Switching Controllers: Realization, Initialization and Stability

Jonathan Paxman

Churchill College

Control Group

Department of Engineering

University of Cambridge

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For Ciaran

Preface

I would like to express my deep gratitude to my supervisor Dr Vinnicombe. His engineering intuition and mathematical rigour have been an outstanding example to me. Our discussions have always been interesting and fruitful.

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Most especially I wish to thank my wife Rachael. Her love, support and patience has been boundless.

Declaration: As required by University Statute, I hereby declare that this dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text. I also declare that the length of this thesis is not more than 60,000 words.

Jonathan Paxman

Cambridge, 2003

Abstract

In the design of switching control systems, the choice of realizations of controller transfer matrices and the choice of initial states for controllers (at switching times) are of critical importance, both to the stability and performance of the system.

Substantial improvements in performance can be obtained by implementing controllers with appropriate realizations. We consider observer form realizations which arise from weighted optimizations of signals prior to a switch. We also consider realizations which guarantee stability for arbitrary switches between stabilizing controllers for a linear plant.

The initial value problem is of crucial importance in switching systems, since initial state transients are introduced at each controller transition. A method is developed for determining controller states at transitions which are optimal with respect to weighted closed-loop performance. We develop a general Lyapunov theory for analyzing stability of reset switching systems (that is, those switching systems where the states may change discontinuously at switching times). The theory is then applied to the problem of synthesizing controller reset relations such that stability is guaranteed under arbitrary switching. The problem is solved via a set of linear matrix inequalities.

Methods for choosing controller realizations and initial states are combined in order to guarantee stability and further improve performance of switching systems.

Keywords: bumpless transfer, antiwindup, coprime factorization, switching control, controller conditioning, stability.

Contents

Notation and Abbreviations

Notation

$\mathbb{R}$	field of real numbers
$\mathbb{R}^+$	field of non-negative real numbers
$\mathbb{Z}$	field of integers
$\mathbb{Z}^+$	field of non-negative integers
$\mathbb{C}$	field of complex numbers
$\mathscr{H}_\infty$	Hardy ∞ space
$\in$	belongs to
$\forall$	for all
$\exists$	there exists
$:=$	defined to be equal to
$\square$	end of proof
$\|x\|$	the norm of vector x, assumed to be the Euclidean norm where no other is specified
A^*	the conjugate transpose of matrix A, or where A is real valued, the transpose
$A^\dagger$	left matrix pseudo inverse of A, $A^\dagger = (A^*A)^{-1}A^*$
$\min\limits_t(x)$	minimum of x with respect to t
$\operatorname{argmin}\limits_t(x)$	the value of t which minimises x
$\implies$	implies
$\iff$	if and only if
$\dot{x}$	the derivative of x with respect to time $\frac{dx}{dt}$
$f : X \to Y$	A function f mapping a set X into a set Y

Abbreviations

LTI	linear time invariant
LMI	linear matrix inequality
SISO	single input single output
AWBT	antiwindup bumpless transfer
IMC	internal model control
CAW	conventional antiwindup
CLF	Common Lyapunov Function
QLF	Quadratic Lyapunov Function
CQLF	Common Quadratic Lyapunov Function

Chapter 1

Introduction

Switching control is one way of dealing with design problems in which the control objectives, or system models are subject to change. It is common, for example to design linear controllers for a number of different linearized operating points of a nonlinear plant, and then seek to switch between the controllers in a sensible way. Switching control may also be appropriate when the plant is subject to sudden changes in dynamic behaviour (for example gear changes in some engine and motor control problems).

A general switching architecture is illustrated in figure 1.1. The (linear or nonlinear) process to be controlled is P, for which N controllers have been designed. A high level controller or switching supervisor S governs the switching process, based on measurements of the plant input and output, and an external command signal h, which may or may not be related to the reference r. The *reset* mechanism allows for the possibility that the controllers may be reinitialized in some way at each switching transition.

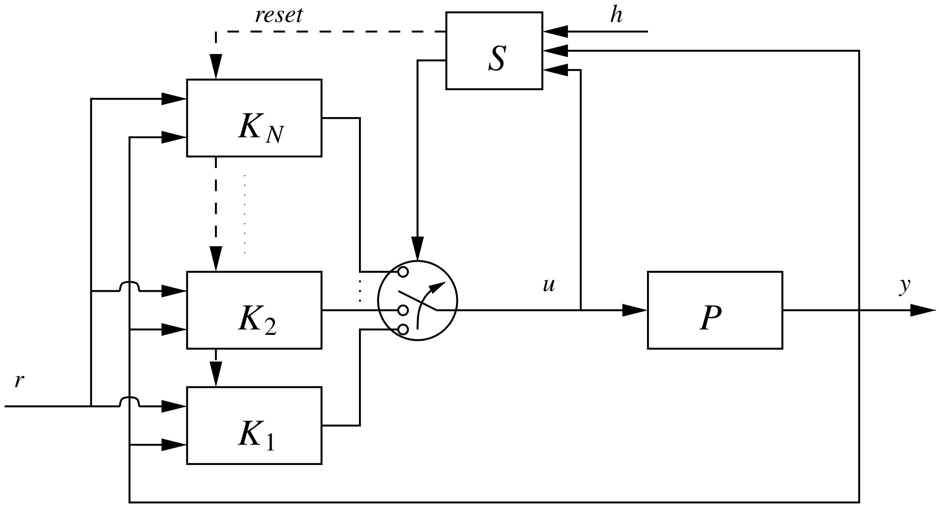


Figure 1.1: General switching architecture

A switching architecture has certain advantages over other multi-controller approaches such as gain scheduling. A family of controllers employed for gain scheduling must have the same order and structure, whereas a switching family require only that the input and output dimensions are consistent. There can also be benefits arising from the transients introduced by a hard switch. In observer switching architectures, information is provided about the plant mode which allows the correct controller to be applied with greater certainty.

Transient signals caused by hard switching can also be a burden on performance. If the controller realization and initialization strategy are poorly designed for switching, substantial transient signals caused by the switching can degrade performance and lead to instability.

The choice of controller realization is a much more important matter for switching systems than for other control problems. In a single (non-switching) ideal control loop, the realization of the controller ceases to be relevant once initial state transients have died down. In a switching architecture however, the realization has an effect every time a switch takes place. The realization is also important when other nonlinearities such as saturation are present.

For similar reasons, the initialization of controllers is important when considering a switching architecture. Should switching controllers (if same order) share state space? Should they retain previous values? Or should they be reinitialized to zero or some other value at each switch? These questions are vitally important to the performance of a switching system.

Stability of switching systems is not a simple matter, even when everything is linear and ideal. It is possible to switch between two stabilizing controllers for a single linear plant in such a way that an unstable trajectory results. In such circumstances, it is possible to ensure stability by choice of appropriate controller realizations, or by sensible choices of controller initializations (or both).

This dissertation is primarily concerned with realizing and initializing controllers for switching systems in ways which ensure stability and enhance performance.

In general we will consider the idealized scenario of a single linear plant and a family of stabilizing linear controllers. The switching signal will usually be assumed to be unknown. We focus on the choice of realization of the controllers and the initialization (or reset) strategy for the controllers. Our methods are generally independent of the controller design and switching strategy design, and therefore fit very well in a four step design process.

i. Controller design

ii. Controller realization

iii. Reset strategy design

iv. Switching strategy design

Controller transfer matrices are designed using conventional linear methods and then the realizations and reset strategies are determined in order to guarantee stability and minimize performance degradation due to the switching process. Switching strategy design is the last step, allowing for the possibility of real-time or manual switching. The switching strategy may be implemented via a switching supervisor or high level controller as illustrated in figure 1.1. The switching strategy in some applications may be determined manually. We examine briefly the problem of switching strategy, considering the calculation of minimum dwell times for switching systems to ensure stability given fixed controller realizations (with or without resets).

1.1 Bumpless transfer and conditioning

The so-called *bumpless transfer* problem has received considerable attention in the literature. The term usually refers to situations where we wish to carry out smooth (in some sense) transitions between controllers. The precise definition of a bumpless transfer is not universally agreed.

Some authors (see [1,29] for example) refer to a bumpless transfer as one where continuous inputs result in continuous outputs regardless of controller transitions. This definition is not often very useful, since controller transitions can cause very large transient signals in the outputs, even when the signals remain continuous (for example if the plant has a high frequency roll off). The definition is also unhelpful if the system is discrete-time.

Other authors (see [13,55,56] for example) refer to bumpless transfer when the difference between signals produced by (and driving) the on-line and off-line controllers are minimal in some sense. Our methods introduced in chapter 4 are based on similar definitions of bumpless transfer.

The third approach is to consider that bumpless transfer occurs if the transients introduced by the controller transitions are minimal in some sense. This definition, or one similar to it has been considered by Graebe and Ahlèn [18] and also by Peng, Hanus and coauthors [49,23]. The latter authors sometimes use the term *conditioned transfer* to distinguish between this property and one where continuity is the aim. Our methods in chapter 5 and subsequent stabilization methods in chapter 6 are connected with this idea.

Example 1.1.1. Consider the following plant transfer function

$$P = \frac{1}{s(s^2 + 0.2s + 1)}. \tag{1.1}$$

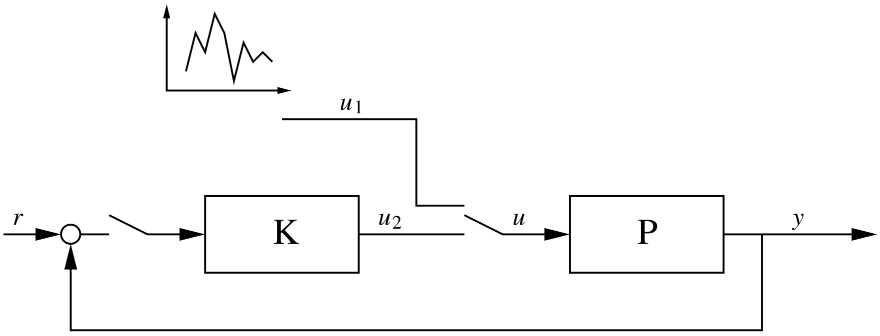


Figure 1.2: Generalized input substitution

The controller is a suboptimal robust controller. The designed continuous controller is

$$K = \left[\begin{array}{c|c} \begin{bmatrix} 0.8614 & -0.1454 & -0.0798 \\ 0.1183 & 0.9705 & -0.4647 \\ 0.0042 & 0.0641 & 0.6094 \end{bmatrix} & \begin{bmatrix} -0.0081 \\ 0.3830 \\ 0.3260 \end{bmatrix} \\ \hline \begin{bmatrix} -1.4787 & -0.9166 & -1.1978 \end{bmatrix} & [0] \end{array}\right]. \tag{1.2}$$

We consider here the regulator problem. That is, we have reference input $r = 0$.

We use the above plant and controller, discretized with a sampling time $T = 0.05$. We drive the plant open loop by a random noise signal (sampled with period 10) u_1 and switch to the output u_2 of controller K at time $k = 160$. The system is set up as illustrated in figure 1.2. No conditioning is applied to the controller prior to the switch, so the controller state remains zero until the switching instant. Figure 1.3 shows the plant input and output for the system as described.

We can clearly observe a substantial transient which follows the switch (particularly in the plant input).

One measure of the "size" of the transient, is the $l_2[n, \infty)$ norm of the signal $\begin{bmatrix} u \\ y \end{bmatrix}$, where n is the switching time. In this example the norm is 17.46. We will return to this example later, showing the results of some conditioning and initialization schemes.

1.2 Controller initialization in a switching architecture

The controller initial value problem is often not an important consideration in conventional control frameworks. When a single controller is used, the effect of the initial state of the controller is typically limited to the initial transients of the system. Provided that the controller is sensibly designed, a zero initial state is usually quite acceptable. Furthermore, optimal

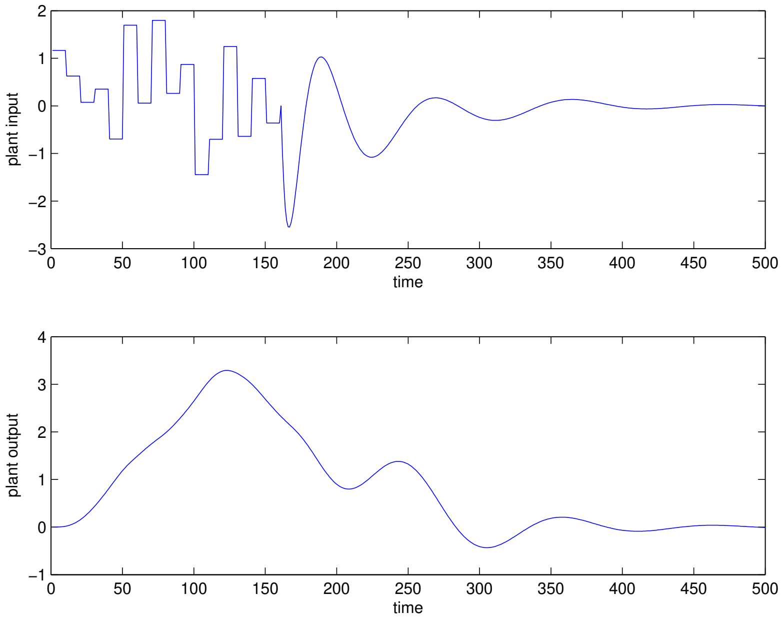


Figure 1.3: Unconditioned response to switch at k=160

initialization of controllers may be impossible if the initial state of the plant is not known. In an output feedback scenario, it may be necessary to run the controller for some time before an observer is able to capture the plant state with a reasonable error margin.

In a switching architecture however, the initial value problems can be extremely important. If new controllers are periodically introduced into the feedback loop, then transients signals due to the initial states of the controllers will occur at each transition. These transient signals can substantially degrade the performance of the resulting systems. If the plant dynamics are assumed to be remain the same, (or at least slowly time-varying) then we may have a great deal of information about the plant states when we switch to a new controller. This information may then be exploited in the solution of the initial value problem at each transition.

1.3 Overview

Chapter 2

Some preliminary material required in the rest of the thesis is introduced. We define some mathematical terminology and notation, particularly associated with dynamical systems and hybrid systems in particular.

We review Lyapunov stability theory and in particular some results concerning the existence of Lyapunov functions and also admissibility of non-smooth Lyapunov functions.

The notion *reset switching system* is introduced, referring to a switching system where the states may change discontinuously at switching boundaries. We also use the term *simple switching system* to refer to switching systems where the states remain continuous across switching boundaries.

We present a brief summary of approaches to the bumpless transfer problem in the literature, including some unified frameworks, and review connections with the anti-windup problem.

We briefly describe the general state estimation problem and present the continuous and discrete-time Kalman filter equations. We also note that the Kalman filter equations can be derived in a purely deterministic context, following the work of Bertsekas and Rhodes [5].

Chapter 3

A review of results concerning the stability of simple switching systems is presented. We introduce results concerning stability (under arbitrary switching) guaranteed by the existence of a common Lyapunov function for the component systems(by Barmish [4] and others), and in particular the converse result by Dayawansa and Martin [10].

We study the Multiple Lyapunov function approach to the study of stability of switching systems introduced by Branicky [7].

We introduce the notion of minimum dwell-time as a means for ensuring stability of switching systems as introduced by Morse [46]. Some tighter dwell-time results are presented based on quadratic Lyapunov functions.

Two results are presented which guarantee the existence of realizations of stabilizing controllers such that the switching system is guaranteed to be stable under arbitrary switching.

Chapter 4

We introduce a bumpless transfer approach which is based on calculation of controller states at transitions which are compatible with input-output pairs in the graph of the controller (to be switched on) and close in some sense to the observed signals.

The solution is presented initially as a weighted least squares optimization. We note that the solution can be implemented as a on observer controller with an optimal Kalman observer gain. Thus the solution can be implemented by an appropriate choice of controller realization without requiring resets to the controller states.

Chapter 5

We introduce an alternative approach to the bumpless transfer problem which is based on choosing controller states which minimize explicitly the transients at each controller transition. The finite horizon problem is solved via a least squares minimization. The infinite horizon solution is solved via Lyapunov equations. The infinite horizon solution is thus also equivalent to minimization of Lyapunov functions with respect to the controller states. A weighted solution can be employed to account for known average dwell times.

Chapter 6

We study the stability of reset switching systems. We introduce a necessary and sufficient Lyapunov theorem for reset switching systems to be stable under arbitrary switching. We study a number of important consequences of this result, including conditions which guarantee stability for linear reset switching systems under arbitrary switching.

Sufficient LMI conditions allow for the synthesis of reset relations which guarantee stability. While such stabilizing resets do not always exist for given controller realizations, we show that these results can be combined with known stabilizing realizations with guaranteed stability, and a substantial improvement in performance.

Chapter 7

We summarize the conclusions of the thesis, and review the original contributions.

Chapter 2

Preliminaries

This chapter introduces some of the basic mathematical terms and definitions which are used in this thesis, along with some fundamental results which are required for proofs in later chapters. Where proofs are omitted, textbook references are provided.

2.1 Spaces and functions

Definitions and results relating to this section may be found in any elementary analysis text such as [38,6,51].

Definition 2.1.1. A *metric space* is a pair (X, d), where X is a set and d is a function from $X \times X$ to $\mathbb{R}^+$ satisfying

- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$ for all $x, y \in X$
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$

$d(x, y)$ is referred to as the *distance* between x and y. d is referred to as the *metric* on X. The third condition above is the *triangle inequality*.

Definition 2.1.2. A *normed space* is a pair $(X, \|.\|)$, where X is a vector space over $\mathbb{R}$ or $\mathbb{C}$, and $\|.\|$ is a function from X to $\mathbb{R}^+$ satisfying

- $\|x\| = 0 \iff x = 0$
- $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in X$, and $\lambda \in \mathbb{R}$

- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

$\|x\|$ is referred to as the *norm* of vector x, and may be thought of as the 'length' of the vector in a generalized sense. The third condition above is the *triangle inequality*.

Every normed space is also a metric space with the induced metric $d(x, y) = \|x - y\|$.

Definition 2.1.3. A sequence $\{x_n\}$ in a metric space X is *Cauchy* if for all $\epsilon \in \mathbb{R}^+$, there exists $N \in \mathbb{Z}^+$ such that $\min\{i, j\} > N$ implies $d(x_i, x_j) < \epsilon$.

Definition 2.1.4. A metric space X is *complete* if every Cauchy sequence in X converges to an element of X. A complete normed space is also called a *Banach* space.

A complete space is essentially one which has no 'holes' in it.

In a normed space X, the *sphere* of radius r about a point x_0 is

$$S_r(x_0) = \{x \in X : \|x - x_0\| = r\}.$$

The (closed) *ball* of radius r about a point x_0 is

$$B_r(x_0) = \{x \in X : \|x - x_0\| \leq r\}.$$

The *open ball* of radius r about a point x_0 is

$$D_r(x_0) = \{x \in X : \|x - x_0\| < r\}.$$

An open ball about a point x_0 is sometimes called a *neighbourhood* of x_0.

Definition 2.1.5. A function between metric spaces $f : X \to Y$ is called *continuous* at a point x, if $f(x_k) \to f(x)$ whenever $x_k \to x$. Equivalently, f is continuous if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon.$$

f is continuous on X if it is continuous at every point in X. If δ of the inequality depends only on ϵ, and not on x, then the function is *uniformly continuous*.

Definition 2.1.6. A function between metric spaces $f : X \to Y$ satisfies the *Lipschitz condition* on a domain Ω if there exists a constant $k \geq 0$ such that

$$d(f(x), f(y)) \leq kd(x, y) \quad \forall x, y \in \Omega.$$

f is *globally Lipschitz* if the condition is satisfied for the whole of X.

This property is sometimes referred to as *Lipschitz continuity*, and it is in fact a stronger condition than uniform continuity.

The following definitions concern scalar functions of a Banach space $V : X \to \mathbb{R}$ or on a Banach space and time $W : X \times \mathbb{R}^+ \to \mathbb{R}$ ($W : X \times \mathbb{Z}^+ \to \mathbb{R}$ in the discrete-time case). Let Ω be a closed bounded region in X, and x be an element of X.

Definition 2.1.7. A scalar function $V(x)$ is *positive semi-definite* (resp. *negative semi-definite*) in Ω if, for all $x \in \Omega$,

- $V(x)$ has continuous partial derivatives with respect to x
- $V(0) = 0$
- $V(x) \geq 0$ (resp. $V(x) \leq 0$)

Definition 2.1.8. A scalar function $V(x)$ is *positive definite* (resp. *negative definite*)in Ω if, for all $x \in \Omega$,

- $V(x)$ has continuous partial derivatives with respect to x
- $V(0) = 0$
- $V(x) > 0$ (resp. $V(x) < 0$) if $x \neq 0$

Definition 2.1.9. A (time dependent) scalar function $W(x, t)$ is *positive semi-definite* (resp. *negative semi-definite*) in Ω if, for all $x \in \Omega$ and all t,

- $W(x, t)$ has continuous partial derivatives with respect to it's arguments
- $W(0, t) = 0$ for all t
- $W(x, t) \geq 0$ (resp. $W(x, t) \leq 0$) for all t

Definition 2.1.10. A (time dependent) scalar function $W(x, t)$ is *positive definite* (resp. *negative definite*) in Ω if, for all $x \in \Omega$ and all t,

- $W(x, t)$ has continuous partial derivatives with respect to it's arguments
- $W(0, t) = 0$ for all t
- $W(x, t) > 0$ (resp. $W(x, t) < 0$) for all t

Definition 2.1.11. A scalar function $W(x, t)$ is *decrescent* in Ω if there exists a positive definite function $V(x)$ such that for all $x \in \Omega$, and all t

$$W(x, t) \leq V(x).$$

This property is also referred to as "W admits an infinitely small upper bound", or "W becomes uniformly small". It is equivalent to saying that W can be made arbitrarily small by choosing x sufficiently close to 0. Any time-invariant positive definite function is decrescent.

Definition 2.1.12. A scalar function $V(x)$ is *radially unbounded* in Ω if $V(x) \to \infty$ as $\|x\| \to \infty$.

Definition 2.1.13. A (time dependent) scalar function $W(x, t)$ is *radially unbounded* in Ω if there exists a positive definite radially unbounded function $V(x)$ such that

$$W(x, t) \geq V(x) \quad \forall t.$$

That is, $W(x, t)$ tends to infinity uniformly in t as $\|x\| \to \infty$.

2.2 Dynamical systems

Broadly speaking, a dynamical system involves the motion of some objects through a space as a function of time. We use the notion *state* to represent some parameter or set of parameters which completely capture the position and behaviour of a system at any one point in time.

To describe a dynamical system completely, we must define carefully the *phase space*, or *state space* of the system - or the set of admissible values for the state. The phase space of a dynamical system is typically a Banach space. We must define the *time space* of the system, which is typically $\mathbb{R}^+$ (continuous-time) or $\mathbb{Z}^+$ (discrete-time). We then must have some way of describing the evolution of the system from one point in time to the next. In the systems we consider, the evolution of the state is usually described by families of differential equations (continuous-time case) or difference equations (discrete-time case). When we consider hybrid or switching systems, we also use *reset relations* in describing state evolution. We may also introduce an *input* which externally influences the behaviour of the system, the input being taken from a specified *input space*.

Consider systems described by continuous first-order ordinary differential equations

$$\dot{x}(t) = f(x(t), t, u(t)) \quad x(t) \in X, \quad t \in \mathbb{R}^+, \quad u(t) \in U, \tag{2.1}$$

or by continuous first-order difference equations

$$x(k+1) = f(x(k), k, u(k)) \quad x(k) \in X, \quad k \in \mathbb{Z}^+, \quad u(k) \in U. \tag{2.2}$$

X is the phase space of the system, and U the input space.

We may also define an output $y(t) \in Y$ (resp $y(k) \in Y$) with the *output space* Y a Banach space, and y defined by the equations

$$y(t) = f(x(t), t, u(t)) \quad x(t) \in X, \quad t \in \mathbb{R}^+, \quad u(t) \in U, \tag{2.3}$$

or

$$y(k) = f(x(k), k, u(k)) \quad x(k) \in X, \quad k \in \mathbb{Z}^+, \quad u(k) \in U. \tag{2.4}$$

In the systems we consider, the phase (and input and output) spaces are continuous ($X = \mathbb{R}^n$), or discrete ($X = \mathbb{Z}^n$), or some combination ($X = \mathbb{R}^n \times \mathbb{Z}^m$). The time space is $\mathbb{R}^+$ in the continuous-time case, or $\mathbb{Z}^+$ in the discrete-time case.

In the following sections, we will occasionally omit the discrete-time version of a result or definition, where the discrete-time counterpart is completely analogous to the continuous-time version.

If $u(t) = 0$ for all t, a system is referred to as *unforced*, or *free*, and may be represented by the unforced equations

$$\dot{x}(t) = f(x(t), t) \quad x(t) \in X, \quad t \in \mathbb{R}^+, \tag{2.5}$$

or

$$x(k+1) = f(x(k), k) \quad x(k) \in X, \quad k \in \mathbb{Z}^+. \tag{2.6}$$

Note that a system with a fixed known input $u(t)$ can also be thought of as a free system described by the equations above, with $u(t)$ being implicit in the function $f(x(t), t)$. Where $u(t)$ is explicit we refer to a *forced* system.

The solution of an unforced system with given initial conditions x_0 and t_0 is known as the *trajectory* of the system, and may be denoted by $\phi(t, x_0, t_0)$.

Existence and uniqueness of trajectories defined in this way can be guaranteed by ensuring that the right hand side of the differential (resp difference) equation satisfies a Lipschitz condition (see for example [28])

$$\| f(x, t), f(y, t) \| \leq k \| x - y \| .$$

We shall assume this condition for all vector fields considered in this thesis.

A dynamical system so described is called stationary if the functions f above do not depend explicitly on t (resp k). An unforced stationary system is sometimes called *autonomous*, and may be described by time invariant equations

$$\dot{x}(t) = f(x(t)), \tag{2.7}$$

or

$$x(k+1) = f(x(k)). \tag{2.8}$$

2.3 Hybrid dynamical systems

The word 'Hybrid' has come to characterize classes of dynamical systems which combine continuous and discrete dynamics. In particular, the state of a hybrid system usually has both discrete and continuous components. Typically $X = \mathbb{R}^n \times \mathbb{Z}$ (noting that $\mathbb{Z}^m$ is equivalent to $\mathbb{Z}$).

We introduce some classes of Hybrid systems which include the switching controller systems that we consider in later chapters.

The evolution of the 'continuous-valued' states of the system are governed by ordinary differential equations (or difference equations), while the discrete state is governed by some discrete valued function.

Consider the family of systems

$$\dot{x}(t) = f_i(x(t)), \quad i \in I, \quad x(t) \in \mathbb{R}^n. \tag{2.9}$$

where I is some index set (typically discrete valued).

Now define a piecewise constant *switching signal* $\sigma(t)$

$$\sigma(t) = i_k \quad t_k \le t < t_{k+1}, \quad i_k \in I \tag{2.10}$$

for some sequence of times $\{t_k\}$ and indices $\{i_k\}$ ($k \in \mathbb{Z}^+$). We assume that $t_k < t_{k+1}$ and $i_k \neq i_{k+1}$ for all k. We will call $\{t_k\}$ the *switching times* of the system. We may also use the term *switching trajectory* for σ.

$\sigma(t)$ is known as a *non-zeno* signal, if there are finitely many transitions in any finite time interval. Signals need to be non-zeno in order that trajectories are well defined for all time. Furthermore, we shall call $\sigma(t)$ *strongly non-zeno* if the ratio of the number of transitions in a finite time interval to the length of the time interval has a fixed upper bound. In general sensible switching strategies will result in strongly non-zeno switching signals. We will usually assume switching signals to be strongly non-zeno.

Now we may describe the following *simple switching system*

$$\begin{aligned} \dot{x}(t) &= f_{\sigma(t)}(x(t)), \\ \sigma(t) &= i_k, \qquad \forall \quad t_k \le t < t_{k+1}, \quad i_k \in I, \quad k \in \mathbb{Z}^+. \end{aligned} \tag{2.11}$$

That is, the evolution of the continuous state of the system is described by the vector field f_{i_k} in the interval $[t_k, t_{k+1})$. The *discrete state* of the system may be thought of as the value of the function $\sigma(t)$ at any given time t.

A simple switching system with a fixed switching signal $\sigma(t)$ may be thought of simply as a time varying continuous system described by a single time-varying vector field (such as 2.5). For a more general problem, we examine classes of admissible switching signals.

Another class of hybrid systems allows the state x to change discontinuously at switching times. This allows us to describe for example physical systems with instantaneous collisions (see for example [57]). In our case, we use this description for plant/controller systems with switching controllers, where the controller state may be reset in some manner at each switch (such as to minimize transients of a weighted output of the closed loop system).

Let us introduce a family of functions

$$g_{i,j} : \mathbb{R}^n \to \mathbb{R}^n, \qquad i, j \in I.$$

The functions $g_{i,j}$ describe the discontinuous change in state at the transition times t_k.

Then a *reset switching system* may be described by the equations

$$\begin{aligned} \dot{x}(t) &= f_{\sigma(t)}(x(t)), \\ \sigma(t) &= i_k, \qquad \forall \ \ t_k \le t < t_{k+1}, \quad i_k \in I, \quad k \in \mathbb{Z}^+, \\ x(t_k^+) &= g_{i_k, i_{k-1}}(x(t_k^-)). \end{aligned} \tag{2.12}$$

We call the functions $g_{i,j}$ *reset relations* between the discrete states i and j. If $g_{j,i_1} = g_{j,i_2}$ for each $i_1, i_2 \in I$ (that is, the reset only depends on the new state), then we may use the shorthand notation g_j.

We may also consider hybrid systems where the state space is not necessarily the same in each discrete state. For example in switching controller systems, it is possible that alternative controllers for the system are different order, or that linear models of the behaviour of a nonlinear plant are different order at various set points. In this case, we consider the family of systems

$$\dot{x}_i(t) = f_i(x_i(t)), \quad i \in I, \quad x_i(t) \in \mathbb{R}^{n_i}, \tag{2.13}$$

and the *multiple state-space switching system* may be described as follows

$$\dot{x}_{\sigma(t)}(t) = f_{\sigma(t)}(x_{\sigma(t)}(t)), \tag{2.14}$$

$$\sigma(t) = i_k \quad t_k \le t < t_k + 1, \quad i_k \in I, \quad k \in \mathbb{Z}^+, \tag{2.15}$$

$$x_{i_k}(t_k^+) = g_{i_k, i_{k-1}}(x_{i_{k-1}}(t_k^-)). \tag{2.16}$$

Note that in this case, the reset relations $g_{i,j}$ are required, since the state cannot be continuous across switches when the state order changes.

2.4 Stability

An *equilibrium state* of an unforced dynamical system, is a state x_e such that $f(x_e, t) = 0$ for all t. Thus if the initial state of the system is x_e, the trajectory of the system will remain at

x_e for all time. A trajectory $\phi(t, x_e, t_0)$ is sometimes referred to as an *equilibrium trajectory*, or *equilibrium solution*.

There are a large number of definitions available for the stability of a system. For unforced systems, definitions generally refer to the stability of equilibria - specifically to the behaviour of trajectories which start close to the equilibrium (sometimes called *perturbed trajectories*). For forced systems, stability usually refers to the relationship between output input functions.

We refer here to a few of the most useful stability definitions for unforced systems, beginning with those discussed by Aleksandr Mihailovich Lyapunov in his championing work on stability theory first published in 1892 [33].

A number of texts provide good references for this material, including [60,21,28].

Consider the dynamical system

$$\dot{x}(t) = f(x(t), t) \quad x(t) \in X, t \in \mathbb{R}^+,$$

with equilibrium state x_e so that $f(x_e, t) = 0$ for all t.

Definition 2.4.1. The equilibrium state x_e is called *stable* if for any given t_0 and $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x_0 - x_e\| < \delta \implies \|\phi(t, x_0, t_0) - x_e\| < \epsilon$$

for all $t > t_0$.

This property is also sometimes referred to as stable in the sense of Lyapunov. It essentially means that perturbed trajectories always remain bounded.

Definition 2.4.2. The equilibrium state x_e is called *convergent*, if for any t_0 there exists $\delta_1 > 0$ such that

$$\|x_0 - x_e\| < \delta_1 \implies \lim_{t \to \infty} \phi(t, x_0, t_0) = x_e.$$

That is, for any $\epsilon_1 > 0$ there exists a $T > t_0$ such that

$$\|x_0 - x_e\| < \delta_1 \implies \|\phi(t, x_0, t_0) - x_e\| < \epsilon_1$$

for all $t > T$.

We say that the perturbed trajectories *converge* to the equilibrium state. Note that convergence does not imply stability nor vice-versa.

Definition 2.4.3. The equilibrium state x_e is called *asymptotically stable* if it is both stable and convergent.

If this property holds for all $x_0 \in X$ (not just in a neighbourhood of x_e), we say that the state is *globally asymptotically stable*. We can also then say that the system is globally asymptotically stable.

Definition 2.4.4. A dynamical system is called *bounded*, or *Lagrange stable* if, for any x_0, t_0 there exists a bound B such that

$$\|\phi(t, x_0, t_0)\| < B$$

In definition 2.4.1, δ may generally depend on t_0 as well as ϵ.

Definition 2.4.5. If the equilibrium state x_e is stable, and the δ (of definition 2.4.1) depends only on ϵ, then we can say that the equilibrium x_e is *uniformly stable*.

Definition 2.4.6. If an equilibrium state is convergent, and the δ_1 and T of definition 2.4.2 are independent of t_0, then the state is known as *uniformly convergent*.

Definition 2.4.7. If an equilibrium state is bounded, and the B of definition 2.4.4 are independent of t_0, then the state is known as *uniformly bounded*.

Uniform boundedness and uniform stability are equivalent for linear systems.

Definition 2.4.8. If an equilibrium state is uniformly stable and uniformly convergent, then it is *uniformly asymptotically stable*.

Definition 2.4.9. If an equilibrium state is uniformly stable, uniformly bounded, and globally uniformly convergent, then it is *globally uniformly asymptotically stable*.

We also say that the system itself is globally uniformly asymptotically stable. Note that uniform stability and uniform convergence are not sufficient to guarantee uniform boundedness (see for example [60]).

Definition 2.4.10. An equilibrium state x_e is called *exponentially stable* if the norm of trajectories may be bounded above by a exponential function with negative exponent. That is, for all t_0, x_0 there exist scalars $a > 0$, $b > 0$ such that

$$\|\phi(t, x_0, t_0) - x_e\| < a\,\|x_0 - x_e\|\, e^{-b(t-t_0)}$$

for all $t > t_0$.

Definition 2.4.11. If, in addition the scalars a and b may be found independently of t_0 and x_0, then the state is called *uniformly exponentially stable*.

Exponential stability implies asymptotic stability, but not vice-versa. For instance the function

$$x(t) = \frac{x_0}{t+1}$$

converges asymptotically to the origin, but is not bounded above by a decaying exponential function. For linear systems however, exponential stability is equivalent to asymptotic stability [60].

In general, we may transform systems with an equilibrium state x_e into an equivalent system with the equilibrium state at the origin. Thus we may discuss the stability of the origin for an arbitrary dynamical system without loss of generality. Additionally, it is possible to transform a system with an *equilibrium trajectory* (not defined here) into one with an equilibrium state at the origin - though time invariance may not be preserved in such a transformation [30].

2.4.1 Stability of hybrid systems

When we discuss hybrid systems, it is necessary to be clear precisely in what sense stability is meant. It is possible to discuss the stability of the continuous and the discrete states, either separately or together.

In this thesis, we only consider hybrid systems where the switching signal is determined by some external process. We thus do not comment on stability with respect to the discrete state. Instead, we may investigate the stability of the continuous states of the system with respect to a particular switching signal, or a class of switching signals.

For the simple switching systems discussed in section 2.3, and for a specific switching signal $\sigma(t)$ we may apply the stability definitions of this section largely without alteration. When discussing stability of an equilibrium x_e, we must assume that x_e is an equilibrium point for each vector field f_i (if there is no such common equilibrium, the system can be at best bounded).

Thus we may say that a switching system is stable for switching signal $\sigma(t)$. Similarly, we may discuss the stability of a reset switching system for a particular switching signal.

In this thesis, we are generally concerned with the concept of stability over large classes of switching signals. For instance we may wish to guarantee stability of a switched system for any signal $\sigma \in S$ where S is a specified class of signals.

Where we refer to stability for arbitrary switching sequences, we generally mean for all strongly non-zeno sequences. Stability is only a sensible question when switching signals are non-zeno. The question of ensuring signals are non-zenon is one of switching strategy design, which is not generally considered in this thesis. Where we do consider restrictions on switching signals by specifying minimum dwell times, switching signals are automatically non-zeno.

2.5 Lyapunov functions for stability analysis

In 1892, Aleksandr Mihailovich Lyapunov presented a doctoral thesis at the University of Kharkov on "The general problem of the stability of motion" [33]. The central theorems of that thesis have formed the foundation of most stability analysis and research in the century since. Lyapunov's original thesis and related works have been translated into French and English, and have been reprinted many times and in many different forms. A special issue of the International Journal of Control celebrating the 100th anniversary of the work [36] contains the English translation of the thesis, along with a biography and bibliography. This issue has been published separately as [34]. An earlier book [35] contains English translations of subsequent related work by Lyapunov and Pliss.

The main body of this work and subsequent work concerned the notion of a Lyapunov function. Lyapunov exploited an apparently very simple idea. Suppose a dynamical system has an invariant set (we are usually concerned with equilibrium points, but we can also discuss periodic orbits or something more complicated). One way to prove that the set is stable is to prove the existence of a function bounded from below which decreases along all trajectories not in the invariant set. A Lyapunov function is, in effect a generalized form of dissipative energy function. The utility of Lyapunov function methods is primarily in the fact that it is not necessary for explicit knowledge of the system trajectories - the functions can often be devised from knowledge of the differential equations. In the linear case, the method is systematic, whereas in the nonlinear case a certain degree of artifice is often required.

We will introduce briefly here the main theorem concerning Lyapunov's so called *direct method* on the connection between Lyapunov functions and stability. We will also discuss some work (primarily from the 1950's and 60's) on converse Lyapunov theorems - that is, showing the existence of Lyapunov functions for certain types of stable systems.

The following is close to the literal text of Lyapunov's theorem (see [36]). Notes in square brackets are added by the author of this thesis.

Theorem 2.5.1. *If the differential equations of the disturbed motion are such that it is possible to find a* [positive or negative] *definite function V, of which the derivative $\dot{V}$ is a function of fixed sign opposite to that of V, or reduces identically to zero* [that is, semi-definite]*, the undisturbed motion* [equilibrium point] *is stable* [in the sense of Lyapunov].

An extension which is presented by Lyapunov as a remark to the theorem, is as follows

Remark 2.5.1. If the function V, while satisfying the conditions of the theorem, admits an infinitely small upper limit *[that is, V is decrescent]*, and if it's derivative represents a definite function, we can show that every disturbed motion, sufficiently near the undisturbed motion, approaches it asymptotically.

Proofs of the theorem may be found in several textbooks (see for example [60,21,28]).

2.5.1 Converse theorems

A large body of work on the existence of Lyapunov functions for stable systems appeared in the literature in the postwar period, roughly when the work of Lyapunov began to attract widespread attention outside of the Soviet Union.

The first converse result is due to Persidskii in 1933, proving the existence of a Lyapunov function for a (Lyapunov) stable set of differential equations in $\mathbb{R}^n$ (see [21] and contained references).

It should be noted that the theorem of Lyapunov (in the case of a strictly decreasing function) yields not just asymptotic stability, but in fact uniform asymptotic stability. It is therefore impossible to prove a converse of the asymptotic stability theorem in it's original form - the result must be strengthened. It was not until the concept of uniform stability had been clearly defined that converse theorems for asymptotically stable systems could be found. Massera [39,40] was the first to note this link, and achieved the first converse results for asymptotic stability. Malkin [37], Hahn [21,20], Krasovskii [30] and Kurzweil [31] have all made substantial contributions to various versions of converse Lyapunov theorems.

The proof of the converse theorem is easiest in the case of uniform exponential stability - not just uniform asymptotic stability (these properties are equivalent in the linear systems case), and it will be sufficient for our purposes to consider converse theorems of exponentially stable equilibria. In dynamical systems theory many stable equilibria of interest are exponentially stable if they are asymptotically stable (for example any linearizable asymptotically stable equilibrium).

We present here the main converse result of interest, based on [28, Theorem 3.12].

Theorem 2.5.2. *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(t, x)$$

where $f : D \times [0, \infty) \to \mathbb{R}^n$ is Lipschitz continuous, and continuously differentiable on D some neighbourhood of the origin. Assume that the equilibrium point is uniformly exponentially stable. That is, there exist positive constants k and γ such that

$$\|x(t)\| \leq \|x(t_0)\| \, ke^{-\gamma(t-t_0)}$$

for all $t > t_0$ and $x(t_0) \in D$.

Then, there exists a function $V : D \times [0, \infty) \to \mathbb{R}^+$ that is positive definite, decrescent and radially unbounded. V is continuous with continuous partial derivatives, and

$$\dot{V} = \frac{\partial V}{\partial t} + \nabla V f(t, x) \le -c \|x\|^2$$

for some positive constant c (that is V is strictly decreasing on trajectories of f).

If the origin is globally exponentially stable and $D = \mathbb{R}^n$, then the function V is defined and has the above properties on $\mathbb{R}^n$. If the system is autonomous (that is $f(x, t) = f(x)$) then V can be chosen independent of t ($V(x, t) = V(x)$).

Proof. Let $\phi(t, x_0, t_0)$ denote the solution of the dynamical system with initial condition x_0 at time t_0. For $x_0 \in D$, we know that $\phi(t, x_0, t_0)$ for all $t > t_0$.

Define the function $V(x, t)$ as follows

$$V(x, t) = \int_t^\infty \phi^*(\tau, x, t)\phi(\tau, x, t)d\tau.$$

To prove that V is positive definite, decrescent and radially unbounded, we need to show the existence of constants c_1 and c_2 such that

$$c_1 \|x\|^2 \le V(x, t) \le c_2 \|x\|^2$$

Since we have exponential bounds on the system trajectories, we have

$$\begin{aligned} V(x, t) &= \int_t^\infty \|\phi(\tau, x, t)\|^2 d\tau \\ &\le \int_t^\infty k^2 e^{-2\gamma(\tau - t)} d\tau \|x\|^2 = \frac{k^2}{2\gamma} \|x\|^2 \end{aligned}$$

Suppose the Lipschitz constant of f is L. Then we have

$$\|\dot{x}\| \le L \|x\|,$$

so

$$\|\phi(\tau, x, t)\|^2 \ge \|x\|^2 e^{-2L(\tau - t)},$$

and

$$V(x, t) \ge \int_t^\infty e^{-2L(\tau - t)} d\tau \|x\|^2 = \frac{1}{2L} \|x\|^2.$$

So we may choose $c_1 = 1/2L$, and $c_2 = k^2/2\gamma$. Thus we have shown V is positive definite, decrescent and radially unbounded.

Now let us consider the value of V at a point corresponding to state x at time t, and at a point on the same trajectory at time $t + T$.

$$\begin{aligned} V(x,t)-&V(\phi(t+T,x,t),t+T) \\ &= \int_t^\infty \|\phi(\tau,x,t)\|^2\, d\tau - \int_{t+T}^\infty \|\phi(\tau,\phi(t+T,x,t),t+T)\|^2\, d\tau \\ &= \int_t^{t+T} \|\phi(\tau,x,t)\|^2\, d\tau \\ &\le \frac{T\,\|x\|^2}{2} \end{aligned}$$

And by taking the limit as $T \to 0$, we obtain

$$\dot{V}(x,t) \le -\frac{\|x\|^2}{2}.$$

That is, V is strictly decreasing on trajectories of f.

Suppose the system is autonomous - then $\phi(t, x_0, t_0)$ depends only on $(t - t_0)$. Say $\phi(t, x_0, t_0) = \psi(x_0, t - t_0)$. Then

$$\begin{aligned} V(x,t) &= \int_t^\infty \psi^*(x,\tau-t)\psi(x,\tau-t)d\tau \\ &= \int_0^\infty \psi^*(x,s)\psi(x,s)ds \end{aligned}$$

which is independent of t, so $V(x,t) = V(x)$. □

The type of construction employed in this proof clearly depends on the uniform exponential stability of the equilibrium. Unfortunately this does not allow us to form a fully necessary and sufficient theorem, since the existence of strictly decreasing Lyapunov functions guarantees only that the equilibrium is uniformly asymptotically stable. It is possible to prove the converse theorem in the uniformly asymptotically stable case, but the proof is considerably more complex. See [28,21] for the appropriate results.

2.5.2 Non-smooth Lyapunov functions

Lyapunov arguments for stability can be applied without the candidate function V being necessarily continuously differentiable. Provided that the function is strictly decreasing along trajectories of the vector field, we can relax the requirement that the V be everywhere continuously differentiable. Convexity and continuity are in fact sufficient.

This extension of Lyapunov's approach appears in the work of Krasovskii [30] in the context of systems containing bounded time delays. There, the vector fields have the form

$$\dot{x}(t) = f(x(t-\tau))$$

with $\tau \in [0, \tau_m]$ for some fixed τ_m. The desired Lyapunov function construction involves a supremum over τ, and hence is not necessarily smooth at every point.

The required generalization of Lyapunov's theorem involves replacing the time derivative

$$\dot{V} = \frac{\partial V}{\partial t} + \nabla V f(t, x) \leq -c \|x\|^2$$

(since ∇V does not exist everywhere) by the one-sided derivative in the direction of the vector field $y = f(x)$

$$\lim_{\Delta t \to 0^+} \frac{V(x + \Delta t y) - V(x)}{\Delta t}.$$

This approach is also used by Molchanov and Pyatnitskii [43,44] in the context of differential inclusions, and is easily adapted to switching systems.

2.6 Bumpless transfer and connections to the anti-windup problem

Much of the bumpless transfer literature has emerged from connections with the study of systems with saturating actuators.

Consider the illustration of figure 2.1. In the anti-windup problem, the Φ block represents the saturation nonlinearity

$$\hat{u} = \begin{cases} u & |u| < 1 \\ \text{sgn}(u) & |u| \geq 1 \end{cases}.$$

Note that an arbitrary saturation may be rewritten as a unity saturation by appropriately scaling the rest f the system.

In the bumpless transfer problem, Φ represents a switching nonlinearity, where $\hat{u} = u$ while the controller is switched on, and some external signal otherwise.

Both problems are characterized by a desire to keep the signal $\hat{u}$ as close as possible to u when the nominal behaviour is not observed (that is, when the saturation is active or when the controller under consideration is off-line).

The approaches generally involve an additional feedback term from $\hat{u}$, the output of the nonlinearity. Note in the anti-windup case, that this involves the controller containing an internal model of the saturation, since typically the actual output of an actuator is not measured.

A controller which has been modified to account for saturation or switching is often referred to as a *conditioned* controller (a term coined in this context by Hanus [23]). An important property of the conditioned controller is that it retains nominal behaviour- that is, the closed

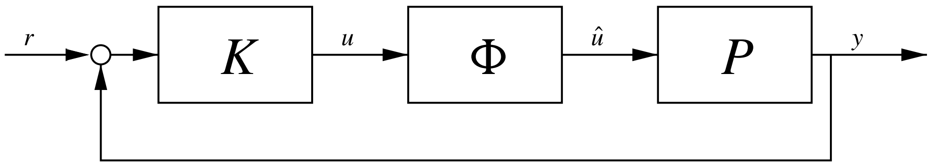


Figure 2.1: General switched or saturating system with no conditioning

loop transfer functions of the nominal system $\Phi = I$ are the same when the nominal controller is replaced by the conditioned controller.

The conditioned controller can usually be expressed as

$$\hat{K} : \quad \begin{bmatrix} e \\ \hat{u} \end{bmatrix} \to u \qquad \text{where } e = r - y.$$

Below we describe several established techniques for dealing with controller switching and actuator saturation. Equations where given are discrete-time, but the extensions to continuous-time are usually trivial.

2.6.1 Conventional antiwindup

Conventional antiwindup is a scheme which grew out of the "anti-reset windup" approach to the anti-windup problem. An additional feedback term X is introduced, feeding from the error $\hat{u} - u$ to the controller input. In the saturation problem, this means the term X is only active when the actuator is saturated, and acts (when correctly designed) to keep u and $\hat{u}$ as close as possible. In the switching problem, X is only active when the controller is off-line, and acts to keep the output of the off-line controller u as close as possible to the output of the online controller $\hat{u}$.

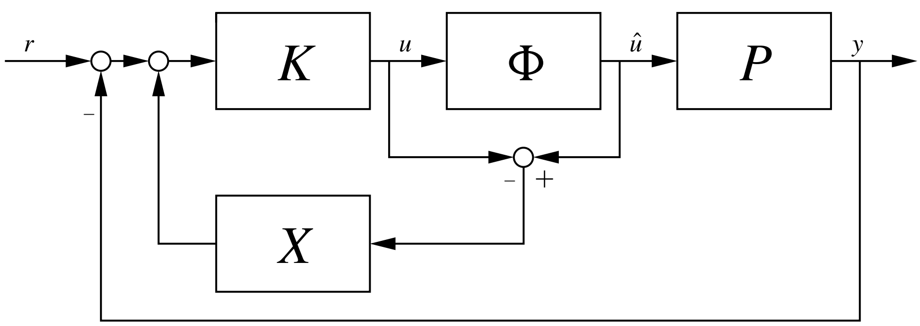


Figure 2.2: Conventional antiwindup

From figure 2.2 in the discrete-time case, we have the modified controller equations as follows

$$x_{k+1} = Ax_k + B(r_k - y_k + X(\hat{u}_k - u_k)), \tag{2.17}$$

$$u_k = Cx_k + D(r_k - y_k + X(\hat{u}_k - u_k)). \tag{2.18}$$

$$\tag{2.19}$$

Provided $(I + DX)$ is nonsingular, we may write

$$u_k = (I + DX)^{-1}Cx_k + (I + DX)^{-1}D(r_k - y_k) + (I + DX)^{-1}DX\hat{u}_k.$$

Now substituting (2.6.1) into (2.17) we obtain

$$\begin{aligned} x_{k+1} &= (A - BX(I + DX)^{-1}C)x_k + (B - BX(I + DX)^{-1}D)(r_k - y_k) \\ &\quad + (BX - BX(I + DX)^{-1}DX)\hat{u}_k \\ &= (A - BX(I + DX)^{-1}C)x_k + B(I + XD)^{-1}(r_k - y_k) + BX(I + DX)^{-1}\hat{u}_k. \end{aligned}$$

Thus we can write the conventional antiwindup controller in the following state space form

$$\hat{K} = \left[\begin{array}{c|cc} A - BX(I + DX)^{-1}C & B(I + XD)^{-1} & BX(I + DX)^{-1} \\ \hline (I + DX)^{-1}C & (I + DX)^{-1}D & (I + DX)^{-1}DX \end{array}\right]. \tag{2.20}$$

The Conventional antiwindup conditioning scheme when applied to a switching control system, may be interpreted as a tracking control design problem. When the controller in question is off-line, the feedback gain X acts as a tracking controller for the "plant" K. The "reference" input is $\hat{u}$ with output u, and a disturbance input of $r - y$ at the "plant" input. This interpretation is illustrated in figure 2.3.

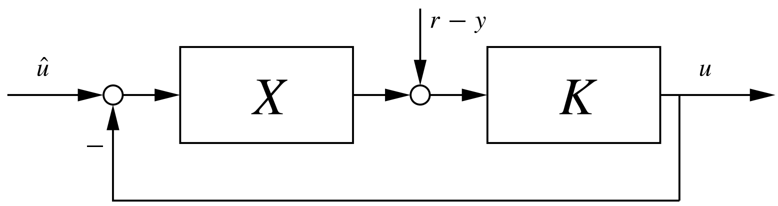


Figure 2.3: Conventional antiwindup as a tracking problem

This is the approach taken by Graebe and Ahlén [19], also including an additional pre-compensator F_L applied to the signal $\hat{u}$. The presence of a non-identity pre-compensator however requires that the conditioning scheme be switched off when the controller is switched in. This requires a precise knowledge of the switching times, and may not be appropriate in some applications.

2.6.2 Hanus conditioned controller

This conditioning technique was developed by Hanus *et al.* [23,22]. The interpretation of the problem is a lack of consistency between the controller state and the plant input during saturation, or prior to a controller switch.

Consistency is restored by applying modified signals to the controller such that the controller output is identical to the plant input.

For the unconditioned controller we have:

$$x_{k+1} = Ax_k + B(r_k - y_k), \tag{2.21}$$

$$u_k = Cx_k + D(r_k - y_k), \tag{2.22}$$

$$\hat{u}_k = \Phi(u_k). \tag{2.23}$$

Hanus introduces a hypothetical "realizable reference". That is, a signal r^r which if applied to the reference input would result in a plant input $u = \hat{u}$. So

$$x_{k+1} = Ax_k + B(r_k^r - y_k), \tag{2.24}$$

$$\hat{u}_k = Cx_k + D(r_k^r - y_k). \tag{2.25}$$

Combining the above, we obtain

$$\hat{u}_k - u_k = D(r_k^r - r_k),$$

and assuming D is nonsingular

$$r^r = r + D^{-1}(\hat{u}_k - u_k). \tag{2.26}$$

Now combining equations (2.22), (2.24), (2.25) and (2.26) we obtain the following conditioned controller equations

$$x_{k+1} = (A - BD^{-1}C)x_k + BD^{-1}\hat{u}_k, \tag{2.27}$$

$$u_k = Cx_k + D(r_k - y_k), \tag{2.28}$$

$$\hat{u}_k = \Phi(u). \tag{2.29}$$

Note that for a controller

$$K = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

with D nonsingular, we can represent K^{-1} as follows [62]

$$K^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right].$$

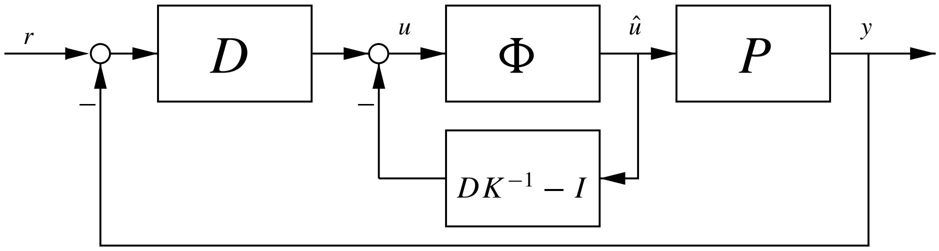


Figure 2.4: Hanus conditioned controller

Thus the Hanus conditioned controller may be implemented as shown in figure 2.4.

The conditioned controller may be expressed in the following state space form

$$\hat{K} = \left[\begin{array}{c|cc} A - BD^{-1}C & 0 & BD^{-1} \\ \hline C & D & 0 \end{array}\right]. \tag{2.30}$$

Note that the Hanus conditioned controller is restricted to controllers with nonsingular D matrix and stable zeros. Also, the design is inflexible in that there are no tuning parameters to allow the conditioning to be varied to suit the performance requirements.

2.6.3 Internal Model Control

The internal model control (IMC) structure [45], though not designed specifically with anti-windup in mind has been shown to have properties conducive to antiwindup [9,29,12], in the case where the plant is open loop stable.

The structure is shown in figure 2.5, where P_M is the known plant model, and K_M is the modified controller, taking feedback from the plant *error* rather than the plant output.

$$K_M = K(I + P_M K)^{-1}$$

where K is the linear controller designed for the linear system $\Phi = I$.

2.6.4 Observer based schemes

An alternative approach to restoring the consistency of controller states and plant input, as suggested by Åström and Rundqwist [2] is to introduce an observer into the controller, observing from the plant input and output. The resulting system attempts to maintain the controller in states consistent with the observed plant signals.

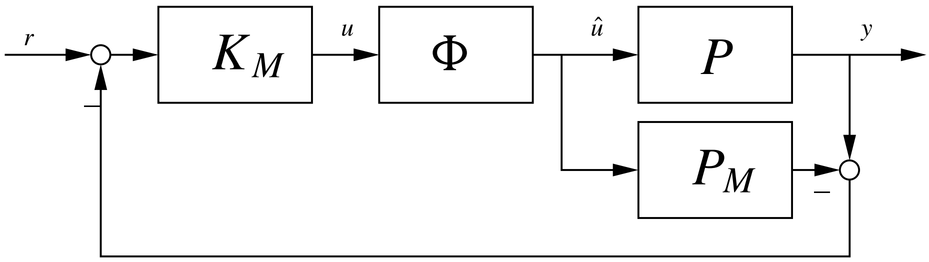


Figure 2.5: Internal Model Control Structure

The observer form for the controller, with observer gain H is defined as follows

$$x_{k+1} = Ax_k + B(r_k - y_k) + H(\hat{u}_k - Cx_k - D(r_k - y_k)), \tag{2.31}$$

$$u_k = Cx_k + D(r_k - y_k), \tag{2.32}$$

$$\hat{u}_k = \Phi(u_k), \tag{2.33}$$

so we can write

$$\hat{K} = \left[\begin{array}{c|cc} A - HC & B - HD & H \\ \hline C & D & 0 \end{array}\right]. \tag{2.34}$$

When $\hat{u}_k = u_k$ we simply have the linear controller

$$K = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

When $\hat{u}_k \neq u_k$, the controller state is updated according to the observed plant input $\hat{u}$. The observer controller structure is shown in figure 2.6.

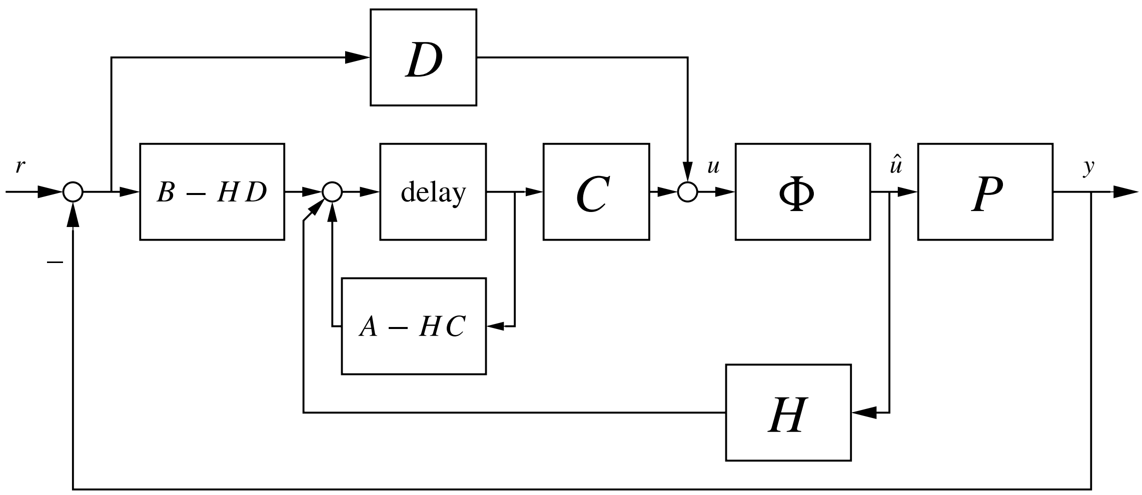


Figure 2.6: Observer based scheme

The observer form for an antiwindup bumpless transfer controller admits a coprime factorization form as shown in figure 2.7, where $K = V^{-1}U$ is a coprime factorization of the controller. We may write $\hat{K}$ for the observer controller as

$$\hat{K} = \begin{bmatrix} U & I - V \end{bmatrix}, \tag{2.35}$$

where

$$V = \left[\begin{array}{c|c} A - HC & -H \\ \hline C & I \end{array}\right], \quad \text{and} \tag{2.36}$$

$$U = \left[\begin{array}{c|c} A - HC & B - HD \\ \hline C & D \end{array}\right]. \tag{2.37}$$

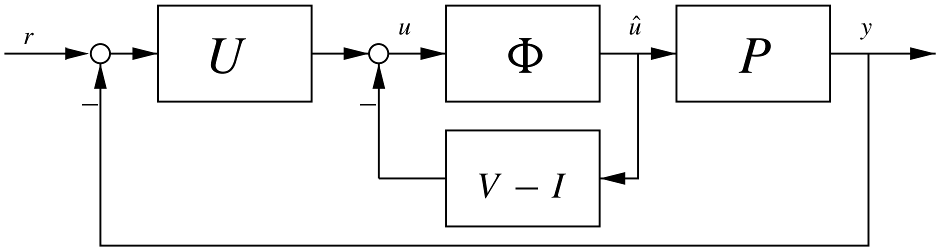


Figure 2.7: Coprime factorization form

2.6.5 Unifying standard techniques

Åström [2,3], Campo [9], Walgama and Sternby [58] have exploited the observer property inherent in a number of the standard schemes in order to generalize them. The Hanus conditioned controller (2.30) for example is an observer controller with $H = BD^{-1}$.

A number of other bumpless transfer schemes (mostly variations on the schemes already examined) can also be represented in this observer form, including the Internal Model Control structure [9,29].

To include a greater number of antiwindup schemes, Campo [9] included an extra parameter $I - H_2$, feeding from $\hat{u}$ directly to u.

Campo's scheme represents antiwindup controllers in the form

$$\hat{K} = \begin{bmatrix} U & I - V \end{bmatrix},$$

where

$$V = \left[\begin{array}{c|c} A - H_1 C & -H_1 \\ \hline H_2 C & H_2 \end{array}\right], \qquad \text{and}$$

$$U = \left[\begin{array}{c|c} A - H_1 C & B - H_1 D \\ \hline H_2 C & H_2 D \end{array}\right].$$

This allows the inclusion of the conventional antiwindup scheme, with parameters $H_1 = BX(I + DX)^{-1}$, and $H_2 = (I + DX)^{-1}$. Note however, that we can only guarantee that V and U are coprime when $H_2 = I$. Note that when $H_2 \neq I$, an algebraic loop is introduced around the nonlinearity which may cause computational difficulties.

2.6.6 Coprime factorization approach

Miyamoto and Vinnicombe [42,41] characterize antiwindup controllers by the coprime factorization form shown in figure 2.9.

$K = V_0^{-1} U_0$ is some coprime factorization of the controller, and all coprime factors are parameterized as

$$V = QV_0 \qquad \text{and} \tag{2.38}$$

$$U = QU_0, \tag{2.39}$$

where Q and Q^{-1} are stable ($Q(s), Q^{-1}(s) \in \mathscr{H}_\infty$ in continuous time case). The antiwindup problem is then formulated as design of the parameter Q.

We choose U_0 and V_0 such that

$$V_0 M_0 + U_0 N_0 = I, \tag{2.40}$$

where $P = N_0 M_0^{-1}$ is the normalized coprime factorization of the plant. With this choice, Q for some of the schemes discussed so far is as shown in table 2.1 [41].

Note that some of these representations are only truly coprime in somewhat restricted circumstances. The unconditioned controller must be open loop stable for U and V to be coprime when implemented directly ($Q = V_0^{-1}$). The Hanus controller must have invertible D matrix and stable zeros (as noted earlier). The IMC controller must have an open loop stable plant model. These restrictions however tend to correspond to the circumstances under which antiwindup properties are reasonable.

Note that when the nonlinearity Φ is a unit saturation, then by rewriting the saturation as a deadzone plus unity we obtain the equivalent picture shown in figure 2.8. Δ is the deadzone

Table 2.1: Q for selected antiwindup schemes

scheme	Q
unconditioned	V_0^{-1}
CAW	$(V_0 + U_0 X)^{-1}$
Hanus	DU_0^{-1}
IMC	M

nonlinearity

$$\Delta : d = \begin{cases} 0 : & |u| < 1 \\ \mathrm{sgn}(u)(|u| - 1) : & \text{otherwise} \end{cases}$$

Since the deadzone always has gain less than one, it is possible to apply the small gain theorem [62]. That is, if the gain G_{ud} from d to u is less than one and the nominal system ($\Delta = 0$) is stable, then the perturbed system is also stable.

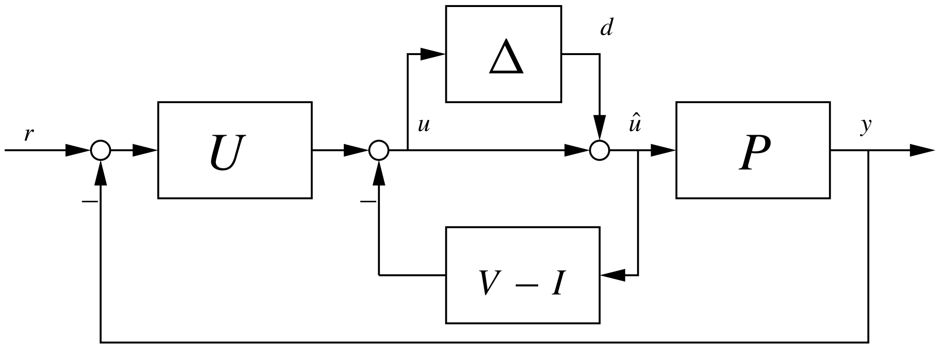


Figure 2.8: Equivalent figure

From figure 2.8, and applying the bezout identity we find that

$$\begin{aligned} G_{ud} &= (-UP - V + I)(UP + V)^{-1} \\ &= -I + M_0 Q^{-1}. \end{aligned}$$

The magnitude of the transfer functions $G_{\hat{u}d}$ and G_{yd} from d to $\hat{u}$ and y provides some indication of the effect of the saturation on system performance. We can write these transfer functions as

$$\begin{aligned} G_{\hat{u}d} &= P(I + G_{ud}) \\ &= N_0 Q^{-1}, \\ G_{yd} &= I + G_{ud} \\ &= M_0 Q^{-1}. \end{aligned}$$

In Miyamoto and Vinnicombe then, the design approach is to make the gains $G_{\hat{u}d}$ and G_{yd} small (in terms of a weighted $\mathscr{H}_\infty$ norm), while ensuring that the $\mathscr{H}_\infty$ norm of G_{ud} remains less than 1. We will see in section 3.4.2 that this approach also results in guaranteed stability and performance in a system with switching controllers and saturating actuators.

It should be noted that each of the schemes examined so far (including the generalized schemes above), when applied to the bumpless transfer problem are equivalent to choosing an appropriate controller state x_K at each controller transition. This is clear, as each of the schemes listed behaves according to the linear controller design immediately when the controller in question is switched on.

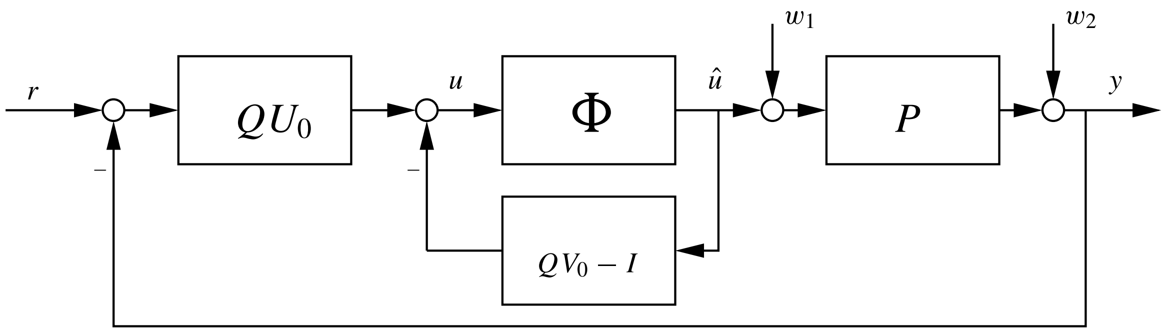


Figure 2.9: Coprime factorization based scheme

2.7 Filtering and estimation

Estimation problems in control usually require estimating the state of a dynamical system from noisy measurements of input and output data. The *filtering* estimation problem specifically requires calculation of an estimate of the 'current' state using data up to and including the current time.

The Kalman filter [27] is an example of an optimal filter. That is, the estimate $\hat{x}$ of the current state x is optimal with respect to the cost function $J = E((x - \hat{x})^T(x - \hat{x}))$, the expectation of the squared error.

We will present here the Kalman filter equations for discrete and continuous-time time-varying state-space systems, with inputs and outputs corrupted by zero-mean Gaussian noise. The initial state estimate is also assumed to be taken from a Gaussian distribution with known mean and variance.

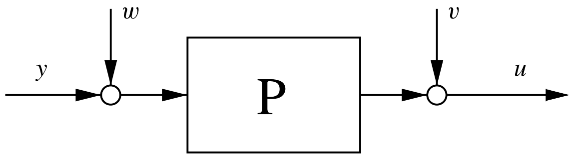


Figure 2.10: Plant subject to input and output noise

2.7.1 Discrete-time equations

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k (u_k + w_k), \\ y_k &= C_k x_k + D_k (u_k + w_k) + v_k. \end{aligned} \tag{2.41}$$

Consider the discrete time system with input and output noise illustrated in figure 2.10, and represented by equations (2.41). Note that more generally we may consider different B matrices for noise input w_k, and control input u_k, however for our purposes it is sufficient to consider noise injected at the input.

We shall assume that the input u_k, and output y_k are known without error. A_k, B_k, C_k and D_k are known for all k. The expected value and covariance of the initial state are known:

$$E(x_0) = \hat{x}_0 = \mu_0, \tag{2.42}$$

$$E\left[(x_0 - \mu_0)(x_0 - \mu_0)^T\right] = P_0 = \Psi. \tag{2.43}$$

The input and output noise are assumed to be uncorrelated, and have known statistics.

$$E(w_k) = E(v_k) = 0, \tag{2.44}$$

$$E\left(v_k v_j^T\right) = R_k \delta(k-j), \tag{2.45}$$

$$E\left(w_k w_j^T\right) = Q_k \delta(k-j), \tag{2.46}$$

$$E\left(v_k x_j^T\right) = E\left(w_k x_j^T\right) = 0 \quad \forall j, k. \tag{2.47}$$

$\delta(k-j)$ is the Dirac delta function ($\delta(k-j) = 1$ when $j = k$, 0 otherwise). We shall assume that Q_k, and R_k are positive definite. In addition, we shall assume that A_k has full rank, B_k has full column rank, and that C_k has full row rank.

We wish to produce an estimate $\hat{x}_k$ of the state at time t_k such that the expectation value $E((x-\hat{x})^T(x-\hat{x}))$ is minimized. We can rewrite this as the trace of the covariance estimate matrix P_k.

$$\begin{aligned} P_k &:= E\left(\left(x-\hat{x}\right)\left(x-\hat{x}\right)^T\right), \\ E\left(\left(x-\hat{x}\right)^T\left(x-\hat{x}\right)\right) &= \text{Trace } P_k. \end{aligned} \tag{2.48}$$

Given the state estimate at the previous time step $\hat{x}_{k-1}$, a natural initial estimate for $\hat{x}_k$ is:

$$\hat{x}_k^{'} = A_{k-1}\hat{x}_{k-1} + B_{k-1}u_{k-1}. \tag{2.49}$$

Note that we have not yet used the output measured at time t_k. We can then apply a correction to the state estimate based on the error between the actual output and predicted output. So a sensible estimate is:

$$\begin{aligned}\hat{x}_k &= \hat{x}_k^{'} + K_k\left[y_k - C_k\hat{x}_k^{'} - D_k u_k\right]\\ &= \left(A_{k-1} - K_kC_kA_{k-1}\right)\hat{x}_{k-1} + \left(B_{k-1} - K_kC_kB_{k-1}\right)u_{k-1} + K_ky_k - K_kD_ku_k,\end{aligned} \tag{2.50}$$

where the gain K_k is to be determined.

Theorem 2.7.1. *Consider the linear discrete time system* (2.41)*, and the linear state estimator* (2.50)*. The optimal gain K_k, which minimizes the least squares estimation error $E\left[(x_k - \hat{x}_k)(x_k - \hat{x}_k)\right]^T = \mathrm{Trace}\, P_k$ is:*

$$K_k = P_kC_k^T\left(D_kQ_kD_k^T + R_k\right)^{-1} \tag{2.51}$$

where

$$P_k = \left(A_{k-1}P_{k-1}A_{k-1}^T + B_{k-1}Q_{k-1}B_{k-1}^T\right)^{-1} + C_k^T\left(D_kQ_kD_k^T + R_k\right)^{-1}C_k \tag{2.52}$$

Proof. The proof of the theorem in this particular form appears in [47], adapted from proofs in Sorenson [52], and Stengel [54]. □

Hence, the Kalman filter can be implemented in recursive form with the three equations

$$\hat{x}_k = \left(A_{k-1} - K_kC_kA_{k-1}\right)\hat{x}_{k-1} + \left(B_{k-1} - K_kC_kB_{k-1}\right)u_{k-1} + K_ky_k - K_kD_ku_k, \tag{2.53}$$

$$K_k = P_kC_k^T\left(D_kQ_kD_k^T + R_k\right)^{-1}, \tag{2.54}$$

$$P_k = \left(A_{k-1}P_{k-1}A_{k-1}^T + B_{k-1}Q_{k-1}B_{k-1}^T\right)^{-1} + C_k^T\left(D_kQ_kD_k^T + R_k\right)^{-1}C_k, \tag{2.55}$$

along with the boundary conditions $P_0 = \Psi$ and $\hat{x}_0 = \mu_0$.

The equations may be simplified further if the system observed, and the noise statistics are time-invariant. In that case the equations reduce to

$$\hat{x}_k = (A - KCA)\hat{x}_{k-1} + (B - KCB)u_{k-1} + Ky_k - KDu_k, \tag{2.56}$$

$$K = PC^T\left(DQD^T + R\right)^{-1}, \tag{2.57}$$

$$P = \left(APA^T + BQB^T\right)^{-1} + C^T\left(DQD^T + R\right)^{-1}C. \tag{2.58}$$

Since the matrices K and P are now constant, they may be precomputed allowing for very simple and computationally efficient observer form implementation of the filter.

2.7.2 Continuous-time equations

Now consider the estimation problem for a continuous-time plant. The signals of figure 2.10, and the state-space equations are now continuous-time.

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)\left(u(t) + w(t)\right), \\ y(t) &= C(t)x(t) + D(t)\left(u(t) + w(t)\right) + v(t). \end{aligned} \tag{2.59}$$

Once again, the expected value μ_0 and covariance Ψ of the initial state are known, as are the statistics of the input and output noise.

$$E\left(w(t)\right) = E\left(v(t)\right) = 0, \tag{2.60}$$

$$E\left(v(t)v^T(s)\right) = R(t)\delta(t-s), \tag{2.61}$$

$$E\left(w(t)w^T(s)\right) = Q(t)\delta(t-s), \tag{2.62}$$

$$E\left(v(t)x^T(s)\right) = E\left(w(t)x^T(s)\right) = 0 \quad \forall \ t, s. \tag{2.63}$$

Under these assumptions, we can again compute the equations for the optimal (with respect to the expectation of the squared state error) Kalman filter.

The resulting equations (not derived here) are:

$$\dot{\hat{x}}(t) = \left(A(t) - K(t)C(t)\right)\hat{x}(t) + \left(B(t) - K(t)D(t)\right)u(t) + K(t)y(t), \tag{2.64}$$

$$K(t) = P(t)C^T(t)\left(D(t)Q(t)D^T(t) + R(t)\right)^{-1}, \tag{2.65}$$

$$\begin{aligned} \dot{P}(t) = {} & P(t)A^T(t) + A(t)P(t) + B(t)Q(t)B^T(t) \\ & - P(t)C^T(t)\left(D(t)Q(t)D(t)^T + R(t)\right)^{-1}C(t)P(t), \end{aligned} \tag{2.66}$$

along with the boundary conditions $P(0) = \Psi$ and $\hat{x}(0) = \mu_0$.

In the time-invariant case, the equations are:

$$\dot{\hat{x}}(t) = (A - KC)\hat{x}(t) + (B - KD)u(t) + Ky(t), \tag{2.67}$$

$$K = PC^T\left(DQD^T + R\right)^{-1}, \tag{2.68}$$

$$0 = PA^T + AP + BQB^T - PC^T\left(DQD^T + R\right)^{-1}CP. \tag{2.69}$$

2.8 The deterministic filtering problem

We now consider an alternative perspective to the standard form of the state estimation. Rather than making stochastic assumptions about the noise signals w and v, we simply find the state-estimate which corresponds to minimization of a quadratic cost function of the noise signals and the initial state. We show that with appropriate identifications, the result of this *deterministic filtering* problem is identical to the optimal Kalman filter estimate with stochastic assumptions.

This approach was first used by Bertsekas and Rhodes in [5], estimating the state of a dynamical system with no control input.

2.8.1 Continuous-time equations

Consider the arrangement of figure 2.10, described by equations (2.59).

In the deterministic filtering problem, we wish to find the estimate $\hat{x}(t)$ of the state $x(t)$, which corresponds with initial state $x(t_0)$, and noise signals $w(t)$ and $v(t)$ which minimize the quadratic cost function

$$J(t) = (x(t_0) - \mu_0)^T \Psi (x(t_0) - \mu_0) + \int_{t_0}^{t} \left(w^T(s) Q(s) w(s) + v^T(s) R(s) v(s) \right) ds \quad (2.70)$$

Theorem 2.8.1. *Assume that the signals u and y are known and defined over the interval $[t_0, t]$. Then, the optimal estimate $\hat{x}(t)$ of the state of system* (2.59) *at time t with respect to the cost function* (2.70) *may be obtained by solving the differential system*

$$\dot{\hat{x}}(t) = (A(t) - K(t)C(t))\, \hat{x}(t) + (B(t) - K(t)D(t))\, u(t) + K(t) y(t), \quad (2.71)$$

$$K(t) = P(t) C^T(t) \left(D(t) Q(t) D^T(t) + R(t) \right)^{-1}, \quad (2.72)$$

$$\begin{aligned} \dot{P}(t) = {} & P(t) A^T(t) + A(t) P(t) + B(t) Q(t) B^T(t) \\ & - P(t) C^T(t) \left(D(t) Q(t) D(t)^T + R(t) \right)^{-1} C(t) P(t), \end{aligned} \quad (2.73)$$

with the boundary conditions $P(0) = \Psi$ and $\hat{x}(0) = \mu_0$.

This result (and the discrete time version) is obtained directly from Bertsekas and Rhodes main result in [5]. The approach is to treat the problem as an optimal tracking problem in reverse time. The noise signal $w(t)$ is treated as the 'control' input, and the boundary condition is on the initial state rather than the final error. A direct dynamic programming solution is also contained in [14].

2.8.2 Discrete-time equations

The discrete-time solution is obtained in similar fashion to the continuous case. We solve an optimization problem with respect to the cost function

$$J_k = (x_0 - \mu_0)^T \Psi (x_0 - \mu_0) + \sum_{i=0}^{k} \left(w_i^T Q_i w_i + v_i^T R_i v_i \right) \quad (2.74)$$

Theorem 2.8.2. *Assume that the signals u and y are known and defined over the interval* $[0, k]$. *Then, the optimal estimate* $\hat{x}_k$ *of the state of system* (2.41) *at time k with respect to the cost function* (2.74) *may be obtained by solving the difference equations*

$$\hat{x}_k = (A_{k-1} - K_k C_k A_{k-1})\,\hat{x}_{k-1} + (B_{k-1} - K_k C_k B_{k-1})\,u_{k-1} + K_k y_k - K_k D_k u_k, \quad (2.75)$$

$$K_k = P_k C_k^T \left(D_k Q_k D_k^T + R_k\right)^{-1}, \quad (2.76)$$

$$P_k = \left(A_{k-1} P_{k-1} A_{k-1}^T + B_{k-1} Q_{k-1} B_{k-1}^T\right)^{-1} + C_k^T \left(D_k Q_k D_k^T + R_k\right)^{-1} C_k, \quad (2.77)$$

with the boundary conditions $P_0 = \Psi$ *and* $\hat{x}_0 = \mu_0$.

These results allow us to use the simplicity of Kalman filter implementations to solve problems which can be described in terms of two-norm noise minimization. We apply the results to the problem of optimal initial state selection for switching controllers in chapter 4.

Chapter 3

Stability of simple switching systems

Stability is a key issue when we consider switching systems. The hybrid nature of switching systems means that we must consider the interaction between discrete and continuous dynamics. It is not enough to merely ensure that all of the component continuous-time systems are stable. Stability problems generally fall into two broad categories.

In this chapter, we consider stability issues relating to simple switching systems. That is, systems where the state remains continuous across switching boundaries.

The strongest results find conditions which guarantee that the system remains stable under arbitrary switching. Such conditions allow the switching supervisor to be designed independently of the systems (or controllers) themselves.

If it is not possible to guarantee stability for all switching signals, then we would like to determine some class of signals for which the system remains stable or at the very least find a particular switching signal for which the system is stable.

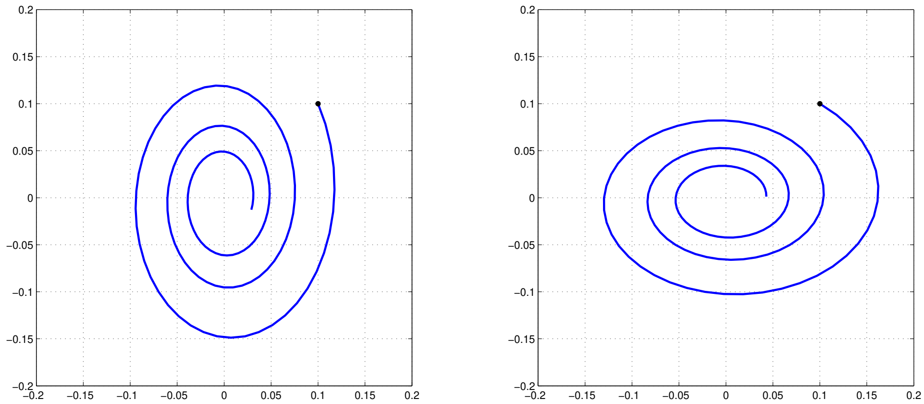
In this chapter, our main interest is in stability under arbitrary switching. It is by no means a trivial problem. It does not suffice to ensure that each of the component systems is stable (or, in a control context that all of the alternative controllers are stabilizing), though it is obviously necessary. It is not difficult to construct examples of unstable trajectories achieved by switching between stable systems (or conversely of stable trajectories by switching between unstable systems).

Consider the following example (similar to examples in [26, 32]).

Example 3.0.1. Consider two stable continuous-time component systems as follows:

$$\dot{x} = A_1 x \qquad\qquad \dot{x} = A_2 x \tag{3.1}$$

$$A_1 = \begin{bmatrix} -0.1 & 1 \\ -2 & -0.1 \end{bmatrix} \qquad\qquad A_2 = \begin{bmatrix} -0.1 & 2 \\ -1 & -0.1 \end{bmatrix} \tag{3.2}$$



(a) f_1 trajectory (b) f_2 trajectory

Figure 3.1:

$f_1 = A_1 x$, and $f_2 = A_2 x$ are both stable dynamical systems. Figure 3.1 shows trajectories for f_1 and f_2 respectively, with the initial condition $x = [0.1 \quad 0.1]^*$.

Suppose we switch between these systems according to regions in the state space. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then let us switch at the boundaries $x_1 = 0$ and $x_2 = 0$ such that $f_1 = A_1 x$ is active in quadrants 2 and 4 ($x_1 x_2 < 0$) of the state space, and $f_2 = A_2 x$ active in quadrants 1 and 3 ($x_1 x_2 \geq 0$). Figure 3.2 shows the resulting unstable trajectory.

We can construct a similar example in a plant and controller context as follows.

Example 3.0.2. Consider the feedback arrangement depicted in figure 3.3. Plant P (with zero through term), and controllers K_1 and K_2 are given by a particular state space representations

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \qquad K_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \qquad K_2 = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$$

Plant G has state x_G, and controller K_i has state x_{Ki} ($i = \{1, 2\}$).

Then, we have a switching system between the closed loop systems

$$\begin{bmatrix} \dot{x_G} \\ \dot{x_{Ki}} \end{bmatrix} = \begin{bmatrix} A + BD_iC & BC_i \\ B_iC & A_i \end{bmatrix} \begin{bmatrix} x_G \\ x_{Ki} \end{bmatrix} \qquad i = 1, 2. \tag{3.3}$$

Now if we choose

$$G = \left[\begin{array}{c|c} 0.5 & 1 \\ \hline 1 & 0 \end{array}\right] \qquad K_1 = \left[\begin{array}{c|c} -0.1 & -1 \\ \hline 2 & -0.6 \end{array}\right] \qquad K_2 = \left[\begin{array}{c|c} -0.6 & -1.5 \\ \hline 1.5 & -0.1 \end{array}\right],$$

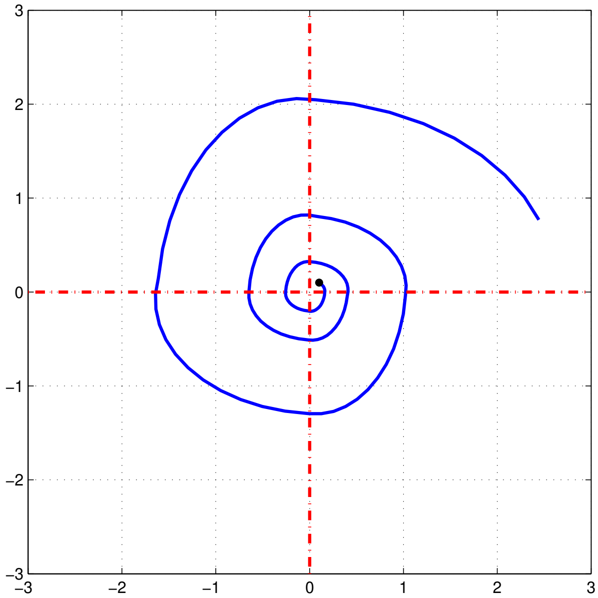


Figure 3.2: switching trajectory

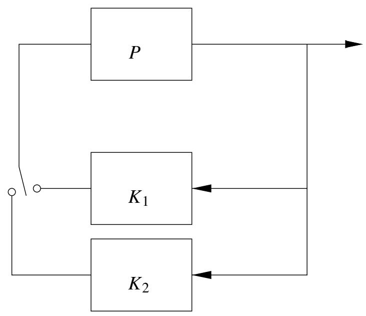
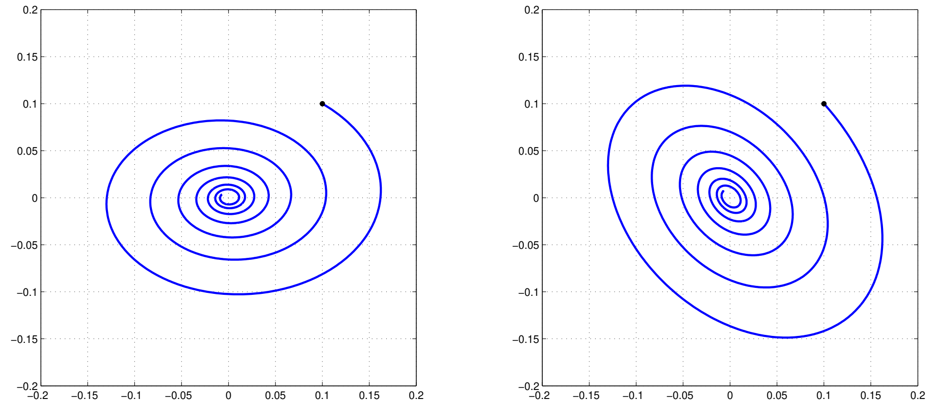


Figure 3.3: Feedback arrangement for example 3.0.2



(a) f_1 trajectory (b) f_2 trajectory

Figure 3.4:

we obtain the closed loop equations

$$\begin{bmatrix} \dot{x}_G \\ \dot{x}_{K1} \end{bmatrix} = \begin{bmatrix} -0.1 & 2 \\ -1 & -0.1 \end{bmatrix} \begin{bmatrix} x_G \\ x_{K1} \end{bmatrix} \quad , \text{ and} \tag{3.4}$$

$$\begin{bmatrix} \dot{x}_G \\ \dot{x}_{K2} \end{bmatrix} = \begin{bmatrix} 0.4 & 1.5 \\ -1.5 & -0.6 \end{bmatrix} \begin{bmatrix} x_G \\ x_{K2} \end{bmatrix}. \tag{3.5}$$

Thus the closed loop dynamics are stable foci about the origin, with the same direction of flow, and orientation 45° apart. Figure 3.4 shows the respective closed loops applied without switching, and with initial states $\begin{bmatrix} x_G & x_{Ki} \end{bmatrix}^* = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}^*$.

Suppose we employ an autonomous switching scheme such that controller K_1 is employed when the plant state satisfies the constraint $|x| \leq 0.1$, and K_2 otherwise. Then trajectories beginning from initial states close to the origin will converge, however for large enough initial conditions, the trajectory will converge to a limit cycle. Figure 3.5 shows the switching trajectory beginning with initial state $\begin{bmatrix} x_G & x_{Ki} \end{bmatrix}^* = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}^*$ $(i = 2)$.

In this chapter we will consider stability issues for simple switching systems By that we mean switching systems in which the state is continuous across switching times.

3.1 Common Lyapunov functions for switching systems

It is a relatively straightforward observation that, given a family of vector fields

$$\dot{x}(t) = f_i(x(t)), \quad i \in I, \quad x(t) \in \mathbb{R}^n \tag{3.6}$$

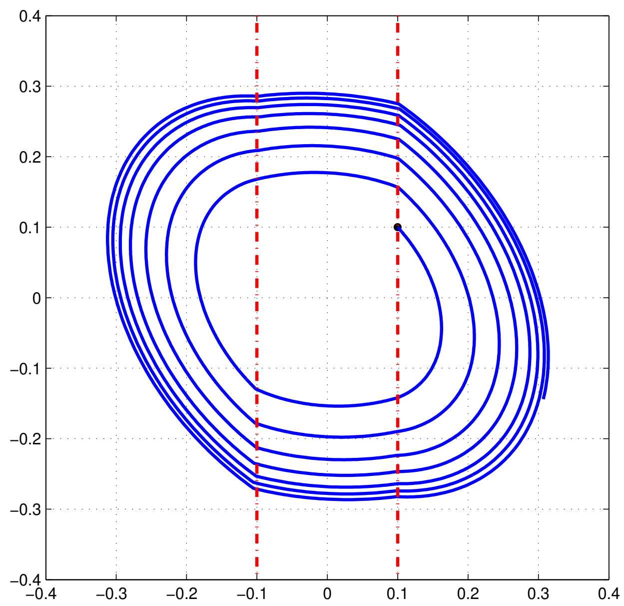


Figure 3.5: switching trajectory

and a family of switching signals $\sigma(t) \in S$, that the switching system

$$\begin{aligned} \dot{x}(t) &= f_{\sigma(t)}(x(t)) \\ \sigma(t) &= i_k, \qquad \text{for} \quad t_k \le t < t_{k+1}, \quad i_k \in I, \quad k \in \mathbb{Z}^+. \end{aligned} \tag{3.7}$$

will be stable if there exists a *common Lyapunov function* (CLF) V which is positive and non-increasing on trajectories of any of the component systems f_i.

This observation has been made by Barmish [4] and others, often in a somewhat wider context, such as for uncertain systems where the system may vary arbitrarily among a family of vector fields, or for differential inclusions [44]. The sufficient condition is essentially a direct consequence of Lyapunov's theorem for nonlinear systems.

The converse problem is substantially more complex, but it has been proved that the existence of a common Lyapunov function is a necessary condition for exponential stability of a simple switching system.

In the linear context, when the vector fields have the form

$$\dot{x}(t) = A_i x(t), \quad i \in I, \quad x(t) \in \mathbb{R}^n,$$

and the switching system is exponentially stable for all switching signals. Molchanov and Pyatnitskii [44] have shown the existence of a quasi-quadratic Lyapunov function

$$V(x) = x^* L(x) x$$

where $L(x) = L^*(x) = L(\tau x)$ for all $x \in \mathbb{R}^n$ and $\tau \in \mathbb{R}^+$. This function is not necessarily continuously differentiable, but is Lipschitz continuous and strictly convex.

Dayawansa and Martin [10] have proved a similar necessity theorem, but have additionally shown that a continuously differentiable Lyapunov function V may be obtained by smoothing a similarly obtained quasi-quadratic function.

Theorem 3.1.1 ([10]). *The linear simple switching system is uniformly exponentially stable if and only if there exists a continuously differentiable function* $V : \mathbb{R}^m \to \mathbb{R}$ *such that* $\dot{V}(x) = \nabla V(x) A_i x$ *is negative definite for all* $i \in I$.

It is important to note that it is not in general possible to construct common quadratic Lyapunov functions. Dayawansa gives an example of a uniformly exponentially stable switching system (with two linear component systems) which does not admit a common quadratic Lyapunov function.

If the index set I is finite, and a common quadratic Lyapunov function exists, it can be found by solving for $P > 0$ and $Q > 0$ the system of inequalities $A_i^* P + P A_i < -Q$ for all $i \in I$.

Dayawansa and Martin also extend the result to nonlinear systems which are globally asymptotically stable and locally uniformly exponentially stable.

3.2 Multiple Lyapunov functions

Peleties and DeCarlo [48], and Michael Branicky in [7] introduce a somewhat different approach for analysing stability of a switching system. The basic idea is to find *Lyapunov-like* functions V_i for each of the component vector fields, which are positive definite and decreasing whenever the i'th loop is active. We then ensure stability for a family of switching signals S by ensuring that V_i is non-increasing on an appropriate sequence of switching times.

The multiple Lyapunov function stability condition is, in a sense weaker than the common Lyapunov function approach when applied to the problem of stability for all switching signals. The multiple function approach can however be more tractable in some cases, especially when considering restricted classes of switching signals when stability cannot be guaranteed for arbitrary switching.

In considering the multiple Lyapunov function approach, we shall modify the definition of switching signals to include the system initial conditions. This allows for the study of stability for systems where the set of possible switching signals at any point is dependent upon the state of the system at that point. For example, we may consider piecewise affine hybrid system under this framework (see for example [26]).

Definition 3.2.1. An *anchored* switching signal is a pair (x_0, σ) where $x_0 \in \mathbb{R}^n$ is an initial state, and $\sigma : \mathbb{R}^+ \to I$ is a (non-zeno) piecewise constant switching signal.

We shall let the class of anchored switching signals under consideration be S, and assume the family of vector fields is finite (let $I = \{1, 2, \ldots, N\}$). Fix a particular signal $(x_0, \sigma) \in S$, with associated switching times $\{t_k\}$ and indices $\{i_k\}$ ($\sigma(t) = i_k$ when $t_k \leq t < t_{k+1}$).

Consider the family of vector fields 3.6, or the equivalent discrete-time family

$$x(k+1) = f_i(x(k)), \qquad i \in I$$

Given a particular switching sequence σ, let $\sigma|_n$ be the sequence of endpoints of the intervals for which the n'th system is active. That is:

$$\sigma|_n = \{t_k : \quad i_k = n, \text{or} \quad i_{k-1} = n\}$$

The *interval completion* of a time sequence $T = t_0, t_1, t_2, \ldots$ is the set

$$\mathcal{I}(T) = \bigcup_{k=0}^{\infty} (t_{2k}, t_{2k+1})$$

Thus, $\mathcal{I}(\sigma|_n)$ is the set of times for which the n'th system is active.

Let $\mathcal{E}(T) = \{t_0, t_2, t_4, \ldots\}$ denote the *even sequence* of T. Then, $\mathcal{E}(\sigma|_n)$ is the set of times at which the n'th system is engaged.

Definition 3.2.2. Given a strictly increasing set of times T in $\mathbb{R}$, a function V is said to be *Lyapunov-like* for a hybrid trajectory $\phi_\sigma(.)$ over a time sequence T if it satisfies the following:

- V is a continuous, positive definite, and radially unbounded function with continuous partial derivatives
- $V(0) = 0$
- $\dot{V}(x(t)) \leq 0$ for all $t \in \mathcal{I}(T)$
- V is monotonically non-increasing on $\mathcal{E}(T)$

The following is a version of Branicky's theorem. We will not prove the result here, since an extension of the theorem is proved in chapter 6.

Theorem 3.2.1 ([7]). *Consider a set of vector fields* (3.6)*. Let S be the set of anchored switching sequences associated with the system.*

If there exist functions V_i such that over all anchored switching sequences $(x_0, \sigma) \in S$, V_i is Lyapunov-like for the hybrid trajectory over $\sigma|_i$, then the switching system (3.7) *is stable in the sense of Lyapunov.*

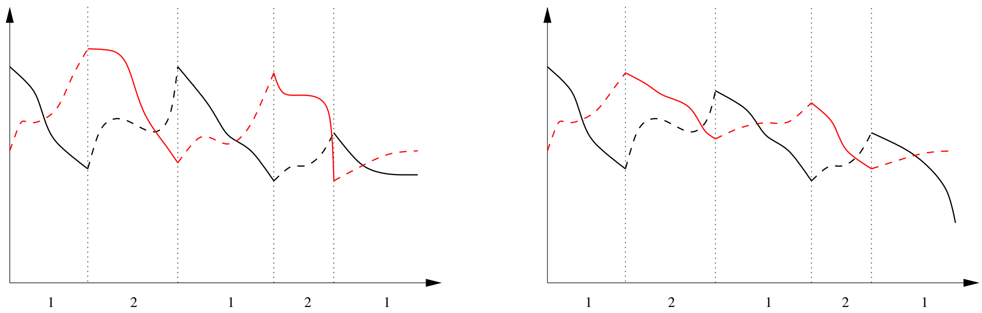


Figure 3.6: Multiple Lyapunov functions for $N = 2$

Additionally, the system is asymptotically stable if for some ϵ_i and δ_i

$$\dot{V}_i(x(t)) < -\epsilon_i \|x(t)\|^2$$

for t in $\mathcal{I}(\sigma|_i)$, and V_i are strictly decreasing on $\mathcal{E}(\sigma|_i)$:

$$V_i(x(t_k)) - V_i(x(t_j)) < -\delta_i \left\|x(t_j)\right\|^2, \tag{3.8}$$

where $t_j < t_k$, and $i_j = i_k = i$.

We will refer to (3.8) as the *sequence decreasing* condition. This is not precisely the same as the asymptotic stability conditions presented by Branicky [7] and Pelleties [48], but it is perhaps the most useful expression for our purposes.

Figure 3.6 illustrates a possible sequence of multiple Lyapunov functions for a 2-switched system which satisfies the (respectively Lyapunov stability and asymptotic stability) conditions for theorem 3.2.1. The black lines denote V_1, and the red lines denote V_2. The solid lines show the active Lyapunov function.

The multiple Lyapunov function approach may be used in order to restrict the set of admissible switching signals in order to ensure stability.

3.3 Dwell time switching

3.3.1 If we are switching between stable vector fields, it is fairly obvious that stability of a simple switching system can be guaranteed if we switch sufficiently slowly. That is, we find can find a τ such that if we *dwell* at each vector field for at least time τ then stability is guaranteed. This approach to stabilizing switching systems is known as *dwell time* switching, and was introduced by Morse in [46]. The selected τ is known as the system *dwell time*.

The approach there is as follows. Each component vector field is stable, so there exist functions V_i for each $i \in I$ which satisfy the inequalities

$$a_i \|x\|^2 \leq V_i(x) \leq b_i \|x\|^2$$

and

$$\nabla V_i(x) f_i(x) \leq -c_i \|x\|^2$$

for some constants a_i, b_i and c_i. Therefore we can write

$$\nabla V_i(x) f_i(x) \leq -\lambda_i V_i(x)$$

where $\lambda_i = c_i/b_i$. This gives the exponential upper bound on V_i of

$$V_i(x(t_0 + \tau) \leq e^{-\lambda_i \tau} V_i(x(t_0)),$$

provided that $\sigma(t) = i$ for $t \in [t_0, t_0 + \tau)$

Now consider beginning with the i_k'th loop active at some time t_k, and switching to the i_{k+1}'th loop at time t_{k+1}. This results in the set of inequalities

$$\frac{a_{i_k}}{b_{i_{k+1}}} V_{i_{k+1}}(x(t_{k+1})) \leq a_{i_k} \|x(t_{k+1})\|^2 \leq V_{i_k}(x(t_{k+1})) \leq e^{-\lambda_{i_k}(t_{k+1}-t_k)} V_{i_k}(x(t_k)).$$

Hence, if the time gap $(t_{k+1} - t_k)$ satisfies

$$t_{k+1} - t_k > \frac{-1}{\lambda_{i_k}} \ln\left(\frac{a_{i_k}}{b_{i_{k+1}}}\right)$$

then the sequence non-increasing condition

$$V_{i_{k+1}}(x(t_{k+1})) < V_{i_k}(x(t_k))$$

is satisfied for that switch. Thus if we choose the dwell time τ to be

$$\tau \geq \sup_{i,j}\left(\frac{-1}{\lambda_i} \ln\left(\frac{a_i}{b_j}\right)\right),$$

then the sequence non-increasing condition can be satisfied for all possible switches.

This approach is obviously potentially conservative, both because of the 'spherical' bounds used in the calculations, and because the choice of Lyapunov function is arbitrary. In the following section, we examine a refinement of the technique for the case of quadratic Lyapunov functions.

3.3.1 Dwell times for linear simple switching systems

Here we consider an approach to dwell time switching which makes use of the elliptical bounds on trajectories which are provided when we have quadratic Lyapunov functions.

Let us illustrate the principal idea for $N = 2$. Suppose we have Lyapunov functions $V_1 = x^* P_1 x$, and $V_2 = x^* P_2 x$, where $P_1 > 0$ and $P_2 > 0$ are solutions to Lyapunov equations

$$A_1^* P_1 + P_1 A_1 = -Q_1$$
$$A_2^* P_2 + P_2 A_2 = -Q_2$$

for some positive definite Q_1 and Q_2.

Let the initial conditions be x_0 at time t_0, with $\sigma(t_0) = 1$. Now we know that the trajectory is bounded by the level set $x^* P_1 x = x_0 P_1 x_0$ while vector field 1 is active. Suppose we switch from loop 1 to loop 2 at time t_1, and then back to loop 1 at time t_2. If A_1 is stable, then we know that

$$x(t_1)^* P_1 x(t_1) < x_0^* P_1 x_0$$

From the matrix Q_1, it is possible to derive an exponential upper bound on the decay of V_1 after some time τ.

Lemma 3.3.1. *Suppose $V(x(t)) = x(t)^* P x(t)$ where $P > 0$ is the solution to the Lyapunov equation*

$$A^* P + PA = -Q$$

for some A and $Q > 0$. Then the value of the function at time $t_0 + \tau$ satisfies the bound

$$V(x(t_0 + \tau)) \leq e^{-\lambda\tau} V(x(t_0))$$

where λ is the minimum eigenvalue of the matrix $P^{-1}Q$.

Proof. The bound is calculated, by calculating the maximum value of the derivative $\dot{V}$ on a particular level curve of V.

Consider the level curve $x^* P x = a$. The maximum value of $\dot{V}$ on this curve, is

$$-\min_x (x^* Q x) \text{ such that } x^* P x = a.$$

Geometrically, this corresponds to finding the number b such that the level curve $x^* Q x = b$ fits exactly inside the level curve $x^* P x = a$ (see figure 3.7).

Consider a transformation $z = Tx$ such that the level curve $x^* P x = a$ is transformed into the circle $z^* z = a$. This is obtained by the Cholesky factorization $P = T^* T$, and it exists precisely when P is positive definite.

So now we wish to find

$$\min_z \left(z (T^*)^{-1} Q T^{-1} z \right) \text{ such that } z^* z = a.$$

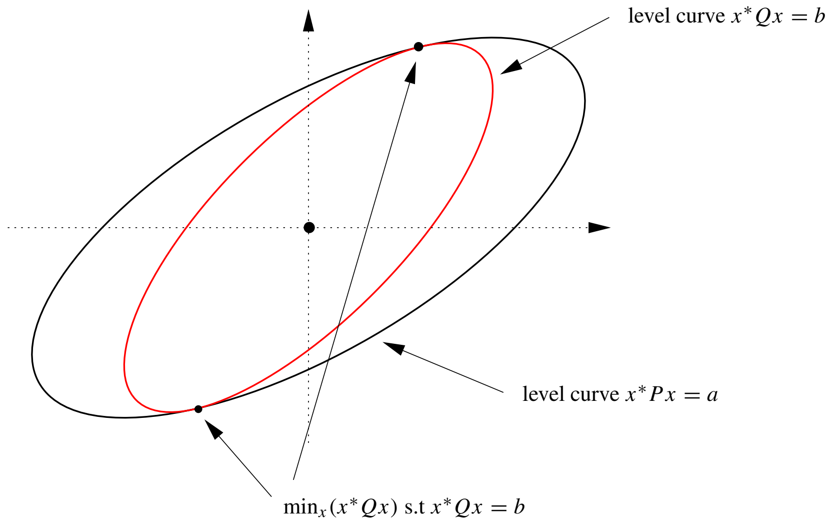


Figure 3.7: Geometrical illustration

This is simply a times the minimum eigenvalue of the matrix $(T^*)^{-1}QT^{-1}$ (or equivalently, the minimum eigenvalue of $P^{-1}Q$). Thus we now have the bound

$$\dot{V} \leq -\lambda V$$

where λ is the minimum eigenvalue of the matrix $P^{-1}Q$ (since eigenvalues are transpose-invariant). Hence

$$V(x(t_0+\tau) \leq e^{-\lambda\tau}V(x(t_0))$$

□

Note that this calculation, based on elliptical bounds is strictly less conservative than the spherical bounds considered in the previous section.

Thus at the time t_1, we have the bound

$$x(t_1)^*P_1x(t_1) \leq e^{-\lambda_1(t_1-t_0)}x_0^*P_1x_0$$

where λ_1 is calculated using lemma 3.3.1. We can also now calculate the maximum value of the Lyapunov function V_2 on the level curve

$$x^*P_1x = e^{-\lambda_1(t_1-t_0)}x_0^*P_1x_0,$$

using a similar technique to that employed in lemma 3.3.1. That is,

$$\max_x(x^*P_2x) \quad \text{such that} \quad x^*P_1x = e^{-\lambda_1(t_1-t_0)}x_0^*P_1x_0$$
$$= k_{12}e^{-\lambda_1(t_1-t_0)}x_0^*P_1x_0,$$

where k_{12} is the maximum eigenvalue of $P_1^{-1}P_2$.

Thus we now have the bound

$$V_2(x(t_1)) \leq k_{12}e^{-\lambda_1(t_1-t_0)}V_1(x(t_0)).$$

We continue in the same fashion through the decay of V_2 to t_2, and at the switch back to loop 1, resulting in the bound

$$V_1(x(t_2)) \leq k_{12}k_{21}e^{-\lambda_1(t_1-t_0)}e^{-\lambda_2(t_2-t_1)}V_1(x(t_0)),$$

where k_{ij} is the maximum eigenvalue of $P_i^{-1}P_j$, and λ_i is the minimum eigenvalue of $P_i^{-1}Q_i$. Thus we can guarantee the sequence decreasing condition if the dwell time τ is chosen such that

$$k_{12}k_{21}e^{-(\lambda_1+\lambda_2)\tau} < 1.$$

That is, we use

$$\tau > \frac{1}{\lambda_1 + \lambda_2} \ln\left(k_{12}k_{21}\right).$$

In the general N loop case, the required dwell time is a supremum of such a form, but over arbitrary sequences of switches.

Let us define $\mathscr{L}[N]$ to be the set of all cycles of elements of $\{1, 2, \ldots, N\}$. That is, $l \in \mathscr{L}[N]$ is a finite sequence beginning and ending with the same element, with no repeated element (except for the first/last). Note that $\mathscr{L}[N]$ can be thought of as the set of all cycles of K_N, the complete graph with N vertices (see [17] for example). If we can ensure that each V_i is decreasing on all such cycles, then stability may be guaranteed via theorem 3.2.1.

We shall use the notation $i \in l$ to mean 'the element i appears in the cycle l'.

As a slight abuse of notation, we shall use $ij \in l$ to mean the sequence $\{i, j\}$ appears in l in that particular order. When l is a cycle of switching sequence indices, this means that a switch from state i to state j occurs in the switching sequence.

Consider a finite family of stable linear vector fields

$$\dot{x} = A_i x,$$

with corresponding Lyapunov functions $V_i = x^*P_ix$ for $i \in \{1, 2, \ldots, N\}$. The $P_i > 0$ satisfy the Lyapunov equations

$$A_i^*P_i + P_iA_i = -Q_i$$

for $Q_i > 0$.

Theorem 3.3.2. *Let k_{ij} be the maximum eigenvalue of $P_i^{-1}P_j$, and λ_i the minimum eigenvalue of $P_i^{-1}Q_i$.*

If we choose τ such that

$$\tau > \sup_{l\in\mathscr{L}[N]} \left(\frac{1}{\sum_{i\in l}\lambda_i} \ln\left(\prod_{ij\in l} k_{ij} \right)\right), \tag{3.9}$$

then the simple switching system

$$\begin{aligned} \dot{x} &= A_{\sigma(t)}x, \\ \sigma(t) &= i_k, \quad \forall \;\; t_k \le t < t_{k+1}, \quad i_k \in \{1,2,\ldots,N\}, \quad k \in \mathbb{Z}^+, \end{aligned} \tag{3.10}$$

is guaranteed to be stable for all switching sequences with a minimum dwell time of τ.

Proof. Consider a particular switching sequence

$$\sigma(t) = i_k, \qquad \text{for} \;\; t_k \le t < t_{k+1},$$

where the sequence of switching times satisfies $t_{k+1} - t_k > \tau$ for all k. Suppose the switching sequence contains a loop (possibly with repeated elements) l, beginning with the m'th vector field. Then, the 'gain' of V_m from the beginning to the end of the loop is bounded above by

$$\left(\prod_{ij\in l} k_{ij}\right) e^{-\left(\sum_{i\in l}\lambda_i\right)\tau}.$$

This is clear, since we have already shown that k_{ij} represents the maximum gain from one Lyapunov function to another at a switch, and $e^{-\lambda_i\tau}$ represents the minimum decay in the value of V_i while in the i'th state.

Furthermore, for any arbitrary loop (which may contain repeated elements), this expression can be factored into similar expressions for non-repeating loops. For example

$$k_{12}k_{23}k_{32}k_{21}e^{-(\lambda_1+\lambda_2+\lambda_3+\lambda_2)\tau} = \left(k_{12}k_{21}e^{-(\lambda_1+\lambda_2)\tau}\right)\left(k_{23}k_{32}e^{-(\lambda_2+\lambda_3)\tau}\right).$$

Hence, if τ is chosen such that

$$\tau > \left(\frac{1}{\sum_{i\in l}\lambda_i} \ln\left(\prod_{ij\in l} k_{ij} \right)\right)$$

for each non-repeating loop l, then

$$\left(\prod_{ij\in l} k_{ij}\right) e^{-\left(\sum_{i\in l}\lambda_i\right)\tau} < 1$$

for each non-repeating loop, and hence also for repeating loops. Thus the decreasing condition from theorem 3.2.1 for asymptotic stability is satisfied, and the switching system is asymptotically stable. □

There is clearly still some conservatism in this result, arising from the fact that it is dependent on particular choices of P_i.

An upper bound on this minimum dwell-time (but a less complex calculation) is the choice

$$\tau > \sup_{i \in l} \left(\frac{1}{\lambda_i} \right) \sup_{ij \in l} \left(\ln \left(k_{ij} \right) \right). \tag{3.11}$$

This will in fact ensure the inequality

$$V_{i_{k+1}}(x(t_{k+1})) < V_{i_k}(x(t_k))$$

is satisfied for all k. This is a stronger condition than our decreasing condition in theorem 3.2.1, but is the same as the condition for asymptotic stability used in [7].

3.4 Choice of controller realizations

Switching between stabilizing controllers for a given plant does not in general ensure stability of the resulting trajectory. It is possible, however to choose realizations for stabilizing controllers such that stability can be guaranteed.

3.4.1 IMC approach

Recent work by Hespanha and Morse [25] uses an Internal Model Control framework in order to choose realizations for given controller such that stability of the switching system is guaranteed.

The result is based on the following lemma, and the Youla parameterization of stabilizing controllers [62].

Lemma 3.4.1. *Given any finite family of asymptotically stable transfer matrices $\mathcal{P} = \{P_i : i \in I\}$ with McMillan degree no larger than n, and any $n \times n$ symmetric positive definite matrix Q, there exist stabilizable and detectable realizations $\{A_i, B_i, C_i, D_i\}$ for each $P_i \in \mathcal{P}$ such that*

$$QA_i + A_i^* Q < 0$$

for all $i \in I$.

Proof. Suppose $\{\bar{A}_i, \bar{B}_i, \bar{C}_i, \bar{D}_i\}$ is any realization for P_i. Then, we can find matrices Q_i satisfying

$$Q_i A_i + A_i^* Q_i = -X_i \tag{3.12}$$

for each i, with X_i positive definite. In fact, we can choose Q_i such that $Q^{-\frac{1}{2}}Q_i^{\frac{1}{2}}$ is full rank for each i. Given positive definite matrices Q and Q_i, we can write $Q = (Q^{\frac{1}{2}})^* Q^{\frac{1}{2}}$ and $Q_i = (Q_i^{\frac{1}{2}})^* Q_i^{\frac{1}{2}}$. Define

$$A_i = Q^{-\frac{1}{2}} Q_i^{\frac{1}{2}} \bar{A}_i Q_i^{-\frac{1}{2}} Q^{\frac{1}{2}}, \qquad B_i = Q^{-\frac{1}{2}} Q_i^{\frac{1}{2}} \bar{B}_i, \qquad C_i = \bar{C}_i Q_i^{-\frac{1}{2}} Q^{\frac{1}{2}}, \qquad D_i = \bar{D}_i$$

Now we can rewrite equation (3.12) as

$$(Q^{-\frac{1}{2}} Q_i^{\frac{1}{2}})^* (QA_i + A_i^* Q)(Q^{-\frac{1}{2}} Q_i^{\frac{1}{2}}) = -I,$$

and hence

$$QA_i + A_i^* Q = -(Q_i^{-\frac{1}{2}} Q^{\frac{1}{2}})^* Q^{-\frac{1}{2}} Q^{\frac{1}{2}} < 0.$$

□

This result means that for any family of stable linear dynamical systems, we can construct realizations such that a simple switched system will be stable for any switching signal σ. Obviously, in a plant/controller framework, it is only possible to choose the realization of the controller and not the plant. Hespanha overcomes this difficulty by using an Internal Model Control framework [45], constructing a controller which contains an internal realization of a particular closed loop transfer function, and a model of the plant. The realization of the model of the closed loop transfer function can then be chosen according to lemma 3.4.1, and stability will be guaranteed provided that all the models are exact.

3.4.2 Choice of coprime factorizations

We can consider another approach to the design of stabilizing switching control systems. This approach is based on the ideas in Miyamoto and Vinnicombe [42] in the context of the anti-windup problem.

Suppose we have a plant G, and a set of stabilizing controllers K_i. We may choose a right coprime factorization of the plant $G = NM^{-1}$, and left coprime factorizations of the controllers $K_i = V_i^{-1} U_i$, such that for each i the bezout identity

$$V_i M + U_i N = I$$

is satisfied [62, section 5.4]. Furthermore given any Q such that $Q, Q^{-1} \in \mathscr{RH}_\infty$, the factorizations $G = \tilde{N}\tilde{M}^{-1}$, and $K_i = \tilde{V}_i^{-1}\tilde{U}_i$ also satisfy the bezout identities

$$\tilde{V}_i \tilde{M} + \tilde{U}_i \tilde{N} = I,$$

where $\tilde{N} = NQ$, $\tilde{M} = MQ$, $\tilde{U}_i = Q^{-1} U_i$, and $\tilde{V}_i = Q^{-1} V_i$.

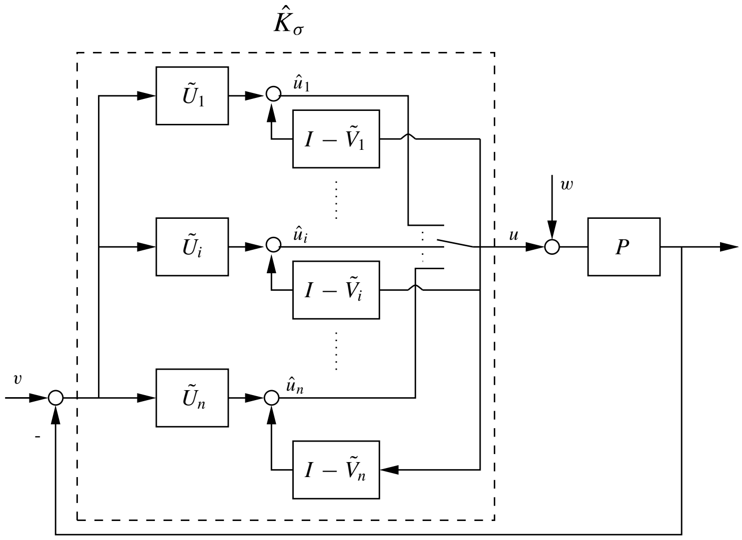


Figure 3.8: Switching arrangement

A particular choice of Q for a controller factorization can also be thought of as a particular choice for the plant factorization (via Q), or *vice versa*. In the switching controller case, this is true provided that all of the controllers have the same choice of Q.

Now consider the coprime factor switching arrangement in figure 3.8. The switching connection is such that $u(t) = \hat{u}_{\sigma(t)}$, where $\sigma(t)$ is the switching signal governing the controller selection. The signals u, v, and w are common to the loops. We can think of this system as a plant P in a feedback loop with the augmented controller $\hat{K}_\sigma$.

Note that for each loop i, we have

$$\begin{aligned}
\hat{u}_i &= (I - \tilde{V}_i)u - \tilde{U}_i P u - \tilde{U}_i P w + \tilde{U}_i v \\
&= (I - \tilde{V}_i - \tilde{U}_i \tilde{N} \tilde{M}^{-1})u - \tilde{U}_i P w + \tilde{U}_i v \\
&= (M - \tilde{V}_i M - \tilde{U}_i \tilde{N})\tilde{M}^{-1} u - \tilde{U}_i P w + \tilde{U}_i v \\
&= (I - \tilde{M}^{-1})u - \tilde{U}_i P w + \tilde{U}_i v.
\end{aligned}$$

Since $u = \hat{u}_\sigma$, we can write

$$\begin{aligned}
u &= (I - \tilde{M}^{-1})u - \tilde{U}_\sigma P w + \tilde{U}_\sigma v \\
&= -\tilde{M}\tilde{U}_\sigma P w + \tilde{M}\tilde{U}_\sigma v \\
&= -\tilde{M}(\tilde{U}_\sigma \tilde{N})\tilde{M}^{-1} w + \tilde{M}\tilde{U}_\sigma v \\
&= -\tilde{M}(I - \tilde{V}_\sigma \tilde{M})\tilde{M}^{-1} w + \tilde{M}\tilde{U}_\sigma v \\
&= -(I - \tilde{M}\tilde{V}_\sigma)w + \tilde{M}\tilde{U}_\sigma v,
\end{aligned}$$

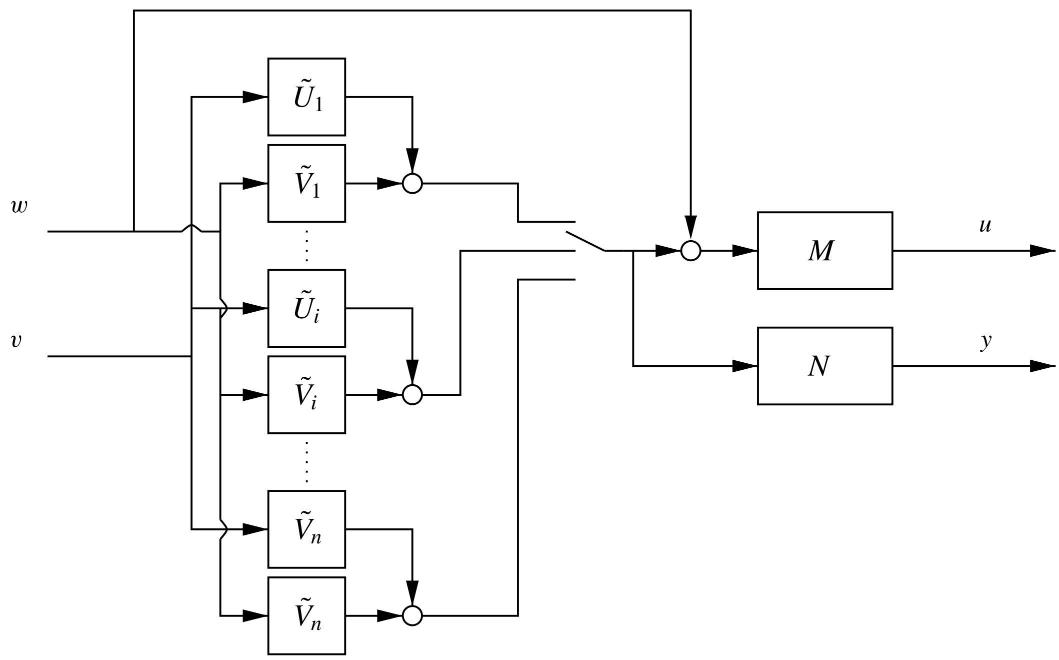


Figure 3.9: Alternative view

and

$$\begin{aligned} y &= P(u+w) \\ &= \tilde{N}\tilde{M}^{-1}((-(I-\tilde{M}\tilde{V}_\sigma)w + \tilde{M}\tilde{U}_\sigma v) + w) \\ &= \tilde{N}\tilde{V}_\sigma w + \tilde{N}\tilde{U}_\sigma v. \end{aligned}$$

We assume that the signals w and v are bounded with bounded two norm, and we know all of the coprime factors are stable. Then the signals $\tilde{V}_\sigma w$, $\tilde{V}_\sigma v$, $\tilde{U}_\sigma w$, and $\tilde{U}_\sigma v$ will all be bounded with bounded two norm. Hence u and y are bounded with bounded two norm, and the switching system is stable for all admissible switching sequences.

We can write these closed-loop relationships in the compact form

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} -(I-\tilde{M}\tilde{V}_\sigma) & \tilde{M}\tilde{U}_\sigma \\ \tilde{N}\tilde{V}_\sigma & \tilde{N}\tilde{U}_\sigma \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}.$$

The arrangement can be illustrated as shown in figure 3.9. The stability of this switching system is guaranteed since $\tilde{M}$, $\tilde{N}$, and each $\tilde{U}_i$ and $\tilde{V}_i$ are stable. Note that the states of the controllers evolve identically irrespective of which controller is active.

The implementation of this arrangement requires all of the controllers (in coprime factor form) to be running even when not active. This corresponds closely with the coprime factor based schemes discussed in section 2.6 for both the antiwindup and bumpless transfer problems. The total state-space required has order equal to the sum of the orders of the coprime factor controllers plus the order of the plant.

In fact, we will see later (section 6.4) that adding a reset structure to such controllers can remove the need to run all of the off-line controllers, and also improve the performance of the system.

In Miyamoto and Vinnicombe [42] and later work by Crawshaw and Vinnicombe [], optimal anti-windup conditioning schemes are derived as a function of a chosen coprime factorization of the plant, and implemented as a controller factorization. The result is that the Youla parameter Q depends only on the plant, and so can be applied to any given LTI controller.

3.5 Concluding remarks

We have discussed a number of important issues relating to the stability of simple switching systems.

We have noted that a necessary and sufficient condition for stability of switching systems under arbitrary switching is the existence of common Lyapunov functions.

Related work by Branicky shows that stability of a switching system over a given family of switching signals may be proved via *multiple Lyapunov functions*. This leads us to a means of computing an upper bound on the minimum dwell time required to ensure stability of switching systems (with fixed realizations).

We have observed that it is possible to ensure stability of a switching controller system by choosing appropriate realizations for the controllers. Both of the methods discussed require non-minimal implementations of the controllers in general. The IMC approach of Hespanha generally requires controllers to be order $2n_G + n_{Ki}$, where n_G and n_{Ki} are the minimal orders of the plant and controllers respectively. The coprime factor approach requires controllers which are order $\max\{n_G, n_{Ki}\}$, however it also requires all controllers to be running when off-line (increasing the effective order by a factor of N, the number of controllers). We will see in chapter 6 however, that we can remove the need to run all controllers by choosing an appropriate initial state when controllers are switched on, while still retaining the stability property (and further improving performance).

Chapter 4

Controller conditioning for switching

In this chapter we consider a performance problem with respect to a single controller switch. The problem is solved via an optimization, which involves finding the controller state at a switching time which most closely corresponds in some sense with the plant inputs and outputs observed prior to the switch. We solve finite and infinite horizon optimizations in order to obtain appropriate controller initializations in the switching context. We also show that the solutions to these optimizations may, for certain weighting structures, be implemented via a Kalman filter observer of the controller using the deterministic filtering results of chapter 2.

4.1 Controller state selection via a noise minimization problem

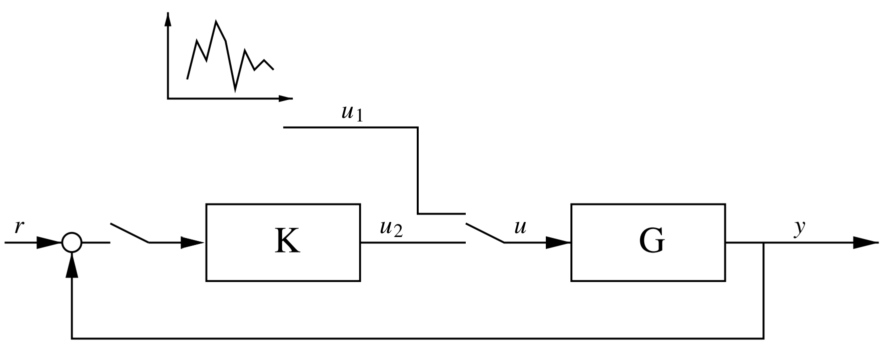


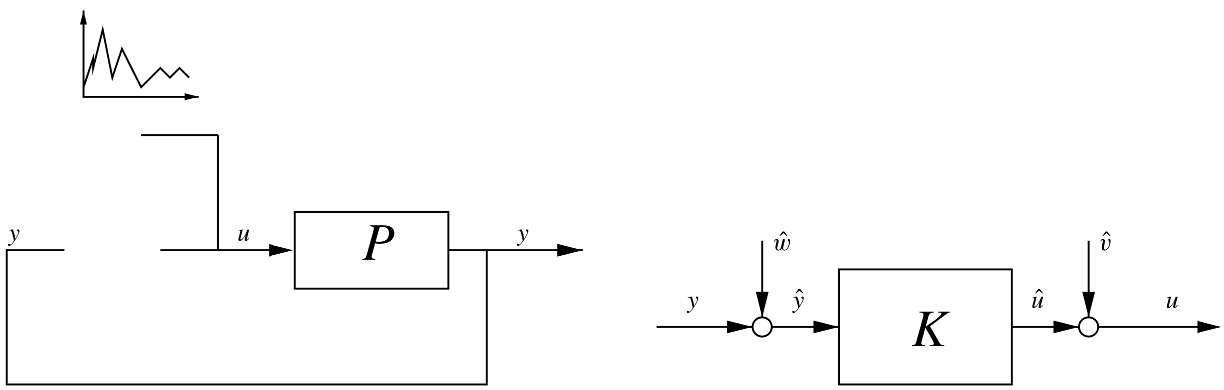
Figure 4.1: Generalized input substitution

Consider the general input substitution for the system illustrated in figure 4.1. We assume that the plant is driven initially by the signal u_1, and switches at some fixed time to the closed loop control signal u_2 generated by the known controller K in feedback with the plant P. The signal u_1 might come from manual control (a pilot), or an alternative automatic controller. Initially, we will consider the regulator problem with $r = 0$. Later we will show how our results fit into a reference tracking framework.

How do we determine a controller state at the switching time which will result in a smooth transfer from the input u_1 to the controller input u_2? One way of interpreting transient signals caused by controller switching, is that they are due in some sense to the mismatch between the controller state at the switch, and the actual plant signals prior to the switch. If the signals lie in the graph of the controller, and the controller is initialized correctly, there would be *no* transient due to the switch at all.

A method of controller state selection therefore, is to measure the plant signals in the time leading up to the switch, and find signals compatible with the off-line controller (in the graph of the controller), which are *close* in some sense to the observed signals. The controller state corresponding to the "nearest" signals is then selected when the controller is switched on.

In other words, recast the setup of figure 4.2(a) into that of figure 4.2(b) (where u and y represent identical signals). We then find the hypothetical signals $\hat{u}$ and $\hat{y}$ prior to the switch such that $\left\| \begin{matrix} \hat{w} \\ \hat{v} \end{matrix} \right\|_2$ is minimized (or some similar weighted norm).



(a) System prior to switch

(b) Alternative view

Figure 4.2:

4.2 Discrete-time explicit solution

Consider now the discrete-time case, and assume that we switch from the external control signal to the known feedback controller K at time $k = n$. We can now write the equations for the hypothetical controller in the loop prior to $k = n$ as follows:

$$\begin{aligned} x_{k+1} &= Ax_k + B\hat{y}_k, \\ \hat{u}_k &= Cx_k + D\hat{y}_k, \end{aligned} \tag{4.1}$$

with $x_k \in \mathbb{R}^s$, $\{u_k, \hat{u}_k, v_k\} \in \mathbb{R}^p$, and $\{y_k, \hat{y}_k, w_k\} \in \mathbb{R}^q$. Iterating these equations, we can write x_{n-k} and y_{n-k}, with $k \geq 0$ in terms of the state at the switching instant x_n.

$$\begin{aligned} x_{n-1} &= A^{-1}x_n - A^{-1}B\hat{y}_{n-1}, \\ x_{n-r} &= A^{-r}x_n - \begin{bmatrix} A^{-r}B & A^{-r+1}B & \ldots & A^{-1}B \end{bmatrix} \begin{bmatrix} \hat{y}_{n-1} \\ \hat{y}_{n-2} \\ \vdots \\ \hat{y}_{n-r} \end{bmatrix}, \\ \hat{u}_{n-r} &= Cx_{n-r} + D\hat{y}_{n-r} \\ &= CA^{-r}x_n - \begin{bmatrix} CA^{-r}B & CA^{-r+1}B & \ldots & CA^{-1}B - D \end{bmatrix} \begin{bmatrix} \hat{y}_{n-1} \\ \hat{y}_{n-2} \\ \vdots \\ \hat{y}_{n-r} \end{bmatrix}. \end{aligned}$$

Let us denote by U_r, Y_r, $\hat{U}_r$ and $\hat{Y}_r$ the r step truncations of signals u, y, $\hat{u}$ and $\hat{y}$ prior to the switching time n expressed in *stacked* form

$$U_r = \begin{bmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_{n-r} \end{bmatrix} \qquad Y_r = \begin{bmatrix} y_{n-1} \\ y_{n-2} \\ \vdots \\ y_{n-r} \end{bmatrix} \qquad \hat{U}_r = \begin{bmatrix} \hat{u}_{n-1} \\ \hat{u}_{n-2} \\ \vdots \\ \hat{u}_{n-r} \end{bmatrix} \qquad \hat{Y}_r = \begin{bmatrix} \hat{y}_{n-1} \\ \hat{y}_{n-2} \\ \vdots \\ \hat{y}_{n-r}, \end{bmatrix}$$

and we can now write the (hypothetical) controller output $\hat{U}_r$ in terms of the state at time $k = n$ and the (hypothetical) controller input $\hat{Y}_r$

$$\hat{U}_r = \Gamma_r x_n - T_r \hat{Y}_r. \tag{4.2}$$

Γ_r is the r'th step truncation of the reverse time impulse response, and T is the r by r

Toeplitz matrix defined below

$$\Gamma_r = \begin{bmatrix} CA^{-1} \\ CA^{-2} \\ \vdots \\ CA^{-r} \end{bmatrix},$$

$$T_r = \begin{bmatrix} CA^{-1}B - D & 0 & \dots & & 0 \\ CA^{-2}B & CA^{-1}B - D & & & 0 \\ \vdots & & \ddots & & \vdots \\ CA^{-r}B & CA^{-r+1}B & \dots & CA^{-2}B & CA^{-1}B - D \end{bmatrix}.$$

Now we may define a weighted quadratic cost function

$$J_r = \left\| W \begin{bmatrix} U_r - \hat{U}_r \\ Y_r - \hat{Y}_r \end{bmatrix} \right\|^2 = \begin{bmatrix} U_r - \hat{U}_r \\ Y_r - \hat{Y}_r \end{bmatrix}^* W^* W \begin{bmatrix} U_r - \hat{U}_r \\ Y_r - \hat{Y}_r \end{bmatrix}, \tag{4.3}$$

and find the values of $\hat{U}_r$, $\hat{Y}_r$, and x_n which achieve the minimum. r defines the 'horizon' of the optimization, and may be selected appropriately.

W is an arbitrary weighting matrix of appropriate dimension, which may be used to scale signals in an appropriate way. It may also be used for example to include a 'forgetting factor', which gives a higher priority to recently observed signals.

We may for instance, wish to optimize with respect to

$$J_r = \alpha_1 \left(u_{n-1} - \hat{u}_{n-1}\right)^2 + \alpha_2 \left(u_{n-2} - \hat{u}_{n-2}\right)^2 + \ldots + \alpha_r \left(u_{n-r} - \hat{u}_{n-r}\right)^2 \\ + \beta_1 \left(y_{n-1} - \hat{y}_{n-1}\right)^2 + \ldots + \beta_r \left(y_{n-r} - \hat{y}_{n-r}\right)^2 \geq 0,$$

where

$$\alpha_1 \geq \alpha_2 \geq \ldots \alpha_r > 0, and \\ \beta_1 \geq \beta_2 \geq \ldots \beta_r > 0.$$

This weight may be represented using the cost function (4.3), with

$$W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} > 0,$$

where

$$W_1 = \operatorname{diag}\left(\sqrt{\alpha_1}, \sqrt{\alpha_2}, \ldots, \sqrt{\alpha_r}\right), \\ W_2 = \operatorname{diag}\left(\sqrt{\beta_1}, \sqrt{\beta_2}, \ldots, \sqrt{\beta_r}\right).$$

Theorem 4.2.1. *For the discrete time dynamical system defined in* (4.1) *and given finite signals U_r and Y_r, then the controller state x_n which solves the minimum noise optimization with respect to the cost function* (4.3) *and corresponding $\hat{Y}_r$ are given by:*

$$\begin{bmatrix} \hat{Y}_r \\ x_n \end{bmatrix} = \left(W \begin{bmatrix} -T_r & \Gamma_r \\ I & 0 \end{bmatrix} \right)^{\dagger} W \begin{bmatrix} U_r \\ Y_r \end{bmatrix}. \tag{4.4}$$

Proof.

$$\begin{aligned} J_r &= \left\| W \begin{bmatrix} U_r - \hat{U}_r \\ Y_r - \hat{Y}_r \end{bmatrix} \right\|^2 \\ &= \left\| W \begin{bmatrix} U_r + T_r Y_r - \Gamma x_n \\ Y_r - \hat{Y}_r \end{bmatrix} \right\|^2 \\ &= \left\| W \begin{bmatrix} U_r \\ Y_r \end{bmatrix} - W \begin{bmatrix} -T_r & \Gamma_r \\ I & 0 \end{bmatrix} \begin{bmatrix} \hat{Y}_r \\ x_n \end{bmatrix} \right\|^2, \end{aligned}$$

so by standard least squares optimization (see appendix A)

$$\underset{\begin{bmatrix} \hat{Y}_r \\ x_n \end{bmatrix}}{\operatorname{argmin}} J = \left(W \begin{bmatrix} -T_r & \Gamma_r \\ I & 0 \end{bmatrix} \right)^{\dagger} W \begin{bmatrix} U_r \\ Y_r \end{bmatrix}, \tag{4.5}$$

and

$$\min_{\begin{bmatrix} \hat{Y}_r \\ x_n \end{bmatrix}} J = \begin{bmatrix} U_r \\ Y_r \end{bmatrix}^* W^* \left(I - W \begin{bmatrix} -T_r & \Gamma \\ I & 0 \end{bmatrix} \left(W \begin{bmatrix} -T_r & \Gamma \\ I & 0 \end{bmatrix} \right)^{\dagger} \right) W \begin{bmatrix} U_r \\ Y_r \end{bmatrix}. \tag{4.6}$$

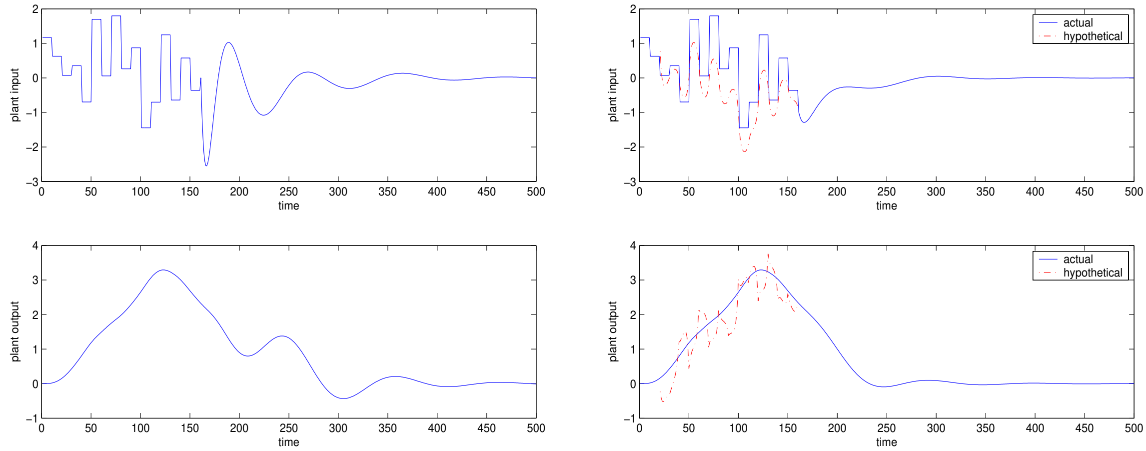
□

4.2.1 Scheme applied to basic example

Consider again the example of section 1.1.1. Using the least squares optimization procedure illustrated above, we can select the controller initial state. The results are shown in figure 4.3.

The solid lines show the actual plant input and output for the simulation. The dotted lines show the hypothetical controller input and output which lie closest to the observed signals in the sense of the unweighted cost function (4.3). We can clearly see an improvement in the transient response of the plant following the switch.

The norm of the signal following the switch is 12.92, compared to 17.46 for the unconditioned case. The improvement is evident to different degrees in examples with different plants, and different prior signals.



(a) Unconditioned response (b) Conditioned response

Figure 4.3: Example

4.3 Kalman filter implementation

Suppose we wish to use a block diagonal weighting matrix

$$W = \operatorname{diag}\begin{bmatrix} N_{n-1} & N_{n-2} & \dots & N_{n-r} & V_{n-1} & \dots & V_{n-r} \end{bmatrix}, \tag{4.7}$$

where each $N_i \in \mathbb{R}^{p\times p}$ and $V_j \in \mathbb{R}^{q\times q}$ is symmetric and positive definite.

In this case, the cost function (4.3) becomes

$$J = \sum_{i=n-r-1}^{n-1} \left(w_i^* W_i^2 w_i + v_i^* V_i^2 v_i \right).$$

By making the identifications $x_0 := x_{n-r-1}$, $R_k = (V_k^2)^{-1}$, $Q_k = (N_k^2)^{-1}$ and $\Psi^{-1} = 0$, we can form the identical cost function of the Deterministic Kalman Filter described in section 2.8. As a result, we may directly apply a Kalman filter in order to obtain the controller state at the switch which minimizes the hypothetical noise signals w and v.

Theorem 4.3.1. *The solution* (4.4) *to the weighted optimization with weighting matrix* (4.7) *is equivalent to the solution $\hat{x}_n$ given by a Kalman filter (initialized at time $n-r-1$ observing the (hypothetical) noisy controller of figure 4.2(b) with "input" U_r and "output" Y_r. The filter has the parameters $R_k = (V_k^2)^{-1}$, $Q_k = (N_k^2)^{-1}$ and $\Psi^{-1} = 0$.*

In this Kalman filter interpretation there is a free choice over the initial state of the filter, which is the state at the beginning of the optimization horizon x_{n-r-1}.

4.4 Implementation issues

We now have two distinct ways of implementing this solution to the bumpless transfer problem. One is via the controller reset defined by theorem 4.2.1, the other via the Kalman filter implementation defined by theorem 4.3.1. We describe here briefly some of the differences in the two approaches.

Clearly, the reset method requires direct access to the controller states, and ability to manipulate these states. If we require an analogue implementation of the controller, then the Kalman filter observer form implementation would be required.

If the reset method is used with a fixed optimization horizon length, the matrix

$$\left(W \begin{bmatrix} -T_r & \Gamma_r \\ I & 0 \end{bmatrix} \right)^{\dagger} W$$

may be precomputed, so that the only on-line computation is a matrix multiplication with matrices of the order of twice the optimization horizon. If the optimization horizon is a significant length however, there may be significant computational advantages to implementing the Kalman filter, since the order of the computations would be much lower.

There is a subtle distinction between the optimization horizons of the two methods. The reset method has a fixed (and moving) horizon, always optimizing over a fixed time prior to the switch. In the (finite horizon) Kalman filter case, the optimization will extend back to the point at which the filter is switched on.

4.5 Time invariant equations

Suppose the controller to be switched on is time-invariant. Then, we may let $A_k = A$, $B_k = B$, $C_k = C$, and $D_k = D$. Assume also that the optimization weighting is time-invariant.

Then, the solutions which are optimal in the infinite horizon may be obtained via the time-invariant Kalman filter equations described in section 2.7. Though the switching performance may be somewhat inferior for an infinite horizon optimization, the implementation of the time invariant Kalman filter is very straightforward. The time invariant Kalman filter is easily implemented in analogue hardware, allowing a wide array of applications for which it may be more difficult to implement a time varying filter. The implementation of the time invariant Kalman filter for conditioning of the controller state, is equivalent to the observer controller implementation of figure 4.4, with constant observer gain $H = AK$. Let also $L = (DQD^* + R)$. Φ is the switching nonlinearity.

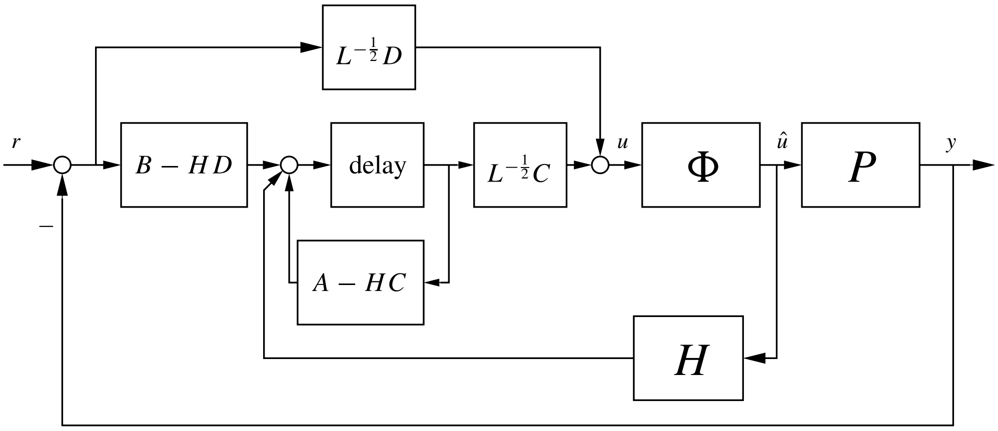


Figure 4.4: Kalman filter observer controller

Figure 4.5 shows the results of the time invariant Kalman filter implemented for example 1. The norm of the input/output signals after the switch is 12.97, compared to 12.87 for the optimization scheme with horizon 90 (and time varying Kalman filter), and compared to 17.46 for the unconditioned case.

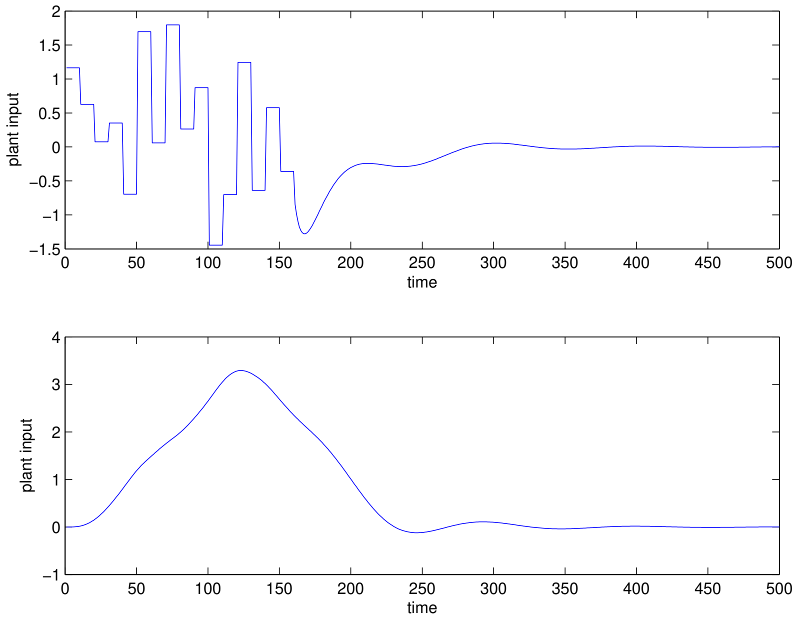


Figure 4.5: Time invariant Kalman filter scheme

Convergence of the optimization scheme to the time invariant Kalman filter state selection is illustrated in figure 4.6. The plot shows the norm of the difference between the controller state x_n selected by the Kalman filter, with the state selected by the optimization scheme (time varying filter) for different horizon lengths.

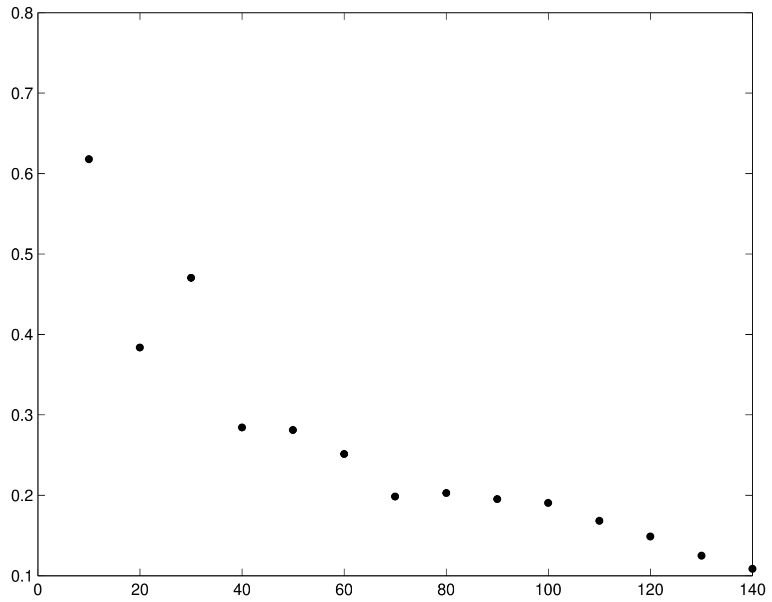


Figure 4.6: Convergence of optimization scheme to time invariant Kalman filter

4.6 Coprime factor representation

The time-invariant Kalman filter observer form also may be represented as a left coprime factorization of the controller $K = V^{-1}U$, where

$$[U \quad V] = \left[\begin{array}{c|cc} A - HC & B - HD & -H \\ \hline L^{-\frac{1}{2}}C & L^{-\frac{1}{2}}D & L^{-\frac{1}{2}} \end{array} \right]$$

The switch between controllers occurs in the $I - V$ feedback loop as shown in figure 4.7. This method of switching corresponds closely with the coprime factorization based anti-windup methods (see section 2.6), with the nonlinearity (saturation or switch) placed in the $I - V$ feedback loop. This is an intuitively sensible arrangement, since the stability of the coprime factors means that they will be stable if left to run open loop.

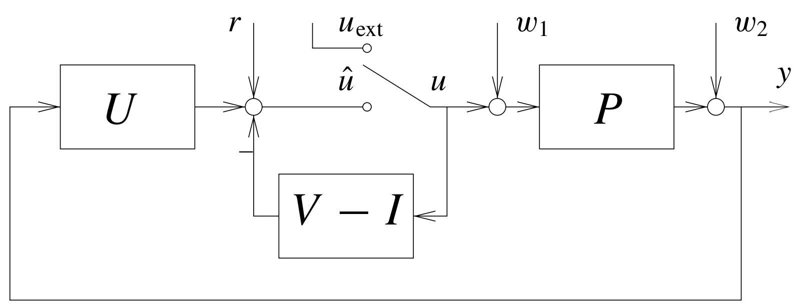


Figure 4.7: Coprime factorization form

4.6.1 Special case weighting

Consider a time varying Kalman filter applied to a discrete-time LTI controller

$$K = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

in the infinite horizon, with $D = 0$.

The basic recursion equations are as follows

$$P_k^{-1} = \left(AP_{k-1}A^* + Q_{k-1}\right)^{-1} + C^* \left(R_k\right)^{-1} C, \tag{4.8}$$

$$K_k = P_k C^* \left(R_k\right)^{-1}. \tag{4.9}$$

Suppose we choose a particular set of weighting matrices

$$Q_{k-1} = \frac{\hat{Q}}{f_k}, \qquad R_k = \frac{\hat{R}}{f_k},$$

with $\hat{Q}$, and $\hat{R}$ some constant matrices, and f_k a scalar function of k. Such a weighting function may be described in the block diagonal form of equation (4.7).

Then we can write the state covariance matrix P_k as follows

$$\begin{aligned} P_k^{-1} &= \left(AP_{k-1}A^* + \frac{\hat{Q}}{f_k}\right)^{-1} + C^* \left(\frac{\hat{R}}{f_k}\right)^{-1} C \\ &= f_k \left(Af_k P_{k-1} A^* + \hat{Q}\right)^{-1} + f_k C^* \left(\hat{R}\right)^{-1} C, \\ (f_k P_k)^{-1} &= \left(A(f_k P_{k-1})A^* + \hat{Q}\right)^{-1} + C^* \hat{R}^{-1} C. \end{aligned}$$

Now suppose we choose the f_k such that

$$\frac{f_k}{f_{k+1}} = \lambda,$$

with λ a constant. Now define

$$\hat{P}_k = f_k P_k.$$

Then we have

$$\begin{aligned} \hat{P}_k^{-1} &= \left(A(\frac{f_k}{f_{k-1}}\hat{P}_{k-1})A^* + \hat{Q}\right)^{-1} + C^* \hat{R}^{-1} C \\ &= \left(A(\lambda \hat{P}_{k-1})A^* + \hat{Q}\right)^{-1} + C^* \hat{R}^{-1} C, \end{aligned}$$

and

$$\begin{aligned} K_k &= P_k C^* (R_k)^{-1} \\ &= \frac{\hat{P}_k}{f_k} C^* (\frac{R}{f_k})^{-1} \\ &= \hat{P}_k C^* \hat{R}^{-1}. \end{aligned}$$

So we have an equivalent Kalman filtering problem with recursion equations

$$\hat{P}_k^{-1} = \left(\hat{A}\hat{P}_{k-1}\hat{A}^* + \hat{Q} \right)^{-1} + C^* \hat{R}^{-1} C, \tag{4.10}$$

$$K_k = \hat{P}_k C^* \hat{R}^{-1}. \tag{4.11}$$

where $\hat{A} = \sqrt{\lambda} A$.

Now since $\hat{A}$, $\hat{Q}$ and $\hat{R}$ are constant then provided

$$\left[\begin{array}{c|c} \hat{A} & B \\ \hline C & 0 \end{array} \right]$$

is detectable, the Kalman filter equations (4.10) and (4.11) will converge to the time invariant equations

$$\hat{P}^{-1} = \left(\hat{A}\hat{P}\hat{A}^* + \hat{Q} \right)^{-1} + C^* \hat{R}^{-1} C, \tag{4.12}$$

$$K = \hat{P} C^* \hat{R}^{-1}. \tag{4.13}$$

Thus, we can in fact implement an exponentially weighted infinite horizon optimization problem with a time invariant Kalman filter with modified A matrix $\hat{A} = \sqrt{\lambda} A$, and modified noise covariances $\hat{Q}$ and $\hat{R}$.

4.6.2 The reference problem

Let us now consider the general reference tracking problem ($r \neq 0$).

The methods of this chapter are based on the idea of keeping the state of the off-line controllers compatible with signals which are close to the actual plant input and output. From this perspective, it does not actually matter in determining the appropriate controller state whether the current controller is operating with reference or not. The difficulty arises when we wish to apply a reference to a new controller after a switch- we may observe potentially large transient signals simply due to the reference itself.

The solution is to gently apply a new reference by filtering smoothly from an appropriately chosen previous reference. We do not use the 'true' reference, since, even if it were available

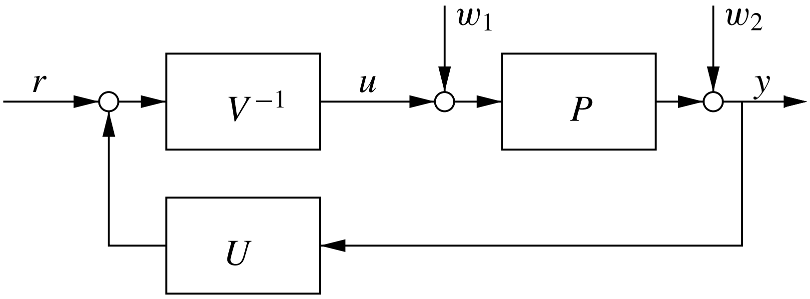


Figure 4.8: reference framework

it may not be an appropriate choice. Instead, we can calculate a hypothetical reference r which is compatible with the new controller and the prior plant input and output.

Suppose we rewrite the system prior to the switch in the form shown in figure 4.8, instead of 4.2(b). We have introduced the hypothetical reference signal in between the coprime factors of the controller. Then we may define the following problem:

Given signals u and y, find a signal r compatible with such signals in terms of figure 4.8. From the figure, we can easily see that

$$r = Vu - Uy$$

satisfies this condition.

Now comparing this expression with figure 4.7 prior to the switch, we can see that $u - \hat{u} = Vu - Uy$, which is precisely the hypothetical r required. That is, if a reference input $\tilde{r}$ is to be applied after the switch, then the closed loop behaviour of the switched system with controller implemented as shown in figure 4.7 will be identical to what would have been observed if the new controller was in operation for *all* time with the reference

$$r = \begin{cases} Vu - Uy & \text{prior to switch} \\ \tilde{r} & \text{after switch} \end{cases}. \tag{4.14}$$

Note that if we implement a new reference arbitrarily, we will get a discontinuity of signals due to the effective discontinuity of the reference. We can correct for this discontinuity by using a reference filter to provide a smooth transition from the hypothetical reference r prior to the switch, to the new desired reference $\tilde{r}$.

This reference filtering approach was applied successfully to achieve smooth transitions for engine control problems by R. G. Ford as reported in [15], and [16].

4.7 Concluding remarks

The approach to the switching problem in this chapter is to find controller states which correspond to signals which are close to those observed.

We find the optimal choice of controller state via a weighted least squares optimization. We find that for a block diagonal weighting structure, the solution can also be obtained via a Kalman filter observer of the controller. This leads to a more compact and efficient solution. In the time-invariant and infinite horizon (with constant or exponential weighting) case we obtain a time-invariant Kalman filter, which may be implemented as a controller in coprime factor form.

In the tracking problem, we can apply a reference filter to reduce transients due to the introduction of new references at switching times.

Note that this method does not require knowledge of the plant model or of plant states, since the method is based on observing the controller model. This makes it particularly suitable for applications where the plant is highly uncertain, or where accurate plant state estimates are difficult to obtain.

Chapter 5

Controller initialization: optimal transfer

If the future behaviour of a system is predictable in a meaningful way, it makes sense for us to consider choosing the controller state at a switch such that the transient behaviour of the plant input and output are minimized in some sense. We may define a generalized (and possibly weighted) output, and minimize the energy of this output (or it's transients) after the switch with respect to the initial state of the controller.

We begin by considering a single switch between two linear controllers. The plant is linear with known dynamics, and we consider initially a zero reference (regulator) problem.

After a switch to a controller K, we have the situation illustrated in figure 5.1.

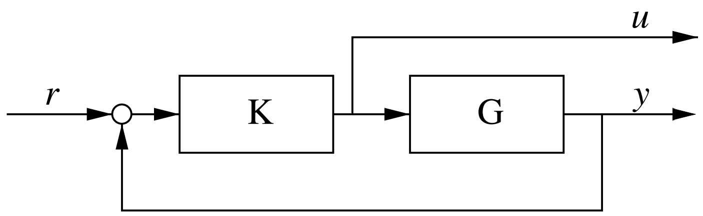


Figure 5.1: System after switch

We will solve this as an initial value problem on the state of K, minimizing an energy function of the 'outputs' u and y.

5.1 Discrete time

Let $k \in \mathbb{Z}^+$ be the discrete time variable. Then the plant input and output are $u(k) \in \mathbb{R}^p$ and $y(k) \in \mathbb{R}^q$.

Suppose G and K have the following discrete-time state-space representations:

$$G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right] \qquad K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]. \tag{5.1}$$

Let G have state $x_G(t) \in \mathbb{R}^{n_G}$ and K state $x_K(t) \in \mathbb{R}^{n_K}$. Let $n = n_G + n_K$.

To ensure the absence of an algebraic loop, we shall assume that

$$D_G D_K = D_K D_G = 0. \tag{5.2}$$

Define the following quadratic cost function

$$V(k) = \sum_{i=k}^{\infty} \big(u(i)^* u(i) + y(i)^* y(i)\big) \tag{5.3}$$

$V(k)$ represents the 'energy' in the signals u and y from the time k onwards. If k is a switching time, then we wish to find the controller state $x_K(k)$ such that this function is minimized.

Let us define $x(k) = \begin{bmatrix} x_G(k) \\ x_K(k) \end{bmatrix}$, and $\tilde{y}(k) = \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$. Then by the results of section A.3.2 in appendix A, we can write the closed loop equations

$$x(k+1) = Ax(k) + Br(k), \tag{5.4}$$

$$\tilde{y}(k) = Cx(k) + Dr(k). \tag{5.5}$$

where

$$A = \begin{bmatrix} A_G + B_G D_K C_G & B_G C_K \\ -B_K C_G & A_K + B_K D_G C_K \end{bmatrix}, \tag{5.6}$$

$$B = \begin{bmatrix} B_G D_K \\ B_K \end{bmatrix}, \tag{5.7}$$

$$C = \begin{bmatrix} D_K C_G & C_K \\ C_G & -D_G C_K \end{bmatrix}, \tag{5.8}$$

$$D = \begin{bmatrix} D_K \\ 0 \end{bmatrix}. \tag{5.9}$$

Now define the partitions $A = [A_1 \quad A_2]$, and $C = [C_1 \quad C_2]$, where

$$A_1 = \begin{bmatrix} A_G + B_G D_K C_G \\ -B_K C_G \end{bmatrix}, \tag{5.10}$$

$$A_2 = \begin{bmatrix} B_G C_K \\ A_K + B_K D_G C_K \end{bmatrix}, \tag{5.11}$$

$$C_1 = \begin{bmatrix} D_K C_G \\ C_G \end{bmatrix}, \tag{5.12}$$

$$C_2 = \begin{bmatrix} C_K \\ -D_G C_K \end{bmatrix}. \tag{5.13}$$

Iterating the closed loop equations, we can write $\tilde{y}(k+i)$ in terms of $x(k)$ as follows:

$$\tilde{y}(k+i) = CA^i x(k) + Dr(k+i) + CBr(k+i-1) + \ldots + CA^{i-1}Br(k) \tag{5.14}$$

so we can write:

$$\tilde{Y} = \Gamma x(k) + TR = \Gamma_1 x_G(k) + \Gamma_2 x_K(k) + TR, \tag{5.15}$$

where

$$\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ CA_1 & CA_2 \\ CAA_1 & CAA_2 \\ \vdots & \\ CA^{N-1}A_1 & CA^{N-1}A_2 \end{bmatrix}, \quad \tilde{Y} = \begin{bmatrix} \tilde{y}(k) \\ \tilde{y}(k+1) \\ \vdots \\ \tilde{y}(k+N) \end{bmatrix},$$

$$R = \begin{bmatrix} r(k) \\ r(k+1) \\ \vdots \\ r(k+N) \end{bmatrix}, \quad T = \begin{bmatrix} D & 0 & \ldots & 0 \\ CB & D & & \\ \vdots & & \ddots & \vdots \\ CA^{N-1}B & \ldots & CB & D \end{bmatrix}.$$

5.1.1 Finite horizon solution

Now consider specifically the regulator problem with $r = 0$. Also consider initially the weighted 'finite horizon' cost function

$$V(k) = \sum_{i=k}^{k+N} \left(v_i u(i)^* u(i) + w_i y(i)^* y(i) \right). \tag{5.16}$$

We can also write this function as

$$V(k) = \left\| W\tilde{Y} \right\|^2 \tag{5.17}$$

where $W = \mathrm{diag}(\sqrt{v_k}, \sqrt{w_k}, \sqrt{v_{k+1}}, \sqrt{w_{k+1}}, \ldots, \sqrt{v_{k+N}}, \sqrt{w_{k+N}})$.

Minimizing this $V(k)$ with respect to the controller state $x_K(k)$ is now a straightforward least squares optimization.

Theorem 5.1.1. *Consider the system illustrated in figure 5.1, and represented by equations* (5.1)*. Assume $r(i) = 0 \quad \forall i$. The optimal controller state $x_K(k)$ with respect to the cost function* (5.17) *is given by:*

$$x_K(k) = -\left(W\Gamma_2\right)^{\dagger} W\Gamma_1 x_G(k). \tag{5.18}$$

Proof. from equation (5.15) we obtain the following:

$$W\tilde{Y} = W\Gamma_1 x_G(k) + W\Gamma_2 x_K(k),$$

so

$$V(k) = \|W\Gamma_1 x_G(k) + W\Gamma_2 x_K(k)\|^2,$$

and by a straightforward least squares optimization (see appendix A), we obtain

$$\underset{x_K(k)}{\operatorname{argmin}}\, J = -\left(W\Gamma_2\right)^{\dagger} W\Gamma_1 x_G(k).$$

with

$$\min_{x_K(k)} J = \left\|\left(I - W\Gamma_1\left(W\Gamma_2\right)^{\dagger}\right) W\Gamma_1 x_G(k)\right\|^2.$$

(† denotes matrix left pseudo-inverse) □

5.1.2 Infinite horizon solution

Now consider the unweighted optimization in the infinite horizon as $N \to \infty$. That is, we wish to minimize

$$V(k) = \sum_{i=k}^{\infty} \left(u(i)^* u(i) + y(i)^* y(i)\right) \tag{5.19}$$

or

$$V(k) = \left\|\tilde{Y}\right\|^2 \tag{5.20}$$

where $\tilde{Y} = [\tilde{y}(k), \tilde{y}(k+1), \ldots]^*$

Theorem 5.1.2. *Consider the system illustrated in figure 5.1, and represented by equations* (5.1)*. Assume that the controller K stabilizes the plant G, and $r(i) = 0 \quad \forall i$. Then the optimal controller state $x_K(k)$ with respect to the cost function* (5.19) *is given by:*

$$x_K(k) = \left(C_2^* C_2 + A_2^* P A_2\right)^{-1} . \left(C_2^* C_1 + A_2^* P A_1\right) x_G(k), \tag{5.21}$$

where P is the solution to the discrete time Lyapunov equation

$$A^* P A - P = -C^* C \tag{5.22}$$

Proof. Essentially, we must show that the expression for the optimal $x_K(k)$ in the finite horizon case:

$$x_K(k) = -(\Gamma_2)^{\dagger}\,\Gamma_1 x_G(k), \tag{5.23}$$

converges to the equation (5.21) in the limit as $N \to \infty$.

$$\lim_{N\to\infty} \Gamma_2^{\dagger}\Gamma_1 x_G(k) = \lim_{N\to\infty} \left(\Gamma_2^*\Gamma_2\right)^{-1}\Gamma_2^*\Gamma_1 x_G(k)$$

$$= \left(C_2^*C_2 + A_2^*\left(\sum_{n=0}^{\infty}\left(A^*\right)^n C^*CA^n\right)A_2\right)^{-1}\cdot\left(C_2^*C_1 + A_2^*\left(\sum_{n=0}^{\infty}\left(A^*\right)^n C^*CA^n\right)A_1\right).$$

Let P be defined as follows:

$$P := \sum_{n=0}^{\infty}\left(\left(A^*\right)^n C^*CA^n\right). \tag{5.24}$$

Pre-multiplying by A^*, and post-multiplying by A we obtain

$$A^*PA = \sum_{n=1}^{\infty}\left(\left(A^*\right)^n C^*CA^n\right) = P - C^*C, \tag{5.25}$$

and P exists precisely when the discrete time Lyapunov equation 5.22 has a solution. That is, when A is stable (see for example [62]) or equivalently, when K stabilizes G. □

5.1.3 Lyapunov function interpretation

We can interpret the solution of the zero reference infinite horizon problem as the minimization (with respect to controller state) of a Lyapunov function for the closed loop corresponding to the switched on controller.

From equation 5.15, we can rewrite the cost function 5.20 as

$$V(k) = x^*(k)\Gamma^*\Gamma x(k) \tag{5.26}$$

$$= x^*(k)\left(\sum_{n=0}^{\infty}\left(A^*\right)^n C^*CA^n\right)x(k) \tag{5.27}$$

$$= x^*(k)Px(k) \tag{5.28}$$

where P is the solution to the Lyapunov equation 5.22. Thus V is in fact a positive definite function of the state $x(k)$, and is strictly decreasing since

$$V\left(x(k+1)\right) - V\left(x(k)\right) = x^*(k)\left(A^*PA - P\right)x(k),$$

and

$$A^*PA - P < 0.$$

Our transfer scheme now has the sensible interpretation of choosing the controller state at the switch $x_K(k)$ such that the 'energy' of the state is minimized.

5.1.4 Weighted solution

Consider now the infinite horizon problem with the specific weighted cost function

$$V(k) = \sum_{i=k}^{\infty} \left((a\phi^i u(i))^* (a\phi^i u(i)) + (b\phi^i y(i))^* (b\phi^i y(i)) \right).$$

where a, b and ϕ are positive constants, with $\phi \leq 1$. a and b can be used to normalize the signals u and y against each other, and ϕ can be used in order to weight the optimization more heavily in the moments immediately after the switch.

We can rewrite this cost function as

$$V(k) = \left\| W\tilde{Y} \right\|^2, \text{ where} \tag{5.29}$$

$$W = \text{diag}(\lambda, \phi\lambda, \phi^2\lambda, \phi^3\lambda, \ldots), \text{ with} \tag{5.30}$$

$$\lambda = \begin{bmatrix} aI & 0 \\ 0 & bI \end{bmatrix}, \text{ where } a, b \in \mathbb{R}^+. \tag{5.31}$$

Theorem 5.1.3. *Consider the system illustrated in figure 5.1, and represented by equations* (5.1). *Assume that the controller K stabilizes the plant G, and $r(i) = 0 \ \forall i$. Then the optimal controller state $x_K(k)$ with respect to the weighted cost function* (5.29) *(with W given by* (5.30) *and* (5.31)*) is given by:*

$$x_K(k) = \left(C_2^* C_2 + A_2^* P A_2\right)^{-1} . \left(C_2^* C_1 + A_2^* P A_1\right) x_G(k), \tag{5.32}$$

where P is the solution to the discrete time Lyapunov equation

$$(\phi A)^* P (\phi A) - P = -C^* \lambda^2 C \tag{5.33}$$

Proof. The proof is essentially the same as for theorem 5.1.2.

We must show that the expression for the (weighted) optimal $x_K(k)$ in the finite horizon case:

$$x_K(k) = -\left(W\Gamma_2\right)^{\dagger} W\Gamma_1 x_G(k), \tag{5.34}$$

converges to the equation (5.32) in the limit as $N \to \infty$.

$$\lim_{N\to\infty} \left(W\Gamma_2\right)^{\dagger} W\Gamma_1 x_G(k) = \lim_{N\to\infty} \left(\Gamma_2^* W^2 \Gamma_2\right)^{-1} \Gamma_2^* W^2 \Gamma_1 x_G(k)$$

$$= \left(C_2^* \lambda^2 C_2 + A_2^* \left(\sum_{i=0}^{\infty} \left(A^*\right)^i C^* \lambda^2 \phi^{2i} C A^i \right) A_2 \right)^{-1} .$$

$$\left(C_2^* \lambda^2 C_1 + A_2^* \left(\sum_{i=0}^{\infty} \left(A^*\right)^i C^* \lambda^2 \phi^{2i} C A^i \right) A_1 \right).$$

Let P be defined as follows:

$$P := \sum_{i=0}^{\infty} \left(\left(\phi A^*\right)^i C^* \lambda^2 C \left(\phi A\right)^i \right). \tag{5.35}$$

Pre-multiplying by ϕA^*, and post-multiplying by ϕA we obtain

$$\phi A^* P \phi A = \sum_{i=1}^{\infty} \left(\left(\phi A^*\right)^i C^* \lambda^2 C \left(\phi A\right)^i \right) = P - C^* \lambda^2 C, \tag{5.36}$$

and P exists precisely when the discrete time Lyapunov equation 5.33 has a solution. That is, when ϕA is stable- which is certainly true when K stabilizes G and $\phi \leq 1$

□

The weighted solution is of importance when we consider switching systems where the behaviour of the switching supervisor is time-independent (at least as far as the controller is concerned). That is, the a-priori probability of a switch occurring between times k_0 and k_1 given that it occurs after k_0, depends only on the size of the interval $\Delta k = k_1 - k_0$.

This property is exhibited by the geometric probability distribution

$$P(k) = (1 - \phi)\phi^k,$$

where ϕ is the *failure probability* at any given step. Then, the probability

$$\begin{aligned} P(k_0 \leq k < k_1 | k_0 \leq k) &= \frac{\phi^{k_0} - \phi^{k_1}}{\phi^{k_0}} \\ &= 1 - \phi^{k_1 - k_0} \end{aligned}$$

as required.

We can use such a-priori information about the switching behaviour, by optimizing over the outputs of the current controller weighted by the probability that the controller has not switched already. In this case, the appropriate weighting function is $W(k) = \phi^k$ (the probability that the switch occurs at time step k or later), which fits precisely with the weighted problem solved above.

5.2 Continuous time

Suppose G and K have the following continuous-time state-space representations:

$$G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array} \right] \qquad K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right]. \tag{5.37}$$

Let G have state $x_G(t) \in \mathbb{R}^{n_G}$ and K state $x_K(t) \in \mathbb{R}^{n_K}$. Let $n = n_G + n_K$. Also assume again that

$$D_G D_K = D_K D_G = 0. \tag{5.38}$$

Given signals $u(t) \in \mathbb{R}^p$ and $y(t) \in \mathbb{R}^q$, let us define a quadratic cost function as follows

$$V(t) = \int_t^\infty \big(u(\tau)^* u(\tau) + y(\tau)^* y(\tau)\big)\, d\tau. \tag{5.39}$$

$V(t)$ represents the 'energy' in the signals u and y from the time t onwards. If t is a switching time, then we wish to find the controller state $x_K(t)$ such that this function is minimized.

Define $x(t) = \begin{bmatrix} x_G(t) \\ x_K(t) \end{bmatrix}$, and $\tilde{y}(t) = \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$. Applying again the results of section A.3.2 in appendix A, the closed loop state space equations can be written

$$A = \begin{bmatrix} A_G + B_G D_K C_G & B_G C_K \\ -B_K C_G & A_K + B_K D_G C_K \end{bmatrix}, \tag{5.40}$$

$$B = \begin{bmatrix} B_G D_K \\ B_K \end{bmatrix}, \tag{5.41}$$

$$C = \begin{bmatrix} D_K C_G & C_K \\ C_G & -D_G C_K \end{bmatrix}, \tag{5.42}$$

$$D = \begin{bmatrix} D_K \\ 0 \end{bmatrix}. \tag{5.43}$$

Now define the partitions $A = [A_1 \quad A_2]$, and $C = [C_1 \quad C_2]$, where

$$A_1 = \begin{bmatrix} A_G + B_G D_K C_G \\ -B_K C_G \end{bmatrix}, \tag{5.44}$$

$$A_2 = \begin{bmatrix} B_G C_K \\ A_K + B_K D_G C_K \end{bmatrix}, \tag{5.45}$$

$$C_1 = \begin{bmatrix} D_K C_G \\ C_G \end{bmatrix}, \tag{5.46}$$

$$C_2 = \begin{bmatrix} C_K \\ -D_G C_K \end{bmatrix}. \tag{5.47}$$

Theorem 5.2.1. *Consider the system illustrated in figure 5.1, and represented by equations* (5.37)*. Assume that the controller K stabilizes the plant G, and $r(i) = 0 \quad \forall i$. Then the optimal controller state $x_K(t)$ with respect to the cost function* (5.39) *is given by:*

$$x_K(t) = -P(2,2)^{-1} P(2,1) x_G(t), \tag{5.48}$$

where

$$P = \begin{bmatrix} P(1,1) & P(1,2) \\ P(2,1) & P(2,2) \end{bmatrix} \tag{5.49}$$

is the solution to the continuous-time Lyapunov equation

$$A^*P + PA = -C^*C \tag{5.50}$$

Proof. The cost function (5.39) can be written in terms of $\tilde{y}(t)$

$$V(t) = \int_t^\infty \tilde{y}(\tau)^*\tilde{y}(\tau)d\tau,$$

and when $r = 0$ we may write $\tilde{y}(t+\hat{t})$ in terms of the state $x(t)$ at time t.

$$\begin{aligned} \tilde{y}(t+\hat{t}) &= Ce^{A(t+\hat{t})}x(0) \\ &= Ce^{A\hat{t}}x(t). \end{aligned}$$

So

$$\begin{aligned} V(t) &= \int_t^\infty \tilde{y}(\tau)^*\tilde{y}(\tau)d\tau \\ &= \lim_{\hat{t}\to\infty} \int_t^{t+\hat{t}} \tilde{y}(\tau)^*\tilde{y}(\tau)d\tau \\ &= \lim_{\hat{t}\to\infty} x^*(t)\left(\int_t^{t+\hat{t}} e^{A^*(\tau-t)}C^*Ce^{A(\tau-t)}d\tau\right)x(t) \\ &= x^*(t)\left(\int_0^\infty e^{A^*\tau}C^*Ce^{A\tau}d\tau\right)x(t). \end{aligned}$$

It is well known that

$$P = \int_0^\infty e^{A^*\tau}C^*Ce^{A\tau}d\tau$$

is the solution to the continuous-time Lyapunov equation (5.50) (see appendix A). Partitioning P according to (5.49), we can now write the cost function

$$V(t) = \begin{bmatrix} x_G(t) \\ x_K(t) \end{bmatrix}^* \begin{bmatrix} P(1,1) & P(1,2) \\ P(2,1) & P(2,2) \end{bmatrix} \begin{bmatrix} x_G(t) \\ x_K(t) \end{bmatrix},$$

and the minimum is $P(1,1) - P(1,2)P(2,2)^{-1}P(2,1)$, achieved when $x_K(t) = -P(2,2)^{-1}P(2,1)x_G(t)$ (see appendix A). □

Once again, it is clear that the solution is equivalent to the minimization of a quadratic Lyapunov function (for the closed loop corresponding to the controller switched on) with respect to the controller state x_K.

We may again apply a weighting function to $\tilde{y}$ in order to normalize u against y, and to place a higher emphasis on the signals close to the switching time.

Suppose

$$V(t) = \int_t^{\infty} \left(\left(ae^{-\phi\tau}u(\tau)\right)^* \left(ae^{-\phi\tau}u(\tau)\right) + \left(be^{-\phi\tau}y(\tau)\right)^* \left(be^{-\phi\tau}y(\tau)\right) \right) d\tau \tag{5.51}$$

$$= \int_t^{\infty} (W(\tau)\tilde{y}(\tau))^* (W(\tau)\tilde{y}(\tau))\, d\tau \tag{5.52}$$

where

$$W(t) = e^{-\phi t}\lambda, \qquad \text{and} \quad \lambda = \begin{bmatrix} aI & 0 \\ 0 & bI \end{bmatrix}.$$

Theorem 5.2.2. *Consider the system illustrated in figure 5.1, and represented by equations* (5.37)*. Assume that the controller K stabilizes the plant G, and $r(i) = 0 \quad \forall i$. Then the optimal controller state $x_K(t)$ with respect to the cost function* (5.52) *is given by:*

$$x_K(t) = -P(2,2)^{-1}P(2,1)x_G(t), \tag{5.53}$$

where

$$P = \begin{bmatrix} P(1,1) & P(1,2) \\ P(2,1) & P(2,2) \end{bmatrix} \tag{5.54}$$

is the solution to the continuous-time Lyapunov equation

$$(A - \phi I)^* P + P(A - \phi I) = -C^*\lambda^2 C \tag{5.55}$$

Proof. When $r = 0$, we may write $\tilde{y}(t + \hat{t})$ as before

$$\begin{aligned} \tilde{y}(t + \hat{t}) &= Ce^{A(t+\hat{t})}x(0) \\ &= Ce^{A\hat{t}}x(t). \end{aligned}$$

So

$$\begin{aligned} V(t) &= \int_t^{\infty} (W(\tau)\tilde{y}(\tau))^* (W(\tau)\tilde{y}(\tau))\, d\tau \\ &= \lim_{\hat{t}\to\infty} \int_t^{t+\hat{t}} \left(e^{-\phi\tau}\lambda\tilde{y}(\tau)\right)^* \left(e^{-\phi\tau}\lambda\tilde{y}(\tau)\right) d\tau \\ &= \lim_{\hat{t}\to\infty} x^*(t) \left(\int_t^{t+\hat{t}} e^{(A-\phi I)^*(\tau-t)}C^*\lambda^2 Ce^{(A-\phi I)(\tau-t)} d\tau \right) x(t) \\ &= x^*(t) \left(\int_0^{\infty} e^{(A-\phi I)^*\tau}C^*\lambda^2 Ce^{(A-\phi I)\tau} d\tau \right) x(t), \end{aligned}$$

and since $A - \phi I$ will always be stable when A is stable, it is established that

$$P = \int_0^\infty e^{(A-\phi I)^* \tau} C^* \lambda^2 C e^{(A-\phi I)\tau} d\tau$$

is the solution to the continuous-time Lyapunov equation (5.55) (see appendix A). Partitioning P according to (5.54), we proceed as for the previous proof to show that the minimum of the cost function is $P(1,1) - P(1,2)P(2,2)^{-1}P(2,1)$, achieved when $x_K(t) = -P(2,2)^{-1}P(1,2)x_G(t)$. □

In the continuous-time case, the weighting function can again be used to represent the probability that a switch has not occurred prior to time t. The exponential probability distribution, with probability density function $f(t) = \phi e^{-\phi t}$ has the required property, that the probability $P(t_0 \le t < t_1 | t_0 \le t)$ depends only on $\Delta t = t_1 - t_0$.

$$\begin{aligned} P(t_0 \le t < t_1 | t_0 \le t) &= \frac{\int_{t_0}^{t_1} \left(\phi e^{-\phi t}\right) dt}{\int_{t_0}^{\infty} \left(\phi e^{-\phi t}\right) dt} \\ &= \frac{e^{-\phi t_0} - e^{-\phi t_1}}{e^{-\phi t_0}} \\ &= 1 - e^{-\phi(t_1 - t_0)} \end{aligned}$$

The appropriate weighting function then, is

$$W(\tau) = P(\tau \le t) = e^{-\phi \tau}$$

which again fits into the weighted form considered above.

5.3 Plant state estimation

In practice it will often not be possible to measure the plant state directly. In such cases it will be necessary to estimate the plant state via an observer and to implement the controller reset appropriate to the estimated state. In such cases, we effectively minimize the weighted cost function

$$V = \left\| E\left(W\tilde{Y}\right)\right\|^2, \tag{5.56}$$

where $E(.)$ is the statistical expectation.

In the finite horizon case ($r = 0$) we use equation (5.15), and the cost function becomes

$$\begin{aligned} V &= \left\| E\left(W\tilde{Y}\right)\right\|^2 \\ &= \left\| E\left(W\left(\Gamma_1 x_G(k) + \Gamma_2 x_K(k)\right)\right)\right\| \\ &= \left\| W\Gamma_1 E(x_G(k)) + W\Gamma_2 x_K(k)\right\|, \end{aligned}$$

and hence the form is similar to the previous results, with the actual plant state x_G replaced by the expectation of the plant state $E(x_G)$. Hence the solutions can be obtained by replacing x_G by an optimal estimate $\hat{x}_G$, computed under appropriate statistical assumptions.

5.4 Problems with non-zero reference

The method which we have discussed so far acts by minimizing the energy in the transient component of the generalized output, which results from the 'initial state' of the system at the switch. The method applies equally well when we consider general reference tracking problems, since the 'initial state' component of the output is independent of the component due to the reference. That is, we can write the generalized output at some time $t+T$ in terms of the state at time t, and the reference r as

$$\tilde{y}(t+T) = Ce^{AT}x(t) + Dr(t+T) + \int_0^T Ce^{A(T-\tau)}Br(\tau)d\tau,$$

or equivalently for discrete-time

$$\tilde{y}(k+n) = CA^n x(k) + Dr(k+n) + \sum_{i=1}^{n} CA^{i-1}Br(k+n-i).$$

The $Ce^{AT}x(t)$ (resp $CA^n x(k)$) component above is the initial state transient component of the signal $\tilde{y}$.

Therefore the numerical problem solved when minimizing the initial state transient for a reference problem is the same as that solved for the regulator problem.

If future references are known a-priori, they may be accounted for explicitly. That is, we can minimize the difference between the signal $\tilde{y}$, and the steady state value $\tilde{y}_{\text{s.s}}$. We use the cost function

$$J = \left\| W\left(\tilde{Y} - \tilde{Y}_{\text{s.s}}\right)\right\|, \tag{5.57}$$

where $\tilde{Y}_{\text{s.s}} = [\tilde{y}^*_{\text{s.s}} \dots \tilde{y}^*_{\text{s.s}}]^*$

This is reasonably straightforward in the step reference case. Take the discrete-time problem, and suppose a fixed step reference of r applies after the switch. Then we can easily calculate the steady state value of $\tilde{y}$.

The closed loop equations after the switch are:

$$\begin{aligned} x(k+1) &= Ax(k) + Br, \\ \tilde{y}(k) &= Cx(k) + Dr. \end{aligned}$$

and we can write

$$\begin{aligned}\tilde{y}(k+N) &= Cx(k+N) + Dr \\ &= CA^N x(k) + \left(C \left(\sum_{i=0}^{N-1} A^i \right) B + D \right) r.\end{aligned}$$

Hence in the steady state, as $N \to \infty$ we have

$$\begin{aligned}\tilde{y}_{\text{s.s}} &= \lim_{N\to\infty} \tilde{y}(n+N) \\ &= \lim_{N\to\infty} CA^N + \left(C \left(\sum_{i=0}^{\infty} A^i \right) B + D \right) r.\end{aligned}$$

Assuming a stabilizing controller, this means

$$\lim_{N\to\infty} A^N = 0,$$

and

$$\sum_{i=0}^{\infty} A^i = (I - A)^{-1},$$

so

$$\tilde{y}_{\text{s.s}} = \left(C(I - A)^{-1} B + D \right) r.$$

From equation (5.15) we have

$$\tilde{Y} = \Gamma_1 x_G(k) + \Gamma_2 x_K(k) + Lr,$$

where

$$L = \begin{bmatrix} 0 \\ CB \\ C(I+A)B \\ \vdots \\ C\left(\sum_{i=0}^{N-1} A^i\right) B \end{bmatrix}.$$

The cost function (5.57) is now

$$J = \left\| W\Gamma_1 x_K(k) + W\left(\Gamma_2 x_G(k) + Lr - \tilde{Y}_{\text{s.s}}\right) \right\|,$$

and the optimal choice of $x_K(k)$ with respect to (5.15) is

$$x_K(k) = -(W\Gamma_1)^{\dagger} \left(\Gamma_2 x_G(k) + Lr - \tilde{Y}_{\text{s.s}}\right),$$

following the proof of theorem 5.1.1 almost exactly.

5.5 Examples

5.5.1 Initialization scheme applied to example 1

Returning again to example 1.1.1, we can perform an optimal reset of the controller state at the switch with respect to the cost function 5.29.

First consider the unweighted (infinite horizon) case.

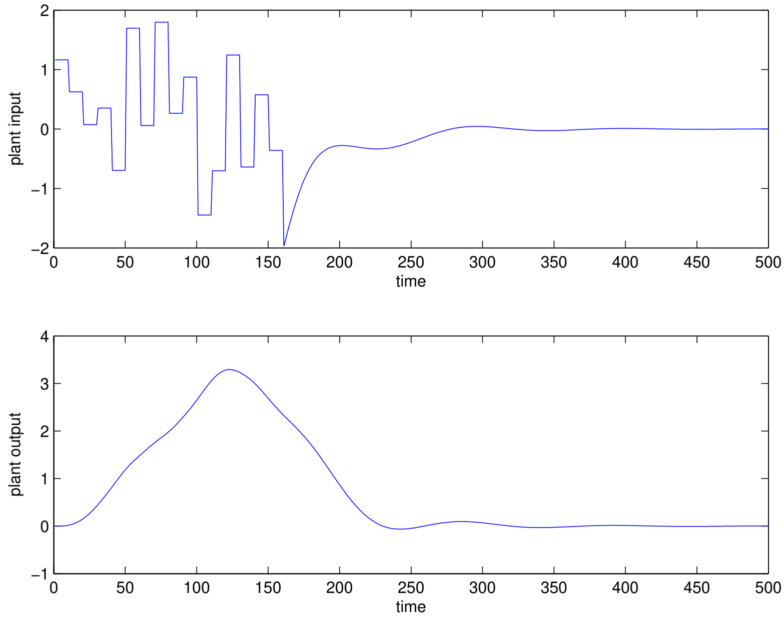


Figure 5.2: Optimal bumpless transfer, finite horizon

The result is a clearly smaller transient, with a signal norm of 12.90 after the switch.

Note that there is a discontinuity immediately following the switch, decaying very rapidly. If we require a smoother transition at the expense of longer decay time, it is possible to manipulate the weighting matrix to achieve the desired results. Figure 5.3 shows the same example, but with a diagonal exponentially weighted cost, from 1 immediately following the switch, to 10^{-4} at the end of the simulation. The result is a reduced discontinuity at the time of the switch, but slightly greater settling time and more oscillatory response.

5.6 Concluding remarks

We have considered here a method of choosing controller states which directly minimizes future transient signals. The problem is solved via weighted least squares in the finite horizon

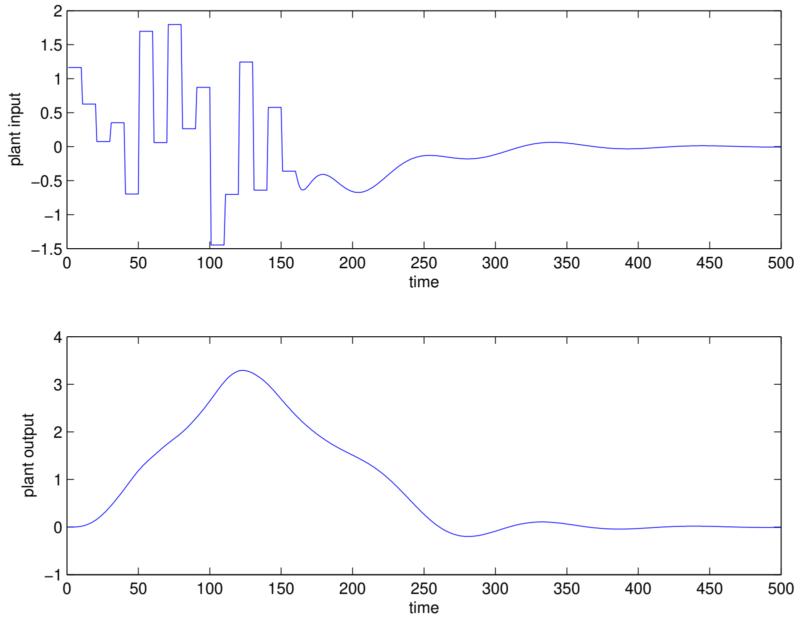


Figure 5.3: Weighted case

case, or via Lyapunov equations in the infinite horizon. The solutions can also be interpreted in terms of minimizing Lyapunov functions for the appropriate closed loop systems.

Although the solutions initially require knowledge of plant states, we may solve similar problems in terms of expected errors to obtain solutions in terms of estimated plant states.

The method requires accurate plant models, since the method is based on the solution of closed loop initial value problems. This is in contrast with the methods of the previous chapter, where no knowledge of the plant model is required.

Chapter 6

Stability of reset switching systems

Controller initialization plays an important role in the stability of switching systems. It is a natural hypothesis to suppose that a sensibly chosen controller reset or bumpless transfer scheme may enhance the stability of switching systems in some sense. Conversely, a poorly chosen controller reset can destabilize an otherwise stable switching system.

In this chapter, we consider the 'reset switching' type of hybrid system. In particular, we are interested in systems in which the state is composed of controller and plant state, of which the plant state is continuous across a switch, and the controller state may change discontinuously according to a reset relation. We investigate a number of stability questions for this type of switching system.

There is firstly an analysis question- given a switching system with specified controller realizations and resets, can stability be guaranteed for all admissible switching signals? If it is not possible to guarantee stability for all switching signals, then we would like to find a class of signals for which stability may be guaranteed.

Secondly, we can consider a synthesis question- given a family of controllers in given realizations, can we devise a system of controller resets such that the switching system is stable for all admissible switching signals? If it is not possible to find such resets for all switching signals, then we can again restrict the set of admissible switching signals so that stabilizing resets may be found.

We consider these questions in this chapter, presenting a necessary and sufficient Lyapunov function result for switching systems with reset. This result gives both a tool for analysing switching systems with given resets, and also for devising stabilizing reset schemes. We also consider an extension of the dwell-time stability results presented in chapter 3 to systems with resets.

Furthermore, we can combine the design of sensible controller resets with realization schemes discussed in chapter 3 in order to guarantee stability for (almost) arbitrary switching signals and improve performance.

6.1 A Lyapunov theorem for reset switching systems

Consider the family of linear vector fields

$$\dot{x}(t) = A_i x(t), \qquad i \in I, \quad x \in \mathbb{R}^n, \tag{6.1}$$

and the linear reset relations

$$g_{i,j} = G_{i,j} x, \tag{6.2}$$

We shall choose an index set I such that the family of matrices A_i forms a compact set.

Let S be the set of all admissible switching signals σ. S is the set of all non-zeno piecewise constant functions taking values from I. Note that the set S is *time invariant* in the following sense: for every signal $\sigma \in S$ and constant τ, there exists another signal $\tilde{\sigma} \in S$ such that $\tilde{\sigma}(t) = \sigma(\tau + t)$ for all $t \in \mathbb{R}^+$.

Theorem 6.1.1. *The reset switching system*

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)} x(t), \\ \sigma(t) &= i_k, \quad \textit{for} \quad t_k \le t < t_{k+1}, \quad i_k \in I, \quad k \in \mathbb{Z}^+, \\ x(t_k^+) &= G_{i_k, i_{k-1}} x(t_k^-), \end{aligned} \tag{6.3}$$

is uniformly asymptotically stable for all admissible switching signals $\sigma \in S$ if and only if there exist a family of functions $V_i : \mathbb{R}^n \to \mathbb{R}$ with the following properties:

- *V_i are positive definite, decrescent and radially unbounded.*
- *V_i are continuous and convex.*
- *There exist constants c_i such that for all x,*

$$\lim_{\Delta t \to 0^+} \left(\frac{V_i(e^{A_i t} x) - V(x)}{\Delta t} \right) \le -c_i \|x\|^2 .$$

- *$V_j(G_{j,i} x) \le V_i(x)$ for all $x \in \mathbb{R}^n$, and $i, j \in I$.*

Proof. (if)

Choose an admissible switching signal $\sigma \in S$. Then, the switching system with resets for this particular signal can be considered to be a linear time varying system with at most finitely many state discontinuities in any finite interval (by the restrictions on S).

If functions V_i exist, satisfying the theorem conditions, then we can construct a time-varying function

$$V_{\sigma(t)}(x(t)) = V_i(x(t)) \text{ when } \sigma(t) = i.$$

Since the functions V_i are decrescent and radially unbounded, we can find a_i and b_i such that

$$a_i \|x\|^2 < V_i(x) < b_i \|x\|^2 .$$

By the third condition on V_i, we know that the V_i are strictly decreasing on trajectories of A_i, and we have the bound

$$\lim_{\Delta t \to 0^+} \left(\frac{V_i(e^{A_i \Delta t} x) - V(x)}{\Delta t} \right) \le -c_i \|x\|^2 .$$

Furthermore, since $V_j(G_{j,i}x) \le V_i(x)$, we know that the function $V_{\sigma(t)}(x(t))$ is non-increasing at the (non-zeno) switching times.

Let $a = \inf_i a_i$, $b = \sup_i b_i$, and $c = \inf_i c_i$. By the compactness of the family A_i, a, b and c must be positive and finite. Hence we can write bounds on the function $V_{\sigma(t)}(x(t))$

$$a \|x(t)\|^2 < V_{\sigma(t)}(x(t)) < b \|x(t)\|^2 ,$$

and

$$\lim_{\Delta t \to 0^+} \left(\frac{V_{\sigma(t)}(e^{A_i \Delta t} x(t)) - V(x(t))}{\Delta t} \right) \le -c \|x(t)\|^2 .$$

Now if we let the initial state be x_0 at time $t = 0$, we have the bound

$$V_{\sigma(t)}(x(t)) < b x_0^2 e^{-\lambda t},$$

where $\lambda = c/a$, and hence

$$\|x\|^2 < \frac{b x_0^2}{a} e^{-\lambda t}.$$

Therefore, the point $x = 0$ is a uniformly asymptotically stable equilibrium of the switching system with resets.

(only if)

The point $x = 0$ is a uniformly asymptotically stable equilibrium of the switching system with resets. Let $\phi_{\sigma(t)}(t, x_0, t_0)$ denote the state of the switching system at time t given initial conditions x_0 at time t_0 and a particular switching signal σ. Since the set S is time invariant, we may assume without loss of generality that $t_0 = 0$. Since both the vector fields and reset relations are linear, we can write the trajectory for a given initial condition as follows:

$$\phi_{\sigma(t)}(t, x_0, 0) = \Phi_{\sigma(t)}(t) x_0,$$

where $\Phi_{\sigma(t)}(t)$ is a 'composite' state transition matrix defined by

$$\Phi_{\sigma(t)}(t) = e^{A_{i_k}(t-t_k)} G_{i_k,i_{k-1}} \dots e^{A_{i_1}(t_2-t_1)} G_{i_1,i_0} e^{A_{i_0} t_1}$$

when $t_k < t < t_{k+1}$.

Now let us define the functions V_i as follows

$$\begin{aligned} V_i(x) &= \sup_{\sigma \in S : \sigma(0)=i} \int_0^\infty \left\| \phi_{\sigma(t)}(t, x, 0) \right\|^2 dt \\ &= \sup_{\sigma \in S : \sigma(0)=i} x^* \left(\int_0^\infty \Phi^*_{\sigma(t)}(t) \Phi_{\sigma(t)}(t) dt \right) x \end{aligned}$$

That is, $V_i(x)$ is the supremum of the two-norm of trajectories beginning at state x with dynamics i. The integrals exist and are bounded since the equilibrium is asymptotically stable (and hence exponentially stable).

For any σ, let

$$Q(\sigma) = \int_0^\infty \Phi^*_{\sigma(t)}(t) \Phi_{\sigma(t)}(t) dt,$$

and let the set of all such $Q(\sigma)$ with $\sigma(0) = i$ be

$$\mathcal{Q}_i = \{Q(\sigma) : \sigma \in S \text{ with } \sigma(0) = i\}.$$

Now denote the closure of $\mathcal{Q}_i$ by $\bar{\mathcal{Q}}_i$. $\mathcal{Q}_i$ is bounded by the exponential stability of the system, so $\bar{\mathcal{Q}}_i$ is compact. Therefore, we can write

$$V_i(x) = \max_Q \{x^* Q x : Q \in \bar{\mathcal{Q}}_i\}.$$

Each function $x^* Q x$ is a continuous map from $\bar{\mathcal{Q}}_i \times \mathbb{R}^n$ to $\mathbb{R}$, so the maximum must be continuous (however, it is not necessarily differentiable).

We can show that the functions V_i are convex as follows. Let

$$x_\mu = \mu x_1 + (1 - \mu) x_0$$

for $\mu \in [0, 1]$. Since positive definite quadratic forms are convex, we have for $Q \in \bar{\mathcal{Q}}_i$,

$$x^*_\mu Q x_\mu \leq \mu x^*_1 Q x_1 + (1 - \mu) x^*_0 Q x_0.$$

Taking the maximum over $\bar{\mathcal{Q}}_i$, we have

$$V_i(x_\mu) \leq \mu V_i(x_1) + (1 - \mu) V_i(x_0),$$

hence the functions V_i are convex. Since they are continuous and convex, the V_i are also Lipschitz continuous.

Now we show that the functions V_i must be strictly decreasing on trajectories of the i'th vector field. We can see this as follows:

$$V_i(x) = \sup_{\sigma(0)=i} x^* \left(\int_0^\infty \Phi^*_{\sigma(t)}(t)\Phi_{\sigma(t)}(t)dt \right) x.$$

The supremum must include all switching signals which have $\sigma(t) = i$ for $0 < t < \tau$, so we have

$$\begin{aligned} V_i(x) &\leq \int_0^\tau \left\| e^{A_i t} x \right\|^2 dt + \sup_{\sigma(\tau)=i} (e^{A_i t}x)^* \left(\int_\tau^\infty \Phi^*_{\sigma(t)}(t)\Phi_{\sigma(t)}(t)dt \right) e^{A_i t} x \\ &= \int_0^\tau \left\| e^{A_i t} x \right\|^2 dt + V(e^{A_i t} x). \end{aligned}$$

By taking τ small, we have

$$\lim_{\tau \to 0} \frac{V_i(x) - V(e^{A_i t} x)}{\tau} \leq \frac{\|x\|^2}{2}.$$

So V_i is strictly decreasing on the i'th vector field. Note that since V_i is in general a quasi-quadratic function, it is not necessarily continuously differentiable.

From the definition of V_i it is clear that

$$V_j(G_{j,i}x) \leq V_i(x),$$

since the supremum

$$\sup_{\sigma(0)=i} \int_0^\infty \left\| \phi_{\sigma(t)}(t, x, 0) \right\|^2 dt$$

clearly includes all those switching trajectories which begin with an almost immediate switch from i to j.

□

The theorem effectively states that stability of the reset switching system depends upon the existence of a family of Lyapunov functions for the separate vector fields such that at any switch on the switching system, it is guaranteed that the value of the 'new' Lyapunov function after the switch will be no larger than the value of the 'old' function prior to the switch.

Note that the standard result on common Lyapunov functions for simple switching systems is a special case of our theorem when the reset matrices $G_{i,j}$ are identity.

In some special cases, it is possible to extend the theorem to the existence of smooth Lyapunov functions. For example, when the reset matrices $G_{i,j}$ are commuting matrices, it is possible to approximate the functions generated by the theorem by smooth functions by applying a smoothing operation in the spherical direction (note that the functions are already smooth in the radial direction).

The existence of Lyapunov fields for the component vector fields clearly implies that the component vector fields must be stable. Indeed this is obvious when you consider the switching signals $\sigma(t) = i$ for all t - that is, the switching signals with no switches are included in the class of switching signals being considered.

The sufficiency part of the theorem is perhaps obvious, since we merely postulate the existence of a time varying piecewise continuous (with respect to t) Lyapunov function $V_{\sigma(t)}(x)$. The necessity part of the theorem however is possibly somewhat surprising.

As was the case when we considered common Lyapunov functions for simple switching systems, we cannot guarantee that the functions V_i are quadratic. The constructive part of the proof only shows existence of quasi-quadratic functions of the form

$$V_i(x) = x^* P(x) x.$$

It still however makes sense to first consider quadratic functions in attempting to prove stability of a switching system. We can write the quadratic version of the theorem as the following sufficient condition.

Corollary 6.1.2. *The reset switching system 6.3 is uniformly asymptotically stable for all admissible switching signals $\sigma \in S$ if there exist a family of matrices $P_i > 0$ with the following properties:*

- $A_i^* P_i + P_i A_i < 0$
- $G_{j,i}^* P_j G_{j,i} - P_i \leq 0$ *for all* $i, j \in I$.

Proof. The sufficiency part of theorem 6.1.1 is clearly satisfied if quadratic functions V_i exist which satisfy the conditions. That is, let

$$V_i(x) = x^* P_i x.$$

Then, V_i is positive definite, decrescent and radially unbounded when $P_i > 0$. V_i is strictly decreasing on trajectories of the i'th vector field when $A_i^* P_i + P_i A_i < 0$, and the condition

$$V_j(G_{j,i} x) \leq V_i(x)$$

is satisfied for all $x \in \mathbb{R}^n$ if and only if

$$G_{j,i}^* P_j G_{j,i} - P_i \leq 0.$$

□

This corollary gives us a criterion which can be easily tested for quadratic Lyapunov functions via a family of linear matrix inequalities. Some care must be taken when dealing with the non-strict inequalities. If the solutions P_i exist within the interior of the non-strict inequalities $G_{j,i}^* P_j G_{j,i} - P_i \leq 0$, then they may be found by testing the strict inequalities

$$G_{j,i}^* P_j G_{j,i} - P_i < 0.$$

If the solutions exist on the boundary of some or all of the non-strict inequalities, then we can test them by introducing slack variables $\delta_{j,i}$. For instance, we test the strict inequalities

$$G_{j,i}^* P_j G_{j,i} - P_i - \delta_{j,i} I < 0,$$

and if it is found that the inequalities hold for an arbitrarily small positive $\delta_{j,i}$, then the non-strict inequality holds. Of course in practice we can only test a small finite $\delta_{j,i}$, so we cannot guarantee arbitrary accuracy.

Remark 6.1.1. Another obvious corollary to theorem 6.1.1, is that the fourth condition must be satisfied in the special case where we switch from state i to j and immediately back to i. That is, a necessary condition for the satisfaction of the theorem is that

$$V_i(G_{i,j} G_{j,i} x) \leq V_i(x).$$

This condition will be useful when we consider plant/controller structures in the following sections.

6.2 Controller initialization

Now we consider a class of resets with a particular structure. We are primarily interested in systems where the component vector fields are made up of plant/controller closed loops. The reset relations we consider then are such that the plant state remains constant across switching boundaries, and the controller state is initialized in some manner. The resets could, for instance be designed according to the performance criteria examined in chapter 5.

Specifically, we consider the N controller switching arrangement illustrated in figure 6.1.

Let $r = 0$, and let the closed loop equations when the i'th controller is in the loop be

$$\dot{x}(t) = A_i x(t) \qquad \text{where} \quad x = \begin{bmatrix} x_G \\ x_K \end{bmatrix}$$

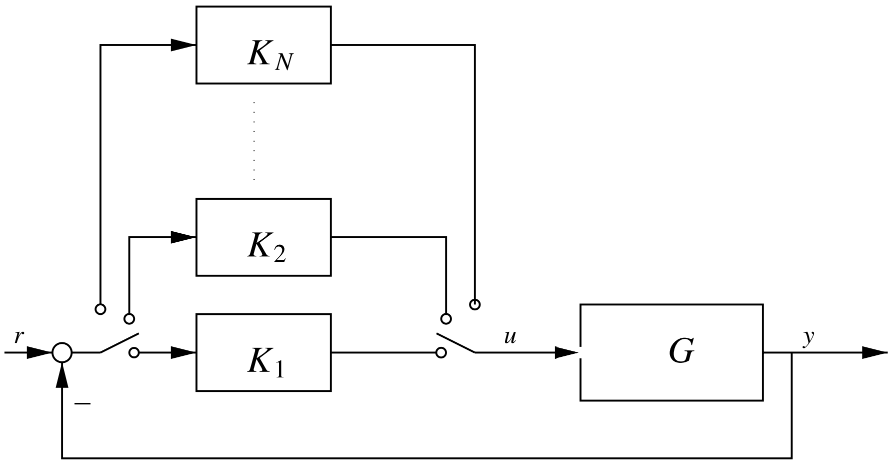


Figure 6.1: Switching system

in the continuous-time case, or

$$x(k+1) = A_i x(k) \qquad \text{where} \quad x = \begin{bmatrix} x_G \\ x_K \end{bmatrix}$$

in the discrete-time case.

If G and K have the following (continuous or discrete-time) state-space representations

$$G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right] \qquad K_i = \left[\begin{array}{c|c} A_{Ki} & B_{Ki} \\ \hline C_{Ki} & D_{Ki} \end{array}\right], \tag{6.4}$$

then the closed loop matrices A_i can be written

$$A_i = \begin{bmatrix} A_i(1,1) & A_i(1,2) \\ A_i(2,1) & A_i(2,2) \end{bmatrix} = \begin{bmatrix} A_G + B_G D_K i C_G & B_G C_K i \\ -B_K i C_G & A_K i + B_K i D_G C_K i \end{bmatrix} \tag{6.5}$$

as outlined in appendix A.

The controller switching occurs according to some piecewise constant switching signal $\sigma : \mathbb{R}^+ \to I$. Where $I = \{1, 2, \ldots N\}$.

$$\sigma(t) = i_k \text{ for } t_k < t \leq t_{k+1}.$$

The plant state is x_G with dimension n_G, and the controllers K_i have states x_{Ki} with dimensions n_K. For simplicity we restrict consideration to controllers of the same dimension, however the results do in fact hold in general for controllers of different dimensions with relatively straightforward modifications.

We define the current controller state to be

$$x_K(t) = x_{Ki}(t) \text{ when } \sigma(t) = i,$$

and the state of the closed loop system is

$$x = \begin{bmatrix} x_G \\ x_K \end{bmatrix}.$$

We now consider resets of the form

$$G_{i,j} = \begin{bmatrix} I & 0 \\ X_{i,j} & 0 \end{bmatrix}, \tag{6.6}$$

where $X_{i,j} \in \mathbb{R}^{n_K \times n_G}$.

The restriction of plant states to be continuous is natural in the switching controller context. The restriction of controller resets to be a function of plant state is considered for several reasons.

- We found in chapter 5 that optimal resets with respect to future performance are a function of plant state. We wish to investigate the stability properties of this form of reset.
- We also know that if a controller reset depending on previous plant *and* controller state satisfies the decreasing condition at switching times (with respect to some appropriate Lyapunov functions), then the minimum with respect to the controller state will also satisfy the condition. While such a function would be independent of previous controller state, it is only guaranteed to be linear when the respective Lyapunov functions are quadratic. Nevertheless, this argument suggests that when stabilizing resets exist, it is likely that stabilizing resets which are a function of plant state only can be achieved.
- Finally, the reset structure 6.6 admits the synthesis techniques developed later in this chapter. The reset synthesis question is much more complicated with a more general structure.

In fact, any such reset structures must minimize some Lyapunov functions in order to guarantee stability under arbitrary switching

Remark 6.2.1. Consider the reset switching system (2.12), with reset matrices with structure given in (6.6). If theorem 6.1.1 is satisfied, then the Lyapunov functions V_i must satisfy the condition

$$\operatorname*{argmin}_{x_K} V_i\left(\begin{bmatrix} x_G \\ x_K \end{bmatrix}\right) = X_{j,i} x_G.$$

To see why this must be true, note that for matrices of structure (6.6), that $G_{j,i}G_{i,j} = G_{j,i}$. Then a necessary condition for asymptotic stability is that

$$V_i\left(\begin{bmatrix} x_G \\ X_{j,i}x_G \end{bmatrix}\right) \leq V_i\left(\begin{bmatrix} x_G \\ x_K \end{bmatrix}\right).$$

Since the inequality must hold for all x_K, it must hold for the minimum over all x_K. So

$$V_i\left(\begin{bmatrix} x_G \\ X_{j,i}x_G \end{bmatrix}\right) = \min_{x_K} V_i\left(\begin{bmatrix} x_G \\ x_K \end{bmatrix}\right)$$

and

$$X_{j,i}x_G = \operatorname*{argmin}_{x_K} V_i\left(\begin{bmatrix} x_G \\ x_K \end{bmatrix}\right).$$

Note that this result links up nicely with the work of chapter 5, where the solutions of single switch performance problems turn out to be (in many cases) minimizations of Lyapunov functions. Here we find that, in order to guarantee stability over arbitrary switching sequences, linear resets are also required to be minimizations of Lyapunov functions.

Similar results can be derived for other reset structures where all points in a subspace of the state-space are mapped to a unique point in that subspace. In those cases again the resets must take the form of minimization over the subspace of some Lyapunov function. The controller/plant structure we consider above is the main structure of concern in this thesis. Simple switching systems are trivial cases where the subspaces are single points.

A further consequence of this observation is that if $G_{i,j}$ are stabilizing resets of the form (6.6) and the arguments

$$\operatorname*{argmin}_{x_K} V_i\left(\begin{bmatrix} x_G \\ x_K \end{bmatrix}\right)$$

are unique, then the matrices $X_{i,j}$ and hence the $G_{i,j}$ can only depend on the index of the new dynamics j. We will write $X_{j,i} = X_j$, and $G_{j,i} = G_j$ subsequently when this is the case.

Note also that the Lyapunov functions which are minimized to obtain stabilizing resets must also be functions which prove stability in theorem 6.1.1.

Now we shall consider the *potentially stabilizing* resets G_i of the form

$$G_i = \begin{bmatrix} I & 0 \\ X_i & 0 \end{bmatrix}, \tag{6.7}$$

where $X_i x_G = \operatorname*{argmin}_{x_K} V_i\left(\begin{bmatrix} x_G \\ x_K \end{bmatrix}\right)$.

Let us define the set of plant states x_G such that the minimum value of V_i with respect to x_K is less than some fixed number k.

Definition 6.2.1. Let $\Omega_i(k)$ be the set defined as follows:

$$\Omega_i(k) = \left\{ x_G : V_i\left(\begin{bmatrix} x_G \\ x_K \end{bmatrix}\right) \leq k \quad \text{for some} \quad x_K \right\}$$

The boundary of Ω_i, $\partial\Omega_i(k)$ is then

$$\partial\Omega_i(k) = \left\{ x_G : V_i\left(\begin{bmatrix} x_G \\ x_K \end{bmatrix}\right) = k \quad \text{for some} \quad x_K \right\}$$

With the specific resets X_i under consideration, this means that $\Omega_i(k)$ can also be thought of as the set of x_G such that

$$V_i\left(\begin{bmatrix} x_G \\ X_i x_G \end{bmatrix}\right) \leq k.$$

Theorem 6.2.1. *Consider the reset switching system* (6.3)*, with reset matrices with structure given in* (6.7)*. The system is asymptotically stable for all switching signals $\sigma \in S$ if and only if there exists a family of functions $V_i : \mathbb{R}^n \to \mathbb{R}$ with the following properties:*

- *V_i are positive definite, decrescent and radially unbounded*
- *V_i are continuous and convex.*
- *There exist constants c_i such that for all x,*

$$\lim_{\Delta t \to 0^+} \left(\frac{V_i(e^{A_i t} x) - V(x)}{\Delta t} \right) \leq -c_i \|x\|^2 .$$

- *V_i are such that*

$$X_i x_G = \operatorname*{argmin}_{x_K} V_i\left(\begin{bmatrix} x_G \\ x_K \end{bmatrix}\right)$$

  for all $x_G \in \mathbb{R}^{n_G}$

-

$$V_j\left(\begin{bmatrix} x_G \\ X_j x_G \end{bmatrix}\right) = V_i\left(\begin{bmatrix} x_G \\ X_i x_G \end{bmatrix}\right)$$

  for all $x_G \in \mathbb{R}^{n_G}$ and $i, j \in I$

for all $x_G \in \mathbb{R}^{n_G}$ and $i, j \in I$

Proof. (if)

Since $X_i x_G$ minimizes V_i with respect to x_K, we can guarantee that

$$V_j\left(\begin{bmatrix} x_G \\ X_j x_G \end{bmatrix}\right) \leq V_i\left(\begin{bmatrix} x_G \\ x_K \end{bmatrix}\right)$$

when x_K is permitted to vary.

(only if)

Consider the case when the loop i is operating, the plant state x_G is in the set $\partial\Omega_i(k)$, and the controller state is such that $x_K = X_i x_G$ (the minimum of V_i is achieved at x_K). That is,

$$V_i\left(\begin{bmatrix} x_G \\ x_K \end{bmatrix}\right) = k.$$

Now let the loop switch from i to j. From the results of theorem 6.1.1, we know that the switching system can only be asymptotically stable if

$$V_j\left(\begin{bmatrix} x_G \\ X_j x_G \end{bmatrix}\right) \leq V_i(x)$$

for all $x \in \mathbb{R}^n$, and $i, j \in I$. Thus, it must be true for the specific case when $x_K = X_i x_G$. So we require

$$V_j\left(\begin{bmatrix} x_G \\ X_j x_G \end{bmatrix}\right) \leq V_i\left(\begin{bmatrix} x_G \\ X_i x_G \end{bmatrix}\right).$$

The converse is also true, so we must have

$$V_j\left(\begin{bmatrix} x_G \\ X_j x_G \end{bmatrix}\right) = V_i\left(\begin{bmatrix} x_G \\ X_i x_G \end{bmatrix}\right)$$

for all x_G. □

This theorem effectively says that for reset switching system (of the form (6.7)) to be asymptotically stable for all switching signals, there must exist Lyapunov functions V_i, such that the level curves have the same 'silhouettes' when projected on to the plant component of the state-space, or equivalently

$$\Omega_j(k) = \Omega_i(k)$$

for all $k > 0$, and for all $i, j \in I$.

We now have quite strict conditions which must be met if a reset switching system is to be stable for all admissible signals σ. It is a relatively straightforward matter to test the condition for all quadratic Lyapunov functions.

6.2.1 Quadratic Lyapunov functions

An immediate consequence of this theorem is that if the plant is first order, and the family of resets X_i are equivalent to the minimization of quadratic Lyapunov functions for the i'th loop, then stability is automatically guaranteed.

Corollary 6.2.2. *Suppose $n_G = 1$, and the reset relations*

$$x_K = X_i x_G$$

are chosen such that X_i minimizes a quadratic Lyapunov function $\hat{V}_i$ for the i'th loop. Then the switching system (6.3) *is asymptotically stable for all switching signals.*

Proof. Since $n_G = 1$, and $\hat{V}_i$ is quadratic, with $\hat{V}_i(0) = 0$, then $\hat{\Omega}_i(1)$ is a closed interval $[-a_i, a_i]$ for some $a_i > 0$. We can now define a new Lyapunov function for the i'th loop

$$V_i(x) = a_i^2 \hat{V}_i(x).$$

Now the set $\Omega_i(1)$ is the closed interval $[-1, 1]$, and $\Omega_i(k)$ is $[-k^2, k^2]$. If we carry out this procedure for each i, we now have a family of Lyapunov functions V_i such that

$$V_j\left(\begin{bmatrix} x_G \\ X_j x_G \end{bmatrix}\right) = V_i\left(\begin{bmatrix} x_G \\ X_i x_G \end{bmatrix}\right),$$

and hence the switching system is asymptotically stable for all switching signals. □

Example 6.2.1. Revisiting example 3.0.1, we may add a reset relation at the switching boundaries such that the function

$$V_i = x^* P_i x$$

is minimized where P_i is the solution to the Lyapunov equation

$$A_i^* P_i + P_i A_i = -I.$$

and i is the index of the controller being switched on. This results in the stable trajectory pictured in figure 6.2.

For plants of more than first order, theorem 6.2.1 is a difficult result to satisfy. It gives an indication of how difficult it can be to guarantee stability of switching systems with arbitrary switching signals. It suggests that it may sometimes be necessary to design the resets specifically for stability rather than simply verify stability for given resets.

To satisfy theorem 6.2.1 via quadratic Lyapunov functions, it is required that the matrices X_i have the form

$$X_i = -P_i(2, 2)^{-1} P_i(2, 1),$$

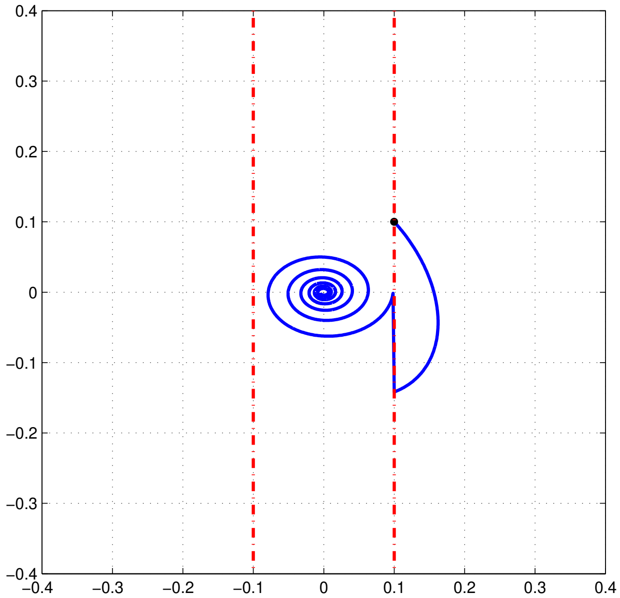


Figure 6.2: switching trajectory with reset relation

where

$$P_i = \begin{bmatrix} P_i(1,1) & P_i(1,2) \\ P_i(2,1) & P_i(2,2) \end{bmatrix}$$

is a positive definite matrix satisfying the Lyapunov inequality

$$A_i^* P_i + P_i A_i < 0$$

This is a clear consequence of theorem 6.2.1, and lemma A.1.1 in appendix A.

Consider the condition

$$V_j\left(\begin{bmatrix} x_G \\ X_j x_G \end{bmatrix}\right) = V_i\left(\begin{bmatrix} x_G \\ X_i x_G \end{bmatrix}\right) \qquad \text{for all } x_G$$

for quadratic functions V_i. That is,

$$\begin{bmatrix} x_G \\ X_j x_G \end{bmatrix}^* P_j \begin{bmatrix} x_G \\ X_j x_G \end{bmatrix} = \begin{bmatrix} x_G \\ X_i x_G \end{bmatrix}^* P_i \begin{bmatrix} x_G \\ X_i x_G \end{bmatrix} \Longrightarrow$$

$$x_G^* \left(P_j(1,1) + P_j(1,2)P_j(2,2)^{-1}P_j(2,1)\right) x_G = x_G^* \left(P_i(1,1) + P_i(1,2)P_i(2,2)^{-1}P_i(2,1)\right) x_G$$

for all x_G, or equivalently

$$P_i(1,1) + P_i(1,2)P_i(2,2)^{-1}P_i(2,1) = P_j(1,1) + P_j(1,2)P_j(2,2)^{-1}P_j(2,1) \qquad (6.8)$$

for each $i \neq j$. So to guarantee stability of the switching system using only quadratic Lyapunov functions, the condition 6.8 must be satisfied for each $i \neq j$.

6.3 Reset synthesis for stability

In the light of the results of theorems 6.1.1 and 6.2.1, a natural question is what to do if designed reset relation do not guarantee stability for all switching sequences. For a particular set of given controllers, we may ask the question of whether a family of reset relations exist which guarantee asymptotic stability for all switching signals.

We shall call such a family of resets a *stabilizing* family of reset relations.

It is a relatively straightforward matter to perform computations on the existence of linear resets which guarantee stability via quadratic Lyapunov functions.

The aim is to find a set of positive definite matrices

$$P_i = \begin{bmatrix} P_i(1,1) & P_i(1,2) \\ P_i(2,1) & P_i(2,2) \end{bmatrix}$$

such that

$$P_i(1,1) - P_i(1,2)P_i(2,2)^{-1}P_i(2,1) = P_j(1,1) - P_j(1,2)P_j(2,2)^{-1}P_j(2,1)$$

for all $j \neq i$, and that the Lyapunov inequalities

$$\begin{aligned} & A_i^* P_i + P_i A_i < 0 \qquad \text{(continuous-time)} \\ \text{or} \quad & A_i^* P_i A_i - P_i < 0 \qquad \text{(discrete-time)} \end{aligned}$$

are satisfied for all i.

Lemma 6.3.1. *For a partitioned matrix*

$$P = \begin{bmatrix} P(1,1) & P(1,2) \\ P(2,1) & P(2,2) \end{bmatrix}$$

let Δ denote the Schur complement of $P(1,1)$ in P. That is,

$$\Delta = P(1,1) - P(1,2)P(2,2)^{-1}P(2,1).$$

Then we can write the inverse of P as follows:

$$P^{-1} = \begin{bmatrix} \Delta^{-1} & -\Delta^{-1}P(1,2)P(2,2)^{-1} \\ -P(2,2)^{-1}P(2,1)\Delta^{-1} & P(2,2) + P(2,2)^{-1}P(2,1)\Delta^{-1}P(1,2)P(2,2)^{-1} \end{bmatrix}$$

Proof. See for example [62]. □

Using lemma 6.3.1, we can now form an equivalent problem in terms of matrices Q_i where $Q_i = P_i^{-1}$.

Define

$$\Delta = P_i(1,1) - P_i(1,2)P_i(2,2)^{-1}P_i(2,1).$$

Δ can be thought of as the inverse of the (1, 1) block of the inverse of P_i, so the equivalent problem is to find positive definite matrices

$$Q_i = \begin{bmatrix} \Delta^{-1} & Q_i(1,2) \\ Q_i(1,2)^* & Q_i(2,2) \end{bmatrix}$$

satisfying

$$\begin{aligned} & Q_i A_i^* + A_i Q_i < 0 \qquad \text{(continuous-time)} \\ \text{or} \quad & A_i Q_i A_i^* - Q_i < 0 \qquad \text{(discrete-time).} \end{aligned}$$

Then the required reset relations are

$$x_K = -P_i(2,2)^{-1}P_i(2,1)x_G,$$

where $P_i = Q_i^{-1}$.

Theorem 6.3.2. *Consider the continuous-time linear plant G, and N controllers K_i defined according to* (6.4)*, and let the closed loop matrices*

$$A_i = \begin{bmatrix} A_i(1,1) & A_i(1,2) \\ A_i(2,1) & A_i(2,2) \end{bmatrix}$$

be defined according to equation (6.5)*.*

There exists a stabilizing family of reset relations when there exist matrices Δ, $Q_i(1,2) \in \mathbb{R}^{n_G \times n_K}$ and $Q_i(2,2) \in \mathbb{R}^{n_{Ki} \times n_K}$ for each $i = \{1, \ldots, N\}$ such that the following system of LMIs is satisfied:

$$\begin{bmatrix} \Phi_i(1,1) & \Phi_i(1,2) \\ \Phi_i(2,1) & \Phi_i(2,2) \end{bmatrix} < 0 \tag{6.9}$$

where

$$\begin{aligned} \Phi_i(1,1) &= \Delta^{-1}A_i(1,1)^* + Q_i(1,2)A_i(1,2)^* + A_i(1,1)\Delta^{-1} + A_i(1,2)Q_i(1,2)^*, \\ S\Phi_i(1,2) &= \Delta^{-1}A_i(2,1)^* + Q_i(1,2)A_i(2,2)^* + A_i(1,1)Q_i(1,2) + A_i(1,2)Q_i(2,2), \\ \Phi_i(2,1) &= Q_i(1,2)^*A_i(1,1)^* + Q_i(2,2)A_i(1,2)^* + A_i(2,1)\Delta^{-1} + A_i(2,2)Q_i(1,2)^*, \\ \Phi_i(2,2) &= Q_i(1,2)^*A_i(2,1)^* + Q_i(2,2)A_i(2,2)^* + A_i(2,1)Q_i(1,2) + A_i(2,2)Q_i(2,2). \end{aligned}$$

The reset relations guaranteeing stability are

$$x_K = -P_i(2,2)^{-1}P_i(2,1),$$

where

$$P_i = \begin{bmatrix} P_i(1,1) & P_i(1,2) \\ P_i(2,1) & P_i(2,2) \end{bmatrix} = \begin{bmatrix} \Delta^{-1} & Q_i(1,2) \\ Q_i(1,2)^* & Q_i(2,2) \end{bmatrix}^{-1}.$$

Proof. We prove the theorem by attempting to find quadratic functions $V_i(x) = x^* P_i x$, and the corresponding resets $X_i = -P_i(2,2)^{-1}P_i(2,1)$ which satisfy theorem 6.2.1. Since we consider only quadratic functions, the necessary and sufficient condition becomes only sufficient.

The LMI conditions (6.9) are simply an expanded version of the Lyapunov inequalities

$$Q_i A_i^* + A_i Q_i < 0$$

where

$$Q_i = \begin{bmatrix} \Delta^{-1} & Q_i(1,2) \\ Q_i(1,2)^* & Q_i(2,2) \end{bmatrix},$$

and

$$A_i = \begin{bmatrix} A_i(1,1) & A_i(1,2) \\ A_i(2,1) & A_i(2,2) \end{bmatrix}.$$

Define $P_i = Q_i^{-1}$. Then the matrices P_i satisfy the Lyapunov inequalities

$$A_i^* P_i + P_i A_i < 0,$$

and also,

$$P_i(1,1) - P_i(1,2)P_i(2,2)^{-1}P_i(2,1) = \Delta$$

for each i. Hence the Lyapunov functions

$$V_i(x) = x^* P_i x$$

satisfy theorem 6.2.1, and asymptotic stability of the reset switching system is proved with the corresponding resets

$$X_i = -P_i(2,2)^{-1}P_i(2,1).$$

□

For the discrete-time case, we require the following lemma.

Lemma 6.3.3. *Let $P \in \mathbb{R}^{n\times n}$ be a positive definite matrix, and $A \in \mathbb{R}^{n\times n}$ any real valued matrix. Then the inequality*

$$A^* P A - P < 0$$

holds if and only if the inequality

$$A P^{-1} A^* - P^{-1} < 0$$

also holds.

Proof. Consider the following matrix, decomposed in two alternative ways:

$$\begin{bmatrix} P^{-1} & A \\ A^* & P \end{bmatrix} = \begin{bmatrix} I & 0 \\ A^* P & I \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & P - A^* P A \end{bmatrix} \begin{bmatrix} I & PA \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} I & AP^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} P^{-1} - AP^{-1}A^* & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} I & 0 \\ P^{-1}A^* & I \end{bmatrix}$$

Since both P and P^{-1} are positive definite, then

$$P - A^* P A > 0 \iff P^{-1} - A P^{-1} A^* > 0$$

□

Theorem 6.3.4. *Consider the discrete-time linear plant G, and N controllers K_i defined according to* (6.4)*, and let the closed loop matrices*

$$A_i = \begin{bmatrix} A_i(1,1) & A_i(1,2) \\ A_i(2,1) & A_i(2,2) \end{bmatrix}$$

be defined according to equation (6.5).

There exists a reset relation for which the switching system is stable for any switching sequence when there exist matrices Δ, $Q_i(1,2) \in \mathbb{R}^{n_G \times n_K}$ and $Q_i(2,2) \in \mathbb{R}^{n_K \times n_K}$ for each $i = \{1, \dots, N\}$ such that the following system of LMIs is satisfied:

$$\begin{bmatrix} \Phi_i(1,1) & \Phi_i(1,2) \\ \Phi_i(2,1) & \Phi_i(2,2) \end{bmatrix} < 0 \tag{6.10}$$

where

$$\begin{aligned}\Phi_1 &= A_i(1,1)^*\Delta^{-1}A_i(1,1) + A_i(2,1)^*Q_i(1,2)^*A_i(1,1)\\ &\qquad + A_i(1,1)^*Q_i(1,2)A_i(2,1) + A_i(2,1)^*Q_i(2,2)A_i(2,1) - \Delta^{-1}\\ \Phi_2 &= A_i(1,1)^*\Delta^{-1}A_i(1,2) + A_i(2,1)^*Q_i(1,2)^*A_i(1,2)\\ &\qquad + A_i(1,1)^*Q_i(1,2)A_i(2,2) + A_i(2,1)^*Q_i(2,2)A_i(2,2) - Q_i(1,2)\\ \Phi_3 &= A_i(1,2)^*\Delta^{-1}A_i(1,1) + A_i(2,2)^*Q_i(1,2)^*A_i(1,1)\\ &\qquad + A_i(1,2)^*Q_i(1,2)A_i(2,1) + A_i(2,2)^*Q_i(2,2)A_i(2,1) - Q_i(1,2)^*\\ \Phi_4 &= A_i(1,2)^*\Delta^{-1}A_i(1,2) + A_i(2,2)^*Q_i(1,2)^*A_i(1,2)\\ &\qquad + A_i(1,2)^*Q_i(1,2)A_i(2,2) + A_i(2,2)^*Q_i(2,2)A_i(2,2) - Q_i(2,2)\end{aligned}$$

The reset relations guaranteeing stability are

$$x_K = -P_i(2,2)^{-1}P_i(2,1),$$

where

$$P_i = \begin{bmatrix} P_i(1,1) & P_i(1,2) \\ P_i(2,1) & P_i(2,2) \end{bmatrix} = \begin{bmatrix} \Delta^{-1} & Q_i(1,2) \\ Q_i(1,2)^* & Q_i(2,2) \end{bmatrix}^{-1}.$$

Proof. The LMI conditions (6.10) are simply an expanded version of the Lyapunov inequalities

$$A_iQ_iA_i^* - Q_i < 0$$

where

$$Q_i = \begin{bmatrix} \Delta^{-1} & Q_i(1,2) \\ Q_i(1,2)^* & Q_i(2,2) \end{bmatrix},$$

and

$$A_i = \begin{bmatrix} A_i(1,1) & A_i(1,2) \\ A_i(2,1) & A_i(2,2) \end{bmatrix}.$$

Define $P_i = Q_i^{-1}$. Then from lemma 6.3.3, the matrices P_i satisfy the Lyapunov inequalities

$$A_i^*P_iA_i - P_i < 0,$$

and also

$$P_i(1,1) - P_i(1,2)P_i(2,2)^{-1}P_i(2,1) = \Delta$$

for each i. Hence the Lyapunov functions

$$V_i(x) = x^*P_ix$$

satisfy theorem 6.2.1, and asymptotic stability of the reset switching system is proved with the corresponding resets

$$X_i = -P_i(2,2)^{-1}P_i(2,1).$$

□

Sometimes theorems 6.9 and 6.10 will not yield a solution for a given set of controllers in specific realizations. In such situations, we can consider alternative realizations of the same controllers in order to ensure stability.

Given a plant P and a set of stabilizing controllers K_i, we can find realizations of the controllers K_i such that a simple switch between the controller is guaranteed to be asymptotically stable, as shown in chapter 3. We can then apply a reset switching scheme in order to improve the performance of the resulting system.

If the K_i are computed using the Hespanha scheme of section 3.4.1, then there will be a common quadratic Lyapunov function for each of the closed loops. Hence there will be quadratic Lyapunov functions with common plant-space projections, and the LMIs of theorems 6.9 and 6.10 are guaranteed to have solutions.

We can also consider implementing the controllers according to the stabilizing coprime factor scheme of section 3.4.2. In that case, we can show that if the closed-loop augmented state spaces (figure 3.8) have CQLF's, then it is always possible to find common plant-space projections of the unaugmented closed-loops (still in coprime factor form).

6.4 Reset switching of coprime factor controllers

If realizations are computed according to the coprime factor scheme of section 3.4.2, then we can think of the controller implementations as being the augmented controllers of figure 3.8. The closed loop state space corresponding to each controller will have a specific structure. The states of the plant and active controller will depend only on themselves- independent of the inactive controllers. The states of the inactive controllers will depend on themselves, as well as on the states of the plant and the current active controller. This can be illustrated in the two controller case by the closed loop representations 6.13, and 6.14 (in theorem 6.4.2), where the plant state is x_1, and the controller states are x_2 and x_3.

Theorem 6.4.2 shows that if the augmented closed loops 6.13, and 6.14 have a CQLF, then the unaugmented loops 6.15, and 6.16 have common projections in plant space, meaning we can apply the reset switching scheme of 6.9 (the discrete-time result follows similarly).

Lemma 6.4.1. *Consider a stable block-triangular dynamical system*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \tag{6.11}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, *and* $x_3 \in \mathbb{R}^{n_3}$ *and a quadratic Lyapunov function*

$$V\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^* \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Then, the function

$$\hat{V}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^* \begin{bmatrix} 2Q_{11} - Q_{13}Q_{33}^{-1}Q_{31} & Q_{21} \\ Q_{12} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is a Lyapunov function for the dynamical system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{6.12}$$

Proof. Let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Since V is a Lyapunov function for (6.11), $V(x)$ is strictly positive and $\dot{V}(x)$ strictly negative for all points $x \neq 0$. Define the functions

$$V_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = V\left(\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}\right)$$

and

$$V_2\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = V\left(\begin{bmatrix} x_1 \\ 0 \\ -Q_{33}^{-1}Q_{31}x_1 \end{bmatrix}\right).$$

Clearly V_1 is strictly positive and strictly decreasing for all $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq 0$ and V_2 is strictly positive and strictly decreasing for all $x_1 \neq 0$. Hence V_2 is positive and decreasing (but not strictly) for all $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq 0$

Now define the function

$$\begin{aligned}\hat{V}\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) &= \begin{bmatrix}x_1\\x_2\end{bmatrix}^* \begin{bmatrix}2Q_{11}-Q_{13}Q_{33}^{-1}Q_{31} & Q_{12}\\ Q_{21} & Q_{22}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\\ &= V_1\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)+V_2\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)\end{aligned}$$

and clearly $\hat{V}$ is strictly positive, and $\dot{\hat{V}}$ is strictly negative on trajectories of (6.11) for all points $\left[\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right] \neq 0$. Since the states x_1 and x_2 are independent of of x_3 in the dynamics of (6.11), then $\hat{V}$ is also strictly positive and strictly decreasing on trajectories of (6.12). Hence the function $\hat{V}$ is a Lyapunov function for the dynamical system (6.12). □

Theorem 6.4.2. *Consider the dynamical systems*

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12} & 0\\ A_{21} & A_{22} & 0\\ A_{31} & A_{32} & A_{33}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}, \tag{6.13}$$

and

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix} = \begin{bmatrix}\tilde{A}_{11} & 0 & \tilde{A}_{13}\\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23}\\ \tilde{A}_{31} & 0 & \tilde{A}_{33}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}, \tag{6.14}$$

Suppose there exists a common quadratic Lyapunov function for the dynamic systems (6.13) *and* (6.14). *Then there exist quadratic Lyapunov functions*

$$V_1\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right), \text{ and } V_2\left(\begin{bmatrix}x_1\\x_3\end{bmatrix}\right)$$

for the systems

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}, \tag{6.15}$$

and

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_3\end{bmatrix} = \begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{13}\\ \tilde{A}_{31} & \tilde{A}_{33}\end{bmatrix}\begin{bmatrix}x_1\\x_3\end{bmatrix} \tag{6.16}$$

respectively, such that

$$\min_{x_2}\left(V_1\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)\right) = \min_{x_3}\left(V_2\left(\begin{bmatrix}x_1\\x_3\end{bmatrix}\right)\right).$$

for all $x_1 \in \mathbb{R}^{n_1}$.

Proof. Let

$$V\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}^*\begin{bmatrix}Q_{11}&Q_{12}&Q_{13}\\Q_{21}&Q_{22}&Q_{23}\\Q_{31}&Q_{32}&Q_{33}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

be the common Lyapunov function for systems (6.13) and (6.14).

Define the functions

$$V_1\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}x_1\\x_2\end{bmatrix}^*\begin{bmatrix}2Q_{11}-Q_{13}Q_{33}^{-1}Q_{31}&Q_{31}\\Q_{12}&Q_{22}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

and

$$V_2\left(\begin{bmatrix}x_1\\x_3\end{bmatrix}\right)=\begin{bmatrix}x_1\\x_3\end{bmatrix}^*\begin{bmatrix}2Q_{11}-Q_{12}Q_{22}^{-1}Q_{21}&Q_{31}\\Q_{13}&Q_{33}\end{bmatrix}\begin{bmatrix}x_1\\x_3\end{bmatrix}$$

By lemma 6.4.1 the function V_1 is a Lyapunov function for (6.15), and function V_2 is a Lyapunov function for (6.16) (by a simple reordering of rows).

By lemma A.2.1 in appendix A,

$$\min_{x_2}\left(V_1\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)\right)=x_1^*(2Q_{11}-Q_{12}Q_{22}^{-1}Q_{21}-Q_{13}Q_{33}^{-1}Q_{31})x_1$$

and

$$\min_{x_3}\left(V_2\left(\begin{bmatrix}x_1\\x_3\end{bmatrix}\right)\right)=x_1^*(2Q_{11}-Q_{12}Q_{22}^{-1}Q_{21}-Q_{13}Q_{33}^{-1}Q_{31})x_1$$

□

Applying reset switching to the coprime factor scheme gives two main advantages. Firstly, the need to run the inactive controllers all the time is eliminated. If there are a large number of controllers in the switching scheme, this may result in a substantial computational benefit. Secondly, the minimization carried out by such a reset scheme carries substantial performance benefits as outlined in chapter 5. The performance benefit is illustrated in example 6.5.1.

6.5 Plant state estimation

In this chapter, we have assumed that we have full state information. The reset rules we have derived have so far assumed that the plant state x_G is known precisely. Obviously we would like to be able to apply the results on controller initialization even when the plant state is merely estimated.

The obvious way to implement a reset switching scheme when the plant is not known precisely is to construct an observer of the plant structure, and to reset according to the observed state $\hat{x}_G$ instead of the actual state x_G. In fact, this leads us to trajectories which are guaranteed to be stable as long as the *nominal* trajectory is stable, and the state estimation error $x_G - \hat{x}_G$ converges to zero exponentially. In fact, we can always construct an exponentially stable observer if the plant is observable.

Suppose

$$\dot{x}(t) = A_i x(t), \qquad i \in I, \quad x \in \mathbb{R}^n \tag{6.17}$$

represent the closed loop equations corresponding to each switching controller, augmented by an exponentially stable plant observer. The state

$$x(t) = \begin{bmatrix} x_G(t) \\ x_K(t) \\ x_{\text{ob}}(t) \end{bmatrix},$$

where x_G is the plant state, x_K the controller state, and $x_{\text{ob}}(t)$ the observer state.

Let G_i be a family of *nominal* reset matrices with structure

$$G_i = \begin{bmatrix} I & 0 & 0 \\ X_i & 0 & 0 \\ 0 & 0 & I \end{bmatrix}.$$

That is, the controller is reset according to the plant state at switching times, while the plant state and observer states remain continuous. Then the reset switching system

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)} x(t), \\ \sigma(t) &= i_k, \qquad \text{for} \quad t_k \le t < t_{k+1}, \quad i_k \in I, \quad k \in \mathbb{Z}^+, \\ x(t_k^+) &= G_{i_k} x(t_k^-) \end{aligned} \tag{6.18}$$

will be referred to as the *nominal* reset switching system, producing the nominal trajectories $x(t)$.

Let $\tilde{G}_i$ be a family of observer based reset matrices with structure

$$\tilde{G}_i = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & X_i \\ 0 & 0 & I \end{bmatrix}.$$

That is, the controller is reset according to the observer state at switching times, while the plant state and observer states remain continuous. Then the reset switching system

$$\begin{aligned} \dot{\tilde{x}}(t) &= A_{\sigma(t)} \tilde{x}(t), \\ \sigma(t) &= i_k, \qquad \text{for} \quad t_k \le t < t_{k+1}, \quad i_k \in I, \quad k \in \mathbb{Z}^+, \\ \tilde{x}(t_k^+) &= \tilde{G}_{i_k} \tilde{x}(t_k^-) \end{aligned} \tag{6.19}$$

will be referred to as the *observer based* reset switching system, producing the trajectories $\tilde{x}(t)$.

In the following theorem, we will use the following notation to denote products of matrices (from the right).

Definition 6.5.1.

$$\prod_{k=1}^{N} A_k = A_N A_{N-1} \dots A_2 A_1.$$

We also use a straightforward lemma to simplify later working.

Lemma 6.5.1. *Given matrices A_i and B_i for $i = \{1, \dots, N\}$ with compatible dimensions, we can write the following expansion*

$$\prod_{i=1}^{N} B_i = \prod_{i=1}^{N} A_i + (B_N - A_N)\left(\prod_{i=1}^{N-1} A_i\right) + \sum_{j=2}^{N-1}\left(\prod_{i=j+1}^{N} B_i(B_j - A_j)\prod_{i=1}^{j-1} A_i\right) + \left(\prod_{i=2}^{N} B_i\right)(B_1 - A_1)$$

Theorem 6.5.2. *If the nominal reset switching system* (6.18) *is stable for all admissible (strongly non-zeno) switching sequences $\sigma(t)$, and the observer error*

$$\varepsilon(t) = x_G(t) - x_{ob}(t)$$

is exponentially stable, then the observer based reset switching system (6.19) *is asymptotically stable for all such $\sigma(t)$.*

Proof. Fix a strongly non-zeno $\sigma(t)$, and initial state $x(t_0) = x_0$. We may assume the two switching systems (6.18) and (6.19) have the same initial state, so $\tilde{x}(t_0) = x_0$. We shall also let $\Delta t_k = t_k - t_{k-1}$ for all $k \in \mathbb{Z}^+$

At any finite time τ, occurring after N controller switches, we can write the state of the nominal system

$$x(\tau) = e^{A_{i_N}(t-t_N)} \prod_{k=1}^{N}\left(G_{i_k} e^{A_{i_{k-1}}(\Delta t_k)}\right) x_0.$$

Similarly, we can write the state of the observer based system

$$\tilde{x}(\tau) = e^{A_{i_N}(t-t_N)} \prod_{k=1}^{N}\left(\tilde{G}_{i_k} e^{A_{i_{k-1}}(\Delta t_k)}\right) x_0$$

By lemma 6.5.1, we can expand this as follows

$$\tilde{x}(\tau) = x(\tau) + e^{A_{i_N}(t-t_N)} \prod_{k=2}^{N} \left(G_{i_k} e^{A_{i_{k-1}}(\Delta t_k)} \right) \left(\left(\tilde{G}_{i_1} - G_{i_1} \right) e^{A_{i_0}(\Delta t_1)} \right) x_0$$
$$+ \sum_{j=2}^{N-1} \left(\prod_{k=j+1}^{N} \left(G_{i_k} e^{A_{i_{k-1}}(\Delta t_k)} \right) \left(\tilde{G}_{i_j} - G_{i_j} \right) e^{A_{i_{j-1}}(\Delta t_j)} \prod_{k=1}^{j-1} \left(\tilde{G}_{i_k} e^{A_{i_{k-1}}(\Delta t_k)} \right) \right) x_0$$
$$+ e^{A_{i_N}(t-t_N)} \left(\tilde{G}_{i_1} - G_{i_1} \right) e^{A_{i_{N-1}}(\Delta t_N)} \prod_{k=1}^{N-1} \left(\tilde{G}_{i_k} e^{A_{i_{k-1}}(\Delta t_k)} \right) x_0.$$

Note that each

$$e^{A_{i_{j-1}}(\Delta t_j)} \prod_{k=1}^{j-1} \left(\tilde{G}_{i_k} e^{A_{i_{k-1}}(\Delta t_k)} \right) x_0$$

is simply $\tilde{x}(t_j)$, and hence

$$\left(\tilde{G}_{i_j} - G_{i_j} \right) e^{A_{i_{j-1}}(\Delta t_j)} \prod_{k=1}^{j-1} \left(\tilde{G}_{i_k} e^{A_{i_{k-1}}(\Delta t_k)} \right) x_0$$

is $X_{i_j}(\tilde{x}_{\mathrm{ob}}(t_j) - \tilde{x}_G(t_j))$. Since the plant observer is the same in the nominal and observer based cases, this is simply $X_{i_j}\varepsilon(t_j)$.

We can now express $\tilde{x}(\tau)$ in terms of $x(\tau)$, and the observer error at each switching time t_j.

$$\begin{aligned} \tilde{x}(\tau) &= x(\tau) + e^{A_{i_N}(t-t_N)} \prod_{k=2}^{N} \left(G_{i_k} e^{A_{i_{k-1}}(\Delta t_k)} \right) X_{i_1}\varepsilon(t_1) \\ &+ \sum_{j=2}^{N-1} \left(\prod_{k=j+1}^{N} \left(G_{i_k} e^{A_{i_{k-1}}(\Delta t_k)} \right) \varepsilon(t_j) \right) + e^{A_{i_N}(t-t_N)} \varepsilon(t_N) \end{aligned} \tag{6.20}$$

Since the nominal system is exponentially stable, there exists $\lambda > 0$ such that

$$\|x(t)\| \leq e^{-\lambda(t-t_0)} \|x(t_0)\|$$

for any t_0, and corresponding initial state $x(t_0)$. Also, since the observer is stable, there exists $\phi > 0$ such that

$$\|\varepsilon(t)\| \leq e^{-\phi(t-t_0)} \varepsilon x(t_0)$$

for any t_0, and corresponding initial error $\varepsilon(t_0)$.

Since the signal $\sigma(t)$ is strongly non-zeno, the ratio of the number of transitions N in a time interval $[0, t)$ to the length of the interval has a fixed upper bound κ.

From equation (6.20), we can now write the inequality

$$\|\tilde{x}(t)\| \leq e^{-\lambda t} \|x_0\| + \sum_{j=1}^{N} \left(e^{-\lambda(t-t_j)} e_{-\phi t_j} \right) \varepsilon(t_0)$$

letting $\psi = \min\{\phi, \lambda\}$ then

$$\|\tilde{x}(t)\| \leq e^{-\lambda t} \|x_0\| + N e^{-\psi(t-t_0)} \varepsilon(t_0),$$

and hence

$$\begin{aligned} \lim_{t\to\infty} \|\tilde{x}(t)\| &\leq \lim_{t\to\infty} e^{-\lambda t} \|x_0\| + \lim_{t\to\infty} \kappa t \left(e^{-\lambda(t-t_j)} e_{-\phi t_j} \right) \varepsilon(t_0) \\ &= 0. \end{aligned}$$

Therefore, the observer based reset switching system (6.19) is asymptotically stable.

□

In fact asymptotic stability here is equivalent to exponential stability. The reset switching system for a fixed $\sigma(t)$ may be thought of as a linear-time varying system, for which asymptotic stability implies exponential stability [60].

Example 6.5.1. Take a second order lightly damped plant

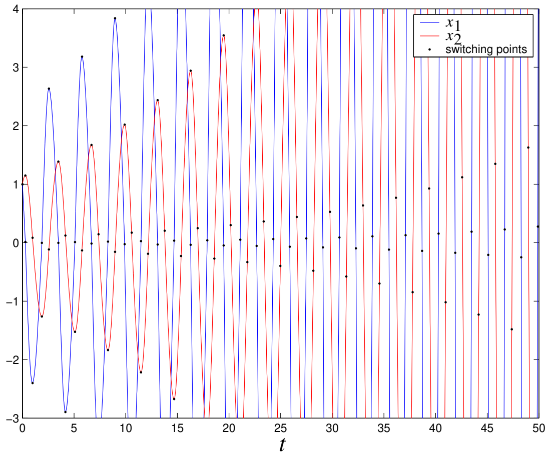
$$P(s) = \frac{1}{s^2 + 0.2s + 1}$$

implemented in controller canonical form

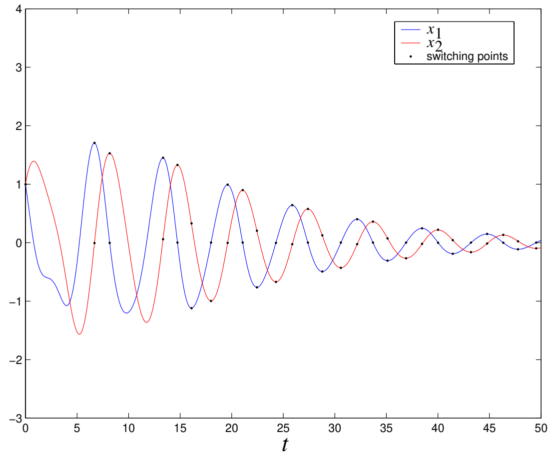
$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -0.2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \end{aligned}$$

and two static stabilizing feedback gains $k_1 = 2$, and $k_2 = 4$. The closed loop equations formed by setting $u = k_1(r - y)$, and $u = k_2(r - y)$ (where r is some reference) are respectively

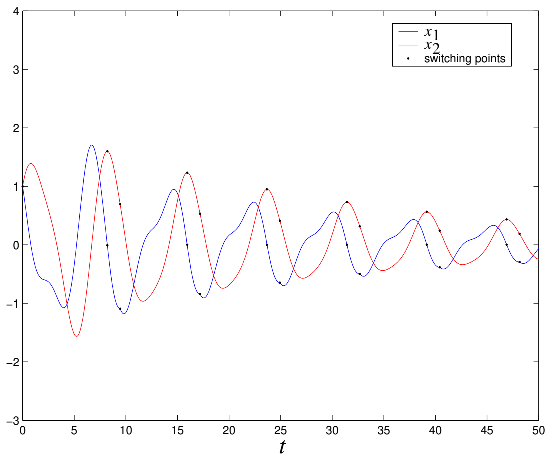
$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -0.2 & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \end{aligned}$$



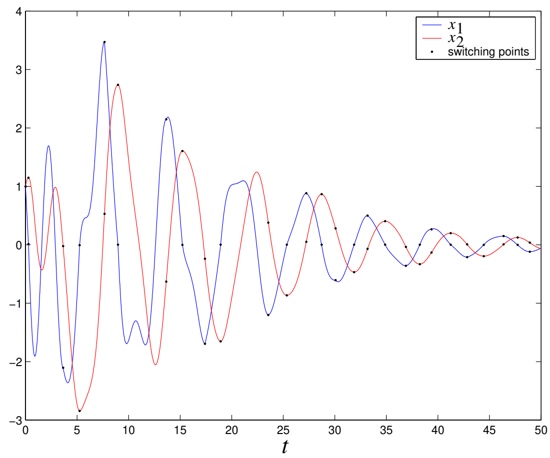
(a) Unstable trajectory



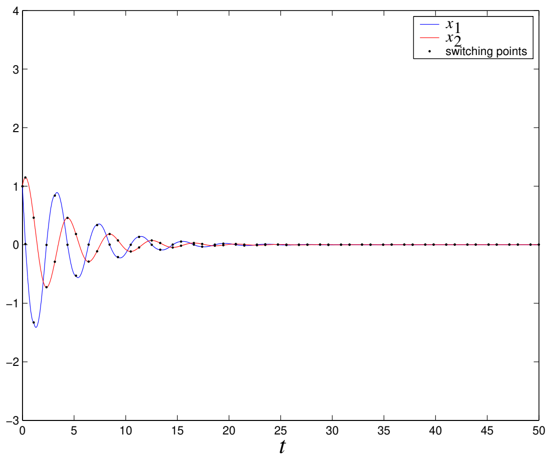
(b) Hespanha realizations (no reset)



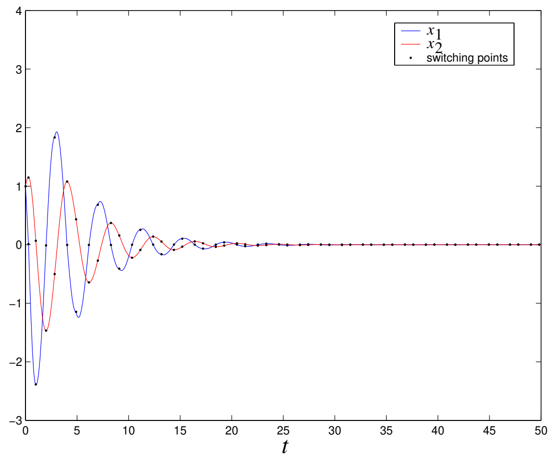
(c) Hespanha realizations (reset to origin)



(d) Coprime factor realization



(e) Coprime factor realizations (Lyapunov based reset full state knowledge)



(f) Coprime factor realizations (Lyapunov based reset, state observer)

Figure 6.3:

and

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.2 & -5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We shall refer to the respective state-space matrices as A_1, A_2, B and C. It is reasonably straightforward to show that while both A_1 and A_2 have eigenvalues in the left half plane, they do not share a common quadratic Lyapunov function.

The switching system defined by

$$\dot{x} = A_{\sigma(t)}x + Bu$$
$$y = Cx$$

is therefore not guaranteed to be stable for all switching signals $\sigma(t)$. Indeed, we can construct a destabilizing signal by switching from k_1 to k_2 when x_2^2 is a maximum (for that loop), and from k_2 to k_1 when x_1^2 is a maximum. This produces the unstable state trajectories shown in figure 6.3(a) from an initial state of $x_1 = x_2 = 1$, and zero reference.

Since the controller is static, we obviously cannot improve stability by resetting controller states! We can, however implement the controllers in a non-minimal form, for which stability may be guaranteed using the results of chapter 3. For this example, we use both the coprime factor approach, and the Hespanha approach.

Hespanha's scheme is implemented by expressing the controller as a model of the desired closed loop (with state transformation to ensure stability) in feedback with a model of the plant. We apply Hespanha's recommended resets (no reset, and total zero reset), and the results is seen in figures 6.3(b), and 6.3(c). Note that stability is obtained, but performance is still poor.

Since the plant is stable, $P = NM^{-1}$ where $N = P$, and $M = I$ is a coprime factorization of the plant. Since the respective closed loops are stable, then the factorizations $k_1 = V_1^{-1}U_1$ and $k_2 = V_2^{-1}U_2$ where

$$U_1(s) = \frac{k_1}{1 + k_1 P(s)} \qquad V_1(s) = \frac{1}{1 + k_1 P(s)}$$
$$U_2(s) = \frac{k_2}{1 + k_2 P(s)} \qquad V_2(s) = \frac{1}{1 + k_2 P(s)},$$

are also coprime, and furthermore the bezout identities $U_1 N + V_1 M = 1$ and $U_1 N + V_1 M = 1$ are satisfied (importantly both with the same factorization of the plant).

When we implement these controllers in the arrangement of figure 3.8 using the same initial condition and switching criterion as before (the non-minimal are initialized to zero), we obtain the stable trajectory shown in figure 3.8. In this case, while the non-minimal controller states do not affect the closed loop transfer functions, they do introduce transient signals at the switching times which ensure the stability of the system. Note however, that the performance is poor and the states take over 50 seconds to converge.

We now apply the results of theorem 6.9 to the loops formed by these non-minimal controllers. We find that there exist as expected, Lyapunov functions of the respective closed loops with common projection into plant-space. Hence we can find a stabilizing controller reset. This results in the stable trajectory shown in figure 6.3(e). Note the performance improvement obtained by using the extra freedom in the controller states at the switching times.

Since the reset scheme as applied for figure 6.3(e) requires full state knowledge, it is not quite a fair comparison with the (non-reset) coprime factor and Hespanha schemes. Therefore, we also implement the results using a plant state observer. The results, shown in figure 6.3(f) show that while performance is slightly worse than the full-state knowledge case, it is still substantially better than any of the other schemes.

6.6 Multiple Lyapunov functions for reset systems

Branicky's multiple Lyapunov function approach can be modified to admit systems in which the state may change discontinuously at switching times.

Theorem 6.6.1. *Consider a set of vector fields $\dot{x} = f_i(x)$ with $f_i(0) = 0$ for all $i \in I$. Let S be the set of anchored switching sequences associated with the system. Let $g_{i,j}$ be reset relations satisfying Lipschitz conditions $\left\|g_{i,j}(x) - g_{i,j}(y)\right\| \leq l_{i,j} \|x - y\|$ for some constants $l_{i,j}$.*

If there exist functions V_i such that over all switching sequences $S \in \mathcal{S}$, V_i is Lyapunov-like for $x_S(.)$ over $S|_i$, then the system is stable in the sense of Lyapunov.

Additionally, the system is asymptotically stable if for some ϵ_i and δ_i

$$\dot{V}_i(x(t)) < -\epsilon_i \|x(t)\|^2 \tag{6.21}$$

for t in $\mathcal{I}(\sigma|_i)$, and V_i are strictly decreasing on $\mathcal{E}(\sigma|_i)$:

$$V_i(x(t_k)) - V_i(x(t_j)) < -\delta_i \left\|x(t_j)\right\|^2, \tag{6.22}$$

where $t_j < t_k$, and $i_j = i_k = i$.

Proof. We begin by proving the case $N = 2$

Let $\epsilon > 0$ be arbitrary. Let $m_i(\alpha)$ denote the minimum value of V_i on $\partial B(\alpha)$. Choose $r_i < R$ such that in $B(r_i)$ we have $V_i < m_i(R)$. Let $r = \min_i(r_i)$. Thus starting in $B(r)$, trajectories of either vector field will remain in $B(R)$.

Let

$$l = \max_{i,j}(l_{j,i} + g_{j,i}(0)),$$

where $l_{j,i}$ is the Lipschitz constant for the reset relation $g_{j,i}$.

Now choose $\rho_i < r/l$ such that in $B(\rho_i)$, $V_i < m_i(r/l)$. Let $\delta = \min_i(\rho_i)$. Now when we begin in $B(\delta)$, we are guaranteed that trajectories will not leave $B(\epsilon)$ after a single switch, and further by the non-increasing condition on Lyapunov-like functions V_i that trajectories will not leave $B(\epsilon)$ in arbitrarily many switches.

To prove for general N, we consider a set of $2N + 2$ concentric circles, constructed similarly such that a trajectory beginning in the central region cannot escape the outer region by switching $(N - 1)$ times through all the alternative vector fields, and hence cannot escape in arbitrary switches via the non-increasing conditions on V_i.

To prove asymptotic stability, choose a particular element $i \in I$. Consider some $t \in \mathcal{I}(\sigma|_i)$. Let $\bar{t}$ denote the largest element in $\mathcal{E}(\sigma|_i)$ with $\bar{t} < t$ (that is, $\bar{t}$ is the last time that the i'th loop was engaged). Let $\underline{t}$ denote the smallest element in $\mathcal{E}(\sigma|_i)$ (that is, $\underline{t}$ is the first time that the i'th loop was engaged). Also let n_t denote the number of elements in $\mathcal{E}(\sigma|_i)$ smaller than t.

Since V_i is positive definite, and radially unbounded, there exist positive constants a_i and b_i such that

$$a_i \|x(t)\|^2 < V_i(x(t)) < b_i \|x(t)\|^2,$$

and from (6.21) we have

$$\dot{V}_i(x(t)) \leq -\epsilon_i a_i V_i(x(t)),$$

so $V_i(x(t))$ has the bound

$$V_i(x(t)) < e^{-(t-\bar{t})\epsilon_i a_i} V(x(\bar{t})).$$

Now from (6.22), we have the inequality

$$\begin{aligned} V_i(x(t_k)) &< V_i(x(t_j)) - \delta_i \left\| x(t_j) \right\|^2 \\ &< (1 - \frac{\delta_i}{b_i}) V_i(x(t_j)) \end{aligned}$$

when $t_j < t_k$ and $i_j = i_k = i$. Without loss of generality, we can choose δ_i such that $\delta_i < b_i$. Hence we can construct iteratively, the bound

$$V_i(x(\bar{t})) < (1 - \frac{\delta_i}{b_i})^{n_t} V_i(x(\underline{t}))$$

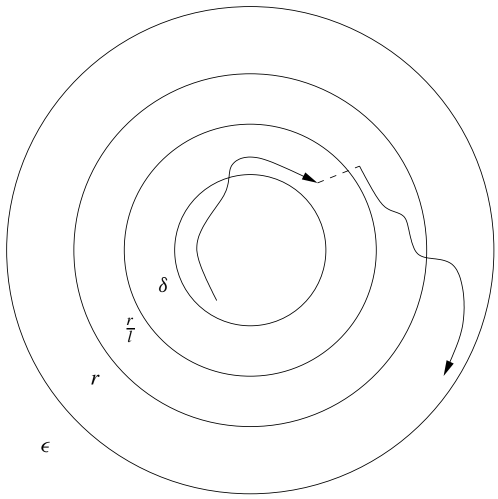


Figure 6.4: Multiple Lyapunov function illustration

and hence we have

$$V_i(x(t)) < e^{-(t-\bar{t})\epsilon_i a_i}(1 - \frac{\delta_i}{b_i})^{n_t} V_i(x(\underline{t})$$
$$< b_i e^{-(t-\bar{t})\epsilon_i a_i}(1 - \frac{\delta_i}{b_i})^{n_t} \left\|x(\underline{t})\right\|^2,$$

and

$$\|x(t)\|^2 < \frac{b_i}{a_i} e^{-(t-\bar{t})\epsilon_i a_i}(1 - \frac{\delta_i}{b_i})^{n_t} \tag{6.23}$$

In considering asymptotic stability, we need only consider those states i such that $t \in \mathcal{I}(\sigma|_i)$ has no upper bound. Hence for t in such states, as $t \to \infty$, either $(t - \bar{t}) \to \infty$, $n_t \to \infty$, or both. Since the inequality (6.23) holds for each $i \in I$, then

$$\lim_{t\to\infty} \|x(t)\|^2 \to 0$$

□

6.7 Dwell times and reset switching systems

If we wish to employ particular realizations for reset switching controllers for which stability is not guaranteed, it may be necessary to introduce a minimum dwell time. We can use the dwell times of section 3.3.1 in order to ensure stability of the switching system. However, we can additionally make use of the form of the reset in order to obtain a less conservative bound on the minimum dwell-time required in order to guarantee stability.

The idea is similar, in that we ensure that over any closed switching sequence beginning (and ending) with the i'th loop, the value of the i'th Lyapunov function decreases strictly.

Equivalently, we ensure that the level curve of the function at the end of the sequence fits strictly inside the level curve at the start.

Suppose we employ resets which minimize Lyapunov functions for the respective closed loops. In this case, we can exploit the minimization of the functions in order to obtain a better bound on the minimum dwell time.

Let

$$x = \begin{bmatrix} x_G \\ x_K \end{bmatrix},$$

where x_G is the plant state, and x_K the controller state.

Given a family of closed loop equations

$$\dot{x} = A_i x,$$

we have the Lyapunov functions $V_i = x^* P_i x$ for each loop $i \in \{1, 2, \dots, N\}$, where $P_i > 0$ is the solution to the Lyapunov equation

$$A_i^* P_i + P_i A_i = -Q_i$$

for some positive definite Q_i.

Furthermore, the controller state resets at each switch (to state i) such that

$$\begin{aligned} x_K &= \min_{x_K} V_i \left(\begin{bmatrix} x_G \\ x_K \end{bmatrix} \right) \\ &= -P_i(2,2)^{-1} P_i(2,1) x_G \end{aligned}$$

where each

$$P_i = \begin{bmatrix} P_i(1,1) & P_i(1,2) \\ P_i(2,1) & P_i(2,2) \end{bmatrix}$$

partitioned appropriately. Let Δ_i denote the Schur complements of $P_i(1,1)$ in each P_i. That is,

$$\Delta_i = P_i(1,1) - P_i(1,2) P_i^{-1}(2,2) P_i(2,1).$$

Now, from lemma 3.3.1 we know that while the i'th loop is active, we have the bound

$$V_i(x(t_0 + \tau)) \leq e^{-\lambda_i \tau} V(x(t_0))$$

where λ_i is the minimum eigenvalue of the matrix $P_i^{-1} Q_i$.

We can compute the maximum gain k_{ij} from V_i to V_j caused by a single switch from loop i to loop j. That is,

$$
\begin{aligned}
k_{ij} &= \max_x \left(\min_{x_K} \left(V_j \left(\begin{bmatrix} x_G \\ x_K \end{bmatrix} \right) \right) \right) \text{ such that } V_i(x) = 1 \\
&= \max_{x_G} (x_G^* \Delta_j x_G) \text{ such that } V_i(x) = 1 \\
&= \max_{x_G} (x_G^* \Delta_j x_G) \text{ such that } x_G^* \Delta_i x_G \leq 1
\end{aligned}
$$

Since the constraint and cost are both quadratic and positive, the maximum must occur on the boundary of the constraint. That is,

$$
\max_{x_G} (x_G^* \Delta_j x_G) \text{ such that } x_G^* \Delta_i x_G = 1,
$$

which is simply the maximum eigenvalue of $\Delta_i^{-1} \Delta_j$.

Theorem 6.7.1. *Let k_{ij} be the maximum eigenvalue of $\Delta_i^{-1} \Delta_j$, and λ_i the minimum eigenvalue of $P_i^{-1} Q_i$.*

If we choose τ such that

$$
\tau > \sup_{l \in \mathscr{L}[N]} \left(\frac{1}{\sum_{i \in l} \lambda_i} \ln \left(\prod_{ij \in l} k_{ij} \right) \right), \tag{6.24}
$$

then the reset switching system

$$
\begin{aligned}
\dot{x} &= A_{\sigma(t)} x, \\
\sigma(t) &= i_k, \quad \forall \;\; t_k \leq t < t_{k+1}, \quad i_k \in \{1, 2, \ldots, N\}, \quad k \in \mathbb{Z}^+, \\
x(t_k^+) &= -P_{i_k}^{-1}(2,2) P_{i_k}(2,1)(x(t_k^-))
\end{aligned} \tag{6.25}
$$

is guaranteed to be stable for all switching sequences with a minimum dwell time of τ.

Proof. The proof is identical to that of theorem 3.3.2, using the straightforward extension of Branicky's theorem to reset switching systems 6.6.1. □

6.8 Concluding remarks

In this chapter we have investigated the relationship between state resets in a switching system, and stability under arbitrary switching.

Theorem 6.1.1 is an important result, as it provides a useful and intuitive extension of Lyapunov theory to reset switching systems, and allows us to analyse stability over arbitrary switching sequences for such systems.

We have made a number of interesting observations about the relationship between stabilizing resets and Lyapunov functions. The most important of these, is that stabilizing resets of the form 6.6 must minimize Lyapunov functions. This observation leads directly to techniques for synthesizing stabilizing resets.

The sufficient LMI conditions in theorems 6.3.2 and 6.3.4 allow us to synthesize linear resets which guarantee stability under arbitrary switching.

Resets which are synthesized in this way are not directly designed for performance, however we have observed that they often give very good performance results. The structure of such resets is the same as those designed for performance over a single switch in chapter 5. When the LMI method has a stabilizing solution, there will generally be many solutions. Therefore there is scope for further improving the performance of these methods by choosing among stabilizing resets, one which maximizes performance in some sense.

Reset synthesis methods may be combined with realization methods in order to guarantee stability, and improve performance as illustrated by example 6.5.1.

Chapter 7

Conclusions

In this thesis we have studied the stability and performance of systems with switching linear controllers.

Our general approach fits into the second and thirds stages of a four stage design procedure for switching systems. That is, we are concerned with the appropriate realization and implementation of controllers, and with the design of appropriate reset relations.

We assume that the transfer matrices of the controllers have been designed without taking into account the switching behaviour of the system. We also generally assume that the design of switching trajectories occurs in a separate stage. Thus we generally design such that the system is guaranteed to be stable under an arbitrary choice of switching signal.

Our consideration of the switching strategy is limited to some small extensions to existing techniques for calculating minimum dwell times for guaranteed stability. Note that minimum dwell times are not necessary if controllers have been realized for stability. Minimum dwell times are only necessary for stability if controller realizations are fixed according to other criteria.

We have observed that given a family of stabilizing controllers, stability may be guaranteed by choosing appropriate controller realizations. This may be achieved by IMC methods introduced by Hespanha, or by coprime factor methods based on the ideas of Miyamoto and Vinnicombe.

The choice of controller realization also has an impact on system performance. In chapter 4, we solve an optimization problem with respect to a single transition which leads to a means of realizing controllers for performance.

Reset design is an important problem in switching system design. We note that the choice of reset relations for switching controllers can have profound effects on both stability and performance. In chapter 5, we designed a reset strategy which is optimal with respect to future performance over a single switch.

In chapter 6, we introduced some new Lyapunov methods for analysing reset switching systems. These methods were then applied to develop an LMI method for synthesizing stabilizing reset relations.

7.1 Summary of main contributions

Chapter 2

We introduce the notion of *reset switching systems* to denote switching systems where the state of the system can change discontinuously at transition times.

Chapter 3

We introduce a new method for calculating minimum dwell times for switching systems, which is less conservative than previous results.

We apply the ideas of Miyamoto and Vinnicombe (from the anti-windup problem) to show that stability under arbitrary switching can be achieved by realizing controllers in appropriate coprime factor form. We note that this result can also provide joint stability under switching and saturation, if the Youla parameter is chosen according to anti-windup results.

Chapter 4

We solve an optimization problem with for a single controller transition. We select the controller state which corresponds to signals which match observed signals most closely with respect to a weighted norm.

We show that this optimal solution is equivalent (with appropriately chosen weights) to a time-varying Kalman filter observing the state of the controller being switched on. The infinite horizon solution is equivalent to a time-invariant filter, and can be implemented in coprime factor form.

We show by example that these results can give a substantial improvement in the performance of a system under switching.

Chapter 5

We solve an alternative optimization problem explicitly over weighted future performance. The solution is equivalent to minimization of a Lyapunov function for the new closed loop. The method is demonstrated by example.

Chapter 6

We introduce a new Lyapunov theorem for the stability under arbitrary switching of *reset switching systems*. A number of consequences of the theorem are noted, including a simple quadratic Lyapunov function extension, and a necessary condition for the theorem.

We also show how the theorem applies when the reset is restricted to a particular structure (continuous plant state, controller reset according to plant). This gives a necessary and sufficient condition for the existence of stabilizing resets of such structure. These results lead to LMI methods for constructing stabilizing resets in continuous or discrete time. The results are shown to hold even when exact plant state information is not available, provided that a stable plant estimator is used.

We show how such methods can be combined with the results of chapter 3 to reduce computational requirements and improve performance, and demonstrate with an interesting illustrative example.

We prove a trivial extension of Branicky's multiple Lyapunov function work to reset switching systems, and use the result to prove an extension of our earlier minimum dwell-time result to reset switching systems.

7.2 Opportunities for further investigation

Chapter 3

When using the coprime factor approach to realizing controllers for stability under arbitrary switching (without saturation), there is freedom available in the choice of Youla parameter Q. Further investigation is needed to find ways in which this freedom may be exploited to further improve performance.

The minimum dwell-time results are dependent on the choice of Lyapunov function (as are existing results), and hence potentially very conservative. Further investigation could lead to methods for choosing Lyapunov functions such that the resulting minimum dwell-times are minimized.

Chapter 6

The main Lyapunov theorem is proved in the restricted scenario of linear systems and linear resets. Further investigation may lead to a more general result for nonlinear systems with nonlinear resets.

When restricting consideration to plant/controller scenarios, the admissible reset is restricted so that the controller is reset as a function of plant state. While this choice is justifiable in general terms, it would be interesting to discover whether the related results hold in the more general scenario where the controller is reset according to both plant and controller states.

In the main LMI results, there are potentially many solutions which yield stabilizing resets. Further work is required to show whether we can use the available freedom to further improve the performance of the resulting scheme.

7.2.1 General

We have taken some idealized viewpoints of the switching problem in order to obtain problems that are solvable. The scenario of a single linear plant controlled by a number of alternative controllers will seldom be the case in practice.

It should be noted however, that many of the results of this thesis trivially extend to cases where the plant switches between linear models of a similar structure (with state continuity), and the controller transitions are synchronized with the plant transitions. In particular, the results of chapters 5, and 6 where the results are based on closed loop dynamics extend in this way.

It is hoped that the results will be useful in a more general scenario, however care must clearly be taken in applying such results outside the scope of the theoretical guarantees.

There is much scope for further investigation of these results, particularly in relation to the robustness of such results under model uncertainty and noisy measurements. We would for example like to be able to make solid guarantees in the case where we have a nonlinear plant which is *covered* by a family of uncertain linear plants, and a family of controllers which each robustly control one of the uncertain linear plants.

Appendix A

Some basic results

A.1 Matrix inversion lemma

Let A be a square matrix partitioned as follows

$$A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Lemma A.1.1. *Suppose A_{11} and A_{22} are nonsingular. Then the following identity holds, when $A_{11} - A_{12}A_{22}^{-1}A_{21}$ is invertible.*

$$\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} = A_{11}^{-1} + A_{11}^{-1}A_{12}\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} A_{21}A_{11}^{-1}. \qquad \text{(A.1)}$$

For a derivation, see [62].

A.2 Least squares optimisation

Lemma A.2.1. *Define the cost function*

$$V = \begin{bmatrix} z \\ x \end{bmatrix}^* \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix},$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} > 0,$$

Then the minimum of V with respect to x is

$$\min_x V = z^* \left(P_{11} - P_{12}P_{22}^{-1}P_{21}\right) z,$$

achieved when

$$x = -P_{22}^{-1} P_{21} z$$

Proof. Completing the square on V, we obtain:

$$\begin{aligned} V &= \begin{bmatrix} z \\ x \end{bmatrix}^* \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} \\ &= z^* P_{11} z + x^* P_{22} x + z^* P_{12} x + x^* P_{21} z \\ &= \left(x + P_{22}^{-1} P_{21} z\right)^* P_{22} \left(x + P_{22}^{-1} P_{21} z\right) + z^* \left(P_{11} - P_{12} P_{22}^{-1} P_{21}\right) z, \end{aligned}$$

and so the minimum is clearly $z^*(P_{11} - P_{12} P_{22}^{-1} P_{21}) z$, achieved when $x = -P_{22}^{-1} P_{21} z$. □

Lemma A.2.2. *Define the cost function*

$$V = \|z - Hx\|^2 .$$

Then, the minimum of V *with respect to* x *is*

$$\min_x V = \left\|(I - HH^\dagger) z\right\|^2 .$$

achieved when

$$x = H^\dagger z,$$

where $H^\dagger$ *is the left pseudo inverse of* H.

Proof. By completing the square on V:

$$\begin{aligned} V &= (z - Hx)^*(z - Hx) \\ &= z^* z + x^* H^* H x - x^* H^* z - z^* H x \\ &= \begin{bmatrix} z \\ x \end{bmatrix}^* \begin{bmatrix} I & -H \\ -H^* & H^* H \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} \\ &= \left[x - \left(H^* H\right)^{-1} H^* z\right]^* \left(H^* H\right) \left[x - \left(H^* H\right)^{-1} H^* z\right] + z^* \left(I - H \left(H^* H\right)^{-1} H^*\right) z. \end{aligned}$$

So the minimum with respect to x is achieved when

$$x = \left(H^* H\right)^{-1} H^* z = H^\dagger z,$$

and the corresponding minimum of V is

$$\min_x V = \left\|(I - HH^\dagger) z\right\|^2 .$$

□

A.3 Closed loop equations

A.3.1 Continuous-time

Consider the feedback arrangement in figure A.1.

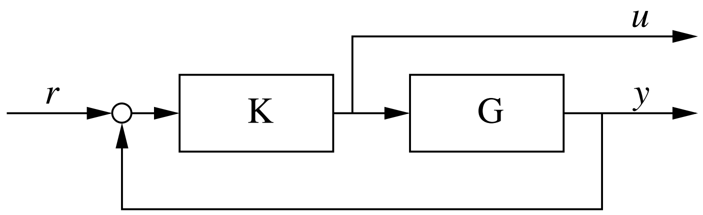


Figure A.1: System after switch

Let the plant state be $x_G(t) \in \mathbb{R}^{n_G}$, and the controller state $x_K(t) \in \mathbb{R}^{n_K}$.

Suppose G and K have the following continuous-time state-space representations:

$$G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right] \qquad K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]. \tag{A.2}$$

Then we can write the following equations

$$\begin{aligned}
\begin{bmatrix} \dot{x}_G \\ \dot{x}_K \end{bmatrix} &= \begin{bmatrix} A_G & 0 \\ 0 & A_K \end{bmatrix}\begin{bmatrix} x_G \\ x_K \end{bmatrix} + \begin{bmatrix} B_G & 0 \\ 0 & -B_K \end{bmatrix}\begin{bmatrix} u \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ B_K \end{bmatrix} r, \\
\begin{bmatrix} u \\ y \end{bmatrix} &= \begin{bmatrix} 0 & C_K \\ C_G & 0 \end{bmatrix}\begin{bmatrix} x_G \\ x_K \end{bmatrix} + \begin{bmatrix} 0 & D_K \\ -D_G & 0 \end{bmatrix}\begin{bmatrix} u \\ y \end{bmatrix} + \begin{bmatrix} D_K \\ 0 \end{bmatrix} r \\
&= \begin{bmatrix} I & -D_K \\ D_G & I \end{bmatrix}^{-1}\begin{bmatrix} 0 & C_K \\ C_G & 0 \end{bmatrix}\begin{bmatrix} x_G \\ x_K \end{bmatrix} + \begin{bmatrix} I & -D_K \\ D_G & I \end{bmatrix}^{-1}\begin{bmatrix} D_K \\ 0 \end{bmatrix} r.
\end{aligned}$$

Now we can write the closed loop state-space equations

$$\begin{aligned}
\begin{bmatrix} \dot{x}_G \\ \dot{x}_K \end{bmatrix} = &\left(\begin{bmatrix} A_G & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_G & 0 \\ 0 & -B_K \end{bmatrix}\begin{bmatrix} I & -D_K \\ D_G & I \end{bmatrix}^{-1}\begin{bmatrix} 0 & C_K \\ C_G & 0 \end{bmatrix}\right)\begin{bmatrix} x_G \\ x_K \end{bmatrix} + \\
&\left(\begin{bmatrix} 0 \\ B_K \end{bmatrix} + \begin{bmatrix} I & -D_K \\ D_G & I \end{bmatrix}^{-1}\begin{bmatrix} D_K \\ 0 \end{bmatrix}\right) r
\end{aligned}, \tag{A.3}$$

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} I & -D_K \\ D_G & I \end{bmatrix}^{-1}\begin{bmatrix} 0 & C_K \\ C_G & 0 \end{bmatrix}\begin{bmatrix} x_G \\ x_K \end{bmatrix} + \begin{bmatrix} B_G & 0 \\ 0 & -B_K \end{bmatrix}\begin{bmatrix} I & -D_K \\ D_G & I \end{bmatrix}^{-1}\begin{bmatrix} D_K \\ 0 \end{bmatrix} r. \tag{A.4}$$

Define the following matrices:

$$A = \begin{bmatrix} A_G & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_G & 0 \\ 0 & -B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ D_G & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_G & 0 \end{bmatrix}, \tag{A.5}$$

$$B = \begin{bmatrix} 0 \\ B_K \end{bmatrix} + \begin{bmatrix} B_G & 0 \\ 0 & -B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ D_G & I \end{bmatrix}^{-1} \begin{bmatrix} D_K \\ 0 \end{bmatrix}, \tag{A.6}$$

$$C = \begin{bmatrix} I & -D_K \\ D_G & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_G & 0 \end{bmatrix}, \tag{A.7}$$

$$D = \begin{bmatrix} I & -D_K \\ D_G & I \end{bmatrix}^{-1} \begin{bmatrix} D_K \\ 0 \end{bmatrix}. \tag{A.8}$$

Then the closed loop equations can be written

$$\begin{bmatrix} \dot{x_G} \\ \dot{x_K} \end{bmatrix} = A \begin{bmatrix} x_G \\ x_K \end{bmatrix} + Br, \tag{A.9}$$

$$\begin{bmatrix} u \\ y \end{bmatrix} = C \begin{bmatrix} x_G \\ x_K \end{bmatrix} + Dr. \tag{A.10}$$

To guarantee the absence of algebraic loops, it will often be assumed that $I + D_K D_G = I + D_G D_K = I$ (usually D_G or D_K zero). When this is the case

$$\begin{bmatrix} I & -D_K \\ D_G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & D_K \\ -D_G & I \end{bmatrix},$$

and the closed loop state-space matrices can be written

$$A = \begin{bmatrix} A_G + B_G D_K C_G & B_G C_K \\ -B_K C_G & A_K + B_K D_G C_K \end{bmatrix}, \tag{A.11}$$

$$B = \begin{bmatrix} B_G D_K \\ B_K \end{bmatrix}, \tag{A.12}$$

$$C = \begin{bmatrix} D_K C_G & C_K \\ C_G & -D_G C_K \end{bmatrix}, \tag{A.13}$$

$$D = \begin{bmatrix} D_K \\ 0 \end{bmatrix}. \tag{A.14}$$

A.3.2 Discrete-time

Let $k \in \mathbb{Z}^+$ be the the discrete-time variable. Let the plant state be $x_G(k) \in \mathbb{R}^{n_G}$, and the controller state $x_K(k) \in \mathbb{R}^{n_K}$.

Suppose G and K have the following discrete-time state-space representations:

$$G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right] \qquad K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]. \tag{A.15}$$

Then in similar fashion to the continuous-time case, we can write the closed loop equations

$$\begin{bmatrix} x_G(k+1) \\ x_K(k+1) \end{bmatrix} = A \begin{bmatrix} x_G(k) \\ x_K(k) \end{bmatrix} + Br(k), \tag{A.16}$$

$$\begin{bmatrix} u(k) \\ y(k) \end{bmatrix} = C \begin{bmatrix} x_G(k) \\ x_K(k) \end{bmatrix} + Dr(k). \tag{A.17}$$

where

$$A = \begin{bmatrix} A_G & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_G & 0 \\ 0 & -B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ D_G & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_G & 0 \end{bmatrix}, \tag{A.18}$$

$$B = \begin{bmatrix} 0 \\ B_K \end{bmatrix} + \begin{bmatrix} B_G & 0 \\ 0 & -B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ D_G & I \end{bmatrix}^{-1} \begin{bmatrix} D_K \\ 0 \end{bmatrix}, \tag{A.19}$$

$$C = \begin{bmatrix} I & -D_K \\ D_G & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_G & 0 \end{bmatrix}, \tag{A.20}$$

$$D = \begin{bmatrix} I & -D_K \\ D_G & I \end{bmatrix}^{-1} \begin{bmatrix} D_K \\ 0 \end{bmatrix}. \tag{A.21}$$

When $I + D_K D_G = I + D_G D_K = I$, these simplify to

$$A = \begin{bmatrix} A_G + B_G D_K C_G & B_G C_K \\ -B_K C_G & A_K + B_K D_G C_K \end{bmatrix}, \tag{A.22}$$

$$B = \begin{bmatrix} B_G D_K \\ B_K \end{bmatrix}, \tag{A.23}$$

$$C = \begin{bmatrix} D_K C_G & C_K \\ C_G & -D_G C_K \end{bmatrix}, \tag{A.24}$$

$$D = \begin{bmatrix} D_K \\ 0 \end{bmatrix}. \tag{A.25}$$

A.4 Continuous-time Lyapunov equation

Theorem A.4.1. *Given a stable matrix A, and $Q > 0$, the solution P to the continuous-time Lyapunov equation*

$$A^* P + PA = -Q \tag{A.26}$$

is given by

$$P = \int_0^\infty e^{A^*\tau} Q e^{A\tau} d\tau, \tag{A.27}$$

Proof. Define the trajectory $x(t)$ as

$$\dot{x}(t) = Ax(t),$$

and let

$$V(t) = x^*(t)Px(t).$$

Then

$$\begin{aligned}\dot{V}(t) &= x^*(t)(A^*P + PA)x(t) \\ &= -x^*(t)Qx(t).\end{aligned}$$

Now we can write

$$\begin{aligned}&\int_0^t \dot{V}(x(\tau))d\tau = V(x(t)) - V(x(0)) \\ \implies -x^*(0)&\left(\int_0^t e^{A^*\tau} Q e^{A\tau} d\tau\right) x(0) = x^*(0)e^{A^*t} P e^{At} x(0) - x^*(0)Px(0).\end{aligned}$$

Since A is stable, we know that

$$\lim_{t\to\infty} e^{At} = 0,$$

so when we take the above integrals to ∞, we obtain

$$P = \int_0^\infty e^{A^*\tau} Q e^{A\tau} d\tau$$

$\square$

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