

SYMMETRIC BUSH-TYPE HADAMARD MATRICES OF ORDER $4m^4$ EXIST FOR ALL ODD m

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(Communicated by John R. Stembridge)

ABSTRACT. Using reversible Hadamard difference sets, we construct symmetric Bush-type Hadamard matrices of order $4m^4$ for all odd integers m .

1. INTRODUCTION

A *Hadamard matrix* of order n is an n by n matrix H with entries ± 1 , such that

$$HH^T = nI_n,$$

where I_n is the identity matrix of order n . It can be easily shown that if n is the order of a Hadamard matrix, then $n = 1, 2$ or $n \equiv 0 \pmod{4}$. The famous Hadamard matrix conjecture states that for every positive integer n divisible by 4, there exists a Hadamard matrix of order n . This conjecture is far from being proved. We refer the reader to [13] for a recent construction of a Hadamard matrix of order 428 (the smallest order for which an example of a Hadamard matrix was not known for many years). In this note, we concentrate on a class of Hadamard matrices of highly specialized form, namely the Bush-type Hadamard matrices.

Let n be a positive integer and let J_{2n} denote the matrix of order $2n$ with all entries being ones. A Hadamard matrix $H = (H_{ij})$ of order $4n^2$, where H_{ij} are $2n \times 2n$ block matrices, is said to be of *Bush-type* if

$$(1.1) \quad H_{ii} = J_{2n} \text{ and } H_{ij}J_{2n} = J_{2n}H_{ij} = 0,$$

for $i \neq j$, $1 \leq i, j \leq 2n$. K. A. Bush [3] proved that the existence of a projective plane of order $2n$ implies the existence of a symmetric Bush-type Hadamard matrix of order $4n^2$. So if one can prove the nonexistence of symmetric Bush-type Hadamard matrices of order $4n^2$, where n is odd, then the nonexistence of a projective plane of order $2n$, where n is odd, will follow. This was Bush's original motivation for introducing Bush-type Hadamard matrices. Wallis [18] showed that $n - 1$ mutually orthogonal Latin squares of order $2n$ lead to a symmetric Bush-type Hadamard matrix of order $4n^2$. Goldbach and Claasen [7] also proved that certain 3-class association schemes can give rise to symmetric Bush-type Hadamard

Received by the editors December 20, 2004 and, in revised form, March 4, 2005.

2000 *Mathematics Subject Classification*. Primary 05B20.

Key words and phrases. Bush-type Hadamard matrix, Hadamard difference set, Hadamard matrix, reversible Hadamard difference set, strongly regular graph.

The second author's research was supported in part by NSF Grant DMS 0400411.

matrices. More recently, Kharaghani and his coauthors [15, 9, 10, 11, 12] rekindled the interest in Bush-type Hadamard matrices by showing that these matrices are very useful for constructions of symmetric designs and strongly regular graphs. Kharaghani [15] conjectured that Bush-type Hadamard matrices of order $4n^2$ exist for all n . While it is relatively easy to construct Bush-type Hadamard matrices of order $4n^2$ for all even n for which a Hadamard matrix of order $2n$ exists (see [14]), it is not easy to decide whether such matrices of order $4n^2$ exist if $n > 1$ is an odd integer. In a recent survey [12], Jungnickel and Kharaghani wrote “Bush-type Hadamard matrices of order $4n^2$, where n is odd, seem pretty hard to construct. Examples are known for $n = 3$, $n = 5$, and $n = 9$ (see [9], [10], and [11] respectively); all other cases are open”. In this note, we will show that symmetric Bush-type Hadamard matrices of order $4m^4$ exist for all odd m .

We first note a relation between symmetric Bush-type Hadamard matrices and strongly regular graphs with certain properties. The following lemma is well known. A weaker form of the lemma appeared in [18]. For convenience of the reader, we provide a proof.

Lemma 1.1. *There exists a symmetric Bush-type Hadamard matrix of order $4n^2$ if and only if there exists a strongly regular graph (SRG in short) with parameters*

$$v = 4n^2, \quad k = 2n^2 - n, \quad \lambda = \mu = n^2 - n,$$

and with the additional property that the vertex set can be partitioned into $2n$ disjoint cliques of size $2n$.

Proof. If $H = (H_{ij})$, where H_{ij} are $2n \times 2n$ block matrices, is a symmetric Bush-type Hadamard matrix of order $4n^2$, then the matrix $A = \frac{1}{2}(J - H)$ is symmetric and satisfies

$$A^2 = n^2 I + (n^2 - n)J.$$

Moreover the $2n \times 2n$ block matrices on the main diagonal of A are all zero matrices. Hence A is the adjacency matrix of an SRG with parameters $v = 4n^2, k = 2n^2 - n, \lambda = \mu = n^2 - n$, and with the additional property that the vertex set can be partitioned into $2n$ disjoint cliques of size $2n$. Conversely, if A is the adjacency matrix of such an SRG, then the matrix $H = J - 2A$ is symmetric and satisfies $H^2 = 4n^2 I$. Since the vertex set of the SRG can be partitioned into $2n$ cliques, each of size $2n$, we may arrange the rows and columns of H so that we can partition H into $H = (H_{ij})$, where the H_{ij} are $2n \times 2n$ block matrices and $H_{ii} = J_{2n}$. It remains to show that $H_{ij}J_{2n} = J_{2n}H_{ij} = 0$ for $i \neq j$, $1 \leq i, j \leq 2n$. Noting that the SRG has the smallest eigenvalue $-n$, we see that the cliques of size $2n$ of the SRG meet the Delsarte bound (sometimes called the Hoffman bound also). By Proposition 1.3.2 [2, p. 10], every vertex in the SRG outside a clique is adjacent to exactly n vertices of the clique. This proves that $H_{ij}J_{2n} = J_{2n}H_{ij} = 0$ for $i \neq j$, $1 \leq i, j \leq 2n$. The proof is complete. \square

2. SYMMETRIC BUSH-TYPE HADAMARD MATRICES FROM REVERSIBLE HADAMARD DIFFERENCE SETS

We start with a very brief introduction to difference sets. For a thorough treatment of difference sets, we refer the reader to [1, Chapter 6]. Let G be a finite group of order v . A k -element subset D of G is called a (v, k, λ) *difference set* in G if the

list of “differences” $d_1 d_2^{-1}$, $d_1, d_2 \in D$, $d_1 \neq d_2$, represents each non-identity element in G exactly λ times. Using multiplicative notation for the group operation, D is a (v, k, λ) difference set in G if and only if it satisfies the following equation in $\mathbb{Z}[G]$:

$$(2.1) \quad DD^{(-1)} = (k - \lambda)1_G + \lambda G,$$

where $D = \sum_{d \in D} d$, $D^{(-1)} = \sum_{d \in D} d^{-1}$, and 1_G is the identity element of G . A subset D of G is called *reversible* if $D^{(-1)} = D$. Note that if D is a reversible difference set, then

$$(2.2) \quad D^2 = (k - \lambda)1_G + \lambda G.$$

If furthermore we require that $1_G \notin D$, then from (2.2) we see that the Cayley graph $\mathbf{Cay}(G, D)$, with vertex set G and two vertices x and y being adjacent if and only if $xy^{-1} \in D$, is an SRG with parameters (v, k, λ, λ) .

The difference sets considered in this note have parameters

$$(v, k, \lambda) = (4n^2, 2n^2 - n, n^2 - n).$$

These difference sets are called *Hadamard* difference sets (HDS), since their $(1, -1)$ -incidence matrices are Hadamard matrices. Alternative names used by other authors are Menon difference sets and H-sets. We will show that reversible HDSs give rise to symmetric Bush-type Hadamard matrices.

Proposition 2.1. *Let D be a reversible HDS in a group G with parameters $(4n^2, 2n^2 - n, n^2 - n)$. If there exists a subgroup $H \leq G$ of order $2n$ such that $D \cap H = \emptyset$, then there exists a Bush-type symmetric Hadamard matrix of order $4n^2$.*

Proof. First note that the Cayley graph $\mathbf{Cay}(G, D)$ is strongly regular with parameters $v = 4n^2 - n$, $k = 2n^2 - n$, $\lambda = \mu = n^2 - n$. The cosets of H in G partition G . Let Hg be an arbitrary coset of H in G . Then any two elements $x, y \in Hg$ are not adjacent in $\mathbf{Cay}(G, D)$ since $xy^{-1} \in H$ and $D \cap H = \emptyset$. Therefore the vertex set of $\mathbf{Cay}(G, D)$ can be partitioned into $2n$ disjoint cliques of size $2n$. By Lemma 1.1, the $(1, -1)$ -adjacency matrix of $\mathbf{Cay}(G, D)$ is a symmetric Bush-type Hadamard matrix of order $4n^2$. \square

Let $G = K \times W$ where $K = \{g_0 = 1, g_1, g_2, g_3\}$ is a Klein four group and W is a group of order n^2 . Each subset D of G has a unique decomposition into a disjoint union $D = \bigcup_{i=0}^3 (g_i, D_i)$ where $D_i \subset W$. Note that if D is a reversible HDS in G , then $(g_i, 1)D$ are also reversible HDSs for all i , $0 \leq i \leq 3$. This observation implies the following.

Proposition 2.2. *Let $K = \{g_0 = 1, g_1, g_2, g_3\}$ be a Klein four group. Let $D = \bigcup_{\ell=0}^3 (g_\ell, D_\ell)$ be a reversible Hadamard difference set in the group $G = K \times W$, where $D_\ell \subseteq W$ and $|W| = n^2$. If there exists a subgroup $P \leq W$ of order n such that $P \cap D_i = P \cap D_j = \emptyset$ for some $i \neq j$, $0 \leq i, j \leq 3$, then there exists a symmetric Bush-type Hadamard matrix of order $4n^2$.*

Proof. Let $E = (g_i, 1)D$. By the above observation, E is a reversible HDS in G . Let $H = (g_0, 1)P \cup (g_i g_j, 1)P$. Then H is a subgroup of G of order $2n$ and $H \cap E = \emptyset$. By Proposition 2.1, E gives rise to a symmetric Bush-type Hadamard matrix of order $4n^2$. \square

3. CONSTRUCTION OF SYMMETRIC BUSH-TYPE HADAMARD MATRICES OF ORDER $4m^4$ FOR ALL ODD m

A symmetric Bush-type Hadamard matrix H of order 4 is exhibited below:

$$H = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

So we will only be concerned with constructions of symmetric Bush-type Hadamard matrices of order $4m^4$ for odd $m > 1$. We will first construct symmetric Bush-type Hadamard matrices of order $4p^4$, where p is an odd prime, from certain $(4p^4, 2p^4 - p^2, p^4 - p^2)$ HDS. To this end, we need to recall a construction of an HDS with these parameters from [19]. Let p be an odd prime and $\text{PG}(3, p)$ be a three-dimensional projective space over $\text{GF}(p)$. We will say that a set C of points in $\text{PG}(3, p)$ is of *type Q* if

$$|C| = \frac{(p^4 - 1)}{4(p - 1)}$$

and each plane of $\text{PG}(3, p)$ meets C in either $\frac{(p-1)^2}{4}$ points or $\frac{(p+1)^2}{4}$ points. For each set X of points in $\text{PG}(3, p)$ we denote by \tilde{X} the set of all non-zero vectors $v \in \text{GF}(p)^4$ with the property that $\langle v \rangle \in X$, where $\langle v \rangle$ is the 1-dimensional subspace of $\text{GF}(p)^4$ generated by v .

Let $S = \{L_1, L_2, \dots, L_{p^2+1}\}$ be a spread of $\text{PG}(3, p)$ and let C_0, C_1 be two sets of type Q in $\text{PG}(3, p)$ such that

$$(3.1) \quad \forall_{1 \leq i \leq s} |C_0 \cap L_i| = \frac{p+1}{2} \text{ and } \forall_{s+1 \leq i \leq 2s} |C_1 \cap L_i| = \frac{p+1}{2},$$

where $s := \frac{p^2+1}{2}$. (We note that if we take S to be the regular spread in $\text{PG}(3, p)$, then examples of type Q sets C_0, C_1 in $\text{PG}(3, p)$ satisfying (3.1) were first constructed in [20] when $p \equiv 3 \pmod{4}$, in [6, 19] when $p = 5, 13, 17$, and in [4] for all odd primes p .) As in [19] we set

$$\begin{aligned} C_2 &:= (L_1 \cup \dots \cup L_s) \setminus C_0, \\ C_3 &:= (L_{s+1} \cup \dots \cup L_{2s}) \setminus C_1. \end{aligned}$$

Note that $C_0 \cup C_2 = L_1 \cup \dots \cup L_s$ and $C_1 \cup C_3 = L_{s+1} \cup \dots \cup L_{2s}$.

Let A (respectively B) be a union of $(s-1)/2$ lines from $\{L_{s+1}, \dots, L_{2s}\}$ (respectively $\{L_1, \dots, L_s\}$). Let $K = \{g_0 = 1, g_1, g_2, g_3\}$ and $W = (\text{GF}(p)^4, +)$. Denote

$$\begin{aligned} D_0 &:= \widetilde{C_0} \cup \widetilde{A}, \\ D_2 &:= \widetilde{C_2} \cup \widetilde{A}, \\ D_1 &:= \widetilde{C_1} \cup \widetilde{B}, \\ D_3 &:= W \setminus (\widetilde{C_3} \cup \widetilde{B}). \end{aligned}$$

Then

$$|D_0| = |D_1| = |D_2| = \frac{p^4 - p^2}{2}, \quad |D_3| = \frac{p^4 + p^2}{2}.$$

By Theorem 2.2 [19] the set

$$D := (g_0, D_0) \cup (g_1, D_1) \cup (g_2, D_2) \cup (g_3, D_3)$$

is a reversible $(4p^4, 2p^4 - p^2, p^4 - p^2)$ difference set in the group $K \times W$.

Pick an arbitrary line, say L_a , from the set $\{L_{s+1}, \dots, L_{2s}\}$ such that $L_a \cap A = \emptyset$. Then $P := \widetilde{L_a} \cup \{0\}$ is a subgroup of W of order p^2 such that $P \cap D_0 = P \cap D_2 = \emptyset$. Now Proposition 2.2 implies that there exists a symmetric Bush-type Hadamard matrix of order $4p^4$. Therefore we have proved

Theorem 3.1. *There exists a symmetric Bush-type Hadamard matrix of order $4p^4$ for every odd prime p .*

In order to build a symmetric Bush-type Hadamard matrix of order $4m^4$ for arbitrary odd $m > 1$ we need to use Turyn's composition theorem [17]. We also need the following simple

Proposition 3.2. *There exists a subgroup $Q \leq W$ of order p^2 such that $Q \subseteq D_3$ and $Q \cap D_1 = \emptyset$.*

Proof. Pick an arbitrary line L_b from $\{L_1, \dots, L_s\}$ such that $L_b \cap B = \emptyset$ and set $Q := \{0\} \cup \widetilde{L_b}$. The conclusion of the proposition follows. \square

Next we recall Turyn's composition theorem. We will use the version as stated in Theorem 6.5 [5, p. 45]. For convenience we introduce the following notation. Let W_1, W_2 be two groups. For $A, B \subseteq W_1$ and $C, D \subseteq W_2$, we define the following subset of $W_1 \times W_2$:

$$\nabla(A, B; C, D) := ((A \cap B) \times C') \cup ((A' \cap B') \times C) \cup ((A \cap B') \times D') \cup ((A' \cap B) \times D),$$

where $A' = W_1 \setminus A$, $B' = W_1 \setminus B$, $C' = W_2 \setminus C$, and $D' = W_2 \setminus D$.

Theorem 3.3 (Turyn [17]). *Let $K = \{g_0, g_1, g_2, g_3\}$ be a Klein four group. Let $E_1 = \bigcup_{i=0}^3 (g_i, A_i)$ and $E_2 = \bigcup_{i=0}^3 (g_i, B_i)$ be reversible Hadamard difference sets in groups $K \times W_1$ and $K \times W_2$, respectively, where $|W_1| = w_1^2$ and $|W_2| = w_2^2$, w_1 and w_2 are odd, $A_i \subseteq W_1$ and $B_i \subseteq W_2$, and*

$$\begin{aligned} |A_0| &= |A_1| = |A_2| = \frac{w_1^2 - w_1}{2}, & |A_3| &= \frac{w_1^2 + w_1}{2}, \\ |B_0| &= \frac{w_2^2 + w_2}{2}, & |B_1| &= |B_2| = |B_3| = \frac{w_2^2 - w_2}{2}. \end{aligned}$$

Define

$$\begin{aligned} E &:= (g_0, \nabla(A_0, A_1; B_0, B_1)) \cup (g_1, \nabla(A_0, A_1; B_2, B_3)) \\ &\cup (g_2, \nabla(A_2, A_3; B_0, B_1)) \cup (g_3, \nabla(A_2, A_3; B_2, B_3)). \end{aligned}$$

Then

$$|\nabla(A_0, A_1; B_0, B_1)| = \frac{w_1^2 w_2^2 + w_1 w_2}{2},$$

$$|\nabla(A_0, A_1; B_2, B_3)| = |\nabla(A_2, A_3; B_0, B_1)| = |\nabla(A_2, A_3; B_2, B_3)| = \frac{w_1^2 w_2^2 - w_1 w_2}{2},$$

and E is a reversible $(4w_1^2 w_2^2, 2w_1^2 w_2^2 - w_1 w_2, w_1^2 w_2^2 - w_1 w_2)$ Hadamard difference set in the group $K \times W_1 \times W_2$.

Proposition 3.4. *With the assumptions as in Theorem 3.3, let $Q \leq W_1$ and $P \leq W_2$ be such that $Q \cap A_2 = \emptyset, Q \subseteq A_3$ and $P \cap B_1 = \emptyset, P \cap B_3 = \emptyset$. Then $(Q \times P) \cap \nabla(A_2, A_3; B_0, B_1) = \emptyset$ and $(Q \times P) \cap \nabla(A_2, A_3; B_2, B_3) = \emptyset$.*

Proof. It follows from the intersections

$$\begin{aligned} Q \cap (A_2 \cap A_3) &= \emptyset, \\ Q \cap (A'_2 \cap A'_3) &= \emptyset, \\ Q \cap (A_2 \cap A'_3) &= \emptyset, \\ Q \cap (A'_2 \cap A_3) &= Q \end{aligned}$$

that $\nabla(A_2, A_3; B_0, B_1) \cap (Q \times P) = Q \times (B_1 \cap P) = \emptyset$ and $\nabla(A_2, A_3; B_2, B_3) \cap (Q \times P) = Q \times (B_3 \cap P) = \emptyset$. \square

Theorem 3.5. *There exists a symmetric Bush-type Hadamard matrix of order $4m^4$ for all odd m .*

Proof. We only need to prove the theorem for odd $m > 1$. Let $K = \{g_0, g_1, g_2, g_3\}$ be a Klein four group. Let p and q be two odd primes, not necessarily distinct, and let $W_1 = (\text{GF}(p)^4, +)$ and $W_2 = (\text{GF}(q)^4, +)$. By the construction before the statement of Theorem 3.1, we can construct a reversible HDS

$$E_1 = (g_0, A_0) \cup (g_1, A_1) \cup (g_2, A_2) \cup (g_3, A_3)$$

in $K \times W_1$ such that

$$|A_0| = |A_1| = |A_2| = \frac{p^4 - p^2}{2}, \quad |A_3| = \frac{p^4 + p^2}{2},$$

and there exists a subgroup $Q \leq W_1$ of order p^2 with the property that $Q \cap A_2 = \emptyset, Q \subset A_3$. (See Proposition 3.2. Note that here the A_i are a renumbering of the D_i ; any renumbering of the D_i still yields a reversible difference set.) Also we can construct a reversible HDS

$$E_2 = (g_0, B_0) \cup (g_1, B_1) \cup (g_2, B_2) \cup (g_3, B_3)$$

in $K \times W_2$ such that

$$|B_0| = \frac{q^4 + q^2}{2}, \quad |B_1| = |B_2| = |B_3| = \frac{q^4 - q^2}{2},$$

and there exists a subgroup $P \leq W_2$ of order q^2 with the property that $P \cap B_1 = \emptyset, P \cap B_3 = \emptyset$ (see the paragraph before the statement of Theorem 3.1). Now we apply Theorem 3.3 to E_1 and E_2 to obtain a reversible HDS

$$\begin{aligned} E &= (g_0, \nabla(A_0, A_1; B_0, B_1)) \cup (g_1, \nabla(A_0, A_1; B_2, B_3)) \\ &\cup (g_2, \nabla(A_2, A_3; B_0, B_1)) \cup (g_3, \nabla(A_2, A_3; B_2, B_3)) \end{aligned}$$

of size $2p^4q^4 - p^2q^2$ in $K \times W_1 \times W_2$. By Proposition 3.4, we have

$$(3.2) \quad (Q \times P) \cap \nabla(A_2, A_3; B_0, B_1) = \emptyset, \quad (Q \times P) \cap \nabla(A_2, A_3; B_2, B_3) = \emptyset.$$

By Proposition 2.2, there exists a symmetric Bush-type Hadamard matrix of order $4(pq)^4$. Now note that $|Q \times P| = p^2q^2$, $|\nabla(A_2, A_3; B_0, B_1)| = |\nabla(A_2, A_3; B_2, B_3)| = \frac{p^4q^4 - p^2q^2}{2}$, and E satisfies property (3.2). We can repeatedly use the above process to produce a reversible HDS satisfying the condition of Proposition 2.2; hence there exists a symmetric Bush-type Hadamard matrix of order $4m^4$ for all odd $m > 1$. The proof is complete. \square

Kharaghani [15, 16] showed how to use Bush-type Hadamard matrices to simplify Ionin's method [8] for constructing symmetric designs. Based on his constructions in [15, 16], we draw the following consequences of Theorem 3.5.

Theorem 3.6. *Let m be an odd integer. If $q = (2m^2 - 1)^2$ is a prime power, then there exist twin symmetric designs with parameters*

$$v = 4m^4 \frac{(q^{\ell+1} - 1)}{q - 1}, \quad k = q^{\ell}(2m^4 - m^2), \quad \lambda = q^{\ell}(m^4 - m^2),$$

for every positive integer ℓ .

Theorem 3.7. *Let m be an odd integer. If $q = (2m^2 + 1)^2$ is a prime power, then there exist Siamese twin symmetric designs with parameters*

$$v = 4m^4 \frac{(q^{\ell+1} - 1)}{q - 1}, \quad k = q^{\ell}(2m^4 + m^2), \quad \lambda = q^{\ell}(m^4 + m^2),$$

for every positive integer ℓ .

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