.

Definition 2.1 (Normal cones). — *Consider a set $Q \subset \mathbb{R}^n$ and a point $\bar{x} \in Q$. Then the* Fréchet normal cone *to Q at $\bar{x}$, denoted $\hat{N}_Q(\bar{x})$, consists of all vectors $v \in \mathbb{R}^n$ satisfying*

$$\langle v, x - \bar{x} \rangle \leqslant o(\|x - \bar{x}\|) \quad \text{for } x \in Q.$$

The limiting normal cone *to Q at $\bar{x}$, denoted by $N_Q(\bar{x})$, consists of all vectors $v \in \mathbb{R}^n$ such that there exist sequences x_i in Q and $v_i \in \hat{N}_Q(x_i)$ satisfying $x_i \to \bar{x}$ and $v_i \to v$. The* Clarke normal cone *to Q at $\bar{x}$ is simply the set $N_Q^c(\bar{x}) := cl\, conv\, N_Q(\bar{x})$.*

By convention, for any point $\bar{x} \notin Q$, we set $\hat{N}_Q(\bar{x}) = N_Q(\bar{x}) = N_Q^c(\bar{x}) = \emptyset$. When Q is a closed convex set, the three normal cones all coincide with the usual convex normal cone of convex analysis, while for a C^1-smooth manifold Q the normal cones coincide with normal spaces in the sense of differential geometry.

Normal cones to graphs of set-valued mappings $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ are naturally associated with generalized differentiation. Here, we should note that in general the limiting normal cone $N_{\operatorname{gph} F}$ provides much finer information about the local behavior of F as opposed to the convexified construction $N^c_{\operatorname{gph} F}$. On the other hand, the results in this paper are strong enough to pertain to the latter, and hence that's the one we mostly focus on. Analogous results for limiting constructions are direct consequences.

DEFINITION 2.2 (Coderivatives). — *Consider a mapping* $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *and a pair* $(\bar{x},\bar{y}) \in \operatorname{gph} F$. *The* Clarke coderivative of F at $\bar{x}$ for $\bar{y}$ *is the set-valued map* $D_c^*F(\bar{x}|\bar{y})\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *defined by*

$$D_c^*F(\bar{x}|\bar{y})(u) := \{v \in \mathbb{R}^n : (v,-u) \in N^c_{\operatorname{gph} F}(\bar{x},\bar{y})\}.$$

The limiting coderivative $D^*F(\bar{x}|\bar{y})$ *is defined analogously.*

When $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is C^1-smooth, then in terms of $\bar{y} := F(\bar{x})$, the coderivative mapping $D^*F(\bar{x}|\bar{y})$ is single-valued and linear, and coincides with the adjoint of the Jacobian $\nabla F(\bar{x})^*$. Analogously to the smooth case, we use the following notation.

DEFINITION 2.3 (Criticality). — *Given a set-valued mapping* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, *we say that a pair* $(\bar{x},\bar{y})$ *in the graph* $\operatorname{gph} F$ *is a* Clarke critical pair *whenever*

$$0 \in D_c^*F(\bar{x}|\bar{y})(u) \quad \text{for some } u \neq 0\,.$$

A vector $\bar{y} \in \mathbb{R}^m$ *is a* Clarke critical value *of* F *if there exists a point* $\bar{x} \in \mathbb{R}^n$ *so that the pair* $(\bar{x},\bar{y})$ *is Clarke critical.*

The coderivatives $D^*F(\bar{x}|\bar{y})$ and $D_c^*F(\bar{x}|\bar{y})$ are positively homogeneous set-valued mappings, hence in particular they admit an outer norm. Unwrapping the notation for ease of reference, we have

$$|D_c^*F(\bar{x}|\bar{y})|^+ = \sup_{\|u\|\leqslant 1} \{\|v\| : v \in D_c^*F(\bar{x}|\bar{y})(u)\}\,,$$

and

$$|D_c^*F^{-1}(\bar{y}|\bar{x})|^+ = \frac{1}{\inf_{\|u\|=1}\{\|v\| : v \in D_c^*F(\bar{x}|\bar{y})(u)\}}\,.$$

In particular $(\bar{x},\bar{y})$ is Clarke critical if and only if $|D_c^*F^{-1}(\bar{y}|\bar{x})|^+ = \infty$. The norms $|D^*F(\bar{x}|\bar{y})|^+$ and $|D^*F^{-1}(\bar{y}|\bar{x})|^+$ admit analogous descriptions.

DEFINITION 2.4 (Asymptotic critical values). — *Given a set-valued map* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, *we say that a vector* $\bar{y}$ *is an* asymptotic Clarke critical value of F on a set $\mathcal{U} \subset \mathbb{R}^n$ *if there exists a sequence* $(x_i,y_i) \in \operatorname{gph} F$ *with* $x_i \in \mathcal{U}$, *such that* y_i *converges to* $\bar{y}$ *and* $|D_c^*F^{-1}(y_i|x_i)|^+$ *tends to infinity.*

It is important to note that the outer norm of the limiting coderivative is very closely related to a pseudo-Lipschitz modulus of the mapping, which will play an important role in Section 6.

DEFINITION 2.5 (Aubin Property). — *A set-valued mapping* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *has the* Aubin property at $\bar{x}$ for $\bar{y} \in F(\bar{x})$ *if the graph*

gph F is locally closed around $(\bar{x},\bar{y})$, and there are neighborhoods X of $\bar{x}$ and Y of $\bar{y}$, along with a constant $\kappa \geqslant 0$ such that

$$F(x') \cap Y \subset F(x) + \kappa\|x' - x\|\mathcal{B}\,, \qquad \text{for all } x, x' \in X.$$

The infimum of κ over all combinations κ, X, and Y so that the condition above holds is the Lipschitz modulus of F at $\bar{x}$ for $\bar{y}$, *and is denoted by* $\operatorname{lip} F(\bar{x}|\bar{y})$.

Provided that the graph of F is closed around $(\bar{x},\bar{y})$, the following relationships hold:

$$\operatorname{lip} F(\bar{x}|\bar{y}) = \left|D^*F(\bar{x}|\bar{y})\right|^+ \leqslant \left|D^*_cF(\bar{x}|\bar{y})\right|^+.$$

The first equality is a celebrated characterization of the Aubin property (see [18], [27] or [31, Thm. 9.40], for example), while the last inequality is immediate from coderivative definitions.

3. Semi-algebraic and o-minimal geometry

A *semi-algebraic* set $Q \subset \mathbb{R}^n$ is a finite union of sets of the form

$$\{x \in \mathbb{R}^n : f_1(x), \ldots, f_k(x) = 0, g_1(x) < 0, \ldots, g_l(x) < 0\},$$

where $f_1, \ldots, f_k$ and $g_1, \ldots, g_l$ are real polynomials in n variables. It follows immediately that the class of semi-algebraic sets is closed under the standard Boolean operations (finite unions/intersections and complementary), while the famous *Tarski–Seidenberg principle* — also known as *quantifier elimination* — shows that semi-algebraicity is preserved under projections.

Semi-algebraic subsets of the real line $\mathbb{R}$ are exactly the finite unions of intervals. This property, known as the *o-minimal* (order-minimal) property, is the basis for an elegant axiomatization of various favorable properties of semi-algebraic sets, culminating with a notion of *definable sets*, or more formally, *sets definable in an o-minimal structure* [14]. This theory allows consideration of much more general sets such as those that are *globally subanalytic*, or sets belonging to the log-exp structure. A slightly more general notion is that of a *tame set* – a set whose intersection with any ball is definable in an o-minimal structure. Typical examples of tame sets which are not definable, are graphs of *real-analytic* functions. Tame sets are the context of the current paper. We do not however formally state what definable and tame sets are here since it would take us far off-field. Indeed, the reader can safely replace the word tame (or definable) by semi-algebraic,

throughout. We point the interested reader to the manuscript [14] or to the short discussion in [24, p. 771].

A key property of definable sets is that they can always be decomposed into a disjoint finite union of smooth (to an arbitrary order) manifolds that fit together in a regular pattern. In what follows p will always denote a finite integer no smaller than one.

DEFINITION 3.1 (Whitney (a)-regular C^p-stratification). — *A* Whitney (a)-regular C^p-stratification *of a set $Q \subset \mathbb{R}^n$ is a partition of Q into finitely many C^p manifolds (called* strata*) satisfying the following compatibility conditions:*

Frontier condition: *For any two strata L and M, the implication*

$$L \cap \operatorname{cl} M \neq \emptyset \implies L \subset (\operatorname{cl} M) \setminus M \quad \text{holds.}$$

Whitney condition (a): *For any sequence of points $\{x_i\}_i$ in a stratum M converging to a point $\bar{x}$ in a stratum L, if some corresponding normal vectors $v_i \in N_M(x_i)$ converge to a vector v, then the inclusion $v \in N_L(\bar{x})$ holds.*

Definable sets always admit Whitney (a)-regular C^p-stratifications for any finite p. The importance of such a result for variational analysts can already be appreciated by observing that the normal cone $N_Q^c(x)$ must be contained in the normal space $N_M(x)$, where M is a stratum containing x in any Whitney (a)-regular C^1-stratifications of Q. We refer the reader to [3, 15, 19] for applications of this fact, and of stratifications more broadly, in Variational Analysis. The forthcoming Theorems 3.2 and 3.3 are in the same spirit.

A set-valued mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ (respectively, a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$) is called *definable* if its graph $\operatorname{gph} F$ is definable. For instance, the functions $|x-y|$ and $\sqrt{x^2+y^4}$ are semi-algebraic, the function $x\,(\sin x)^{-1}$, for $x \in (0,\pi)$, is globally subanalytic, while the function

$$x \mapsto \exp(\sqrt{|x|})\log(|x|+1)$$

is definable in the log-exp structure.

The following is a convenient generalization of the "projection formula" (from [3, Prop. 4]) to the coderivative setting. Henceforth, we use the symbol π_x to denote the coordinate projection $(x,y) \mapsto x$, and the symbol $T_{\mathcal{M}}(x)$ to denote the tangent space to a C^p-manifold $\mathcal{M}$ at x.

THEOREM 3.2 (Extended projection formula). — *Consider a set-valued mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a Whitney (a)-regular C^1-stratification $\{\mathcal{M}_i\}$*

of the graph $\operatorname{gph} F \subset \mathbb{R}^n \times \mathbb{R}^m$. *Then for any pair* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ *in a stratum* $\mathcal{M}_i$, *the orthogonality relation holds:*

$$\operatorname{par} D_c^* F(\bar{x}|\bar{y})(u) \perp \pi_x\Big(T_{\mathcal{M}_i}(\bar{x}, \bar{y})\Big) \qquad \text{for all } u \in \mathbb{R}^m.$$

Proof. — We can suppose without loss of generality that $D_c^* F(\bar{x}|\bar{y})(u)$ is nonempty. Then by definition of the coderivative we have the chain of implications

$$v \in D_c^* F(\bar{x}|\bar{y})(u) \iff (v, -u) \in N^c_{\operatorname{gph} F}(\bar{x}, \bar{y}) \implies (v, -u) \in N_{\mathcal{M}_i}(\bar{x}, \bar{y}).$$

Hence for any tangent vector $(a, b) \in T_{\mathcal{M}_i}(\bar{x}, \bar{y}) \subset \mathbb{R}^n \times \mathbb{R}^m$, we deduce

$$\langle D_c^* F(\bar{x}|\bar{y})(u), a\rangle = \langle u, b\rangle,$$

and consequently $\langle D_c^* F(\bar{x}|\bar{y})(u) - D_c^* F(\bar{x}|\bar{y})(u), a\rangle = 0$, as claimed. □

The *dimension* of a definable set Q is the maximal dimension of any stratum in any Whitney (a)-regular C^1-stratification of Q. It is well-known that this definition is independent of the choice of the stratification. In particular, a definable subset of $\mathbb{R}^m$ has measure zero if and only if it has dimension at most $m-1$. The following is analogous to the main result of [19].

THEOREM 3.3 (Sard). — *Consider a definable set-valued map* $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *with a closed graph. Then the set of Clarke critical values of* F *is a definable set of dimension at most* $m-1$. *Moreover, the set of asymptotic Clarke critical values of* F *on any bounded definable set* $\mathcal{U}$ *is also a definable set of dimension at most* $m-1$.

Proof. — The fact that the set of Clarke critical values is definable follows by standard quantifier elimination. Consider now any Whitney (a)-regular C^p-stratification of $\operatorname{gph} F$. Suppose the equality $|D_c^* F^{-1}(\bar{y}|\bar{x})|^+ = \infty$ holds, or equivalently we have $(0, u) \in N^c_{\operatorname{gph} F}(\bar{x}, \bar{y})$ for some vector $u \neq 0$. Letting L be the manifold (stratum) containing $(\bar{x}, \bar{y})$, we deduce $(0, u) \in N_L(\bar{x}, \bar{y})$. Let $\pi_L\colon L \to \mathbb{R}^m$ be the restriction to L of the canonical projection $\pi\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$. It follows that $\bar{y}$ is a critical value in the classical sense of π_L. Applying the standard smooth Sard's theorem with p sufficiently large, and recalling that the number of strata is finite, we deduce that such critical values $\bar{y}$ have measure zero, as claimed.

Fix now a bounded definable set $\mathcal{U}$ and suppose that there exists a sequence $(x_i, y_i) \in \operatorname{gph} F$ with $x_i \in \mathcal{U}$, such that y_i converges to $\bar{y}$ and $|D_c^* F^{-1}(y_i|x_i)|^+$ tends to infinity. Hence inclusions $(w_i, u_i) \in N^c_{\operatorname{gph} F}(x_i, y_i)$ hold for some vectors w_i, u_i satisfying $\|u_i\| = 1$ and $w_i \to 0$. Since $\mathcal{U}$ is bounded and $\operatorname{gph} F$ is closed, we may suppose $x_i \to \bar{x}$ for some point $\bar{x}$

satisfying $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Let L be a stratum containing $(\bar{x}, \bar{y})$. We may assume, passing to a subsequence, that either $\{(x_i, y_i)\}_i$ is contained in L or it is contained in a stratum $\mathcal{M}$ with $L \subset (\operatorname{cl} \mathcal{M}) \setminus \mathcal{M}$. Passing to the limit (and evoking the Whitney-(a) condition in the second case) we deduce $(0, u) \in N_L(\bar{x}, \bar{y})$ for some $u \neq 0$, that is, $\bar{y}$ is a critical value of π_L. Again applying the standard smooth Sard's theorem with p sufficiently large, we deduce that such values $\bar{y}$ have measure zero, thereby completing the proof. □

Remark 3.4. — Boundedness of the set $\mathcal{U}$ is necessary for the theorem above to hold. This can be immediately seen even in the single-valued setting. Indeed, following [24, p. 776], define $F(x, y) = \frac{x}{y}$ and $\mathcal{U} = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. It follows easily that every $r > 0$ is an asymptotic Clarke critical value of F.

We also note in passing that it is clear from the above proof that instead of assuming that $\operatorname{gph} F$ is closed, we could have assumed that $\operatorname{gph} F$ is only locally closed. The latter means that for every $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, there exists a neighborhood V such that $V \cap \operatorname{gph} F$ is closed.

4. Finite length of bounded trajectories

Our focus is on the trajectory length of the classical *sweeping process*, introduced by Moreau [28]. Given a set-valued mapping $S\colon \mathbb{R} \rightrightarrows \mathbb{R}^n$, called the sweeping set, we consider absolutely continuous curves $\gamma\colon (a, b) \to \mathbb{R}^n$ satisfying the inclusion

$$\dot{\gamma}(r) \in -N^c_{S(r)}\big(\gamma(r)\big) \qquad \text{for a.e. } r \in (a, b)\,. \tag{4.1}$$

See Figure 4.1 below for an illustration.

Notice that provided that $\operatorname{gph} S$ is closed, continuity of γ along with condition (4.1) implies that the points $\gamma(r)$ lie in $S(r)$ for all r.

Moreau's original construction assumed convexity of the sets $S(r)$, in which case the normal cone $N^c_{S(r)}$ becomes the usual normal cone of convex analysis. Convexity will not play a role in our work, however. We first establish the following bound on the speed of the sweeping process.

THEOREM 4.1 (Speed of the generalized sweeping process). — *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a definable mapping with a closed graph and assume that $\gamma\colon (a, b) \to \mathbb{R}$ is* a.e *differentiable and satisfies the "sweeping inclusion"* (4.1)*. Then the following estimate on speed holds:*

$$\|\dot{\gamma}(r)\| \leqslant \big|D^*_c S\big(r|\gamma(r)\big)\big|^+\,, \qquad \text{for a.e. } r \in (a, b)\,. \tag{4.2}$$

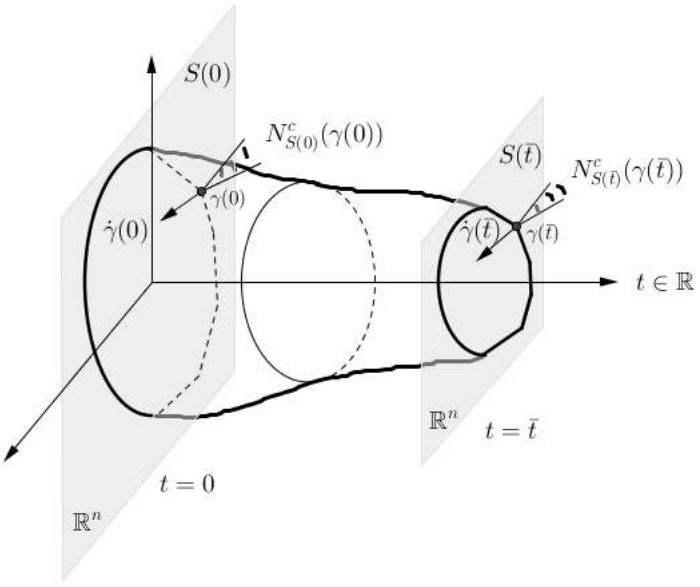

Figure 4.1. Sweeping process, with $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ and $\dot\gamma(t) \in -N^c_{S(t)}(\gamma(t))$

Proof. — Notice that (4.2) is vacuously satisfied when $|D^*S(r|\gamma(r))|^+ = +\infty$, since in this case $|D^*_c S(r|\gamma(r))|^+ \geqslant |D^*S(r|\gamma(r))|^+ = +\infty$. Thus we may limit our attention to parameters r with $|D^*S(r|\gamma(r))|^+ < +\infty$ (that is, S has the Aubin property at r for $\gamma(r)$). Consider the a.e differentiable curve

$$r \mapsto \zeta(r) := (r, \gamma(r)) \qquad r \in (a,b)\,,$$

and observe that (4.1) implies $\zeta(r) \in \operatorname{gph} S$ for a.e. $r \in (a,b)$.

Claim. — For a.e. $r \in (a,b)$ it holds:

$$\langle N^c_{\operatorname{gph} S}(\zeta(r)), \dot\zeta(r)\rangle = 0\,. \tag{4.3}$$

Proof of Claim. — Let $\{\mathcal{M}_i\}$ be a Whitney (a)-regular C^1-stratification of $\operatorname{gph} S$. Fix $r \in (a,b)$ at which ζ is differentiable and let $\mathcal{M}_i$ be the stratum containing $\zeta(r)$. Then either $\dot\zeta(r)$ lies in the tangent space $T_{\mathcal{M}_i}(\zeta(r))$ or there exists an open interval I around r, such that the image $\zeta(I\setminus\{r\})$ does not intersect $\mathcal{M}_i$. It follows quickly that for a.e. $r \in (a,b)$ the implication

$$\zeta(r) \in \mathcal{M}_i \implies \dot\zeta(r) \in T_{\mathcal{M}_i}(\zeta(r)) \quad \text{holds.}$$

On the other hand, for such r, the Whitney–(a) condition implies that $N^c_{\operatorname{gph} S}(\zeta(r))$ is contained in the normal space $N_{\mathcal{M}_i}(\zeta(r))$. Equation (4.3) follows. □

Fix any $r \in (a,b)$ satisfying (4.3). Setting $H := \{r\} \times \mathbb{R}^n$ we have

$$\{r\} \times S(r) = H \cap \operatorname{gph} S\,.$$

Combining this equation with (4.1) we have

$$(1, -\dot\gamma(r)) \in N^c_{\{r\}\times S(r)}(\zeta(r)) = N^c_{H\cap \operatorname{gph} S}(\zeta(r))\,.$$

Since S has the Aubin property at r for $\gamma(r)$, the qualification condition

$$(t,0) \in N_{\operatorname{gph} S}(\zeta(r)) \quad \Longrightarrow \quad t = 0 \qquad \text{holds.}$$

Hence applying the calculus rule [31, Thm. 6.42], we deduce

$$N^c_{H\cap \operatorname{gph} S}(\zeta(r)) \subset N^c_H(\zeta(r)) + N^c_{\operatorname{gph} S}(\zeta(r)) \\ = (\mathbb{R}\times\{0\}) + N^c_{\operatorname{gph} S}(\zeta(r))\,.$$

We conclude that the inclusion

$$(\lambda, -\dot\gamma(r)) \in N^c_{\operatorname{gph} S}(\zeta(r)) \tag{4.4}$$

holds for some $\lambda \in \mathbb{R}$. Appealing then to equation (4.3), we obtain the equality

$$\langle (\lambda, -\dot\gamma(r)), (1, \dot\gamma(r))\rangle = 0\,, \qquad \text{for a.e. } r \in (a,b)\,,$$

and hence

$$\lambda = ||\dot\gamma(r)||^2\,.$$

Plugging this expression back into (4.4) we obtain

$$\Big(||\dot\gamma(r)||^2, -\dot\gamma(r)\Big) \in N^c_{\operatorname{gph} S}(\zeta(r))\,.$$

Observe that in the case $\dot\gamma(r) = 0$, the claimed inequality (4.2) is trivial. Hence we may suppose that this is not the case and deduce that

$$\Big(||\dot\gamma(r)||, \frac{-\dot\gamma(r)}{||\dot\gamma(r)||}\Big) \in N^c_{\operatorname{gph} S}(\zeta(r))\,,$$

which readily yields

$$\big|D^*_c S\big(r|\gamma(r)\big)\big|^+ \geqslant \|\dot\gamma(r)\|\,,$$

as claimed. □

Following the notation of Theorem 4.1, an identical proof shows that if the stronger inclusion

$$\dot{\gamma}(r) \in -N_{S(r)}(\gamma(r)) \qquad \text{holds for a.e. } r \in (a,b)\,.$$

then the stronger relation

$$\|\dot{\gamma}(r)\| \leqslant \operatorname{lip} S\big(r|\gamma(r)\big)$$

holds for almost every r. We note that if S is a Lipschitz continuous mapping, then the authors of [6, 1] show that the "catching up algorithm" produces curves satisfying the above estimate. In contrast, the speed estimate we have just established applies to *all* solutions of the sweeping process in the definable setting.

In light of Theorem 4.1, to understand the length of the solutions of the sweeping process it is essential to consider integrability of the coderivative outer norms $\big|D_c^* S\big(r|\gamma(r)\big)\big|^+$. To this end, we introduce the following definition, much akin to the one considered in [4, §3.3] in the context of subgradient dynamical systems.

Definition 4.2 (Talweg function). — *Consider a set-valued mapping $S : (a,b) \rightrightarrows \mathbb{R}^n$ and let $\mathcal{U}$ be a subset of $\mathbb{R}^n$. Then the* talweg function *of S on $\mathcal{U}$ is the function $\varphi\colon (a,b) \to \overline{\mathbb{R}}$ defined by*

$$\varphi(r) := \sup_{x \in S(r) \cap \mathcal{U}} \Big\{ \big|D_c^* S\big(r|x\big)\big|^+ \Big\}. \tag{4.5}$$

The talweg function of S is rarely infinite.

Lemma 4.3 (Finiteness of the talweg function). — *Consider a definable set-valued mapping $S : (a,b) \rightrightarrows \mathbb{R}^n$, with closed values, and let $\mathcal{U}$ be a bounded subset of $\mathbb{R}^n$. Then there exists $\epsilon > 0$ such that the talweg function of S on $\mathcal{U}$ is not equal to $+\infty$ on $(a, a+\epsilon)$.*

Proof. — Define $\overline{S}\colon [a,b] \rightrightarrows \mathbb{R}^n$ to be the set-valued map whose graph is $\operatorname{cl}\operatorname{gph} S$. Since S has closed values, equality $S(t) = \overline{S}(t)$ holds for all t with an exception of at most finitely many points; see e.g. [15, Prop. 2.30] or [13, Thm. 32]. The result now follows from Theorem 3.3 applied to $F := \overline{S}^{-1}$. □

Next we show that the talweg function of S on a bounded set $\mathcal{U}$ is indeed integrable. The argument is an application of the curve selection lemma on the talweg in the spirit of [4, 20, 24].

THEOREM 4.4 (Integrability of the talweg function). — *For any definable, nonempty-valued, closed-valued mapping $S : (a,b) \rightrightarrows \mathbb{R}^n$ and the talweg function φ of S on a bounded definable set $\mathcal{U}$, the integral*

$$\int_a^b \varphi(r)\,\mathrm{d}r \quad \textit{is finite.}$$

Proof. — For any $r \in (a,b)$, define the set (talweg)

$$\mathcal{V}(r) := \left\{ x \in S(r) \cap \mathcal{U} : \left|D_c^* S\big(r|x\big)\right|^+ \geqslant \frac{1}{2}\varphi(r) \right\}.$$

Notice that φ and $\mathcal{V}$ are definable, and moreover by Lemma 4.3 each set $\mathcal{V}(r)$ is nonempty except for finitely many numbers $r \in (a,b)$. Without loss of generality, assume that the interval (a,b) does not contain such exceptional points. Using the curve selection lemma (e.g. [14, Thm. 1.17]), we obtain a definable curve $r \mapsto \theta(r)$ satisfying $\theta(r) \in \mathcal{V}(r)$ for all $r \in (a,b)$. Since θ is differentiable except for finitely many points, we can assume without loss of generality that θ is differentiable on (a,b).

We claim that the equality

$$t = \left\langle \dot\theta(r), D_c^* S^{-1}(\theta(r)|r)(t) \right\rangle \qquad \text{holds for a.e. } r \in (a,b), \tag{4.6}$$

for which the Clarke coderivative on the right-hand-side is nonempty. To see this, fix a Whitney (a)-regular C^1-stratification of $\operatorname{gph} S$. Then for almost every r, there exists $\varepsilon > 0$ so that the assignment $\tau \mapsto (\tau, \theta(\tau))$ maps the interval $(r-\varepsilon, r+\varepsilon)$ into a single stratum. For such r, fix a stratum $\mathcal{M}$ containing $(r,\theta(r))$. Then clearly the vector $(1,\dot\theta(r))$ is tangent to $\mathcal{M}$ at $(r,\theta(r))$. Therefore by the Whitney condition (a), the Clarke normal cone $N^c_{\operatorname{gph} S}(r,\theta(r))$ is contained in the orthogonal complement to $(1,\dot\theta(r))$. From the definition of the coderivative, we immediately deduce equality (4.6), whenever the Clarke coderivative on the right-hand-side is nonempty.

On the other hand, from (2.1) we have unconditionally

$$\inf_{t=\pm 1} \operatorname{dist}\left(0; D_c^* S^{-1}(\theta(r)|r)(t)\right) = \frac{1}{|D_c^* S(r|\theta(r))|^+}. \tag{4.7}$$

Hence if neither -1 or $+1$ are in the domain of $D_c^* S^{-1}(\theta(r)|r)$, then we have $0 = |D_c^* S(r|\theta(r))|^+ \geqslant \frac{1}{2}\varphi(r)$. On the other hand, for those r satisfying $|D_c^* S(r|\theta(r))|^+ \neq 0$, equations (4.6) and (4.7) immediately imply

$$\begin{aligned} 1 &\leqslant \inf_{t=\pm 1} \left[\operatorname{dist}\left(0; D_c^* S^{-1}(\theta(r)|r)(t)\right)\right] \left\|\dot\theta(r)\right\| \\ &= \frac{\left\|\dot\theta(r)\right\|}{|D_c^* S(r|\theta(r))|^+} \leqslant \frac{2}{\varphi(r)} \left\|\dot\theta(r)\right\|. \end{aligned}$$

Since the curve θ is definable and bounded, it has finite length on (a,b), and consequently the integral $\int_a^b \varphi(r)\,dr$ is finite, as claimed. □

The main result of the paper is now straightforward.

COROLLARY 4.5 (Bounded length of orbits). — *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a definable set-valued mapping having a closed graph and let (a,b) be a possibly unbounded interval of $\mathbb{R}$. Then any absolutely continuous curve $\gamma\colon (a,b) \to \mathbb{R}^n$ satisfying the sweeping inclusion*

$$\dot{\gamma}(r) \in -N^c_{S(r)}\big(\gamma(r)\big) \qquad \text{for a.e. } r \in (a,b),$$

has finite length.

Proof. — First, we may assume that the interval (a,b) is bounded. Indeed, given a trajectory $\gamma : (a,b) \to \mathbb{R}^n$ as above, we consider the semi-algebraic diffeomorphism $\psi : \mathbb{R} \to (-1,1)$ by setting $\psi(t) = \frac{t}{\sqrt{1+t^2}}$. Then the curve $\gamma \circ \psi^{-1}$ is an orbit of the sweeping process $S \circ \psi^{-1}$, defined on $(-1,1)$, and it has the same length as γ.

Without loss of generality, we may also assume that the inclusion $(a,b) \subset \operatorname{dom} S$ holds. Since γ is bounded, there exists a bounded set $\mathcal{U}$ containing the image of γ. Let φ be the talweg of S on $\mathcal{U}$. Then by Theorem 4.1, for a.e. $r \in (a,b)$ we have

$$\|\dot{\gamma}(r)\| \leqslant \big|D^*_c S\big(r|\gamma(r)\big)\big|^+ \leqslant \varphi(r)\,. \tag{4.8}$$

Lemma 4.4 immediately implies the result. □

Remark 4.6 (local monotonicity). — Theorem 4.1 and Corollary 4.5 can be formulated in a slightly more general setting, to incorporate certain strongly monotone perturbations of the trajectory, as considered in [22]. To this end, recall that a mapping $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is*locally α-monotone*, whenever we have

$$\limsup_{\|x-y\|\to 0} \frac{\langle F(x)-F(y), x-y\rangle}{\|x-y\|^2} \geqslant \alpha > 0\,.$$

Let $\gamma\colon (a,b) \to \mathbb{R}^n$ be an absolutely continuous curve and set $\gamma^F := F \circ \gamma$, where either F or $-F$ is locally α-monotone. Assume that $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ is definable with a closed graph, γ^F is absolutely continuous and the sweeping inclusion holds:

$$\dot{\gamma}(r) \in -N^c_{S(r)}\big(\gamma^F(r)\big) \qquad \text{for a.e. } r \in (a,b).$$

Then analogously to Theorem 4.1, for almost all $r \in (a,b)$ we have

$$\|\dot{\gamma}(r)\| \leqslant \frac{1}{\alpha} \cdot \big|D^*_c S\big(r|\gamma^F(r)\big)\big|^+ \leqslant \frac{1}{\alpha}\varphi(r)\,, \tag{4.9}$$

where φ is the talweg given by (4.5) for any subset $\mathcal{U}$ of $\mathbb{R}^n$ containing the image of γ^F. The proof of (4.9) follows the lines of the proof of Theorem 4.1. Indeed, one considers the curve $r \mapsto \zeta^F(r) := (r, \gamma^F(r))$ and eventually deduces

$$\Big(\big\langle \dot\gamma(r), \dot\gamma^F(r)\big\rangle, -\dot\gamma(r)\Big) \in N^c_{\operatorname{gph} S}(\zeta^F(r))\,.$$

Considering again the case $\dot\gamma(r) \neq 0$ (else the claimed inequality is trivial), we get

$$\begin{aligned}\big|D^*_c S\big(r|\gamma^F(r)\big)\big|^+ &\geqslant \Big|\Big\langle \frac{\dot\gamma(r)}{\|\dot\gamma(r)\|}, \dot\gamma^F(r)\Big\rangle\Big| \\ &= \lim_{\epsilon\downarrow 0}\Big|\Big\langle \frac{\gamma(r+\epsilon)-\gamma(r)}{\|\gamma(r+\epsilon)-\gamma(r)\|}, \frac{(F\circ\gamma)(r+\epsilon)-(F\circ\gamma)(r)}{\epsilon}\Big\rangle\Big| \\ &\geqslant \alpha\cdot\|\dot\gamma(r)\|\,,\end{aligned}$$

and the assertion follows.

Remark 4.7 (State-dependent process and ODE). — It is interesting to note that the analogue of Corollary 4.5 easily fails when the sweeping set is state-dependent. Consider any autonomous system of ODEs

$$\dot x = F(x)\,,$$

where F is a semi-algebraic, Lipschitz continuous mapping. Define the semi-algebraic set-valued mapping $S(x) := x + F(x)^\perp$. Then every orbit γ of the ODE is a solution of the state-dependent sweeping process

$$\dot\gamma(t) \in -N_{S(\gamma(t))}(\gamma(t))\,.$$

Consequently, limit cycles and hence bounded orbits of infinite length can easily appear.

5. Desingularization of the coderivative

In this section we record a "desingularization" result for general definable set-valued mappings $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ in the spirit of [24]. Roughly speaking, any such mapping after a "localization" and a reparametrization of its domain can be made to have bounded coderivative norms outside of the critical values of S^{-1}. At the end of the section, we show how our result recovers the desingularization result of Kurdyka [24]. We use this technique then to investigate solvability of the definable sweeping process in the next section. Here's the main desingularization result.

THEOREM 5.1 (Desingularization of the Clarke coderivative). — *Consider a definable set-valued mapping $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ having a closed graph, and let $\mathcal{U}$ be a bounded subset of $\mathbb{R}^n$. Then for any real $a \in \mathbb{R}$ there exists a number $b > a$ and a strictly increasing, continuous function $\Psi\colon [a,b) \to \mathbb{R}$ that is C^1-smooth on (a,b), satisfies $\Psi(a) = a$, and such that:*

$$|D_c^*(S \circ \Psi)(r|x)|^+ \leqslant 1 \qquad \text{for all } r \in (a,b) \text{ and all } x \in S(\Psi(r)) \cap \mathcal{U}.$$

Proof. — If there exists $b > a$ such that the interval (a,b) does not intersect $\operatorname{dom} S$, then there is nothing to prove. Consequently, since S is definable, we may suppose that there exists $b > a$ satisfying the inclusion $(a,b) \subset \operatorname{dom} S$. Let $\varphi\colon (a,b) \to \overline{\mathbb{R}}$ be the talweg of the restriction $S|_{(a,b)}$ on $\mathcal{U}$. Clearly we may assume that φ is continuous on (a,b). If there exists $\varepsilon > 0$ such that φ equals zero on $(a, a+\epsilon)$, then the theorem is trivial yet again. Hence we may suppose that φ is nonzero on the interval (a,b). Define now the function

$$\Phi(r) := a + \int_a^r \varphi(\tau)\,\mathrm{d}\tau \qquad \text{for } r \in [a,b)\,.$$

By Theorem 4.4, the function $\Phi\colon [a,b) \to [a, \Phi(b))$ above is well defined. Moreover Φ is clearly strictly increasing, and C^1-smooth on (a,b) with a nonvanishing derivative.

Consider now the inverse $\Psi := \Phi^{-1}$. Then $\Psi\colon [a, \Phi(b)) \to [a,b)$ is strictly increasing, continuous, and C^1-smooth on $(a, \Psi(b))$. Appealing to [31, Ex. 10.39], for any $\tau \in (a, \Phi(b))$ and any $x \in S(\Psi(\tau)) \cap \mathcal{U}$ we obtain

$$|D_c^*(S \circ \Psi)(\tau|x)|^+ = \frac{|D_c^* S(\Psi(\tau)|x)|^+}{\varphi(\Psi(\tau))} \leqslant 1,$$

as claimed. □

Remark 5.2 (Absolute continuity of Ψ^{-1}). — It is immediate from the proof of Theorem 5.1, that the inverse of the desingularizing function, namely Ψ^{-1}, is guaranteed to be absolutely continuous.

5.1. Sweeping by sublevel sets and gradient descent

We now show how Theorem 5.1 subsumes Kurdyka's seminal desingularization result [24] for C^1 definable functions (see also [3] for a nonsmooth extension). To this end, let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a C^1 definable function and consider the sweeping process associated to sublevel sets

$$\begin{cases} S : \mathbb{R} \rightrightarrows \mathbb{R}^n \\ S(r) := [f \leqslant r]\,. \end{cases} \tag{5.1}$$

Let $t \mapsto x(t)$, for $t \in [0, +\infty)$, be a bounded gradient orbit for f, that is, $\dot{x} = -\nabla f(x)$ with an asymptotic critical value $a := \lim_{t\to+\infty} f(x(t))$, and set $b = f(x(0))$. It follows easily that the mapping

$$t \mapsto q(t) = b - f(x(t))$$

is a diffeomorphism between $(0, \infty)$ and (a, b). Setting $h = q^{-1}$ and $u = x \circ h$ we obtain a curve $u : (a, b] \to \mathbb{R}^n$ with the same image as x and satisfying

$$\dot{u}(r) = -\frac{\nabla f(u(r))}{||\nabla f(u(r))||^2}\,, \qquad \text{for } r \in (a, b]\,.$$

Since equalities $f(u(r)) = r$ and $N_{S(r)}(u(r)) = \mathbb{R}_+ \nabla f(u(r))$ hold, we immediately obtain

$$\dot{u}(r) \in -N_{S(r)}(u(r))\,.$$

That is, the gradient curve $t \mapsto x(t)$, upon reparametrization, is a solution of the sweeping process (5.1). Moreover, an easy computation shows

$$|D_c^* S(f(x)|x)|^+ = \frac{1}{||\nabla f(x)||}\,.$$

Thus the talweg mapping of Definition 4.2 reads

$$\varphi(r) = (\inf\,\{\,||\nabla f(x)|| : f(x) = r,\, x \in \mathcal{U}\})^{-1}$$

and Theorem 4.4, Corollary 4.5, and Theorem 5.1 recover the results of Kurdyka in [24].

6. Existence of solutions

In this section we will be interested in the existence of trajectories generated by the sweeping process. More specifically, given a set-valued mapping $S\colon [0, \eta) \rightrightarrows \mathbb{R}^n$ and a point $x_0 \in S(0)$, we would like to know when there exists a curve $\gamma\colon [0, \eta) \to \mathbb{R}^n$ (appropriately regular) satisfying

$$\left\{ \begin{array}{ll} -\dot{\gamma}(t) \in N^c_{S(t)}(\gamma(t)) & \text{a.e. on } [0, \eta) \\ \gamma(t) \in S(t) & \text{for all } t \in [0, \eta) \\ \gamma(0) = x_0 & \end{array} \right\}. \tag{6.1}$$

In the case that S is Lipschitz continuous with respect to the Pompeiu–Hausdorff distance, a complete answer was provided in [6, Thm. 4.2] and [1, Thm. 3.1]. Here we mean that a mapping $S\colon [0, \eta) \rightrightarrows \mathbb{R}^n$ is *L-Lipschitz continuous* if

$$S(t') \subset S(t) + L|t - t'|\mathcal{B} \qquad \text{for all } t, t' \in [0, \eta)\,.$$

We record below this existence result.

THEOREM 6.1 (Existence of Lipschitz trajectories). — *Let $S\colon [0,\eta) \rightrightarrows \mathbb{R}^n$ be a L-Lipschitz mapping with nonempty, closed values. Then for any $x_0 \in S(0)$, there exists a L-Lipschitz curve $\gamma\colon [0,\eta) \to \mathbb{R}^n$ satisfying* (6.1).

Remark 6.2 (Extensions to the limiting normal cone). — In a very recent paper [17], it was shown that the analogue of Theorem 6.1 holds for definable L-Lipschitz mappings with the limiting normal cone $N_{S(t)}$ replacing the Clarke normal cone $N^c_{S(t)}$. For simplicity, we will state all of our results in the narrower Clarke situation, but an entirely analogous existence theory holds for the limiting case with an identical proof. The only difference is that we must reference the recent manuscript [17] instead of [1, 6] whenever appropriate.

There has been a considerable effort recently to weaken the Lipschitz assumption in the theorem above; see for example [16, 5] and references therein. We will now show that in the definable setting, existence of (at least) piecewise absolutely continuous solutions of (6.1) can be established even when S is not Lipschitz continuous. This will follow by combining Theorem 6.1 with the desingularization techniques developed in the previous sections.

We begin with a local existence result. To this end, note that if S is not Lipschitz continuous, then there is an obvious obstruction to having even a continuous local solution of (6.1) emanating from a point $x_0 \in S(0)$. Indeed, when x_0 lies outside of the outer limit

$$\operatorname*{Limsup}_{t\searrow 0} S(t) := \left\{ \lim_{i\to\infty} x_i : x_i \in S(t_i) \text{ with } t_i \searrow 0 \right\},$$

clearly no such solution can exist. For example, when S corresponds to a sublevel mapping $S(t) = [f \leqslant r_0 - t]$ (for a function $f\colon \mathbb{R}^n \to \mathbb{R}$), such points $x_0 \in [f = r_0] \subset S(0)$ are precisely the local minimizers of f, and no continuous descent curve can emanate from local minimizers.

THEOREM 6.3 (Local existence for the definable sweeping process). — *Consider a definable mapping $S\colon \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ with a closed graph. Then for any $x_0 \in \operatorname{Limsup}_{t\searrow 0} S(t)$, there exists $\epsilon > 0$ and an absolutely continuous curve $\gamma\colon [0,\epsilon) \to \mathbb{R}^n$ satisfying*

$$\left\{ \begin{array}{ll} -\dot\gamma(t) \in N^c_{S(t)}(\gamma(t)) & \text{a.e. on } [0,\epsilon) \\ \gamma(t) \in S(t) & \text{for all } t \in [0,\epsilon) \\ \gamma(0) = x_0 & \end{array} \right\}.$$

Proof. — Fix a closed ball $\mathcal{U}$ in $\mathbb{R}^n$ containing x_0 in its interior, and define the truncation $\widehat{S}(t) := S(t) \cap \mathcal{U}$. Notice, by the assumption $x_0 \in \operatorname{Limsup}_{t\searrow 0} S(t)$ that we have the analogous inclusion $x_0 \in \operatorname{Limsup}_{t\searrow 0} \widehat{S}(t)$. Appealing to definability, we deduce $(0,\epsilon) \subset \operatorname{dom} \widehat{S}$ for some $\epsilon > 0$. By Theorem 5.1, there exists a real number η and a strictly increasing, continuous function $\Psi\colon [0,\eta) \to \mathbb{R}_+$ that is C^1-smooth on $(0,\eta)$, satisfies $\Psi(0) = x_0$, and such that:

$$\operatorname{lip}(\widehat{S} \circ \Psi)(r|x) \leqslant 1 \qquad \text{for all } r \in (0,\eta) \text{ and all } x \in S(\Psi(r)) \cap \mathcal{U}\,.$$

Shrinking η, we may assume that the inclusion $(0,\eta) \subset \operatorname{dom}(\widehat{S} \circ \Psi)$ holds. Appealing to the definition of the Aubin property and the compactness of $\mathcal{U}$, it is easy to see that the mapping $\widehat{S} \circ \Psi$ is locally 1-Lipschitz continuous around any $r \in (0,\eta)$. Hence $\widehat{S} \circ \Psi$ is 1-Lipschitz continuous on the entire interval $(0,\eta)$. Define now the mapping $F\colon [0,\eta) \to \mathbb{R}^n$ given by

$$F(t) := \begin{cases} \widehat{S} \circ \Psi(t) & \text{if } t \in (0,\eta) \\ \operatorname{Limsup}_{r\searrow 0} \widehat{S}(r) & \text{if } t = 0 \end{cases}$$

Notice that F is 1-Lipschitz continuous, has a closed graph, and satisfies $x_0 \in F(0)$. By Theorem 6.1, then there exist a 1-Lipschitz curve $x\colon [0,\eta) \to \mathbb{R}^n$ satisfying

$$\left\{ \begin{array}{ll} -\dot{x}(t) \in N^c_{F(t)}(x(t)) & \text{a.e. on } [0,\eta) \\ x(t) \in F(t) & \text{for all } t \in [0,\eta) \\ x(0) = x_0 & \end{array} \right\}.$$

Since the curve x is 1-Lipschitz, shrinking $\eta > 0$, we may assume that the image of x is contained in the interior of $\mathcal{U}$. Set $\epsilon := \lim_{t\nearrow \eta} \Psi(t)$, and define the curve $\gamma\colon [0,\epsilon) \to \mathbb{R}^n$ by setting $\gamma(r) := x(\Psi^{-1}(r))$. Notice by Remark 5.2 that the inverse Ψ^{-1} is absolutely continuous. Hence γ is absolutely continuous as well, being a composition of a Lipschitz function and an absolutely continuous function. Finally observe that γ satisfies $\gamma(r) = x(\Psi^{-1}(r)) \in F(\Psi^{-1}(r)) \subset S(r)$ for all $r \in [0,\epsilon)$ and $-\dot{\gamma}(r) = \frac{-1}{\psi'(\psi^{-1}(r))}\dot{x}(\Psi^{-1}(r)) \in N^c_{F(\Psi^{-1}(r))}(x(\Psi^{-1}(r))) = N^c_{S(r)}(\gamma(r))$ for a.e. $r \in [0,\epsilon)$. This concludes the proof. □

Next, we will try to maximally extend local solutions of the sweeping process, aiming for a global solution. To this end, we first observe the following.

Corollary 6.4 (Convergence to extrema). — *Consider a definable map $S\colon \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ with a closed graph and a point $x_0 \in S(t_0)$. Then any absolutely continuous curve $\gamma\colon [0,\epsilon) \to \mathbb{R}^n$, having a maximal domain of*

definition, such that

$$\left\{\begin{array}{ll} -\dot{\gamma}(t) \in N^c_{S(t)}(\gamma(t)) & \text{a.e. on } [0,\epsilon) \\ \gamma(t) \in S(t) & \text{for all } t \in [0,\epsilon) \\ \gamma(0) = x_0 & \end{array}\right\},$$

is either unbounded, or has finite length and converges to some point $x_\infty \notin \operatorname{Limsup}_{t \searrow \epsilon} S(t)$.

Proof. — This follows immediately from Theorems 4.5 and 6.3. □

Let us recall that $S\colon [0,\eta) \rightrightarrows \mathbb{R}^n$ is *locally bounded* at t if there exists an open interval I around t such that the image $S(I)$ is a bounded set. We say that S is *inner-semicontinuous* at t if for any $x \in S(t)$ and any sequence $t_i \in [0,\eta)$ converging to t, there exists a sequence $x_i \in S(t_i)$ converging to x. In particular, in the notation of the above theorem, S is not inner-semicontinuous at ϵ, as certified by x_∞.

The following is the main result of the section.

Corollary 6.5 (Global existence). — *Consider a locally bounded, definable mapping* $S\colon [0,\eta) \rightrightarrows \mathbb{R}^n$ *with a closed graph and nonempty values. Then for any* $x_0 \in \operatorname{Limsup}_{t \searrow 0} S(t)$ *there exists a curve* $\gamma\colon [0,\eta) \to \mathbb{R}^n$ *satisfying:*

(1) *There is a partition* $t_0 = 0 < t_1 < \ldots < t_k = \eta$ *of the interval* $[0,\eta)$ *such that* γ *is absolutely continuous on each segment* $[t_i, t_{i+1})$*; and*
(2) *The curve* γ *satisfies:*

$$\left\{\begin{array}{ll} -\dot{\gamma}(t) \in N^c_{S(t)}(\gamma(t)) & \text{a.e. on } [0,\eta) \\ \gamma(t) \in S(t) & \text{for all } t \in [0,\eta) \\ \gamma(0) = x_0 & \end{array}\right\}.$$

When S *is inner-semicontinuous on the entire interval* $[0,\eta)$*, then no partition is needed and we can assure that* γ *is absolutely continuous on the entire interval* $[0,\eta)$.

Proof. — Since the Aubin property implies inner-semicontinuity, by Theorem 3.3 (see also [13]) the mapping S is inner-semicontinuous at every point $t \in [0,\eta)$ outside of some finite set $\mathcal{N}$. By Theorem 6.3, there exists $\epsilon > 0$ and an absolutely continuous curve $\gamma\colon [0,\epsilon) \to \mathbb{R}^n$ satisfying the conditions (6.1). By Zorn's lemma we may maximally extend the domain of γ subject to the system (6.1). Denote the resulting domain by $[0,a)$. By Corollary 6.4 and local boundedness of S, the curve γ converges to some point $x_\infty \notin \operatorname{Limsup}_{t \searrow a} S(t)$. In particular, S is not inner-semicontinuous

at a and therefore we deduce that $a \in \mathcal{N}$. We can now repeat the argument with x_0 being a point in $\operatorname{Limsup}_{t \searrow a} S(t)$. Notice that the latter set is nonempty since S is locally bounded. The process can restart at most finitely many times since $\mathcal{N}$ has finite cardinality. Concatenating the (finitely many) curves obtained in this way yields the result. □

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