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# Symmetric alternating sign matrices

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**Abstract**

In this note we consider completions of $n \times n$ symmetric $(0, -1)$-matrices to symmetric alternating sign matrices by replacing certain 0s with +1s. In particular, we prove that any $n \times n$ symmetric $(0, -1)$-matrix that can be completed to an alternating sign matrix by replacing some 0s with +1s can be completed to a symmetric alternating sign matrix. Similarly, any $n \times n$ symmetric $(0, +1)$-matrix that can be completed to an alternating sign matrix by replacing some 0s with −1s can be completed to a symmetric alternating sign matrix.

## 1 Introduction

An *alternating sign matrix*, abbreviated ASM, is an $n \times n$ $(0, +1, -1)$-matrix such that, ignoring 0s, in each row and column, the +1s and −1s alternate, beginning and ending with a +1. An ASM cannot contain any −1s in rows 1 and $n$ and columns 1 and $n$. The book [1] by Bressoud contains a history of the development of ASMs.

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In [2], there is an investigation of the zero-nonzero patterns of ASMs. The paper [3] considers the problem of completing a $(0, -1)$-matrix to an ASM by replacing some 0s with $+1$s. Each row and column of an ASM contains an odd number of nonzeros with the first and last rows and columns each containing exactly one nonzero and that nonzero is a $+1$. If an ASM (regarded as a square) is subjected to any of the symmetries of a square (the dihedral group), the result is also an ASM.

The simplest examples of ASMs are the permutation matrices. Other examples of ASMs are

$$\left[\begin{array}{c|c|c|c|c|c|c} & & & +1 & & & \\ \hline & & +1 & -1 & & +1 & \\ \hline & +1 & -1 & & +1 & & \\ \hline +1 & -1 & & +1 & & & \\ \hline & & +1 & & & -1 & +1 \\ \hline & +1 & & & -1 & +1 & \\ \hline & & & & +1 & & \end{array}\right] \text{ and } \left[\begin{array}{c|c|c|c|c|c} & & & +1 & & \\ \hline & & +1 & -1 & +1 & \\ \hline & +1 & -1 & +1 & -1 & +1 \\ \hline +1 & -1 & +1 & -1 & +1 & \\ \hline & +1 & -1 & +1 & & \\ \hline & & +1 & & & \end{array}\right].$$

(For visual clarity, we usually block off rows and columns and then suppress the 0s in $(0, +1, -1)$-matrices.)

Our emphasis in this note is on combinatorial properties of symmetric ASMs, of which the preceding two ASMs are examples. Given an $n \times n$ $(0, -1)$-matrix $A$, any matrix $B$ obtained from $A$ by replacing some 0s by $+1$s is a $(+1)$-*completion* of $A$; if $B$ is an ASM, then $B$ is called a $(+1)$-*completion of* $A$ *to an ASM* or an *ASM* $(+1)$-*completion of* $A$. In [3] ASM $(+1)$-completions of $(0, -1)$-matrices (called, simply, ASM completions) were investigated with an emphasis on the so-called bordered-permutation $(0, -1)$-matrices. By an $n \times n$ *bordered-permutation* $(0, -1)$-*matrix* $A$ we mean an $n \times n$ $(0, -1)$-matrix such that the first and last rows and columns contain only zeros, and the submatrix $A[\{2, 3, \ldots, n-1\}|\{2, 3, \ldots, n-1\}]$ obtained by deleting rows and columns 1 and $n$ is $-P$ where $P$ is a permutation matrix. Here we consider $(+1)$-completions of symmetric $(0, -1)$-matrices to symmetric ASMs.

We also consider here completions of an $n \times n$ $(0, +1)$-matrix $A$ to ASMs by replacing some 0s with $-1$s. We call these *ASM* $(-1)$-*completions* . In order that $A$ has an ASM $(-1)$-completion, it is necessary that there be at least one $+1$ in each row and column, only one $+1$ in the first and last rows and columns, and no consecutive $+1$s in a row or column.

**Example 1** Let $A$ be the symmetric bordered-permutation $(0, -1)$-matrix:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | | |
| | | | | | | −1 | |
| | | | | | −1 | | |
| | | | −1 | | | | |
| | | | | −1 | | | |
| | | −1 | | | | | |
| | −1 | | | | | | |
| | | | | | | | |

.

Then it is straightforward to check that $A$ has a unique $(+1)$-completion to an ASM and this $(+1)$-completion is symmetric:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | +1 | |
| | | | | | +1 | −1 | +1 |
| | | | +1 | | −1 | +1 | |
| | | +1 | −1 | +1 | | | |
| | | | +1 | −1 | +1 | | |
| | +1 | −1 | | +1 | | | |
| +1 | −1 | +1 | | | | | |
| | +1 | | | | | | |

.

On the other hand, the symmetric $(0, -1)$-matrix

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | | |
| | | | | | | −1 | |
| | | | −1 | | −1 | | |
| | | −1 | | | | | |
| | | | | | | | |
| | | −1 | | | | | |
| | −1 | | | | | | |
| | | | | | | | |

does not have a $(+1)$-completion to an ASM; it suffices to examine rows 1, 2, and 3.

**Example 2** Consider the $7 \times 7$ symmetric $(0, +1)$-matrix

$A =$

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | +1 | | |
| | | | +1 | | | |
| | | +1 | | | +1 | |
| | +1 | | | | | +1 |
| +1 | | | | +1 | | |
| | | +1 | | | | |
| | | | +1 | | | |

.

The $-1$s in any $(-1)$-completion of $A$ to an ASM must be in the shaded positions. Any $(-1)$-completion of $A$ must have three $-1$s. There are three $(-1)$-completions of $A$, namely, as given below, the matrix $A'$ and its transpose, and the symmetric matrix $A''$:

$A' =$

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | +1 | | |
| | | | +1 | | | |
| | | +1 | −1 | | +1 | |
| | +1 | | | −1 | | +1 |
| +1 | | −1 | | +1 | | |
| | | +1 | | | | |
| | | | +1 | | | |

, $A'' =$

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | +1 | | |
| | | | +1 | | | |
| | | +1 | | −1 | +1 | |
| | +1 | | −1 | | | +1 |
| +1 | | −1 | | +1 | | |
| | | +1 | | | | |
| | | | +1 | | | |

.

In [3] it was shown that every bordered-permutation $(0, -1)$-matrix can be $(+1)$-completed to an ASM. We first show that every $n \times n$ symmetric bordered-permutation $(0, -1)$-matrix can be $(+1)$-completed to a symmetric ASM and obtain a bound on the number of such $(+1)$-completions. There is not an analogue of this result for $(-1)$-completions, since a permutation matrix is already an ASM. Our main results are that (i) if a symmetric $(0, -1)$-matrix has an ASM $(+1)$-completion, then it also has a symmetric ASM $(+1)$-completion, and (ii) if a symmetric $(0, +1)$-matrix has an ASM $(-1)$-completion, then it also has a symmetric ASM $(-1)$-completion.

## 2 Symmetric ASM Completions

**Theorem 3** *Let $A = [a_{ij}]$ be an $n \times n$ symmetric bordered-permutation $(0, -1)$-matrix. Then $A$ has a $(+1)$-completion to a symmetric ASM.*

*Proof.* This theorem will follow from Theorem 7 and the theorem in [3] that every bordered-permutation $(0, -1)$-matrix can be $(+1)$-completed to an ASM. We give a short independent proof.

We use induction on $n$. The theorem is trivial if $n = 2$ or 3. Let $n \geq 4$. Let $k$ be such that $a_{2k} = a_{k2} = -1$. Let $A' = A(2, k|2, k)$ be the symmetric matrix obtained from $A$ by deleting rows and columns 2 and $k$. (This matrix is $(n - 1) \times (n - 1)$ if $k = 2$ and $(n-2) \times (n-2)$ otherwise.) We use for the indices of the row and columns of $A'$ the same indices they had in $A$; thus the index set for rows and columns of $A'$ is $\{1, 2, \ldots, n\} \setminus \{2, k\}$. By induction $A'$ has a $(+1)$-completion $B' = [b'_{ij}]$ to a symmetric ASM. Let $r$ be such that $b'_{1r} = b'_{r1} = +1$.

If $r > k$, let $s$ be the first integer such that $b'_{s,k-1} = b'_{k-1,s} = +1$. We then let $B$ be the matrix which has +1s in all other positions that $B'$ has +1s except for the positions $(1, r), (r, 1), (s, k - 1),$ and $(k - 1, s)$ and, in addition, has +1s in positions $(1, k), (k, 1), (2, k - 1), (k - 1, 2), (2, r), (r, 2), (s, k), (k, s)$. Then $B$ is a symmetric ASM $(+1)$-completion of $A$.

If $r < k$, let $s$ be the first integer such that $b'_{s,k+1} = b'_{k+1,s} = +1$. We then let $B$ be the matrix which has +1s in all other positions that $B'$ has +1s except for the positions $(1, r), (r, 1), (s, k+1)$, and $(k+1, s)$ and, in addition, has +1s in positions $(1, k), (k, 1), (2, k + 1), (k + 1, 2), (2, r), (r, 2), (s, k), (k, s)$. Then $B$ is a symmetric ASM (+1)-completion of $A$. □

We give an example illustrating the inductive proof of Theorem 3.

**Example 4** In this example, $n = 8$ and with the above notation, $k = 4$, $r = 6$, and $s = 5$. Let

$A=$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | | |
| | | | −1 | | | | |
| | | | | | −1 | | |
| | −1 | | | | | | |
| | | | | −1 | | | |
| | | −1 | | | | | |
| | | | | | | −1 | |
| | | | | | | | |

$\rightarrow A' =$

| | | | | | |
|---|---|---|---|---|---|
| | | | | | |
| | | | −1 | | |
| | | −1 | | | |
| | −1 | | | | |
| | | | | −1 | |
| | | | | | |

.

Then, where we have included the row and column indices for clarity, we have

$B' =$

| | 1 | 3 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| 1 | | | | +1 | | |
| 3 | | | +1 | −1 | +1 | |
| 5 | | +1 | −1 | +1 | | |
| 6 | +1 | −1 | +1 | | | |
| 7 | | +1 | | | −1 | +1 |
| 8 | | | | | +1 | |

$\rightarrow$

$B =$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | | | | +1 | | | | |
| 2 | | | +1 | −1 | | +1 | | |
| 3 | | +1 | | | | −1 | +1 | |
| 4 | +1 | −1 | | | +1 | | | |
| 5 | | | | +1 | −1 | +1 | | |
| 6 | | +1 | −1 | | +1 | | | |
| 7 | | | +1 | | | | −1 | +1 |
| 8 | | | | | | | +1 | |

.

If $A$ is a symmetric $(0, -1)$-matrix, then $\pi_s(A)$ denotes the number of (+1)-completions of $A$ to a symmetric ASM. By Theorem 3, if $A$ is also a bordered-permutation $(0, -1)$-matrix, then $\pi_s(A) \geq 1$. We now give an upper bound for $\pi_s(A)$ in general.

**Theorem 5** *Let $A = [a_{ij}]$ be an $n \times n$ bordered symmetric $(0, -1)$-matrix such that $A$ has $a$ $-1$s on the main diagonal and $2b$ $-1$s off the main diagonal. Then*

$$\pi_s(A) \leq \frac{1}{2^{a+b}} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k(n-2k)!k!}. \tag{1}$$

(The number $\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k(n-2k)!k!}$ is the number of $n \times n$ symmetric permutation matrices [5, p. 218].)

*Proof.* Let $k$ be the maximum index of a row of $A$ with a $-1$, and let $l$ be the maximum index of a column with a $-1$ in row $k$. Thus $a_{kl} = a_{lk} = -1$ and we let $B = [b_{ij}]$ be the symmetric matrix obtained from $A$ by replacing $a_{kl}$ and $a_{lk}$ with 0s. We show that $\pi_s(A) \leq \frac{\pi_s(B)}{2}$ by establishing, when $\pi_s(A) \neq 0$, a one-to-two correspondence from the set $C_s(A)$ of symmetric ASM (+1)-completions of $A$ to the set $C_s(B)$ of symmetric ASM (+1)-completions of $B$. We consider two cases depending on whether $k \neq l$ or $k = l$.

Case 1 ($k \neq l$): Let $A' = [a'_{ij}] \in C_s(A)$. There exists $l' > l$ such that $a'_{kl'} = +1$. We choose $l'$ to be the smallest such integer so that $a'_{kp} = 0$ for all $p$ with $l < p < l'$. There also exists $k' > k$ such that $a'_{k'l} = +1$, and we choose $k'$ to be the smallest such integer so that $a'_{ql} = 0$ for all $q$ with $k < q < k'$. We then define $B' = [b'_{ij}]$ to be the matrix obtained from $A'$ by replacing $a'_{kl}, a'_{k'l}, a'_{kl'}$, and also $a'_{lk}, a'_{lk'}, a'_{l'k}$, with 0s, and replacing $a'_{k'l'}$ and $a'_{l'k'}$ (both of which must equal 0) with +1s. The matrix $B'$ is an ASM (+1)-completion of $B$, and the map $f : C_s(A) \to C_s(B)$ given by $A' \to B'$ is injective. In a similar way by choosing the first +1 to the left of $a'_{kl} = -1$ we obtain another injective map $g : C_s(A) \to C_s(B)$. We have that $g(C_s(A)) \cap f(C_s(A)) = \emptyset$. Thus, in the case that $k \neq l$, each (+1)-completion of $A$ gives two (+1)-completions of $B$.

Case 2 ($k = l$): Thus $a_{kk} = -1$ and the principal submatrix $A[k, k+1, \ldots, n | k, k+1, \ldots, n]$ of $A$ determined by rows and columns $k, k+1, \ldots, n$ has a unique $-1$ and this $-1$ is in its $(1, 1)$-position. Let $A' = [a'_{ij}] \in C_s(A)$. In $A'$ there is a unique +1 to the right of $a'_{kk} = -1$, say in column $r$ and a unique +1 below it, so in row $r$. The principal submatrix $A'[k, k+1, \ldots, n | k, k+1, \ldots, n]$ of $A$ is a symmetric permutation matrix with an additional $-1$ in its $(1, 1)$-position. Let $B'$ be the matrix obtained from $A'$ by replacing $a'_{kk} = -1, a_{kr} = +1, a'_{rk} = +1$ with 0s and replacing $a_{rr} = 0$ with +1. Then $B'$ is an ASM (+1)-completion of $B$. In a similar way, we determine in $A'$ the largest integer $p$ with $p < k$ such that $a_{kp} = +1$, and thus $a_{pk} = +1$. Let $B'$ be the matrix obtained from $A'$ by replacing $a'_{kk} = -1$ with +1, replacing $a_{kp} = a_{kr} = a_{rk} = a_{pk} = +1$ with 0s, and replacing $a'_{rp} = a'_{pr} = 0$ with +1. Then $B'$ is an ASM (+1)-completion of $B$. As before we have two injections of $C_s(A)$ into $C_s(B)$ with disjoint images, and thus each (+1)-completion of $A$ gives two (+1)-completions of $B$.

Iterating the above, we see that $\pi_s(A) \leq \frac{\pi_s(C)}{2^{a+b}}$ where $C$ is the $n \times n$ zero matrix.

The number of ASM (+1)-completions of $C$ is the number of symmetric permutation matrices, and the theorem now follows. □

We note that equality occurs in (1) if $A = O$.

In the proof of the next theorem we shall make use of an idea from [3]. Let $A$ be an $n \times n$ $(0, -1)$-matrix and assume that $A$ can be (+1)-completed to an ASM. Let $\sigma(A)$ equal the number of $-1$s in $A$. Let $Z \subseteq \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ be the set of zero positions of $A$. The $-1$s of $A$ partition $Z$ into two families of $(n + \sigma(A))$ sets, the *horizontal partition* $\mathcal{H}(A) = (H_i : 1 \leq i \leq n + \sigma(A))$, consisting of the *horizontal blocks*, and the *vertical partition* $\mathcal{V}(A) = (V_i : 1 \leq i \leq n + \sigma(A))$, consisting of the *vertical blocks*. These are defined as follows: If there are $c_i \geq 0$ $-1$s in row $i$ of $A$, then row $i$ determines the $c_i + 1$ horizontal blocks consisting of those positions occupied by the 0s to the left of the first $-1$, in-between two consecutive $-1$s, and to the right of the last $-1$. The vertical blocks are defined in a similar way. Included in $\mathcal{H}(A)$ is the set of $n$ positions in the first row and the set of $n$ positions in the last row. Included in the vertical partition $\mathcal{V}(A)$ is the set of $n$ positions in the first column and the set of $n$ positions in the last column. Each $H_i \in \mathcal{H}(A)$ and each $V_j \in \mathcal{V}(A)$ intersect in at most one element of $Z$. The bipartite graph $G(A) \subseteq K_{n+\sigma(A),n+\sigma(A)}$ with vertex bipartition $\mathcal{H}(A), \mathcal{V}(A)$ has an edge joining $H_i \in \mathcal{H}(A)$ and $V_j \in \mathcal{V}(A)$ if and only if $H_i \cap V_j \neq \emptyset$ (and thus $|H_i \cap V_j| = 1$). As observed in [3], the matrix $A$ has an ASM (+1)-completion if and only if the bipartite graph $G(A)$ has a perfect matching; more specifically, if $(\{H_i, V_{\theta(i)}\} : 1 \leq i \leq n + \sigma(A))$ is a perfect matching of $G(A)$, where $\theta$ is a permutation of $\{1, 2, \ldots, n + \sigma(A)\}$, then a (+1)-completion of $A$ to an ASM is obtained by replacing the 0s in $A$ in the positions $\{H_i \cap V_{\theta(i)} : 1 \leq i \leq n + \sigma(A)\}$ with +1s.

Now suppose that $A$ is an $n \times n$ symmetric $(0, -1)$-matrix. Then there is a bijection between $\mathcal{H}(A)$ and $\mathcal{V}(A)$ defined by $H_i \to V_i$ where $V_i = \{(s, r) : (r, s) \in H_i\}$ $(i = 1, 2, \ldots n + \sigma(A))$. With subscripts for the blocks in $\mathcal{H}(A)$ and $\mathcal{V}(A)$ as in this bijection, we have that $H_i \cap V_j \neq \emptyset$ if and only if $H_j \cap V_i \neq \emptyset$ $(1 \leq i, j \leq n + \sigma(A))$. Thus the $(n + \sigma(A)) \times (n + \sigma(A))$ biadjacency matrix $C = [c_{ij}]$ of the bipartite graph $G(A)$ is symmetric and can be viewed as the adjacency matrix of a *loopy graph* $G^*(A)$ with vertex set $u_1, u_2, \ldots, u_{n+\sigma(A)}$ ($u_i$ corresponds to both $H_i$ and $V_i$) whose edges are all those pairs $\{u_i, u_j\}$ such that $H_i \cap V_j \neq \emptyset$ (equivalently, $H_j \cap V_i \neq \emptyset$). ($G^*(A)$ may have loops since it is possible that for some $i$, $H_i \cap V_i \neq \emptyset$ giving a loop at $u_i$, and thus we use the common term of loopy graph.) A *perfect matching of a loopy graph* is a collection of pairwise disjoint edges (possibly including loops) such that each vertex occurs on exactly one edge. Such a perfect matching corresponds to a symmetric permutation matrix $P$ such that $P \leq C$ (entrywise). A perfect matching determines positions of $A$ in which to put +1s in order to get a (+1)-completion of $A$ to a symmetric ASM.

**Example 6** Let $n = 5$ and consider the symmetric $(0, -1)$-matrix

$$A = \left[\begin{array}{c|c|c|c|c} & & & & \\ \hline & & -1 & & \\ \hline & -1 & & & \\ \hline & & & -1 & \\ \hline & & & & \end{array}\right]$$

where $\sigma(A) = 3$. Then the loopy graph $G^*(A)$ has 8 vertices and its adjacency matrix $C$ (as usual, only the 1s are shown) is

$C =$

| | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $u_7$ | $u_8$ |
|---|---|---|---|---|---|---|---|---|
| $u_1$ | 1 | 1 | | 1 | | 1 | | 1 |
| $u_2$ | 1 | 1 | | | | | | |
| $u_3$ | | | | | | 1 | | 1 |
| $u_4$ | 1 | | | | | | | |
| $u_5$ | | | | | 1 | 1 | | 1 |
| $u_6$ | 1 | | 1 | | 1 | | | |
| $u_7$ | | | | | | | | 1 |
| $u_8$ | 1 | | 1 | | 1 | | 1 | 1 |

.

The loopy graph $G^*(A)$ has a perfect matching (corresponding to the shaded 1s), equivalently, there exists a symmetric permutation matrix $Q \leq C$ (entrywise), and hence there exists a $(+1)$-completion of $A$ to a symmetric ASM, namely

$$\left[\begin{array}{c|c|c|c|c} & & +1 & & \\ \hline & +1 & -1 & +1 & \\ \hline +1 & -1 & +1 & & \\ \hline & +1 & & -1 & +1 \\ \hline & & & +1 & \end{array}\right].$$

**Theorem 7** *Let $A$ be an $n \times n$ symmetric $(0, -1)$-matrix that has a $(+1)$-completion to an ASM. Then $A$ has a $(+1)$-completion to a symmetric ASM.*

*Proof.* Let the $n \times n$ matrix $B = [b_{ij}]$ be an ASM $(+1)$-completion of $A$ and let

$$q(B) = \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij} - b_{ji}|.$$

Then $q(B)$ is an even integer which counts the number of positions $(i, j)$ with $i \neq j$ such that $b_{ij} + b_{ji} = 1$. If $q(B) = 0$, then $B$ is a symmetric ASM $(+1)$-completion of $A$. Suppose that $q(B) \neq 0$. Since $B$ is an ASM $(+1)$-completion of $A$, the $+1$s of $B$ determine a perfect matching $M$ of the bipartite graph $G(A)$. We consider those $q(B)/2$ edges $\{H_i, V_j\}$ of $M$ such that the position $(p, q)$ of $A$ in their intersection $H_i \cap V_j$ contains a $+1$ but there is not a $+1$ in position $(q, p)$ of $A$ in the intersection

$H_j \cap V_i$ (so there is a $+1$ in the unique position in $H_j \cap V_k$ for some $k \neq j$). These $q(B)/2$ edges determine an asymmetric digraph $D$ (an orientation of a graph), whose vertex set is $\{u_1, u_2, \ldots, u_{n+\sigma(A)}\}$, with no loops and at least one edge, such that any vertex with positive indegree also has positive outdegree, and vice-versa. Thus $D$ has a directed cycle

$$\gamma : u_{i_1} \to u_{i_2} \to \cdots \to u_{i_k} \to u_{i_1}$$

of length $k \geq 2$.

First suppose that the length $k$ of $\gamma$ is even. Then we obtain a new $(+1)$-completion $B'$ of $A$ to an ASM by replacing with 0s, the $+1$s in positions $H_{i_1} \cap V_{i_2}, H_{i_3} \cap V_{i_4}, \ldots, H_{i_{k-1}} \cap V_{i_k}$, and by replacing with $+1$s, the 0s in positions $H_{i_3} \cap V_{i_2}, H_{i_5} \cap V_{i_4}, \ldots, H_{i_{k-1}} \cap V_{i_{k-2}}, H_{i_1} \cap V_{i_k}$. Moreover, $q(B') < q(B)$.

Now suppose that $k$ is odd. Then we claim that there is a vertex $u_r$ of the cycle $\gamma$ such that $H_r \cap V_r \neq \emptyset$, and thus $H_r \cap V_r = \{(s,s)\}$ for some $s$. If not, then for each $i$, $H_i$ consists of positions strictly above the main diagonal or else positions strictly below the main diagonal. A similar conclusion holds for each $V_i$. This implies that $\gamma$ has even length, a contradiction. Thus we may assume that $H_{i_1} \cap V_{i_1} = \{(s,s)\}$ and thus that the entry in $B$ in position $(s,s)$ is 0. Then we obtain a new $(+1)$-completion $B'$ of $A$ to an ASM by replacing the 0s in positions $H_{i_1} \cap V_{i_1}, H_{i_3} \cap V_{i_2}, H_{i_5} \cap V_{i_4}, \ldots, H_{i_k} \cap V_{i_{k-1}}$ with $+1$s and replacing the $+1$s in positions $H_{i_1} \cap V_{i_2}, H_{i_3} \cap H_{i_4}, \ldots, H_{i_{k-2}} \cap V_{i_{k-1}}, H_{i_k} \cap V_{i_1}$ with 0s. Again we have that $q(B') < q(B)$. By repeating this argument, after a finite number of steps, we obtain a symmetric $(+1)$-completion of $A$ to an ASM. □

Another way to formulate Theorem 7 is: Let $A$ be an $n \times n$ ASM whose $-1$s are in a symmetric pattern. Then there is an $n \times n$ symmetric ASM $B$ with $-1$s exactly where $A$ has $-1$s.

We now give two examples illustrating the argument in the proof of Theorem 7 in both the even cycle and odd cycle cases.

**Example 8** Let $A$ be the $7 \times 7$ symmetric $(0,-1)$-matrix and let $B$ be the $7 \times 7$ (non-symmetric) $(+1)$-completion of $A$ to an ASM as shown:

$$A = \left[\begin{array}{c|c|c|c|c|c|c} & & & & & & \\ \hline & -1 & & & & & \\ \hline & & & & & & \\ \hline & & & -1 & & & \\ \hline & & & & & & \\ \hline & & & & & -1 & \\ \hline & & & & & & \end{array}\right] \text{ and } B = \left[\begin{array}{c|c|c|c|c|c|c} & +1 & & & & & \\ \hline +1 & -1 & & +1 & & 0 & \\ \hline & & & 0 & +1 & & \\ \hline & 0 & +1 & -1 & 0 & +1 & \\ \hline & & 0 & +1 & & & \\ \hline & +1 & & 0 & & -1 & +1 \\ \hline & & & & & +1 & \end{array}\right]$$

The positions which are not symmetrically occupied are shaded.

Label the sets in $\mathcal{H}(A)$ in the order of the rows and from left to right, and label the sets in $\mathcal{V}(A)$ in order of the columns and from top to bottom. Then the $10 \times 10$

biadjacency matrix $C$ of the bipartite graph $G(A)$ is

$$C = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c}
1 & 1 & & 1 & 1 & & 1 & 1 & & 1 \\ \hline
1 & & & & & & & & & \\ \hline
 & & 0 & 1 & 1 & & 1 & 1 & & 1 \\ \hline
1 & & 1 & 1 & 1 & & 1 & 1 & & 1 \\ \hline
1 & & 1 & 1 & 0 & & & & & \\ \hline
 & & & & & 0 & 1 & 1 & & 1 \\ \hline
1 & & 1 & 1 & & 1 & 1 & 1 & & 1 \\ \hline
1 & & 1 & 1 & & 1 & 1 & 0 & & \\ \hline
 & & & & & & & & & 1 \\ \hline
1 & & 1 & 1 & & 1 & 1 & & 1 & 1
\end{array}\right].$$

The shaded positions, where we have also shaded the corresponding diagonal elements, determine a directed cycle of even length 6 given by

$$u_3 \to u_5 \to u_4 \to u_7 \to u_6 \to u_8 \to u_3.$$

This directed cycle then determines the symmetric (+1)-completion of $A$ to an ASM given by

$$B' = \left[\begin{array}{c|c|c|c|c|c|c}
 & +1 & & & & & \\ \hline
+1 & -1 & & +1 & & 0 & \\ \hline
 & & & 0 & +1 & & \\ \hline
 & +1 & 0 & -1 & 0 & +1 & \\ \hline
 & & +1 & 0 & & & \\ \hline
 & 0 & & +1 & & -1 & +1 \\ \hline
 & & & & & +1 &
\end{array}\right].$$

**Example 9** Let $A$ be the $9 \times 9$ symmetric $(0, -1)$-matrix

$$A = \left[\begin{array}{c|c|c|c|c|c|c|c|c}
 & & & & & & & & \\ \hline
 & & & & & & & & \\ \hline
 & & & & -1 & & -1 & & \\ \hline
 & & & -1 & & & & & \\ \hline
 & & -1 & & & & & & \\ \hline
 & & & & & & & -1 & \\ \hline
 & & -1 & & & & -1 & & \\ \hline
 & & & & & -1 & & & \\ \hline
 & & & & & & & &
\end{array}\right].$$

Then $A$ has a non-symmetric ASM $(+1)$-completion

$B =$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 0 | | +1 | | |
| | | | | +1 | | 0 | | |
| | | | +1 | −1 | +1 | −1 | +1 | |
| | | +1 | −1 | **0** | | **+1** | | |
| 0 | +1 | −1 | **+1** | | | **0** | | |
| | | +1 | | | | | −1 | +1 |
| +1 | 0 | −1 | **0** | **+1** | | **−1** | +1 | |
| | | +1 | | | −1 | +1 | | |
| | | | | | +1 | | | |

,

where the non-symmetric +1s and their symmetrically opposite 0s have been shaded. Using the same labeling procedure as in Example 8, the digraph $D$ for this example has the directed cycle of length 3

$$u_7 \to u_{13} \to u_9 \to u_7$$

which, by using the fact that the entry in $H_9 \cap V_9 = \{(5,5)\}$ is a 0, then gives the symmetric ASM $(+1)$-completion of $A$

$B' =$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 0 | | +1 | | |
| | | | | +1 | | 0 | | |
| | | | +1 | −1 | +1 | −1 | +1 | |
| | | +1 | −1 | **0** | | **+1** | | |
| 0 | +1 | −1 | **0** | **+1** | | **0** | | |
| | | +1 | | | | | −1 | +1 |
| +1 | 0 | −1 | **+1** | **0** | | **−1** | +1 | |
| | | +1 | | | −1 | +1 | | |
| | | | | | +1 | | | |

.

We now consider an $n \times n$ $(0,+1)$-matrix $A$ with at least one $+1$ in each row and column. In this case we consider the *horizontal partition* $\mathcal{H}^+(A) = (H_i^+ : 1 \leq i \leq p)$ where the $H_i^+$, taken in some order, consist of those positions between two neighboring +1s in a row and, similarly, the *vertical partition* $\mathcal{V}^+(A) = (V_i^+ : 1 \leq i \leq p)$ where the $V_i^+$, taken in some order, consist of those positions between two neighboring +1s in a column. As indicated, for the following reason, the number of sets $p$ in each of the two partitions is the same: Let the row sum vector of $A$ be $(r_1, r_2, \ldots, r_n)$ and let the column sum vector be $S = (s_1, s_2, \ldots, s_n)$. Then the number of sets in the horizontal partition is

$$\sum_{i=1}^{n}(r_i - 1) = \left(\sum_{i=1}^{n} r_i\right) - n = \left(\sum_{i=1}^{n} s_i\right) - n = \sum_{i=1}^{n}(s_i - 1),$$

the same as the number of sets in the vertical partition. Note that if a row (respectively, column) of $A$ contains only one +1, then none of the positions in that row (respectively, column) are in a set of the horizontal partition (respectively, vertical partition). Let $C = [c_{ij}]$ be the $p \times p$ $(0,1)$-matrix where $c_{ij} = 1$ if and only if $H_i^+ \cap V_j^+ \neq \emptyset$ $(1 \leq i, j \leq p)$. The matrix $C$ is the biadjacency matrix of a bipartite graph $BG(C)$ with vertices bipartitioned as $\{H_i^+ : 1 \leq i \leq p\}$ and $\{V_i^+ : 1 \leq i \leq p\}$ with an edge joining $H_i^+$ and $V_j^+$ if and only if $H_i^+ \cap V_j^+ \neq \emptyset$ (and so consists of a single posiition). There will be a $(-1)$-completion of $A$ to an ASM if and only if $BG(C)$ has a perfect matching, equivalently, if and only if there is a permutation matrix $P \leq C$ (entrywise).

If $A$ is symmetric, then the matrix $C$ is a symmetric $(0,1)$-matrix, possibly with 1s on the main diagonal, and so is the adjacency matrix of a a loopy graph $G(C)$. There is a $(-1)$-completion of $A$ to a symmetric ASM if and only if $G(C)$ has a perfect matching (that is, a pairwise disjoint collection of edges and loops meeting all the vertices), that is, a symmetric permutation matrix $P \leq C$ (entrywise).

**Theorem 10** *Let $A$ be an $n \times n$ symmetric $(0,+1)$-matrix that has a $(-1)$-completion to an ASM. Then $A$ has a symmetric $(-1)$-completion to an ASM.*

*Proof.* The technique of the proof is identical to the technique used in the proof of Theorem 7 and so is omitted. □

## 3 Coda

Let $A$ be an $n \times n$ ASM. Then the row sum vector and the column sum vector of $A$ both equal the $n$-vector $(1,1,\ldots,1)$ of all 1s. Let $\mathrm{patt}(A)$ be the $(0,1)$-matrix obtained from $A$ by replacing each entry with its absolute value. Then $\mathrm{patt}(A)$ is the (*combinatorial*) *pattern* of $A$. Because of the alternating sign property of ASMs, the pattern of an ASM uniquely determines the ASM. The pattern $\mathrm{patt}(A)$ of $A$ has a row sum vector $R = (r_1, r_2, \ldots, r_n)$ and a column sum vector $S = (s_1, s_2, \ldots, s_n)$ and it is easy to verify [2] that

$$R, S \leq (1,3,5,7,\ldots,7,5,3,1) \quad \text{(entrywise)}. \tag{2}$$

Let $ASM(R,S)$ denote the set of all ASMs whose pattern has row sum vector $R$ and column sum vector $S$. In a symmetric ASM the row sum vector of its pattern equals its column sum vector. It is an open question to characterize $R$ and $S$ for which $ASM(R,S) \neq \emptyset$; the above conditions (2) are necessary but far from sufficient in general [2]. Let $ASM_{\mathrm{sym}}(R)$ denote the set of all symmetric ASMs whose patterns have row sum vector, and hence column sum vector $R$. If an ASM $A$ has a symmetric pattern, then $A$ is necessarily a symmetric ASM.

The following question is motivated by a theorem of Fulkerson, Hoffman, and McAndrew (see [4] where their theorem is extended to include 1s on the main diagonal) who proved that if there is a $(0,1)$-matrix with row and column sum vector $R$, then there is a symmetric $(0,1)$-matrix with row sum vector, and hence column sum vector, equal to $R$. We have been unable to answer the following ASM analogue of this theorem.

**Question:** Let $A$ be an $n \times n$ ASM whose pattern has row and column sum vector equal to $R$. Is there a symmetric ASM whose pattern has row and column sum vector equal to $R$?

Let $A^+$ be the $(0,1)$-matrix obtained from $A$ by replacing the $-1$s with 0s. Then $A^+$ has row and column sum vector $R^+$ for some $R^+$. Let $A^-$ be the $(0,-1)$-matrix obtained from $A$ by replacing the $+1$s with 0s. Then $A^-$ has row and column sum vector $R^-$ for some $R^-$. By the above theorem, there exists a symmetric $(0,1)$-matrix $B$ with row and column sum vector $R^+$, and there exists a symmetric $(0,-1)$-matrix $C$ with row and column sum vector $R^-$. We have $R^+ + R^- = (1,1,\ldots,1)$, but $B$ and $C$ need not have disjoint patterns. However, even if they did, $B+C$ need not be an alternating sign matrix.

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