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SYMMETRIC CONFERENCE MATRICES OF ORDER $pq^2 + 1$

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Introduction and definitions. A *conference matrix* of order n is a square matrix C with zeros on the diagonal and ± 1 elsewhere, which satisfies the orthogonality condition $CC^T = (n-1)I$. If in addition C is symmetric, $C = C^T$, then its order n is congruent to 2 modulo 4 (see [**5**]). *Symmetric conference matrices* (C) are related to several important combinatorial configurations such as regular two-graphs, equiangular lines, Hadamard matrices and balanced incomplete block designs [**1**; **5**; and **7**, pp. 293–400]. We shall require several definitions.

A *strongly regular graph* with parameters (v, k, λ, μ) is an undirected regular graph of order v and degree k with adjacency matrix $A = A^T$ satisfying

$$(1.1) \quad A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J, \quad AJ = kJ$$

where J is the all one matrix. Note that in a strongly regular graph any two adjacent (non-adjacent) vertices are adjacent to exactly $\lambda(\mu)$ other vertices. An easy counting argument implies the following relation between the parameters v, k, λ and μ;

$$(1.2) \quad k(k - \lambda - 1) = (v - k - 1)\mu.$$

A strongly regular graph is said to be *pseudo-cyclic* (PC) if $v - 1 = 2k = 4\mu$. From (1.1) and (1.2) it is readily deduced that a PC-graph has parameters of the form $(4t + 1, 2t, t - 1, t)$, where $t > 0$ is an integer. We note that the complement of a PC-graph is again pseudo-cyclic with the same parameters as the original graph (though not necessarily isomorphic to it), i.e. both A and $A^c = J - I - A$ satisfy (1.1).

$$(1.3) \quad A^2 = t(J + I) - A, \quad AJ = 2tJ.$$

A *symmetric block design* with parameters (v, k, λ) is a collection of v k-subsets, called *blocks*, of a set of v elements, referred to as *points*, which has a point-block incidence matrix A satisfying

$$(1.4) \quad AA^T = A^TA = (k - \lambda)I + \lambda J, \quad AJ = JA = kJ.$$

In a symmetric block design any two distinct points (blocks) are incident with exactly λ blocks (points). Again, as before, a simple counting argument yields

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a condition on the parameters v, k and λ;

$$(1.5) \qquad k(k-1) = \lambda(v-1).$$

A symmetric block design is said to be *skew-Hadamard* (*SH*) if $v - 1 = 2k$, A has a zero diagonal and $A^T = A^c = J - I - A$. From (1.4) and (1.5) it follows that an *SH*-design has parameters of the form $(4t-1, 2t-1, t-1)$, $t > 0$ and both A and $A^T = A^c$ satisfy (1.4), i.e.

$$(1.6) \qquad AA^T = A^TA = tI + (t-1)J, \quad AJ = JA = (2t-1)J.$$

A given PC-strongly regular graph with parameters $(4t+1, 2t, t-1, t)$ and adjacency matrix A can be uniquely extended to a conference matrix of order $n = 4t + 2$.

$$(1.7) \qquad C = \begin{pmatrix} 0 & j^T \\ j & B \end{pmatrix}, \quad B = 2A - J + I,$$

where j is the all one vector of order $4t + 1$. This is a consequence of the fact that

$$0 + j^Tj = n - 1, \quad 0j^T + j^TB = o^T,$$
$$j0 + Bj = o, \quad jj^T + B^2 = J + (2A - J + I)^2 = (4t+1)I.$$

Conversely, a PC graph can be obtained from a C-matrix by normalizing it to contain one's in the ith row and column except for $c_{ii} = 0$ and by deleting this row and column from C. The resulting matrix B yields a (0, 1)-matrix $A = (B + J - I)/2$ satisfying (1.3). We note that by choosing different rows for normalization we may obtain different nonisomorphic PC-graphs of order $4t + 1$. The set of all PC-graphs derivable from a particular conference matrix C forms a so-called *switching class* of graphs [**5**]. It is readily observed that the entire switching class can be recovered from any of its members via the corresponding C-matrix.

The existence of C-matrices is implied by the existence of PC-graphs associated with C (see [**5**] and [**7**, p. 294]).

THEOREM 1.1. *A necessary condition for the existence of a PC-graph of order* $v = 4t + 1, t > 0$ *is that* v *is a sum of squares of two integers.*

Hence a PC-graph does not exist if the square-free part of $v = 4t + 1$ contains a prime congruent to 3 (mod 4).

There are very few constructions for PC-graphs and for symmetric conference matrices [**7**, pp. 313–319].

THEOREM 1.2 (Paley). *For an odd prime power* $v = p^r$, *let* $a_1, \ldots, a_v$, *be the elements of* $GF(v)$ *numbered so that* $a_v = 0$, $a_{v-i} = -a_i$, $i = 1, \ldots, v-1$. *Define* $B = (b_{ij})$ *by*

$$(1.8) \qquad b_{ij} = \chi(a_j - a_i), \quad 1 \leqq i, j \leqq v,$$

where χ *is the quadratic character of* $GF(v)$, *i.e.* $\chi(0) = 0$, $\chi(x) = 1$ *if* $x = y^2$, *for some* $y \in GF(v)$ *and* $\chi(x) = -1$ *otherwise. Then*

$$BB^T = vI - J, \quad BJ = JB = 0, \tag{1.9}$$

and $A = (B + J - I)/2$ *is the adjacency matrix of a PC-graph if* $v \equiv 1 \pmod 4$ *and the incidence matrix of an SH-design if* $v \equiv 3 \pmod 4$.

The only other known method for constructing PC-graphs employs Kronecker-products of $(-1, 0, 1)$-matrices associated with PC-graphs and SH-designs [**6**].

THEOREM 1.3 (Turyn). *If* v *and* w *are the orders of a PC-graph and SH-matrix respectively, then* v^k *and* w^{2k} *are orders of PC-graphs for any integer* $k > 0$.

So, for example, it is easily verified that if V is a symmetric or skew-matrix of order v satisfying (1.9), then

$$W = V \otimes V + I \otimes J - J \otimes I \tag{1.10}$$

is a symmetric matrix of order v^2 also satisfying (1.9). The smallest PC-graph of non-prime power order, given by formula (1.10) has 225 nodes and is based on a skew-Hadamard design of order 15 (see [**3**]).

This paper is concerned with the construction of a new class of conference matrices of order $pq^2 + 1$, where $q = 4t - 1$ is a prime power and $p = 4t + 1$ is the order of a PC-graph. The construction is based on certain block-regular matrices, formed from the cyclotomic classes of $GF(q)$.

2. Feasible block-regular matrices. The fact that a product of orders of a finite number of PC-graphs is congruent to 1 (mod 4) and satisfies Theorem 1.1 suggests the possibility of extending Turyn's construction from powers to products of PC-graphs. The aim of this Section is to derive necessary conditions for such an extension.

Let v_α and v_β be the orders of two PC-graphs with parameters $(4\alpha + 1, 2\alpha, \alpha - 1, \alpha)$, $(4\beta + 1, 2\beta, \beta - 1, \beta)$ and adjacency matrices $A_\alpha = (a_{ij}^\alpha)$, $A_\beta = (a_{ij}^\beta)$ respectively. Then $(16\alpha\beta + 4\alpha + 4\beta + 1, 8\alpha\beta + 2\alpha + 2\beta, 4\alpha\beta + \alpha + \beta - 1, 4\alpha\beta + \alpha + \beta)$ is an admissible parameter set for a PC-graph of order $v_{\alpha\beta} = v_\alpha v_\beta$. Let $A = (A_{ij})$, $1 \leqq i, j \leqq v_\alpha$ be a block matrix, with blocks $A_{ij} = (a_{kl}^{ij})$, $1 \leqq k, l \leqq v_\beta$ consisting of regular (0, 1)-matrices. Partially motivated by the Kronecker-product construction of Theorem 1.3 we assume that

$$A_{ij}J = JA_{ij} = \begin{cases} 2\beta & \text{if } i = j, \\ x & \text{if } i \neq j \text{ and } a_{ij}^\alpha = 1, \\ y & \text{if } i \neq j \text{ and } a_{ij}^\alpha = 0. \end{cases} \tag{2.1}$$

For A to be the adjacency matrix of a PC-graph it is necessary that $A = A^T$ satisfies relations (1.3). The regularity condition in (1.3) requires that

$$2\beta + 2\alpha x + 2\alpha y = 8\alpha\beta + 2\alpha + 2\beta \tag{2.2}$$

which implies

$$(2.3)\qquad y = 4\beta + 1 - x = v_\beta - x.$$

In order to exploit the quadratic equation in (1.3) we make use of the following elementary fact. If X and Y are both $v \times v$ (0, 1)-matrices such that $XJ = xJ$, $YJ = yJ$, then row sums in the product $Z = XY$, $Z = (z_{ij})$ are all equal to

$$(2.4)\qquad r(Z) = \sum_{j=1}^{v} z_{ij} = xy, \quad i = 1, \ldots, v.$$

From the underlying block-degree structure of A, dictated by the PC-graph with adjacency matrix A_β (see (2.1)), we deduce that

$$(2.5a)\qquad r((A^2)_{ii}) = r\left(\sum_{k=1}^{v_\alpha} A_{ik}A_{ki}\right) = \sum_{k=1}^{v_\alpha} r(A_{ik}A_{ik}) = (2\beta)^2 + 2\alpha x^2 + 2\alpha y^2.$$

If $i \neq j$, then if $a_{ij}{}^\alpha = 1$

$$(2.6a)\qquad r((A^2)_{ij}) = \sum_{k=1}^{v_\alpha} r(A_{ik}A_{kj}) = 2(2\beta x) + (\alpha - 1)x^2 + 2\alpha xy + \alpha y^2,$$

and if $a_{ij}{}^\alpha = 0$

$$(2.7a)\qquad r((A^2)_{ij}) = \sum_{k=1}^{v_\alpha} r(A_{ik}A_{kj}) = 2(2\beta y) + \alpha x^2 + 2\alpha xy + (\alpha - 1)y^2.$$

On the other hand, since by (1.3) $A^2 = t(J + I) - A$, where $t = 4\alpha\beta + \alpha + \beta$, we obtain

$$(2.5b)\qquad r((A^2)_{ii}) = r(t(J + I) - A_{ii}) = (4\alpha\beta + \alpha + \beta)(4\beta + 2) - 2\beta.$$

Similarly, if $i \neq j$ then

$$r((A^2)_{ij}) = r(tJ - A_{ij}) = (4\alpha\beta + \alpha + \beta)(4\beta + 1) - \begin{cases} x & \text{if } a_{ij}{}^\alpha = 1, \quad (2.6b) \\ y & \text{if } a_{ij}{}^\alpha = 0. \quad (2.7b) \end{cases}$$

A comparison of relations (a) with the corresponding relations (b) together with (2.3) lead to three *identical* quadratic equations for x:

$$(2.8)\qquad x^2 - (4\beta + 1)x + (4\beta + 1)\beta = 0,$$

the roots of which are

$$(2.9)\qquad x_1 = x = \tfrac{1}{2}(v_\beta - \sqrt{v_\beta}),\ x_2 = y = \tfrac{1}{2}(v_\beta + \sqrt{v_\beta}).$$

Consequently, the order v_β is a square.

THEOREM 2.1. *A necessary condition for A, defined by* (2.1), *to be the adjacency matrix of a PC-graph of order $v_\alpha \cdot v_\beta$ is that $v_\beta = q^2$ is a square and that $x = q(q - 1)/2$, $y = q(q + 1)/2$.*

We remark that the conditions of Theorem 2.1 generalize to product matrices of the form (2.1) based on strongly regular graphs with $k = 2\mu$. This fact is employed in constructions for other families of regular two-graphs to be reported on elsewhere.

3. Cyclotomic block-matrices. In this section we shall exhibit a family $\mathscr{A}_q$ of regular (0, 1)-matrices serving as building blocks in the construction of PC-graphs. As suggested by Theorem 2.1 each matrix in $\mathscr{A}_q$ will be of order q^2 with row (and column) sums either $(q^2 - 1)/2$ or $q(q - 1)/2$, or $q(q + 1)/2$. In order to utilize the theory of Galois fields we shall assume that q is a prime power.

For a prime power $q = 4t - 1$, $t > 0$, let $a_1, \ldots, a_q$ be the elements of $GF(q)$ numbered as in Theorem 1.2. Define

$$(3.1) \qquad P[a_k] = (p_{ij}^{\ k}), \quad p_{ij}^{\ k} = \begin{cases} 1 & \text{if } a_j - a_i = a_k \\ 0 & \text{otherwise,} \end{cases}$$

where $1 \leqq i, j, k \leqq q$. From the properties of $GF(q)$ it follows that:

(i) $P[a_k]$ are permutation matrices, $1 \leqq k \leqq q$, $P[a_q] = I$.
(ii) If $k \neq l$ and $p_{ij}^{\ k} = 1$ then $p_{ij}^{\ l} = 0$. Consequently, $P[a_1] + \ldots + P[a_q] = J$.
(iii) The matrices $P[a_k]$ form an abelian group under multiplication $P[a_k] \cdot P[a_l] = P[a_m]$, $a_m = a_k + a_l$.

For example, to prove (iii) we assume that for some s, $p_{is}^{\ k} = p_{sj}^{\ l} = 1$, which implies $p_{ij}^{\ m} = 1$. From (3.1) we have $a_s - a_i = a_k$, $a_j - a_s = a_l$ and $a_j - a_i = a_m$. Eliminating a_s from these equations we get $a_m = a_k + a_l$. In fact, the $P[a_k]$'s form a so-called *cyclotomic* association scheme (cf. [**2**; **4**]). Let E and F be matrices of order q and degree $(q - 1)/2$ defined by:

$$(3.2) \qquad E = (e_{ij}), \quad e_{ij} = \begin{cases} 1 & \text{if } \chi(a_j - a_1) = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$(3.3) \qquad F = (f_{ij}), \quad f_{ij} = \begin{cases} 1 & \text{if } \chi(a_j - a_i) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

From Theorem 1.2 it follows that F is the incidence matrix of an SH-design and therefore satisfies (1.6). With help of E and F we are ready to introduce the matrix family $\mathscr{A}_q$ consisting of

$$(3.4) \qquad A_d = (A_{ij}^{\ d}), \quad A_{ij}^{\ d} = \begin{cases} O & \text{if } i = j \\ I + F & \text{if } i \neq j \text{ and } f_{ij} = 1 \\ J - F & \text{otherwise,} \end{cases}$$

$$(3.5) \qquad A_k = (A_{ij}^{\ k}), \quad A_{ij}^{\ k} = FP[g^{2k}(a_i + a_j)], \quad 1 \leqq k \leqq 2t - 1,$$

$$(3.6) \qquad A_{2t} = (A_{ij}^{\ 2t}), \quad A_{ij}^{\ 2t} = EP[a_i],$$

where O is the all zero matrix, g is a primitive element of $GF(q)$ and $1 \leqq i, j \leqq q$. The matrix A_d, based on the Kronecker product construction (1.10), corresponds to a PC-graph of order $q^2 = 4\beta + 1$, $\beta = 2t(2t - 1)$. The block-matrices A_k consist of blocks which are various permutations of F (or E) governed by the quadratic residues of $GF(q)$.

In the rest of this section we shall investigate products of elements in $\mathscr{A}_q$.

LEMMA 3.1. *The matrices* A_d, $A_k \in \mathscr{A}_q$, $1 \leqq k \leqq 2t$, *satisfy*:

$$A_d A_k + A_k A_d = (2t - 1)qJ - A_k, \tag{3.7}$$

$$A_k^2 = A_k A_l = A_k^T A_l = A_k A_l^T = (2t - 1)^2 J, \quad k \neq l. \tag{3.8}$$

Proof. Since F is the incidence matrix of an SH-design we may use (1.6) to obtain

$$F^2 = F(J - I - F^T) = t(J - I) - F. \tag{3.9}$$

We also note, that F can be expressed as a sum of those $P[a_k]$ for which $\chi(a_k) = 1$. Thus,

$$F = \sum_{k=1}^{2t-1} P[g^{2k}], \quad FP[a_l] = P[a_l]F. \tag{3.10}$$

The (i, j)-th blocks of $A_d A_k$ and $A_k A_d$, $1 \leqq k \leqq 2t - 1$, can be computed as follows:

$$\begin{aligned}(A_d A_k)_{ij} &= \sum_{n=1}^{2t-1} \{(I + F)FP[g^{2k}(a_i + a_j + g^{2n})] \\ &\quad + (J - F)FP[g^{2k}(a_i + a_j + g^{2n+1})]\} \\ &= \left\{t(J - I) \sum_{n=1}^{2t-1} P[g^{2(k+n)}] \right. \\ &\quad \left. + [t(J + I) - J + F] \sum_{n=1}^{2t-1} P[g^{2(k+n)+1}]\right\} P[g^{2k}(a_i + a_j)] \\ &= 2t[(2t - 1)J - F]P[g^{2k}(a_i + a_j)] = 2t[(2t - 1)J - A_{ij}^{k}],\end{aligned} \tag{3.11}$$

$$(A_k A_d)_{ij} = (2t - 1)[(2t - 1)J + A_{ij}^{k}]. \tag{3.12}$$

Summation of (3.11) and (3.12) yields (3.7). The case $k = 2t$ can be proved along the same lines.

In order to verify (3.8) let us determine the (i, j)-th block of $A_k A_l$, $1 \leqq k, l \leqq 2t - 1$,

$$\begin{aligned}(A_k A_l)_{ij} &= \sum_{n=1}^{q} FP[g^{2k}(a_i + a_n)]FP[g^{2l}(a_n + a_j)] \\ &= F^2 \sum_{n=1}^{q} P[g^{2k}(a_i + a_n) + g^{2l}(a_n + a_j)] = F^2 \sum_{n=1}^{q} P[a_n'].\end{aligned}$$

We will show that the a_n' are all distinct. Suppose that $a_n' = a_m'$ for some $n \neq m$. This is equivalent to

$$g^{2k}(a_i + a_n) + g^{2l}(a_n + a_j) = g^{2k}(a_i + a_m) + g^{2l}(a_m + a_j).$$

and implies the equation $(a_n - a_m)g^{2k}[1 + g^{2(l-k)}] = 0$. But, $g^{2(l-k)} \neq -1 = g^{2t-1}$ in $GF(q)$, $q = 4t-1$. Thus, $a_n = a_m$ and $n = m$ contrary to our assumption. Consequently, $(A_k A_l)_{ij} = F^2 J = (2t-1)^2 J$. Similar proofs take place for the remaining cases of (3.8).

LEMMA 3.2. *The matrices $A_k \in \mathscr{A}_q$, $1 \leqq k \leqq 2t$, satisfy*

(3.13) $\quad A_k A_k^T = q[tV_k + (t-1)J], \quad A_k^T A_k = q[tW_k + (t-1)J],$

where $V_k = (P[g^{2k}(a_i - a_j)])$, $W_k = (P[-g^{2k}(a_i - a_j)])$, $1 \leqq k \leqq 2t-1$, *and* $V_{2t} = I_q \otimes J_q$, $W_{2t} = J_q \otimes I_q$.

Proof. From (1.6) and (3.10) we obtain for $1 \leqq k \leqq 2t-1$,

$$(A_k A_k^T)_{ij} = \sum_{n=1}^{q} FP[g^{2k}(a_i + a_n)]P[g^{2k}(a_n + a_j)]^T F^T$$

$$= FF^T \sum_{n=1}^{q} P[g^{2k}(a_i - a_j)] = qtP[g^{2k}(a_i - a_j)] + q(t-1)J.$$

The other cases in (3.13) are verified in a similar way.

4. A construction for PC-graphs. Before assembling the adjacency matrix of a PC-graph from the elements of $\mathscr{A}_q$ we require the definition of a skew-Latin square. A Latin square $L = (l_{ij})$ of order $2n + 1$ with symbols $\{0, \pm 1, \ldots, \pm n\}$ is said to be *skew-symmetric* if $l_{ii} = 0$ and $l_{ji} = -l_{ij}$, $1 \leqq i, j \leqq 2n+1$. So, for example, the circulant

$$(4.1) \qquad L = (l_{ij}), l_{ij} = \begin{cases} j - i + p & \text{if } i - j > n \\ j - i - p & \text{if } j - i > n \\ j - i & \text{otherwise,} \end{cases}$$

forms a skew-Latin square of order $p = 2n + 1$. It can be shown that the number of non-equivalent skew-Latin squares grows very rapidly as the order increases.

We are now in a position to state our main results.

THEOREM 4.1. *For $t > 0$, such that $q = 4t - 1$ is a prime power, let $p = 4t + 1$ be the order of a PC-graph with adjacency matrix $\tilde{A} = (\tilde{a}_{ij})$. Then*

$$(4.2) \qquad A = (A_{ij}), \quad A_{ij} = \begin{cases} A_d & \text{if } i = j \\ A(l_{ij}) & \text{if } i \neq j \text{ and } \tilde{a}_{ij} = 1 \\ J - A(l_{ij}) & \text{otherwise,} \end{cases}$$

is the adjacency matrix of a PC-graph of order pq^2 for any skew-Latin square $L = (l_{ij})$ of order p. Here $A_d \in \mathscr{A}_q$ and $A(l_{ij})$, $1 \leqq i, j \leqq p$ are related to the *matrices* $A_k \in \mathscr{A}_q$ *as follows:*

$$(4.3)\qquad A(l_{ij}) = \begin{cases} A_k & \text{if } l_{ij} = k \\ A_k^T & \text{if } l_{ij} = -k \end{cases}, \quad 1 \leqq k \leqq 2t.$$

Proof. We note that A is of the form (2.1) and satisfies the necessary condition stated in Theorem 2.1. It remains to show that A satisfies the quadratic equation in (1.3) with $t' = (pq^2 - 1)/4 = (16t^2 - 4t - 1)t$. We shall make frequent use of the following fact. If X, Y are regular (0, 1)-matrices of order q^2 and degree $q(q-1)/2$ then

$$(4.4)\qquad X(J - Y) = (J - X)Y = \binom{q}{2}J - XY, \quad (J - X)(J - Y) = qJ + XY.$$

Using the same notation as in Section 3 it is easily verified that

$$(4.5)\qquad \sum_{k=1}^{2t} (V_k + W_k) = qI + J.$$

Since $A_d^2 = 2t(2t-1)(J + I) - A_d$ and each row (column) of A contains each of the matrices A_k (or $J - A_k$) and A_k^T (or $J - A_k^T$), $k = 1, \ldots, 2t$, exactly once, then by (4.2), (4.4), (4.5) and Lemma 3.2:

$$(4.6)\qquad (A^2)_{ii} = \sum_{n=1}^{p} A_{in}A_{ni} = \sum_{n=1}^{p} A_{in}A_{in}^T = A_d^2 + 2tqJ + \sum_{k=1}^{2t} (A_kA_k^T + A_k^TA_k) = (16t^2 - 4t - 1)t(J + I) - A_{ii}.$$

If $i \neq j$ then, by Lemma 3.1, if $\tilde{a}_{ij} = 1$,

$$(4.7)\qquad (A^2)_{ij} = \sum_{n=1}^{p} A_{in}A_{jn}^T = A_dA(l_{ij}) + A(l_{ij})A_d + (t-1)(2t-1)^2J + 2t \cdot 2t(2t-1)J + t(2t)^2J = (16t^2 - 4t - 1)tJ - A_{ij},$$

and if $\tilde{a}_{ij} = 0$

$$(4.8)\qquad (A^2)_{ij} = A_d[J - A(l_{ij})] + [J - A(l_{ij})]A_d + t(2t-1)^2J + 2t \cdot 2t(2t-1)J + (t-1)(2t)^2J = (16t^2 - 4t - 1)tJ - A_{ij},$$

where, similarly as in (2.5a)–(2.7a), we employed the given strongly regular PC-graph with adjacency matrix $\tilde{A}$.

The matrices (4.2) can be used to derive many other non-isomorphic solutions of (1.3). To illustrate this derivation process, let

$$(4.9)\qquad A' = (A_{ij}') = (Q_{ij}A_{ij}P_{ij}), \quad P_{ji} = Q_{ij}^T, \quad 1 \leqq i, j \leqq p,$$

where P_{ij}, Q_{ij} are permutation matrices of order q^2 and $A = (A_{ij})$ satisfies (4.2). If we succeed to find P_{ij}, Q_{ij} such that Lemmas 3.1 and 3.2 hold for elements of the corresponding set $\mathscr{A}_k'$, then A' will be the adjacency matrix of a PC-graph of order pq^2. One possible choice for P_{ij}, Q_{ij} is provided by the following:

THEOREM 4.2. *Let A' be given by* (4.9) *with $P_{ij} \in \{P^r, Q_r, r = 1, \ldots, q\}$ if $i = j$ and $P_{ij} = I$ otherwise. Here P is a block-diagonal permutation matrix, $(P)_{kl} = \delta_{kl}P[a_k]$, $1 \leqq k, l \leqq q$ and Q_r maps $A_d{}^r = (P^r)^T A_d P^r$ to its complement $Q_r{}^T A_d{}^r Q_r = (A_d{}^r)^c = J - I - A_d{}^r$. Then A' is the adjacency matrix of a PC-graph of order pq^2, if for any $1 \leqq i < j \leqq p$ the following conditions are satisfied (see Theorem* 4.1 *for notation). If $A_{ii}' = A_d{}^r$, $A_{jj}' = A_d{}^s$ and $l_{ij} = k$ then: if $1 \leqq k \leqq 2t - 1$ then*

$$(4.10)\quad \chi(g^{2k} + rg^0) \geqq 1, \quad \chi(g^{2k} - sg^0) \geqq 0,$$

are either both true or both false, and if $k = 2t$ then

$$(4.11)\quad \chi(g^{2k} - sg^0) \leqq 0,$$

where χ is the quadratic character of $GF(q)$. In case that $k < 0$, (4.10) *and* (4.11) *hold with r and s interchanged. Finally, if either $A_{ii}' = (A_d{}^r)^c$ or $A_{jj}' = (A_d{}^s)^c$, or both are true, then* (4.10) *and* (4.11) *hold with $\geqq$, $\leqq$ replaced by $<$, $>$ in those inequalities involving either r or s, or both r and s respectively.*

Proof. Noting that, by definition (3.4), A_d corresponds to a self-complementary PC-graph of order q^2 we may choose Q_r to be an isomorphism between the graph and its complement. Now, since

$$(4.12)\quad (A_d{}^r)_{ij} = ((P^r)^T A_d P^r)_{ij} = A_{ij}{}^d P[r(a_i - a_j)],$$

calculations similar to those in (3.11) and (3.12) yield

$$(4.13)\quad (A_d{}^r A_k)_{ij} = \Big\{ t(J - I) \sum_{n=1}^{2t-1} P[g^{2n}(g^{2k} + rg^0)] + [t(J + I) - J + F] \sum_{n=1}^{2t-1} P[g^{2n+1}(g^{2k} + rg^0)] \Big\} P[g^{2k}(a_i + a_j)],$$

$$(4.14)\quad (A_k A_d{}^s)_{ij} = \Big\{ t(J - I) \sum_{n=1}^{2t-1} P[-g^{2n}(g^{2k} - sg^0)] + [t(J + I) - J + F] \sum_{n=1}^{2t-1} P[-g^{2n+1}(g^{2k} - sg^0)] \Big\} P[g^{2k}(a_i + a_j)]$$

It is immediately verified that (3.7) holds if either $\chi(g^{2k} + rg^0) = 1$, $\chi(g^{2k} - sg^0) = 0, 1$ or $\chi(g^{2k} + rg^0) = 0, -1$, $\chi(g^{2k} - sg^0) = -1$. The other cases follow along the same lines. The result is a consequence of Lemmas 3.1 and 3.2.

In order to demonstrate the construction techniques of this section we are going to exhibit all non-isomorphic PC-graphs on 45 nodes which can be derived from Theorem 4.1 and Theorem 4.2. For $t = 1$ we have $p = 5$, $q = 3$ and the elements of $GF(3) \cong Z_3$ are numbered so that $a_1 = 1$, $a_2 = 2$ and

$a_3 = 0$. From the defining relations (3.2)–(3.6) applied to $GF(3)$ with primitive element $g = 2$ we obtain:

$$(4.15)\quad A_d = \begin{bmatrix} 000 & 110 & 101 \\ 000 & 011 & 110 \\ 000 & 101 & 011 \\ 101 & 000 & 110 \\ 110 & 000 & 011 \\ 011 & 000 & 101 \\ 110 & 101 & 000 \\ 011 & 110 & 000 \\ 101 & 011 & 000 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 100 & 010 & 001 \\ 010 & 001 & 100 \\ 001 & 100 & 010 \\ 010 & 001 & 100 \\ 001 & 100 & 010 \\ 100 & 010 & 001 \\ 001 & 100 & 010 \\ 100 & 010 & 001 \\ 010 & 001 & 100 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 001 & 001 & 001 \\ 001 & 001 & 001 \\ 001 & 001 & 001 \\ 100 & 100 & 100 \\ 100 & 100 & 100 \\ 100 & 100 & 100 \\ 010 & 010 & 010 \\ 010 & 010 & 010 \\ 010 & 010 & 010 \end{bmatrix}$$

There are two non-equivalent skew-Latin squares of order 5: the circulant matrices with first rows $(0, 1, 2, -2, -1)$ and $(0, 1, -2, 2, -1)$ respectively. The unique PC-graph of order 5 and adjacency matrix $\tilde{A}$ is a pentagon. It can be combined with each of the skew-Latin squares in 4 non-isomorphic ways, corresponding to the labellings (1, 2, 3, 4, 5), (1, 3, 5, 2, 4), (1, 2, 3, 5, 4) and (1, 3, 4, 2, 5). Using a computer analysis we established that all graphs obtained from the first skew-Latin square are isomorphic to those obtained from the second square. Thus, the construction in Theorem 4.1 generates 4 non-isomorphic PC-graphs of order 45 with adjacency matrices A_I-A_{IV} given by:

$$(4.16)\quad A_I = \begin{bmatrix} A_d & A_1 & \bar{A}_2 & \bar{A}_2^T & A_1^T \\ A_1^T & A_d & A_1 & \bar{A}_2 & \bar{A}_2^T \\ \bar{A}_2^T & A_1^T & A_d & A_1 & \bar{A}_2 \\ \bar{A}_2 & \bar{A}_2^T & A_1^T & A_d & A_1 \\ A_1 & \bar{A}_2 & \bar{A}_2^T & A_1^T & A_d \end{bmatrix}, \quad A_{III} = \begin{bmatrix} A_d & A_1 & \bar{A}_2 & A_2^T & \bar{A}_1^T \\ A_1^T & A_d & A_1 & \bar{A}_2 & \bar{A}_2^T \\ \bar{A}_2^T & A_1^T & A_d & \bar{A}_1 & A_2 \\ A_2 & \bar{A}_2^T & \bar{A}_1^T & A_d & A_1 \\ \bar{A}_1 & \bar{A}_2 & A_2^T & A_1^T & A_d \end{bmatrix},$$

$$(4.17)\quad A_{II} = (A_{ij}^{II}), \quad A_{ij}^{II} = \begin{cases} A_d, & i = j \\ \bar{A}_{ij}^{I}, & i \neq j \end{cases}, \qquad A_{IV} = (A_{ij}^{IV}), \quad A_{ij}^{IV} = \begin{cases} A_d, & i = j \\ \bar{A}_{ij}^{III}, & i \neq j \end{cases},$$

where $\bar{A}_k = J - A_k$. Extending these matrices as in (1.7) we obtain 4 non-equivalent conference matrices C_I–C_{IV} of order 46. Both C_I and C_{II} have automorphism groups of order 10 with orbits $(1 \times 1, 1 \times 5, 4 \times 10)$ ($i \times j \Leftrightarrow$ i orbits of size j) representing 6 non-isomorphic PC-graphs per switching class with groups $(1 \times 10, 1 \times 5, 4 \times 1)$ ($i \times j \Leftrightarrow i$ graphs with groups of order j). Both C_{III} and C_{IV} have automorphism groups of order 2 with orbits $(6 \times 1, 20 \times 2)$ representing 26 graphs per switching class with groups $(6 \times 2, 20 \times 1)$. All together we have generated 64 nonisomorphic PC-graphs of order 45, 48 of which have trivial automorphism groups. We remark, that automorphisms of a symmetric conference matrix C are represented by ± 1 permutation matrices P such that $P^T C P = C$.

An exhaustive search for permutation matrix-combinations satisfying the conditions of Theorem 4.2 yields the following sets of diagonal blocks for A':

$$\begin{array}{llll} & (1, 1^c, 3, 2^c, 3) & (1, 3^c, 1^c, 2^c, 2^c) & (1, 3^c, 2, 3^c, 1^c) \\ (4.18) & (1, 2, 2, 1^c, 3) & (2, 1^c, 3, 1, 2) & (3, 1, 1^c, 3, 2^c) \\ & (2, 2, 2, 2, 2) & (2, 2, 1^c, 3, 1) & (3, 1, 2, 2, 1^c) \\ & (3, 3, 3, 3, 3) & (2, 3^c, 1^c, 1, 3^c) & (3, 2^c, 3, 1, 1^c), \end{array}$$

where the value r or r^c of the i-th component indicates that $A_{ii}' = A_d^r$ or $(A_d^r)^c$ respectively. Inserting the diagonal blocks represented by the first column in (4.18) (and the corresponding complementary sets) into A_I and A_{II} we obtain after extension 8 conference matrices of type $(1 \times 1, 1 \times 5, 4 \times 5)$ with groups of order 10 and 8 matrices of type $(6 \times 1, 20 \times 2)$ with groups of order 2. Inserting all sets of (4.18) (and their complements) into A_{III} and A_{IV} we obtain another 48 matrices of the second type. Hence, Theorem 4.2 yields a total of 64 non-equivalent symmetric C-matrices of order 46, generating 1504 non-isomorphic PC-graphs on 45 nodes, 1152 of which have trivial automorphism groups. We note that the PC-graphs (4.16) and (4.17) are included in those obtained from Theorem 4.2.

Many more PC-graphs (and C-matrices) can be constructed by permuting off-diagonal blocks in (4.9). So, for example, by setting

$$(4.19) \quad \begin{aligned} A_{12}{}^I &= A_{12}{}^{III} = (A_{21}{}^I)^T = (A_{21}{}^{III})^T = A_1P^2, \\ A_{52}{}^I &= A_{52}{}^{III} = (A_{25}{}^I)^T = (A_{25}{}^{III})^T = \bar{A}_2P, \end{aligned}$$

in (4.16) and (4.17) we obtain 4 C-matrices with groups of order 3 and orbits $(10 \times 1, 12 \times 3)$ representing 22 PC-graphs per switching class with groups $(10 \times 3, 12 \times 1)$ respectively.

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