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Switching-Algebraic Analysis of Multi-State System Reliability

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Authors' contributions

This work was carried out in collaboration between the two authors. Author AMAR envisioned and designed the study, performed the symbolic and numerical analysis, solved the detailed example and wrote the whole manuscript. Author MAA managed the literature search, prepared the tables, and drew the figures. Both authors read and approved the final manuscript.

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ABSTRACT

Multi-State systems are systems whose outputs are multi-valued (due to multiple levels of capacity or performance) and (possibly) whose inputs are also multi-valued (due to multiple performance levels or multiple modes of failure). These systems are a generalization of binary or dichotomous systems that have binary or two-valued outputs and inputs. The multi-state reliability model generalizes and adapts many of the concepts and techniques of the binary reliability model, and naturally ends up with sophisticated concepts and techniques of its own. This paper explores the possibility of simply analyzing a multi-state system by reformulating or encoding its inputs in terms of binary inputs and evaluating each of its multiple output levels as an individual binary output of these alternative inputs. This means that we dispense with multiple-valued logic in the analysis of a multi-state system, since this system is now analyzed solely via switching algebra (two-valued Boolean algebra). The wealth of tools and techniques of switching algebra are now used (without any modification or adaptation) in the analysis of the multi-state system (at the cost of an expanded input domain). The paper makes its point though the analysis of a standard commodity-supply system, whose multi-valued inputs are expressed in terms of physically-meaningfully binary inputs.

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The analysis is made possible through the use of advanced techniques for deriving probability–ready expressions together with the employment of large-size Karnaugh maps and utilization of multiplication tables, symmetric switching functions, and Boolean quotients. Though the system studied involves twelve binary input variables, its manual analysis is completed successfully herein, yielding results that exactly agree with those obtained earlier *via* automated methods, and are possibly less prone to the notorious effects of round-off errors.

Keywords: System reliability; probability-ready expression; k-out-of-n; switching–algebraic analysis; multi-state system; binary system; Boolean quotient; eight-variable Karnaugh map.

1. INTRODUCTION

Many practical systems and (possibly) their components have more than two states (*i.e.*, operational and failed). On the system level, multiple states can be interpreted as multiple levels of system capacity or performance. On the component level, the multiple states can be interpreted as different performance levels and also as multiple failure modes with each mode having a different impact on the system level performance. These systems are modeled as multi-state systems (MSSs). Prominent among MSSs is the class of coherent MSSs whose cornerstones are the k-out-of-n MSSs. (See Appendix A).

The literature abounds with many research papers on MSSs [1-18]. A significantly large proportion of these papers are devoted to coherent MSSs, and, in particular, to their backbone class of k-out-of-n MSSs [19-22]. Almost every technique used with binary systems has been modified, adapted, or extended for use with MSSs. A plethora of sophisticated concepts and techniques for MSSs have accumulated over the past few decades.

This paper advocates the simple thesis that binary tools do not have to be left behind while handling MSSs. In fact, there is a wealth of these tools, and many of them are pedagogically insightful and computationally powerful. The paper offers a switching-algebraic analysis of a standard multi-state commodity-supply system, in which techniques of switching algebra are solely used without any modification or adaptation. This analysis can be extended to other MSSs of comparable sizes, and might be automated to handle MSSs of lager sizes.

We do not propose to solve general and large MSS problems by reducing them to binary problems. But we have other more modest purposes in mind. These are:

1. To provide a truly independent means to check and verify the somewhat weird and frequently non-transparent and sophisticated MSS solutions,
2. To offer some pedagogical insight on the nature of MSS problems and a justification of the currently used MSS mathematics,
3. To establish a clear and insightful interrelationship between binary modeling and MSS modeling, and
4. To push mathematical tools of binary modeling to their utmost utility, and make the most of them.

The organization of the remainder of this paper is as follows. Section 2 advocates working in the switching (two-valued Boolean) domain despite the multi-valued nature of the pertinent problem. Section 3 offers a physically-meaningful binary (two-valued) description of a typical multi-state system, while Section 4 details the binary analysis of such a system utilizing several important concepts of switching algebra, including those of probability-ready expressions, Boolean quotients, and symmetric switching functions. Section 5 discusses our results for the homogeneous (i. i. d.) case, and verifies that the expectations of the multiple instances of the multi-valued output add identically to 1. Section 6 uses large (8-varaible) Karnaugh maps to verify the analysis in the heterogeneous case. Work in this section constitutes an alternative map method that can be used instead of the preceding algebraic analysis. Section 7 shows that our numerical results exactly agree with those obtained by Tian et al. [19] and later by Mo et al. [22]. Section 7 also suggests that our method might be less prone to the undesirable effects of round-off errors. Section 8 concludes the paper. Three appendices are included to make the paper self–contained. Appendix A provides an ample verbal description for the MSS that is solved throughout this paper. Appendix B reviews several pertinent concepts in reliability theory, while Appendix C briefly describes symmetric switching functions (SSFs) and their utility in characterizing successes of binary k-out-of-n: G systems.

2. ADVANTAGES OF WORKING IN THE SWITCHING DOMAIN

This paper is essentially a sequel and a multi-state extension of earlier work on computing system reliability through working in switching (two-valued Boolean domain) [23-44]. Rushdi and Rushdi [42] list advantages of reliability modeling for binary systems in the Boolean domain. These advantages include easy formulation, useful insight and fallacy avoidance. These advantages are still all valid for handling MSSs. In addition, the utilization of familiar already-existing tools is definitely an asset.

Admittedly, it might be more natural to formulate multi-state reliability problems in term of multi-valued logic rather than binary logic. However, one of the 'good' alternatives for handling problems of multi-valued logic is to reduce them to problems of binary logic. There is already an unsettled debate (extending to areas beyond the scope of reliability), on whether problems of multi-valued logic should be better solved in the multi-valued domain, or should be alternatively replaced by equivalent problems in the binary domain [45]. We reiterate herein our belief that one might prefer one of these two alternatives to the other only as a matter of personal discretion, taste, and background.

3. BINARY DESCRIPTION OF A TYPICAL MULTI-STATE SYSTEM

In this section, we introduce a typical multi-state system that has been proposed and studied by Tian et al. [19] and further studied by Fadhel et al. [21], and Mo et al. [22]. This system is verbally described in Appendix A and is shown in Fig. 1. It is modeled as a multi-state k-out-of-n: G system with n = 4, $k_1 = 4$, $k_2 = 2$, and $k_3 = 3$ (see Appendix B). In a nutshell, the system is a supply system of a certain commodity (*e.g.*, oil, water, energy, transportation traffic, or communication traffic, *etc.*) that employs four pipelines to transport the given commodity from the given source to three sink nodes called stations.

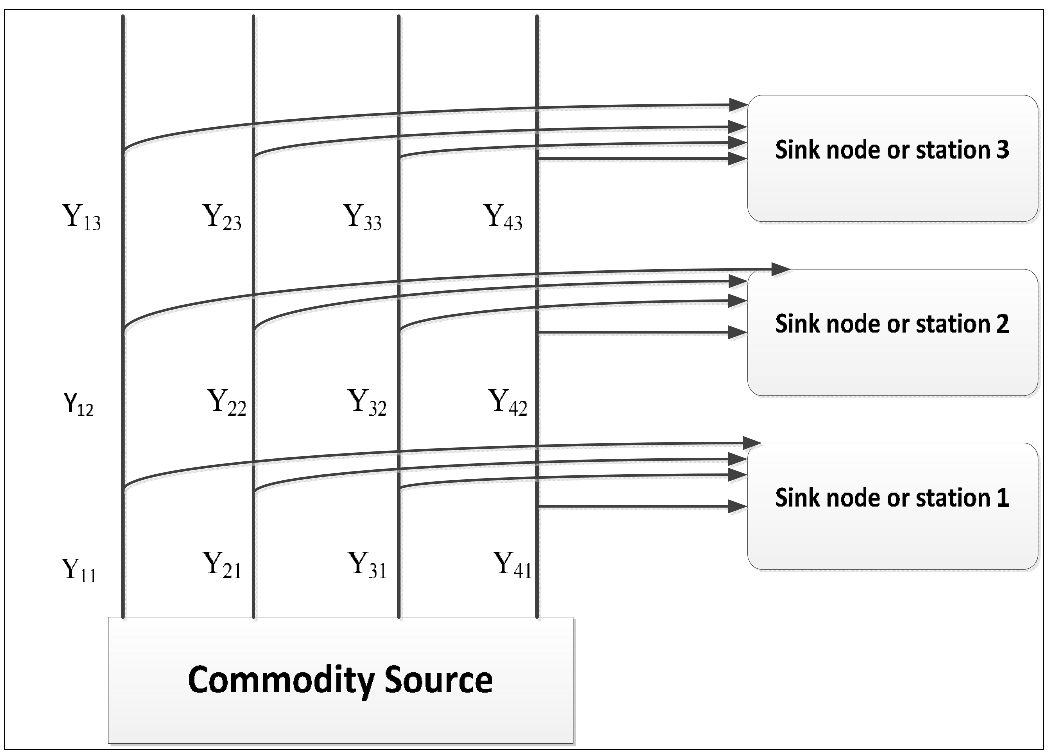

Fig. 1. A commodity-supply system that is modeled as a multi-state k-out-of-n: G system (Adapted from Tian et al. [19]). Here Y_{ij} denotes the success of section j of pipeline i ($1 \leq i \leq 4$, $1 \leq j \leq 3$).

As a multi-state system, the system can be quantified by the multi-state input variables $X_i\{j\}$, $1 \leq i \leq 4$, $0 \leq j \leq 3$, and the multi-state output variable $S\{k\}$, $0 \leq k \leq 3$. These variables are defined as follows

> $X_i\{j\}$= A binary indicator that the commodity can reach *up to* station j through pipeline (route, transmission line, or communication link) number i $(1 \leq i \leq 4,\ 0 \leq j \leq 3)$. In other words, $X_i\{j\}$ indicates that the commodity can reach all stations ℓ $(1 \leq \ell \leq j)$ though pipeline i.

For convenience, the definition above refers to a station 0 (which actually does not exist), but the inclusion of j = 0 allows us to handle the null case in which the commodity cannot reach any of the existing stations. Note that for a specific pipeline i, the set of values $\{X_i\{j\},\ 0 \leq j \leq 3\}$ is an orthonormal set, *i.e.*, one and only one of the variables $X_i\{0\}$, $X_i\{1\}$, $X_i\{2\}$ and $X_i\{3\}$ is 1, while the rest are 0., *i.e.*, for $1 \leq i \leq 4$.

$$X_i\{0\}+ X_i\{1\}+ X_i\{2\}+ X_i\{3\} =1 \tag{1a}$$

$$X_i\{j_1\}^* X_i\{j_2\} = 0 \text{ for } j_1 \neq j_2 \tag{1b}$$

Likewise, we use $S\{k\}$ $\{0 \leq k \leq 3\}$ as a binary indicator that the system can meet the commodity demand *up to* station k *i.e.*, for all stations ℓ $(1 \leq \ell \leq k)$. Again, we note that station 0 does not exist, and hence k = 0 means that the system cannot meet the commodity demand of any existing station.

We now introduce a new set of twelve input binary physically-meaningful variables Y_{ij} $(1 \leq i \leq 4,\ 1 \leq j \leq 3)$ to describe the original multi-state system, where Y_{ij} denotes the success of section j of pipeline i (see Fig. 1). Each of the four-valued variables X_i $\{1 \leq i \leq 4\}$ is now replaced by three binary variables Y_{i1} ,Y_{i2} and Y_{i3} through the relations deduced in Table 1, and graphically depicted in Fig. 2. Note that Fig. 2 is a Karnaugh-map-like structure of three map variables Y_{i1}, Y_{i2} and Y_{i3}, and four exhaustive and mutually exclusive areas depicting the four orthonormal instances of X_i. Table 2 lists direct and inverse relations among expectations of instances of X_i and those of the Y_{ij}'s. The inverse relations are needed for converting the input data of Tian et al. [19] into input data for our purposes.

In passing, we note that we have chosen to express the original multi-valued inputs in terms of physically meaningful binary variables Y_{ij} without insisting on minimizing the number of the new binary inputs. In fact, two binary inputs Z_{i1} and Z_{i2} suffice as a binary reformulation of the four-valued X_i, since we can write (for $1 \leq i \leq 4$)

$$X_i\{0\} = \bar{Z}_{i1}\bar{Z}_{i2} \tag{2}$$

$$X_i\{1\} = \bar{Z}_{i1}Z_{i2} \tag{3}$$

$$X_i\{2\} = Z_{i1}\bar{Z}_{i2} \tag{4}$$

$$X_i\{3\} = Z_{i1}Z_{i2} \tag{5}$$

However, it is very difficult to ascribe physical meaning to the artificially-constructed variables Z_{i1} and Z_{i2}. Moreover, if we use the Z_{ij}'s rather than the Y_{ij}'s, the analysis in Section 4 might become less transparent.

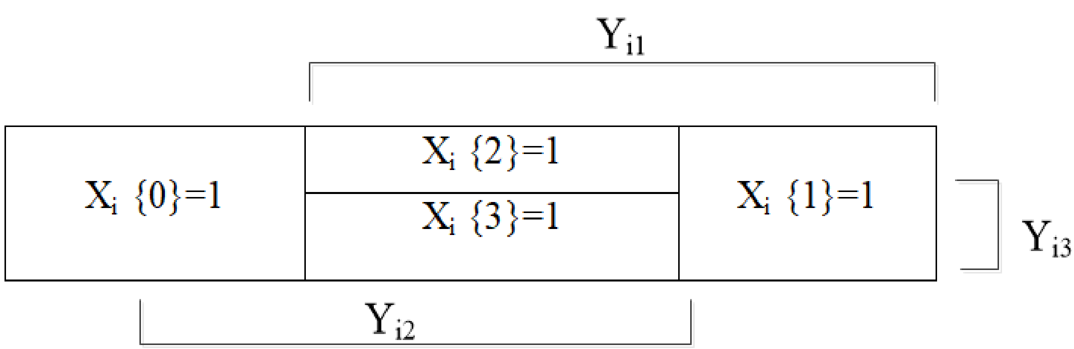

Fig. 2. Relation between members of the orthonormal set $\{X_i(j),\ 1 \leq i \leq 4,\ 0 \leq j \leq 3\}$, and the binary variables $\{Y_{ij}\}$, $1 \leq i \leq 4$, $1 \leq j \leq 3$.

Table 1. Relating the four-valued variable X_i to the three binary variables Y_{i1}, Y_{i2} and Y_{i3}

Situation	Description in terms of X_i	Description in terms of the Y_{ij}'s	Resulting relation
The commodity cannot reach any station	$X_i(0) = 1$	$Y_{i1} = 0$	$X_i\{0\} = \overline{Y}_{i1}$
The commodity reaches station 1 (and no further)	$X_i(1) = 1$	$Y_{i1} = 1$, $Y_{i2} = 0$	$X_i\{1\} = Y_{i1}\,\overline{Y}_{i2}$
The commodity reaches station 2 (and no further)	$X_i(2) = 1$	$Y_{i1} = Y_{i2} = 1$, $Y_{i3} = 0$	$X_i\{2\} = Y_{i1}\,Y_{i2}\,\overline{Y}_{i3}$
The commodity reaches station 3	$X_i(3) = 1$	$Y_{i1} = Y_{i2} = Y_{i3} = 1$	$X_i\{3\} = Y_{i1}\,Y_{i2}\,Y_{i3}$

Table 2. Direct and inverse relations among expectations of instances of X_i and those of the encoding binary variables Y_{ij}'s

Expectations of instances of X_i in terms of those of the Y_{ij}'s	Expectations of Y_{ij}'s in terms of those of instances of X_i
$E\{X_i(0)\} = E\{\overline{Y}_{i1}\}$	$E\{Y_{i1}\} = 1 — E\{X_i(0)\}$
$E\{X_i(1)\} = E\{Y_{i1}\}\,E\{\overline{Y}_{i2}\}$	$E\{Y_{i2}\} = 1 — E\{X_i(1)\}/(1 — E\{X_i(0)\})$
$E\{X_i(2)\} = E\{Y_{i1}\}\,E\{Y_{i2}\}\,E\{\overline{Y}_{i3}\}$	
$E\{X_i(3)\} = E\{Y_{i1}\}E\{Y_{i2}\}\,E\{Y_{i3}\}$ = $1 — E\{X_i(0)\} — E\{X_i(1)\} — E\{X_i(2)\}$	$E\{Y_{i3}\} = 1 — E\{X_i(2)\}/(1 — E\{X_i(0)\} — E\{X_i(1)\})$

Now, we seek a formulation of the four-valued system success variable $S\{k\}$, $\{0 \leq k \leq 3\}$ in terms of binary variables. Again, we sacrifice minimality of the number of variables for the gain of intuitive insight. We use $S_m\,\{1 \leq m \leq 3\}$ to depict the success of station m (that its commodity demand is met). Hence, the four instances of S are given by the relations

$$S\{0\} = \overline{S}_1 \tag{6}$$

$$S\{1\} = S_1\,\overline{S}_2 \tag{7}$$

$$S\{2\} = S_1\,S_2\,\overline{S}_3 \tag{8}$$

$$S\{3\} = S_1\,S_2\,S_3 \tag{9}$$

These relations are demonstrated via the Karnaugh-map-like structure of Fig. 3, which has three map variables S_1, S_2 and S_3 and has four mutually exclusive and exhaustive areas for the four instances of the four-valued variable S.

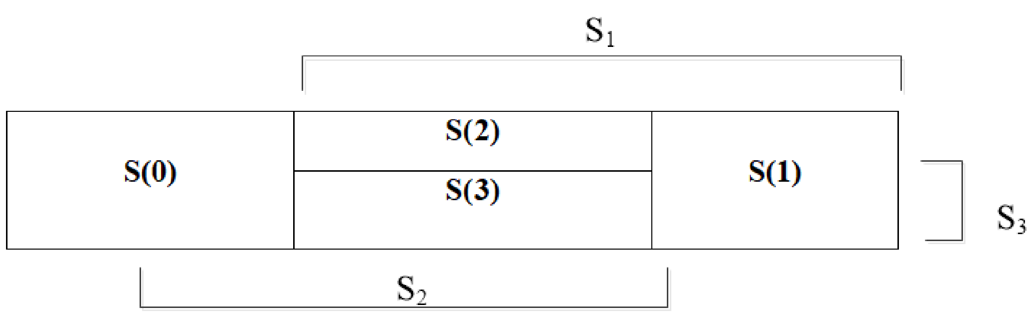

Fig. 3. Relation between the three binary four-valued system success variable $S\{k\}$, $0 \leq k \leq 3$ and the binary station success variables S_m , $1 \leq m \leq 3$.

4. BINARY ANALYSIS OF THE TYPICAL MULTI-STATE SYSTEM

Our objective in this section is to compute the expectations $E\{S(k)\}$ of the four instances ($0 \le k \le 3$) of the four-valued system success. These are to be expressed in terms of the reliabilities $E\{Y_{ij}\}$ of the various pipelines sections. First, we compute the station successes S_1, S_2 and S_3 as shown in Table 3, and further we (digress a little bit to) compute the expectations of these three successes (which represent binary coherent systems). First, we directly obtain

$$E\{S(0)\} = 1 - E\{S_1\} = 1 - p_{11}\, p_{21}\, p_{31}\, p_{41} \quad (10)$$

Using the expression of S_2 in Table 3, we obtain its complement $\overline{S}_2$ via (C.4) and (C.5) as

$$\begin{aligned} \overline{S}_2 &= Sy(\{0, 1\}; Y_{11} Y_{12}, Y_{21} Y_{22}, Y_{31} Y_{32}, Y_{41} Y_{42}) \\ &= Sy(\{3, 4\}; \overline{Y_{11}\, Y_{12}}, \overline{Y_{21}\, Y_{22}}, \overline{Y_{31}\, Y_{32}}, \overline{Y_{41}\, Y_{42}}) \end{aligned} \quad (11)$$

Table 3. Formulas for station successes and their expectations

Station number	General success formula	PRE formula	Expectation formula
1	$S_1 = \wedge_{i=4}^{4} \overline{X}_i\{0\}$ $= \wedge_{i=1}^{4} Y_{i1}$	$S_1 = \wedge_{i=1}^{4} Y_{i1}$	$E\{S_1\} = \wedge_{i=1}^{4} p_{i1}$
2	$S_2 = S_y$ ({2, 3, 4}; $X_1\{2\} \vee X_1\{3\}$, $X_2\{2\} \vee X_2\{3\}$, $X_3\{2\} \vee X_3\{3\}$, $X_4\{2\} \vee X_4\{3\}$) $= S_y$ ({2, 3, 4}, $Y_{11}Y_{12}$, $Y_{21}Y_{22}$, $Y_{31}Y_{32}$, $Y_{41}Y_{42}$)	$S_2 = Z_1 Z_2 \vee Z_1 \overline{Z_2} Z_3 \vee Z_1 \overline{Z_2}\, \overline{Z_3} Z_4 \vee \overline{Z_1} Z_2 Z_3 \vee \overline{Z_1} Z_2 \overline{Z_3} Z_4 \vee \overline{Z_1}\, \overline{Z_2} Z_3 Z_4$ where $Z_i = Y_{i1} Y_{i2}$, $1 \le i \le 4$	$E\{S_2\} = w_1w_2 + w_1(1-w_2)w_3$ $+ w_1(1-w_2)(1-w_3)w_4$ $+ (1-w_1) w_2w_3$ $+ (1-w_1) w_2(1-w_3)w_4$ $+ (1-w_1)(1-w_2) w_3w_4$ where $w_i = E\{Y_{i1} Y_{i2}\}$ $= p_{i1}\, p_{i2}$, $1 \le i \le 4$
3	$S_3 = S_y$ ({3, 4}; $X_1(3)$, $X_2(3)$, $X_3(3)$, $X_4(3)$ $= S_y$ ({3, 4}, $Y_{11} Y_{12} Y_{13}$, $Y_{21} Y_{22} Y_{23}$, $Y_{31} Y_{32} Y_{33}$, $Y_{41} Y_{42} Y_{43}$)	$S_3 = Z_1 Z_2 Z_3 \vee Z_1Z_2\overline{Z_3} Z_4 \vee Z_1 \overline{Z_2} Z_3Z_4 \vee \overline{Z_1} Z_2 Z_3 Z_4$ where $Z_i = Y_{i1} Y_{i2} Y_{i3}$, $1 \le i \le 4$	$E\{S_3\} = w_1w_2w_3 + w_1 w_2 (1-w_3)w_4$ $+ w_1(1-w_2)w_3w_4 + (1-w_1) w_2w_3 w_4$ where $w_i = E\{Y_{i1} Y_{i2} Y_{i3}\}$ $= p_{i1}\, p_{i2}\, p_{i3}$, $1 \le i \le 4$

Hence, the instant S{1} is given by

$$\begin{aligned} S\{1\} &= S_1\overline{S_2} = Y_{11}Y_{21}Y_{31}Y_{41}\, \overline{S_2} \\ &= Y_{11}Y_{21}Y_{31}Y_{41}\ (\overline{S_2} / Y_{11}Y_{21}Y_{31}Y_{41}) \\ &= Y_{11}Y_{21}Y_{31}Y_{41}\ Sy(\{3, 4\}, \overline{Y_{12}}, \overline{Y_{22}}, \overline{Y_{32}}, \overline{Y_{42}}) \end{aligned} \quad (12)$$

where we made some simplifications using property (B.3) of the Boolean quotient (See Appendix B). Using results of Appendix C, we rewrite $S_1\overline{S}_2$ in PRE form as

$$S\{1\} = Y_{11}Y_{21}Y_{31}Y_{41} (\overline{Y}_{12}\overline{Y}_{22}\overline{Y}_{32} \vee \overline{Y}_{12}\overline{Y}_{22}Y_{32}\overline{Y}_{42} \vee \overline{Y}_{12}Y_{22}\overline{Y}_{32}\overline{Y}_{42} \vee Y_{12}\, \overline{Y}_{22}\overline{Y}_{32}\overline{Y}_{42}) \quad (13)$$

which transforms directly, on a one-to-one basis, into the expectation

$$S\{1\}\} = p_{11}\, p_{21}\, p_{31}\, p_{41}\, (\, q_{12}\, q_{22}\, q_{32} + q_{12}\, q_{22}\, p_{32}\, q_{42} + q_{12}\, p_{22} q_{32}\, q_{42} + p_{12}\, q_{22}\, q_{32}\, q_{42}\,) \quad (14)$$

Similarly, we obtain the products S_1S_2 and S_1S_3 as

$$\begin{aligned} S_1S_2 &= Y_{11}Y_{21}Y_{31}Y_{41}\ Sy(\{2, 3, 4\}; Y_{11}Y_{12}, Y_{21}Y_{22}, Y_{31}Y_{32}, Y_{41}Y_{42}) \\ &= Y_{11}Y_{21}Y_{31}Y_{41}\ (Sy(\{2, 3, 4\}; Y_{11}Y_{12}, Y_{21}Y_{22}, Y_{31}Y_{32}, Y_{41}Y_{42}) / Y_{11}Y_{21}Y_{31}Y_{41}) \\ &= Y_{11}Y_{21}Y_{31}Y_{41}\ Sy(\{2, 3, 4\}; Y_{12}, Y_{22}, Y_{32}, Y_{42}) \end{aligned} \quad (15)$$

$$S_1S_3 = Y_{11}Y_{21}Y_{31}Y_{41}\ Sy(\{3, 4\}; Y_{11}Y_{12} Y_{13}, Y_{21}Y_{22} Y_{23}, Y_{31}Y_{32} Y_{33}, Y_{41}Y_{42} Y_{43})$$

$$= \ Y_{11}Y_{21}Y_{31}Y_{41} \ (Sy(\{3, 4\}; Y_{11}Y_{12}\, Y_{13}, Y_{21}Y_{22}\, Y_{23}, Y_{31}Y_{32}\, Y_{33}, Y_{41}Y_{42}\, Y_{43}) \,/\, Y_{11}Y_{21}Y_{31}Y_{41})$$
$$= Y_{11}Y_{21}Y_{31}Y_{41} \ Sy(\{3, 4\}; Y_{12}\, Y_{13}, Y_{22}\, Y_{23}, Y_{32}\, Y_{33}, Y_{42}\, Y_{43}) \qquad (16)$$

We now observe that ($S_1S_3 \le S_1S_2$) since

$$Sy(\{3, 4\}; Y_{12}\, Y_{13}, Y_{22}\, Y_{23}, Y_{32}\, Y_{33}, Y_{42}\, Y_{43})$$
$$= Y_{12}\, Y_{13}\, Y_{22}\, Y_{23}\, Y_{32}\, Y_{33} \vee Y_{12}\, Y_{13}\, Y_{22}\, Y_{23}\, Y_{42}\, Y_{43}$$
$$\vee\ Y_{12}\, Y_{13}\, Y_{32}\, Y_{33}\, Y_{42}\, Y_{43} \vee Y_{22}\, Y_{23}\, Y_{32}\, Y_{33}\, Y_{42}\, Y_{43}$$
$$\le Y_{12}\, Y_{22} \vee Y_{12}\, Y_{32} \vee Y_{12}\, Y_{42} \vee Y_{22}\, Y_{32} \vee Y_{22}\, Y_{42} \vee Y_{32}\, Y_{42}$$
$$= Sy(\{3, 4\}; Y_{12}, Y_{22}, Y_{32}, Y_{42})$$

Therefore, the final expression for S{3} is

$$S\{3\} = S_1S_2\, S_3 = (S_1S_2)\,(S_1S_3) = S_1S_3$$
$$= Y_{11}Y_{21}Y_{31}Y_{41} \ Sy(\{3, 4\}; Y_{12}\, Y_{13}, Y_{22}\, Y_{23}, Y_{32}\, Y_{33}, Y_{42}\, Y_{43}) \qquad (17)$$

which can be recast in the PRE form

$$S\{3\} = Y_{11}Y_{21}Y_{31}Y_{41}\,(Z_1\, Z_2\, Z_3 \vee Z_1\, Z_2\, \bar{Z}_3 Z_4 \vee Z_1\, \bar{Z}_2\, Z_3\, Z_4$$
$$\vee\ \bar{Z}_1 Z_2\, Z_3\, Z_4\,) \qquad (18)$$

where

$$Z_i = Y_{i2}\, Y_{i3}\,, \ \ 1 \le i \le 4 \qquad (19)$$

and finally we obtain the expectation E{S{3}} as

$$E\{S\{3\}\} = p_{11}\, p_{21}\, p_{31}\, p_{41}\,(\, w_1w_2w_3 + w_1w_2\,(1 - w_3)\, w_3w_4 \ + (1 - w_1)\, w_2w_3w_4) \qquad (20)$$

where

$$w_i = E\{Y_{i2}\ Y_{i3}\} = p_{i2}\, p_{i3}\,, \quad 1 \le i \le 4 \qquad (21)$$

Now, the instance S{2} of the multi-state four-valued variable S is (by virtue of (B.3))

$$S\{2\} = S_1\, S_2\, \bar{S}_3 = S_1\,(\, S_2\, \bar{S}_3 \,/\, S_1\,) = S_1\,(S_2 \,/\, S_1\,)\,(\,\bar{S}_3 \,/\, S_1) \qquad (22)$$

where (S_2 / S_1) is obtained from (15) and (B.3) as

$$(S_2 \,/\, S_1\,) = Sy(\{2, 3, 4\}; Y_{12}\,, Y_{22}\,, Y_{32}\,, Y_{42}) \qquad (23)$$

whose PRE form is

$$(S_2 / S_1\,) = Y_{12}\, Y_{22} \vee Y_{12}\, Y_{32}\, \bar{Y}_{22} \vee Y_{12}Y_{42}\, \bar{Y}_{22}\bar{Y}_{32}$$
$$\vee\ Y_{22}\, Y_{32}\bar{Y}_{12} \vee Y_{22}Y_{42}\, \bar{Y}_{12}\bar{Y}_{32} \vee Y_{32}Y_{42}\bar{Y}_{12}\bar{Y}_{22} \qquad (24)$$

From Table 3, we can write $\bar{S_3}$ as

$$\bar{S}_3 = Sy(\{0, 1, 2\}; Y_{11}\, Y_{12}\, Y_{13}, Y_{21}\, Y_{22}\, Y_{23}, Y_{31}\, Y_{32}\, Y_{33}, Y_{41}\, Y_{42}\, Y_{43}) \qquad (25)$$

and hence its quotient with respect to $S_1 = Y_{11}Y_{21}Y_{31}Y_{41}$ is

$$(\,\bar{S}_3 \,/\, S_1\,) = Sy\,(\{0, 1, 2\}; Y_{12}\, Y_{13}\,, Y_{22}\, Y_{23}, Y_{32}\, Y_{33}, Y_{42}\, Y_{43})$$
$$= Sy(\{\, 2, 3, 4\}; \overline{Y_{12}\ Y_{13}}, \overline{Y_{22}\ Y_{23}}, \overline{Y_{32}\, Y_{33}}, \overline{Y_{42}\ Y_{43}}) \qquad (26)$$

which is given be the PRE form

$$(\bar{S_3} \,/\, S_1\,) = \overline{Y_{12}\ Y_{13}}\ \overline{Y_{22}\ Y_{23}} \vee \overline{Y_{12}\ Y_{13}}\ \overline{Y_{32}\, Y_{33}}\ Y_{22}\, Y_{23}$$

$$
\begin{aligned}
&\vee\ \overline{Y_{12}\ Y_{13}}\ \ \overline{Y_{42}\ Y_{43}}\ Y_{22}\,Y_{23}\,Y_{32}\,Y_{33} \vee \overline{Y_{22}\ Y_{23}}\ \ \overline{Y_{32}\,Y_{33}}\ Y_{12}\,Y_{13} \\
&\vee\ \overline{Y_{22}\ Y_{23}}\ \ \overline{Y_{42}\ Y_{43}}\ Y_{12}\,Y_{13}\,Y_{32}\,Y_{33} \\
&\vee\ \overline{Y_{32}\,Y_{33}}\ \ \overline{Y_{42}\ Y_{43}}\ Y_{12}\,Y_{13}\,Y_{22}\,Y_{23} \\
&= (\overline{Y}_{12} \vee Y_{12}\overline{Y}_{13})(\overline{Y}_{22} \vee Y_{22}\overline{Y}_{23}) \\
&\vee (\overline{Y}_{12} \vee Y_{12}\overline{Y}_{13})(\overline{Y}_{32} \vee Y_{32}\overline{Y}_{33})\,Y_{22}\,Y_{23} \\
&\vee (\overline{Y}_{12} \vee Y_{12}\overline{Y}_{13})(\overline{Y}_{42} \vee Y_{42}\overline{Y}_{43})\,Y_{22}\,Y_{23}\,Y_{32}\,Y_{33} \\
&\vee (\overline{Y}_{22} \vee Y_{22}\overline{Y}_{23})(\overline{Y}_{32} \vee Y_{32}\overline{Y}_{33})\,Y_{12}\,Y_{13} \\
&\vee (\overline{Y}_{22} \vee Y_{22}\overline{Y}_{23})(\overline{Y}_{42} \vee Y_{42}\overline{Y}_{43})\,Y_{12}\,Y_{13}\,Y_{32}\,Y_{33} \\
&\vee (\overline{Y}_{32} \vee Y_{32}\overline{Y}_{33})(\overline{Y}_{42} \vee Y_{42}\overline{Y}_{43})\,Y_{12}\,Y_{13}\,Y_{22}\,Y_{23} \qquad (27)
\end{aligned}
$$

Table 4 is used for ANDing (multiplying) the PRE form (S_2 / S_1) in (24) and the PRE form ($\overline{S}_3$ / S_1) in (27) to produce a PRE form of ($S_2\overline{S}_3$ / S_1). For convenience, each loop (collection of cells) in Table 4 is labelled by a certain integer number that we call a loop-characterizing integer. The final result for the expectation of S{2} is given by

$$
\begin{aligned}
&E\{S\{2\}\} = E\{S_1\}\ E\{S_2\overline{S_3} / S_1\} \\
&= p_{11}\,p_{21}\,p_{31}\,p_{41}\,(p_{12}\,p_{22}\,q_{13}\,q_{23} + p_{12}\,p_{22}\,q_{13}(q_{32} + p_{32}q_{33})\,p_{23} \\
&+ p_{12}\,p_{22}q_{13}(q_{42} + p_{42}q_{43})\,p_{23}\,p_{32}\,p_{33} + p_{12}\,p_{22}\,q_{23}(q_{32} + p_{32}q_{33})\,p_{13} \\
&+ p_{12}\,p_{22}q_{23}(q_{42} + p_{42}q_{43})\,p_{13}\,p_{32}\,p_{33} \\
&+ p_{12}\,p_{22}(q_{32} + p_{32}q_{33})(q_{42} + p_{42}q_{43})\,p_{13}\,p_{23} \\
&+ p_{12}p_{32}q_{22}q_{13} + p_{12}\,p_{32}\,q_{22}q_{33}\,p_{13} \\
&+ p_{12}\,p_{32}q_{22}(q_{42} + p_{42}q_{43})\,p_{13}\,p_{33} + p_{12}\,p_{42}\,q_{22}q_{32}q_{13} \\
&+ p_{12}\,p_{42}\,q_{22}q_{32}\,p_{13} + p_{22}\,p_{32}q_{12}\,q_{23} + p_{22}\,p_{32}q_{12}q_{33}\,p_{23} \\
&+ p_{22}\,p_{32}q_{12}(q_{42} + p_{42}q_{43})\,p_{23}\,p_{33} + p_{22}\,p_{42}\,q_{12}q_{32}q_{23} \\
&+ p_{22}\,p_{42}q_{12}q_{32}\,p_{23} + p_{32}\,p_{42}\,q_{12}q_{22} \qquad (28)
\end{aligned}
$$

5. THE HOMOGENEOUS CASE

In this case of independent identically-distributed (*i.i.d.*) binary components Y_{ij} , all components share the same reliability

$$E\{Y_{ij}\} = p \qquad (1 \leq i \leq 4,\ 1 \leq j \leq 3). \qquad (29)$$

The *i.i.d.* reliability of the three stations become

$$E\{S_1\} = P^4 \qquad (30)$$

$$E\{S_2\} = 6p^6 - 8p^6 + 3p^8 \qquad (31)$$

$$E\{S_3\} = 4p^9 - 3p^{12} \qquad (32)$$

Fig. 4 demonstrates the change of each of these expectations versus p for p Є [0.0, 1.0]. The quantity $E\{S_1\}$ represents a series system whose reliability polynomial is a monomial of a type-I graph through the two points (0.0, 0.0) and (1.0, 1.0) and with no inflection point within the interval (0.0, 1.0). Each of the quantities. $E\{S_2\}$ and $E\{S_3\}$ has the typical S-shape (type II) curve of a coherent system, which passes through (0.0, 0.0) and (1.0, 1.0) and has a single inflection point within the interval (0.0, 1.0).

The four possible values of the multi-state system has (*i.i.d.*) expectations

$$E\{S(0)\} = 1 - p^4 \qquad (33)$$

$$E\{S(1)\} = p^4\,(4q^3 - 3q^4) = p^4 - 6p^6 + 8p^7 - 3p^8 \qquad (34)$$

$$E\{S(2)\} = 6p^6 - 8p^7 + 3p^8 - 4p^{10} + 3p^{12} \qquad (35)$$

$$E\{S(3)\} = p^{10}\,[1 + 3(1 - p^2)] = 4p^{10} - 3p^{12} \qquad (36)$$

which add to 1 for all p Є [0,1]. Fig. 5 shows a plot of E{S{0}}, E{S{1}}, E{S(2)}and E{S(3)} versus p for p Є [0,1]. The figure shows that S{0} behaves like a coherent binary failure while S{3} acts like a coherent binary success. Both S{1} and S{2} have a general non-coherent behavior, which somewhat mimics that of a k-to-l-out-of-n: G system [39].

Table 4. Multiplication (ANDing) table that multiplies the PRE forms of (S_2 / S_1) and ($\overline{S_3}$ / S_1) to produce a PRE form of ($S_2\overline{S_3}$ / S_1)

$\wedge$	$(\bar{Y}_{12} \vee Y_{12}\bar{Y}_{13})(\bar{Y}_{22} \vee Y_{22}\bar{Y}_{23})$	$(\bar{Y}_{12} \vee Y_{12}\bar{Y}_{13})(\bar{Y}_{32} \vee Y_{32}\bar{Y}_{33}) Y_{22} Y_{23}$	$(\bar{Y}_{12} \vee Y_{12}\bar{Y}_{13})(\bar{Y}_{42} \vee Y_{42}\bar{Y}_{43}) Y_{22} Y_{23} Y_{32} Y_{33}$	$(\bar{Y}_{22} \vee Y_{22}\bar{Y}_{23})(\bar{Y}_{32} \vee Y_{32}\bar{Y}_{33}) Y_{12} Y_{13}$	$(\bar{Y}_{22} \vee Y_{22}\bar{Y}_{23})(\bar{Y}_{42} \vee Y_{42}\bar{Y}_{43}) Y_{12} Y_{13} Y_{32} Y_{33}$	$(\bar{Y}_{32} \vee Y_{32}\bar{Y}_{33})(\bar{Y}_{42} \vee Y_{42}\bar{Y}_{43}) Y_{12} Y_{13} Y_{22} Y_{23}$
$Y_{12} Y_{22}$	1 $Y_{12} Y_{22} \bar{Y}_{13} \bar{Y}_{23}$	2 $Y_{12} Y_{22}\bar{Y}_{13}(\bar{Y}_{32} \vee Y_{32}\bar{Y}_{33}) Y_{23}$	3 $Y_{12} Y_{22}\bar{Y}_{13}(\bar{Y}_{42} \vee Y_{42}\bar{Y}_{43}) Y_{23} Y_{32} Y_{33}$	4 $Y_{12} Y_{22}\bar{Y}_{23}(\bar{Y}_{32} \vee Y_{32}\bar{Y}_{33}) Y_{13}$	5 $Y_{12} Y_{22}\bar{Y}_{23}(\bar{Y}_{42} \vee Y_{42}\bar{Y}_{43}) Y_{13} Y_{32} Y_{33}$	6 $Y_{12} Y_{22}(\bar{Y}_{32} \vee Y_{32}\bar{Y}_{33})(\bar{Y}_{42} \vee Y_{42}\bar{Y}_{43}) Y_{13} Y_{23}$
$Y_{12} Y_{32}\bar{Y}_{22}$	7 $Y_{12} Y_{32}\bar{Y}_{22}\bar{Y}_{13}$	0	0	8 $Y_{12} Y_{32} \bar{Y}_{22}\bar{Y}_{33} Y_{13}$	9 $Y_{12} Y_{32}\bar{Y}_{22}(\bar{Y}_{42} \vee Y_{42}\bar{Y}_{43}) Y_{13} Y_{33}$	0
$Y_{12} Y_{42}\bar{Y}_{22}\bar{Y}_{32}$	10 $Y_{12} Y_{42} \bar{Y}_{22}\bar{Y}_{32}\bar{Y}_{13}$	0	0	11 $Y_{12} Y_{42} \bar{Y}_{22}\bar{Y}_{32} Y_{13}$	0	0
$Y_{22} Y_{32}\bar{Y}_{12}$	12 $Y_{22} Y_{32}\bar{Y}_{12}\bar{Y}_{23}$	13 $Y_{22} Y_{32}\bar{Y}_{12}\bar{Y}_{33} Y_{23}$	14 $Y_{22} Y_{32}\bar{Y}_{12}(\bar{Y}_{42} \vee Y_{42}\bar{Y}_{43}) Y_{23} Y_{33}$	0	0	0
$Y_{22} Y_{42}\bar{Y}_{12}\bar{Y}_{32}$	15 $Y_{22} Y_{42} \bar{Y}_{12}\bar{Y}_{32}\bar{Y}_{23}$	16 $Y_{22} Y_{42}\bar{Y}_{12}\bar{Y}_{32} Y_{23}$	0	0	0	0
$Y_{32} Y_{42}\bar{Y}_{12}\bar{Y}_{22}$	17 $Y_{32} Y_{42} \bar{Y}_{12}\bar{Y}_{22}$	0	0	0	0	0

6. KARNAUGH-MAP VERIFICATION

Despite the large number of input variables involved (twelve variables), we are able to verify our results by utilizing Boolean quotients and 8-variable Karnaugh maps. First we note that the four instances of S form an orthonormal set. In particular, we have

$$S(1) \vee S(2) \vee S(3) = \overline{S(0)} = Y_{11}Y_{21}Y_{31}Y_{41} \quad (37)$$

and, hence, in terms of Boolean quotients, we have

$$S(1) / Y_{11}Y_{21}Y_{31}Y_{41} \vee S(2) / Y_{11}Y_{21}Y_{31}Y_{41} \vee S(3) / Y_{11}Y_{21}Y_{31}Y_{41} = 1 \quad (38)$$

The identity (38) is independent of the four variables Y_{11}, Y_{21}, Y_{31}, Y_{41} and hence it involves only eight of the twelve Y_{ij} variables, This means that any of the three Boolean quotients in (38) is a function of the eight variables $\{Y_{i2}, Y_{i3}\}$, $1 \le i \le 4$. Fig. 6 shows an 8-variable Karnaugh map which shades the cells in which S{1} / $Y_{11}Y_{21}Y_{31}Y_{41}$ is asserted, while Fig. 7 shows a similar map which shades the cells for in which S{3} / $Y_{11}Y_{21}Y_{31}Y_{41}$ is equal to 1. Fig. 8 is a verification of the identity (38). It borrows shadings (in light grey and dark blue, respectively) for the two Boolean quotients S{1} / $Y_{11}Y_{21}Y_{31}Y_{41}$ and S{3} / $Y_{11}Y_{21}Y_{31}Y_{41}$ from Figs. 6 and 7. The remaining cells in Fig. 8 represent S{2} / $Y_{11}Y_{21}Y_{31}Y_{41}$ as obtained in Table 4. We use various colors in Fig. 8 to label the 17 entries in Table 4, each identified by the integer assigned in Table 4. For example, entry number 1 in Table 4 is $Y_{12}\ Y_{22}\ \bar{Y}_{13}\ \bar{Y}_{23}$ is depicted in Fig. 8 by a square of 16 cells in light yellow that is distinguished by the loop-characterizing integer 1. Fig. 8 nicely verifies the identity (38). It also shows that our PRE representation of S{2} / $Y_{11}Y_{21}Y_{31}Y_{41}$ is almost minimal, but not perfectly minimal. In fact, we could have used the Karnaugh map in Fig. 8 (albeit with difficulty) to derive (and not only verify) an expression for S{2} / $Y_{11}Y_{21}Y_{31}Y_{41}$. The amp would have produced an equivalent (but slightly better) version of the results in Table 4, for which the two loops 10 and 11 are combined into a single loop and likewise the two loops 15 and 16 are also combined into a single loop.

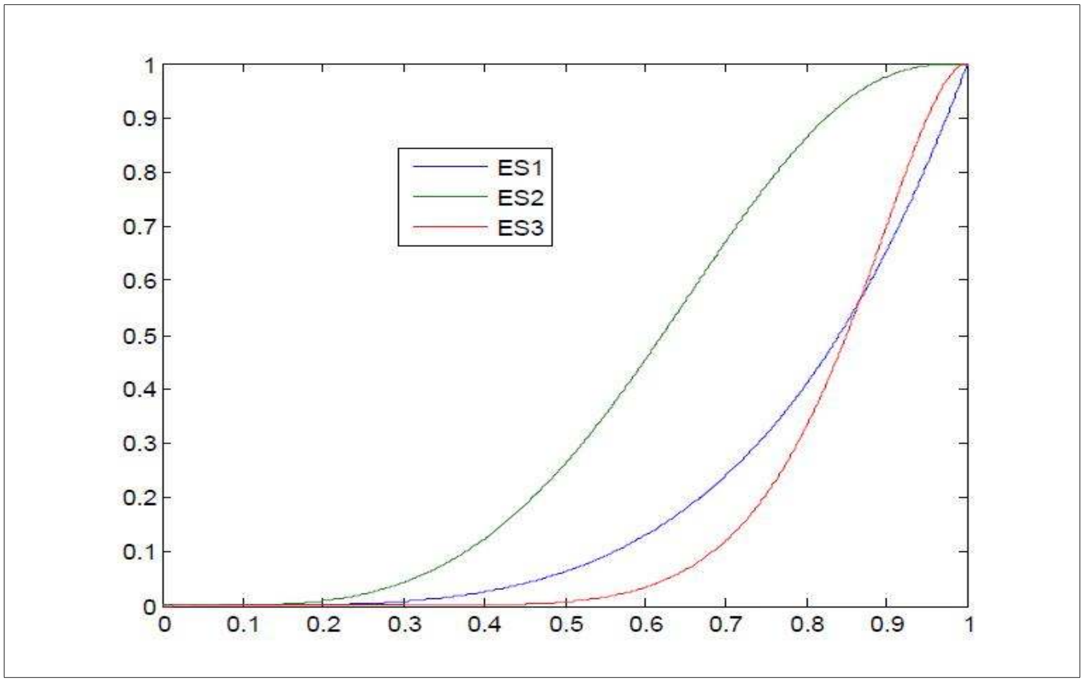

Fig. 4. Three graphs of E{S_1}(type-I), E{ S_2 }(type-II) and E{S_3}(type-II) versus p.

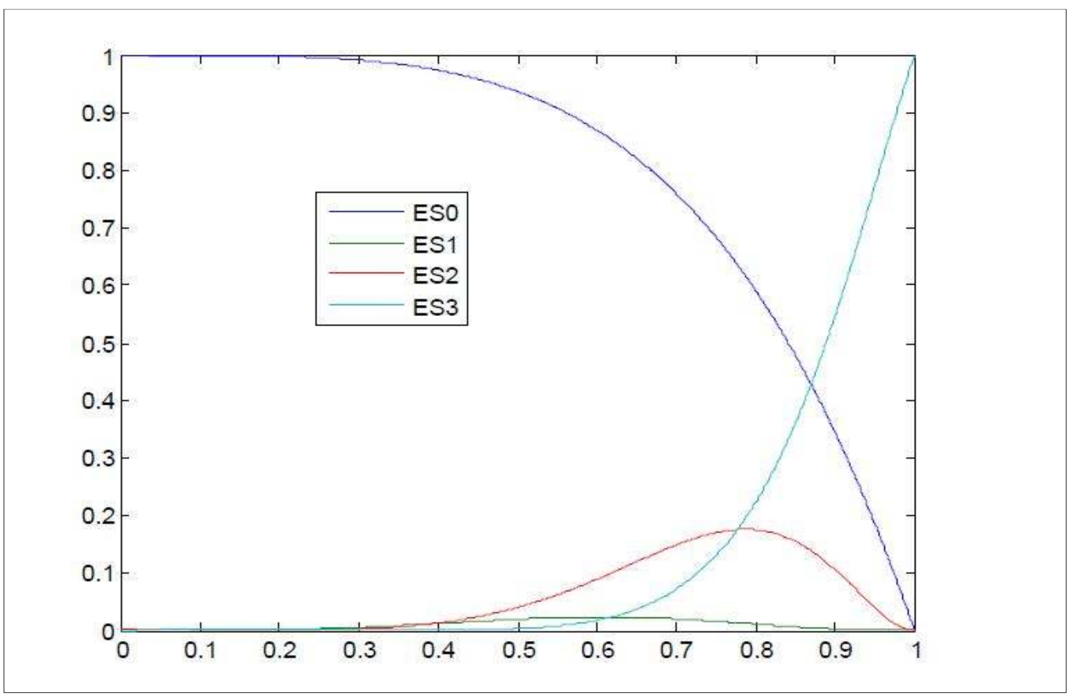

Fig. 5. A plot of E{S(0)}, E{S(1)}, E{S(2)} and E{S(3)} versus p for p Є [0,1].

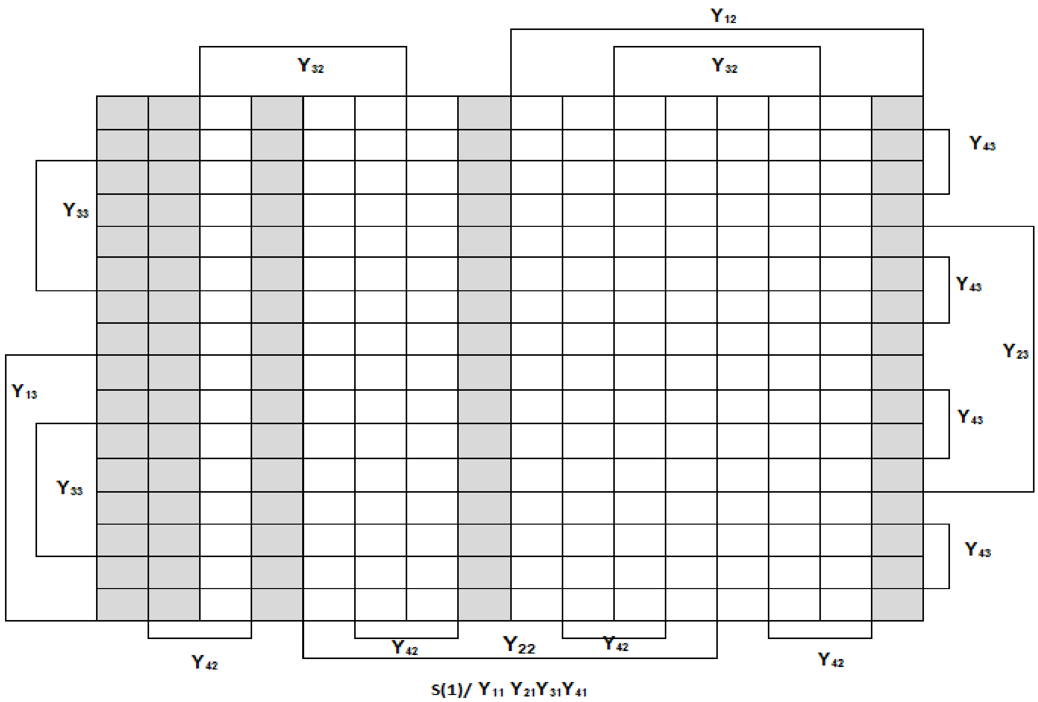

Fig. 6. An 8-variable Karnaugh map with the shaded cells depicting the Boolean quotient $S(1) / Y_{11} Y_{21}Y_{31}Y_{41}$

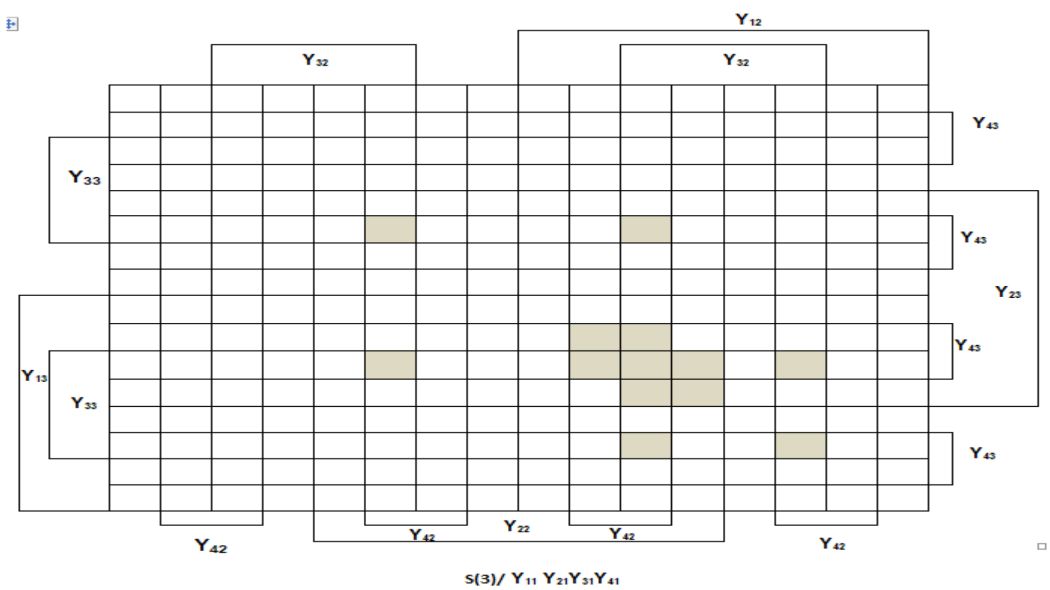

Fig. 7. An 8-variable Karnaugh map with the shaded cells indicating the Boolean quotient $S(3) / Y_{11} Y_{21}Y_{31}Y_{41}$

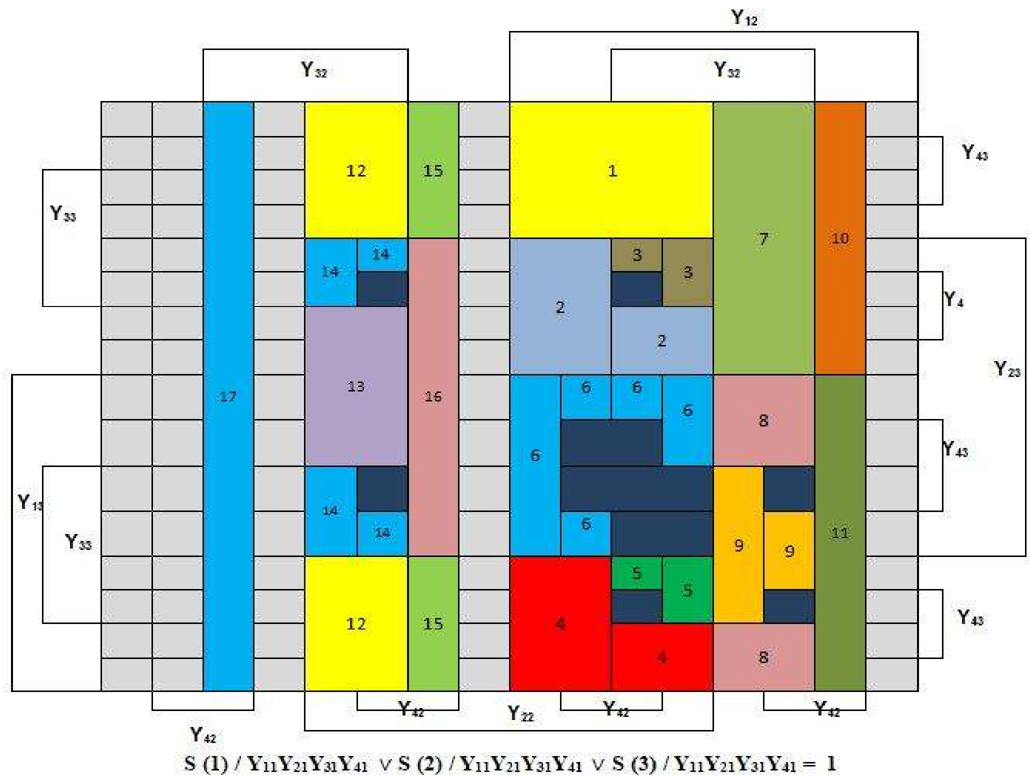

Fig. 8. An 8-variable Karnaugh map verifying the identity (38) . Asserted cells for $S(1) / Y_{11} Y_{21}Y_{31}Y_{41}$ (Fig. 6) are shaded in light grey and those for $S(3) / Y_{11} Y_{21}Y_{31}Y_{41}$ are marked in dark blue. Other colors in the map label loops for the 17 entries in Table 4, each identified by its assigned integer in Table 4.

7. COMPARISON WITH PREVIOUS WORK

The problem handled herein was solved via multi-state techniques by Tian et al. [19] and later by Mo et al. [22]. Both teams of authors used as inputs a certain matrix **P**, which is equivalent to the following input expectations

$$[E\{X_i\{j\}\}] = \begin{bmatrix} .050 & .0950 & .0684 & .7866 \\ .050 & .0950 & .0684 & .7866 \\ .030 & .0776 & .0446 & .8478 \\ .030 & .0776 & .0446 & .8478 \end{bmatrix} \quad (1 \le i \le 4,\ 0 \le j \le 3) \qquad (39)$$

which can be translated (*via* Table 2) to the following input expectations, for ($1\le i \le 4,\ 1 \le j \le 3$)

$$[E\{Y_{ij}\}] = \begin{bmatrix} .950000000000000 & .900000000000000 & .920000000000000 \\ .950000000000000 & .900000000000000 & .920000000000000 \\ .970000000000000 & .920000000000000 & .950022411474675 \\ .970000000000000 & .920000000000000 & .950022411474675 \end{bmatrix} \qquad (40)$$

$$[E\{\overline{Y_{ij}}\}] = \begin{bmatrix} .050000000000000 & .100000000000000 & .080000000000000 \\ .050000000000000 & .100000000000000 & .080000000000000 \\ .030000000000000 & .080000000000000 & .049977588525325 \\ .030000000000000 & .080000000000000 & .049977588525325 \end{bmatrix} \qquad (41)$$

Table 5. Comparison of the present results with those in earlier work

Expectation of	Tian et al.[19]	Mo et al.[22]	Our results
S(0)	0.1508	0.150838	0.150837750000000
S(1)	0.0023	0.002282	0.002282548128000
S(2)	0.0892	0.089181	0.089180866435691
S(3)	0.7577	0.757699	0.757698835436309
Total	1.0000	1.000000	1.000000000000000

Table 5 compares our results for this specific input with the earlier results of Tian et al. [19] and later by Mo et al. [22]. The three sets of results are essentially the same, despite the existence of differences of precision. Though a precision of four significant digits would suffice in practical situations, we have deliberately used an *exaggerated* precision of fifteen significant digits so as to make sure round-off errors in our calculations are definitely negligible. This exaggerated precision is really unwanted, but it could be beneficial in assessing the effect of round-off errors in any comparable future computation.

In passing, we observe that the Karnaugh map proved to be a handy and powerful tool for our current application. Other related uses of the Karnaugh map (beyond its conventional use in digital design) are also available (see, *e. g.*, [28, 29, 31, 32, 45-49]). The variant of the map used herein is the Conventional Karnaugh Map (CKM). Other important map versions include the Variable-Entered Karnaugh Map (VEKM) (see, *e. g.* , [28, 29, 31, 34, 45]), and the Multi-Valued Karnaugh Map (MVKM) (see, *e. g.* , [45]).

8. CONCLUSIONS

This paper demonstrated how MSS reliability can be handled *via* switching-algebraic tools. A classical MSS problem was manually analyzed by reformulating its multi-valued inputs as equivalent physically-meaningful binary variables. The paper resorted only to binary concepts and tools including those of probability-ready expressions, Boolean quotients, disjointness, shellability, Boolean multiplications, and relatively large Karnaugh maps. Results obtained are not only satisfactory but replicate earlier results with more precision.

This paper is admittedly somewhat long. Our justification for this is that we strived to make the paper a self-contained pedagogical tutorial. We tried to give detailed and clear explanations whenever needed, and to establish a clear and insightful interrelationship between binary modeling and MSS modeling. Our results provide a truly independent means to check and verify future solutions of a standard MSS problem. A significant contribution of the paper is that, while handling MSS reliability modeling, it successfully pushed mathematical tools of binary reliability modeling to their utmost utility.

Though the Conventional Karnaugh Map (CKM) used herein would have sufficed to completely solve the problem at hand, we opted for an algebraic solution, and employed the CKM in a verification role only. We avoided the need to construct a 12-variable Karnaugh map for certain functions by choosing to represent Boolean quotients of these functions, thereby reducing our task to one of constructing an eight-variable Karnaugh map, which (albeit relatively large) was reasonably manageable. An alternative way to handle the mapping of our 12-variable functions is to use the Variable-Entered Karnaugh Map (VEKM). Yet another map method (for handling the MSS reliability problem) is a method employing a Multi-Valued Karnaugh Map (MVKM).

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COMPETING INTERESTS

The authors have declared that no competing interests exist.

REFERENCES

1. Levitin G, Lisnianski A, Ushakov I. Reliability of multi-state systems: A historical overview. In Mathematical and

Statistical Methods in Reliability. 2003;123-137.
2. Lisnianski A, Levitin G. Multi-state system reliability: Assessment, optimization and applications. World Scientific Publishing Company. 2003;6.
3. Zuo MJ, Tian Z, Huang HZ. An efficient method for reliability evaluation of multistate networks given all minimal path vectors. IIE transactions. 2007;39(8):811-817.
4. Natvig B. Multistate systems reliability: Theory with applications. John Wiley & Sons, New York, NY, USA; 2010.
5. Liu Y, Lin P, Li YF, Huang HZ. Bayesian reliability and performance assessment for multi-state systems. IEEE Transactions on Reliability. 2015;64(1):394-409.
6. Yu H, Yang J, Peng R, Zhao Y. Reliability evaluation of linear multi-state consecutively-connected systems constrained by M consecutive and N total gaps. Reliability Engineering & System Safety. 2016;150:35-43.
7. Xiao H, Shi D, Ding Y, Peng R. Optimal loading and protection of multi-state systems considering performance sharing mechanism. Reliability Engineering & System Safety. 2016;149:88-95.
8. Chen YL, Chang CC, Sheu DF. Optimum random and age replacement policies for customer-demand multi-state system reliability under imperfect maintenance. International Journal of Systems Science. 2016;47(5):1130-1141.
9. Chiacchio F, D'Urso D, Manno G, Compagno L. Stochastic hybrid automaton model of a multi-state system with aging: Reliability assessment and design consequences. Reliability Engineering & System Safety. 2016;149:1-13.
10. George-Williams H, Patelli E. A hybrid load flow and event driven simulation approach to multi-state system reliability evaluation. Reliability Engineering & System Safety. 2016;152:351-367.
11. Yi X, Dhillon BS, Mu HN, Zhang Z, Hou P. Reliability analysis method for multi-state repairable systems based on goal oriented methodology. In ASME 2016 International Mechanical Engineering Congress and Exposition American Society of Mechanical Engineers. 2016;V014T14A003-V014T14A003.
12. Hu T, Yin D, Chen T. Analysis of multi-state warm standby system reliability model with repair priority. In Industrial Engineering and Engineering Management (IEEM). 2017 IEEE International Conference. 2017;2112-2118. IEEE.
13. Jiang T, Liu Y. Parameter inference for non-repairable multi-state system reliability models by multi-level observation sequences. Reliability Engineering & System Safety. 2017;166:3-15.
14. Eryilmaz S. Reliability analysis of multi-state system with three-state components and its application to wind energy. Reliability Engineering & System Safety. 2018;172:58-63.
15. Mi J, Li YF, Peng W, Huang HZ. Reliability analysis of complex multi-state system with common cause failure based on evidential networks. Reliability Engineering & System Safety. 2018;174:71-81.
16. Ren Y, Zeng C, Fan D, Liu L, Feng Q. Multi-state reliability assessment method based on the MDD-GO Model. IEEE Access. 2018;6:5151-5161.
17. Wang G, Duan F, Zhou Y. Reliability evaluation of multi-state series systems with performance sharing. Reliability Engineering & System Safety. 2018;173: 58-63.
18. Tao T, Zio E, Zhao W. A novel support vector regression method for online reliability prediction under multi-state varying operating conditions. Reliability Engineering & System Safety. 2018;177: 35-49.
19. Tian Z, Zuo MJ, Yam RC. Multi-state k-out-of-n systems and their performance evaluation. IIE Transactions. 2008;41(1): 32-44.
20. Zhao X, Cui L. Reliability evaluation of generalised multi-state k-out-of-n systems based on FMCI approach. International Journal of Systems Science. 2010;41(12): 1437-1443.
21. Fadhel SF, Alauldin NA, Ahmed YY. Reliability of dynamic multi-state oil supply system by structure function. International Journal of Innovative Research in Science, Engineering and Technology. 2014;3(6): 13548-13555.
22. Mo Y, Xing L, Amari SV, Dugan JB. Efficient analysis of multi-state k-out-of-n system. Reliability Engineering & System Safety. 2015;133:95-105.
23. Hurley RB. Probability maps, IEEE Transactions on Reliability. 1963;R-12(3): 39-44.

24. Bennetts RG. On the analysis of fault trees. IEEE Transactions on Reliability. R-1975;24(3):175-185.
25. Abraham JA. An improved algorithm for network reliability. IEEE Transactions on Reliability. R-1979;28(1):58-61.
26. Dotson W, Gobien J. A new analysis technique for probabilistic graphs. IEEE Transactions on Circuits and Systems. CAS-1979;26(10):855-865.
27. Bennetts RG. Analysis of reliability block diagrams by Boolean techniques. IEEE Transactions on Reliability. R-1982;31(2): 159-166.
28. Rushdi AM. Symbolic reliability analysis with the aid of variable-entered Karnaugh maps. IEEE Transactions on Reliability. R-1983;32(2):134-139.
29. Rushdi AM, Al-Khateeb DL. A review of methods for system reliability analysis: A Karnaugh-map perspective. Proceedings of the First Saudi Engineering Conference, Jeddah, Saudi Arabia. 1983;1:57-95.
30. Rushdi AM. How to hand-check a symbolic reliability expression. IEEE Transactions on Reliability. R-1983;32(5):402-408.
31. Rushdi AM. Overall reliability analysis for computer-communication networks. Proceedings of the Seventh National Computer Conference, Riyadh, Saudi Arabia. 1984;23-38.
32. Rushdi AM. On reliability evaluation by network decomposition. IEEE Transactions on Reliability. R-1984;33(5):379-384. Corrections: ibid. R-1985;34(4):319.
33. Rushdi AM, Goda AS. Symbolic reliability analysis via Shannon's expansion and statistical independence. Microelectronics and Reliability. 1985;25(6):1041-1053.
34. Rushdi AM. Map derivation of the minimal sum of a switching function from that of its complement. Microelectronics and Reliability. 1985;25:1055-1065.
35. Rushdi AM. Utilization of symmetric switching functions in the computation of k-out-of-n system reliability. Microelectronics and Reliability. 1986;26(5):973-987.
36. Rushdi AM. A switching-algebraic analysis of consecutive-k-out-of-n: F systems. Microelectronics and Reliability. 1987; 27(1):171-174.
37. Rushdi AM. A switching-algebraic analysis of circular consecutive-k-out-of-n: F systems. Reliability Engineering & System Safety. 1988;21(2):119-127.
38. Rushdi AM. Reliability of k-out-of-n Systems, Chapter 5 in Misra, K. B. (Editor), New Trends in System Reliability Evaluation. Fundamental Studies in Engineering, Elsevier Science Publishers, Amsterdam, the Netherlands. 1993;16: 185-227.
39. Rushdi AM. Partially-redundant systems: Examples, reliability, and life expectancy. International Magazine on Advances in Computer Science and Telecommunications. 2010;1(1):1-13.
40. Rushdi MAM, Ba-Rukab OM, Rushdi AM. Multidimensional recursive relation and mathematical induction techniques: The case of failure frequency of k-out-of-n systems. Journal of king Abdulaziz University: Engineering Science. 2016; 27(2):15-31.
41. Rushdi AMA, Alturki AM. Computation of k-out-of-n system reliability via reduced ordered binary decision diagrams. British Journal of Mathematics & Computer Science. 2017;22(3):1-9.
42. Rushdi AM, Rushdi MA. Switching-algebraic analysis of system reliability. Chapter 6 in M. Ram and P. Davim (Editors). Advances in Reliability and System Engineering. Cham, Switzerland: Springer International Publishing. 2017; 139-161.
43. Rushdi AMA, Alturki AM. Unification of mathematical concepts and algorithms of k-out-of-n system reliability: A perspective of improved disjoint products. Journal of Engineering Research. 2018;6(4):1-31.
44. Rushdi AMA, Al-Amoudi MA. Recursively-defined combinatorial functions: The case of binomial and multinomial coefficients and probabilities. Journal of Advances in Mathematics and Computer Science. 2018;27(4):1-16.
45. Rushdi AMA. Utilization of Karnaugh maps in multi-value qualitative comparative analysis. International Journal of Mathematical, Engineering and Management Sciences (IJMEMS). 2018; 3(1):28-46.
46. Rushdi AMA, Badawi RMS. Karnaugh map utilization in coincidence analysis. Journal of King Abdulaziz University: Faculty of Computers and Information Technology. 2017;6(1-2):37-44.
47. Rushdi AMA, Badawi RMS. Karnaugh map utilization in Boolean analysis: The case of war termination. Journal of Qassim University: Engineering and Computer sciences. 2017;10(1):53-88.

48. Rushdi RA, Rushdi AM. Karnaugh-map utility in medical studies: The case of Fetal Malnutrition. International Journal of Mathematical, Engineering and Management Sciences (IJMEMS). 2018; 3(3):220-244.
49. Rushdi AM, Zagzoog S, Balamesh AS. Derivation of a scalable solution for the problem of factoring an n-bit integer. Journal of Advances in Mathematics and Computer Science. 2018;29(7).
50. Brown FM. Boolean Reasoning: The logic of Boolean equations. Kluwer Academic Publisher, Boston, MA, USA; 1990.
51. Caldwell SH. The recognition and identification of symmetric switching functions. Transactions of the American Institute of Electrical Engineers, Part I: Communication and Electronics. 1954; 73(2):142-147.
52. Marcus MP. The detection and identification of symmetric switching functions with the use of tables of combinations. IRE Transactions on Electronic Computers. 1956;4:237-239.
53. Cunkle CH. Symmetric Boolean functions. The American Mathematical Monthly. 1963;70(8):833-836.
54. Arnold RF, Harrison MA. Algebraic properties of symmetric and partially symmetric Boolean functions. IEEE Transactions on Electronic Computers. EC-1963;12(3):244-251.
55. Born RC, Scidmore AK. Transformation of switching functions to completely symmetric switching functions. IEEE Transactions on Computers. C-1968; 17(6):596-599.
56. Kim BG, Dietmeyer DL. Multilevel logic synthesis of symmetric switching functions. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems. 1991;10(4):436-446.
57. Canteaut A, Videau M. Symmetric Boolean functions. IEEE Transactions on Information Theory. 2005;51(8):2791-2811.
58. Maurer PM. Symmetric Boolean functions. International Journal of Mathematics, Game Theory, and Algebra. 2015;24(2/3):159-202.
59. Cusick TW. Hamming weights of symmetric Boolean functions. Discrete Applied Mathematics. 2016;215:14-19.
60. Ball MO, Provan JS. Bounds on the reliability polynomial for shellable independence systems. SIAM Journal on Algebraic Discrete Methods. 1982;3(2): 166-181.
61. Ball MO, Provan JS. Disjoint products and efficient computation of reliability. Operations Research. 1988;36(5):703-715.
62. Colbourn CJ. Combinatorial aspects of network reliability. Annals of Operations Research. 1991; 33(1):1-15.
63. Boros E, Crama Y, Ekin O, Hammer PL, Ibaraki T, Kogan A. Boolean normal forms, shellability, and reliability computations. SIAM Journal on Discrete Mathematics. 2000;13(2):212-226.
64. Bruni R. On the orthogonalization of arbitrary Boolean formulae. Advances in Decision Sciences. 2005;2005(2):61-74.
65. Crama Y, Hammer PL. Boolean Functions: Theory, algorithms, and applications. Cambridge University Press; 2011.
66. Rushdi AMA, Hassan AK. Reliability of migration between habitat patches with heterogeneous ecological corridors. Ecological Modelling. 2015;304:1-10.
67. Rushdi AMA, Hassan AK. An exposition of system reliability analysis with an ecological perspective. Ecological Indicators. 2016;63:282-295.
68. Rushdi AM, Alturki AM. Novel representations for a coherent threshold reliability system: A tale of eight signal flow graphs. Turkish Journal of Electrical Engineering & Computer Sciences. 2018; 26(1):257-269.

APPENDIX A: A VERBAL DESCRIPTION OF THE SOLVED EXAMPLE

Consider the commodity-supply system discussed in [19,21,22]. As shown in Fig.1, a certain commodity-supply system is delivered from the commodity source to three stations through four commodity pipelines. Both the system and each pipeline have four states, which are defined in Table (A.1). The states of each pipeline is defined according to which station the commodity supply will be able to reach *via* this pipeline, and the states of the system is defined according to whether the demands of *up to* a certain station can be met. Different stations have different demands on the commodity. Station 1 requires at least four pipelines working to meet its demand; Station 2 requires at least two pipelines working to meet its demand; Station 3 requires at least three pipelines working to meet its demand. Thus, this commodity supply system can be regarded as a multi-state k-out-of-n system with $n = 4$, $M = 3$, $k_1 = 4$, $k_2 = 2$, and $k_3 = 3$.

Table (A.1). Description of component/ system states of the commodity-supply system analyzed herein.

State of pipeline i	Meaning	System state	Meaning
0	Commodity cannot reach any station	0	No commodity demand of any station is met
1	Commodity can reach up to station 1 via pipeline i	1	System can meet the commodity demand of up to station 1
2	Commodity can reach up to station 2 via pipeline i	2	System can meet the commodity demand of up to station 2
3	Commodity can reach up to station 3 via pipeline i	3	System can meet the commodity demand of up to station 3

APPENDIX B: USEFUL PERTINENT CONCEPTS

Probability-Ready Expressions: A probability-Ready Expression (*RRE*) [42] is an expression in the switching (Boolean) domain that can be directly transformed, on a one-to-one basis, to its Real or Probability Transform by replacing switching (Boolean) indicators by their statistical expectations, and also replacing logical multiplication and addition (ANDing and ORing) by their arithmetic counterparts. A switching expression is a *PRE* expression if

a) all *ORed* terms are disjoint, and
b) all *ANDed* sums are statistically independent.

Boolean Quotient: Given a Boolean function f and a term t, the Boolean quotient of f with respect to t, denoted by (f/t), is defined to be the function formed from f by imposing the constraint {t = 1} explicitly [50], *i.e.,*

$$f/t = [f]_{t=1}, \tag{B.1}$$

The Boolean quotient is also known as a ratio, a subfunction, or a restriction. Brown [50] and Rushdi & Rushdi [42] list several useful properties of Boolean quotients. A fundamental property of the Boolean quotient states that a term ANDed with a function is equal to the term ANDed with the Boolean quotient of the function with respect to the term, namely.

$$t \wedge f = t \wedge (f/t) \tag{B.2}$$

If the term t is a factor of the function f *(i. e.* , $f = t \wedge g$), then (B.2) takes the simpler form (frequently utilized in this paper)

$$f = t \wedge (f/t) \tag{B.3}$$

A Multi-State Coherent System

A coherent MSS is a system possessing the three properties:

1. **Causality** : The system is in state "0" if all of its components are in state "0", and the system is in state "M" (the highest possible state) if all of its components are in state "M", *i.e.,*

   $$S(\mathbf{0}) = 0 \text{ and } S(\mathbf{M}) = M . \tag{B.4}$$

2. **Monotonicity** : The system state is non-decreasing with the increase of each component state, *i.e.* ,

   $$S(\mathbf{X}) \geq S(\mathbf{Y}) \text{ if } \mathbf{X} \geq \mathbf{Y} . \tag{B.5}$$

3. **Relevancy**: No system component is a dummy one , *i.e.* , each system component i has at least one instance in which it produces a change in system state, *i.e.,* $S(\mathbf{X} \mid X_i = j_1) > S(\mathbf{X} \mid X_i = j_2)$ when $j_1 > j_2$ for a certain value $\mathbf{X} / X_i$ of inputs other than X_i

A Multi –State k-out-of-n system: There are different definitions for a multi-state k-out-of-n system that ensure it is a coherent system. Here, we follow reference [22], which states that "An n-component coherent multi-state system is called k-out-of-n: G system if $S(\mathbf{X}) \geq j$ ($1 \leq j \leq M$) whenever at least k_m components are in state m or above for all m such that $1 \leq m \leq j$". A multi-state k-out-of-n: G system is called a decreasing k-out-of-n: G system if $k_1 > k_2 > \ldots > k_M$. The dual of a multi-state k-out-of-n: G system is the multi-state k-out-of-n: F system.

APPENDIX C: SUCCESS OF A K-OUT-OF-N: G SYSTEMS

A symmetric switching function (SSF) [35, 37, 51-59]

$$f = Sy(\mathbf{A}; \mathbf{X}) = Sy(\{a_1, a_2, \ldots, a_m\}; X_1, X_2 \ldots, X_n) \quad (C.1)$$

is specified *via* its characteristic set

$$\mathbf{A} = \{a_1, a_2, \ldots, a_m\} \subseteq \{0, 1, 2, \ldots, n\} \quad (C.2)$$

and its inputs $\mathbf{X} = [X_1, X_2, \ldots, X_n]^T$. This function has the value 1 iff

$$\sum_{i=1}^{n} X_i = a_i, \ 1 \le i \le m < (n+1) \quad (C.3)$$

and has the value 0 otherwise, The complement $\bar{f}$ of the above SSF has a characteristic set defined by the set difference

$$\bar{\boldsymbol{A}} = \{0, 1, 2, \ldots, n\} — \{a_1, a_2, \ldots, a_m\} \quad (C.4)$$

The SSF f in (C.1) can be expressed in terms of complemented arguments

$\bar{\boldsymbol{X}} = [\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n]$ for a complemented characteristic set given by

$\{n — a_m, \ldots, n — a_2, n — a_1\}$, *i.e.*,

$$f = Sy(\{n — a_m, \ldots, n — a_2, n — a_1\}; \bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n) \quad (C.5)$$

The success S (k, n, **X**) of a k-out-of-n: G systems is a monotonically non-decreasing symmetric switching function of a characteristic set {k, k+1, …, n}, *i.e.,* it is given by [35, 38] :

$$S(k, n, \mathbf{X}) = Sy(\{k, k+1, \ldots, n\}; \mathbf{X}) \quad (C.6)$$

where Sy(A,**X**) is a symmetric switching function of characteristic set A. Rushdi [35] showed that S (k, n, **X**) has the same minimal sum and complete sum, with $\binom{n}{k}$ prime implicants, given by

$$S(k, n, \mathbf{X}) = \vee X_{i1} X_{i2} \ldots X_{ik} \quad (C.7)$$

Where the ORing in (C.7) is taken over subsets of size k of the set of first n positive integers, *i. e.*, $\{i_1, i_2, \ldots, i_k\} \subseteq \{1,2,\ldots, n\}$. In other words, the set $\{i_1, i_2, \ldots, i_k\}$ consists of k elements selected from the set of first n positive integers, where order does not matter and repetition is not allowed. Rushdi [35] also showed that S (k, n, **X**) can be written in a disjoint sum-of-products S_{dis} (k, n, **X**) form without increasing the number $\binom{n}{k}$ of implicants in (C.7), a property later designated as shellability [60-68]. Rushdi and Alturki [43] showed that in going from S to S_{dis} there is $\binom{k-1}{0} = 1$ term that remains intact, and there are $\binom{k}{1} = k$ terms, which are each augmented with a single complemented literal. In general there are $\binom{l-1}{k-1}$ terms that are each augmented with n complemented literals, where $k \le l \le n$, where

$$\binom{n}{k} = \sum_{l=k}^{n} \binom{l-1}{k-1} \quad (C.8)$$

Fig. C.1 demonstrates the symmetric non-decreasing function representing S(2, 4, **Z**). The minimal sum (or complete sum) for this function is :

$$Sy(\{2, 3, 4\}, \mathbf{Z}) = Z_1 Z_2 \vee Z_1 Z_3 \vee Z_1 Z_4 \vee Z_2 Z_3 \vee Z_2 Z_4 \vee Z_3 Z_4 \quad (C.9)$$

This function is a disjunction of $\binom{4}{2}$ = 6 prime implicants, each being a product of two nucomplemented literals. This function is shellable and has a disjoint PRE form given by

$$Sy(\{2, 3, 4\}; \mathbf{Z}) = Z_1 Z_2 \vee Z_1\overline{Z_2} Z_3 \vee Z_1 \overline{Z_2}\,\overline{Z_3} Z_4 \vee \overline{Z_1} Z_2 Z_3 \vee \overline{Z_1}Z_2\overline{Z_3}Z_4 \vee \overline{Z_1}\overline{Z_2}Z_3Z_4 \quad (C.10)$$

In going from (C.9) to (C.10), the number of terms remains the same (6), the first term remains intact, k = 2 terms are each augmented with a single augmented literal and $\binom{4-1}{2-1} = 3$ terms are each augmented with two complemented literals.

Fig. C.2 replicates the demonstration in Fig. C.1 for S(3, 4, **Z**) = Sy({ 3, 4}; **Z).** The minimal sum (or complete sum) for this function is

$$Sy(\{3, 4\}; Z) = Z_1 Z_2 Z_3 \vee Z_1 Z_2 Z_4 \vee Z_1 Z_3 Z_4 \vee Z_2 Z_3 Z_4 \quad (C.11)$$

This function is a disjunction of $\binom{4}{3}$ = 4 prime implicants, each being a product of three uncomplemented literals. Again, this function is shellable and has a disjoint PRE form given by

$$Sy(\{3, 4\}; \mathbf{Z}) = Z_1 Z_2 Z_3 \vee Z_1Z_2\overline{Z_3} Z_4 \vee Z_1 \overline{Z_2} Z_3Z_4 \vee \overline{Z_1} Z_2 Z_3Z_4 \quad (C.12)$$

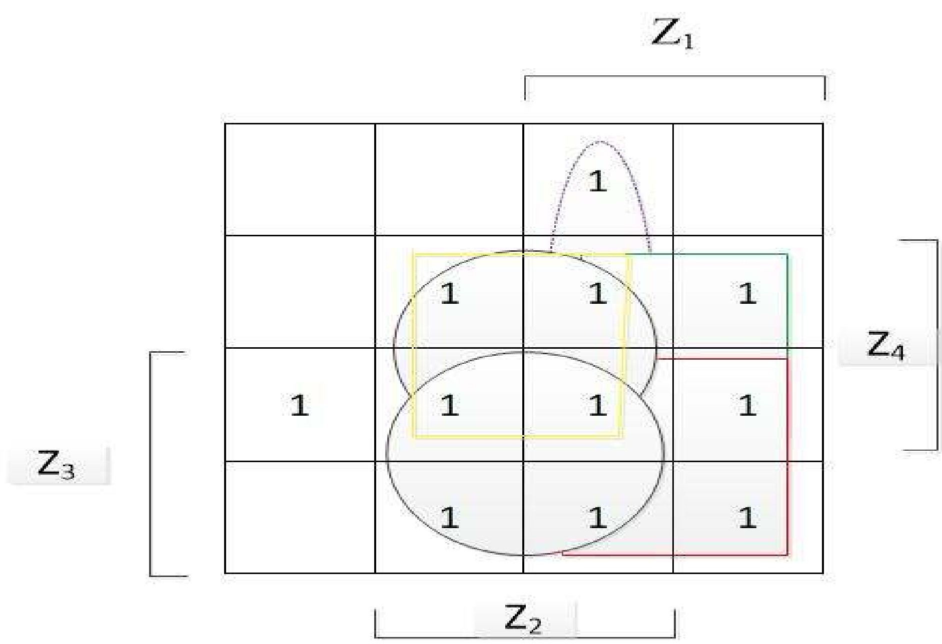
In going from (B.6) to (B.7), the number of terms remains the same (4), the first term remains as it is while the $\binom{4-1}{2-1} = 3$ other terms are each augmented with a complemented literal.

Finally, we use Fig. C. 3 to show the Karnaugh map for S(0) given by

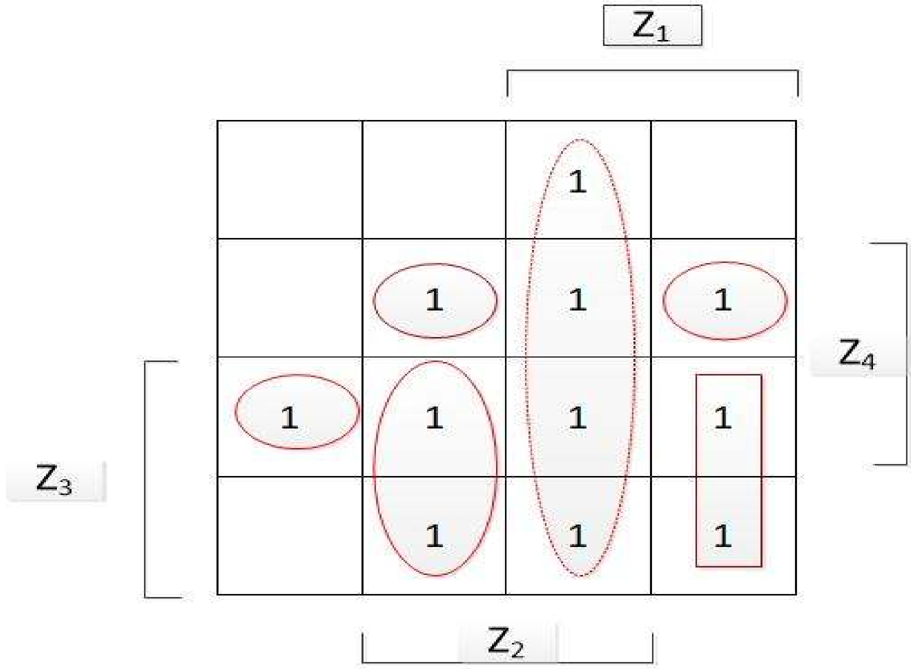
$$S(0) = Sy(\{0, 1, 2, 3\}; \overline{Y}_{11} \vee \overline{Y}_{21} \vee \overline{Y}_{31} \vee \overline{Y}_{41}) \quad (C.13)$$

The map uses four loops to cover the 1 entries as a sum of four products

$$S(0) = \overline{Y}_{11} \vee \overline{Y}_{21} \vee \overline{Y}_{31} \vee \overline{Y}_{41} \quad (C.14)$$

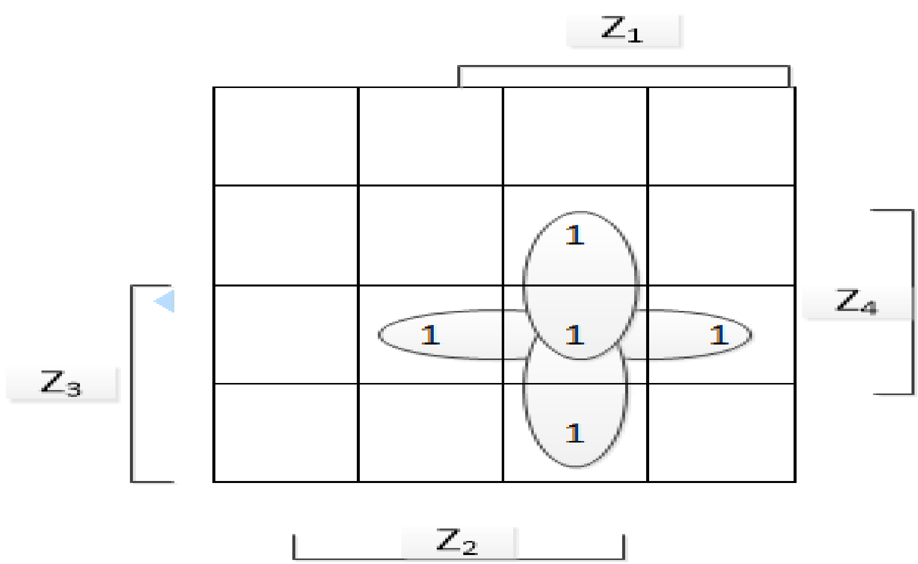


(a) Sy({2, 3, 4}; Z) = $\mathbf{Z_1 Z_2 \vee Z_1 Z_3 \vee Z_1 Z_4 \vee Z_2 Z_3 \vee Z_2 Z_4 \vee Z_3 Z_4}$

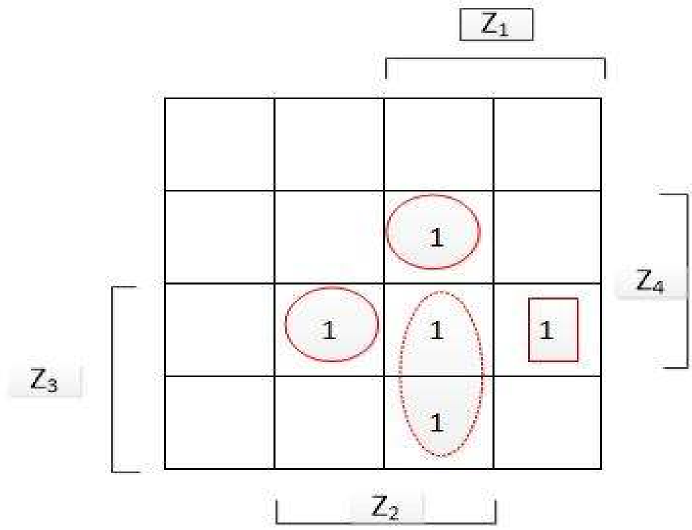


(b) $Sy(\{2, 3, 4\}; Z) = Z_1 Z_2 \vee Z_1\overline{Z_2}Z_3 \vee Z_1\overline{Z_2}\,\overline{Z_3}Z_4 \vee \overline{Z_1}Z_2 Z_3 \vee \overline{Z_1}Z_2\overline{Z_3}Z_4 \vee \overline{Z_1}\,\overline{Z_2}Z_3 Z_4$

Fig. C.1. Karnaugh maps for the symmetric monotonically non-decreasing function representing at least 2 good component out of 4. The map in (a) has overlapping loops and the map in (b) has disjoint ones.

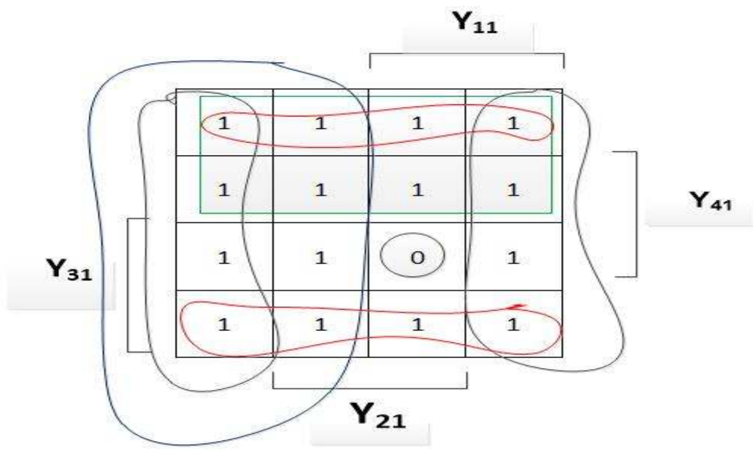


(a) $Sy(\{3, 4\}; Z) = Z_1 Z_2 Z_3 \vee Z_1 Z_2 Z_4 \vee Z_1 Z_3 Z_4 \vee Z_2 Z_3 Z_4$



(b) Sy({3, 4}; Z) = $Z_1 Z_2 Z_3 \vee Z_1 Z_2 \overline{Z_3} Z_4 \vee Z_1 \overline{Z_2} Z_3 Z_4 \vee \overline{Z_1} Z_2 Z_3 Z_4$

Fig. C.2. Karnaugh maps for the symmetric monotonically non-decreasing function representing at least 3 good component out of 4. The map in (a) has overlapping loops and the map in (b) has disjoint ones.



$$S(0) = \overline{Y}_{11} \vee \overline{Y}_{21} \vee \overline{Y}_{31} \vee \overline{Y}_{41}$$

Fig. C.3. The Karnaugh map for S(0), which is expressed as a single essential prime indicate as a product of sums or, equivalently, as four essential prime implicants as a sum of products.

The map also uses a single loop to cover the 0 entry as a product of a single sum, thereby producing the same expression in (C.14).

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