ORIGINAL PAPER

SYMMETRIC FUNCTIONS OF BINARY PRODUCTS OF TRIBONACCI LUCAS NUMBERS AND ORTHOGONAL POLYNOMIALS

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Abstract. *In this paper, we will recover the new generating functions of some products of Tribonacci Lucas numbers and orthogonal polynomials. The technic used her is based on the theory of the so called symmetric functions.*

Keywords: *symmetric functions; generating functions; Tribonacci Lucas numbers; orthogonal polynomials.*

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1. INTRODUCTION

Recently, Fibonacci and Lucas numbers have investigated very largely and authors tried to developed and give some directions to mathematical calculations using these types of special numbers [1-4]. One of these directions goes through to the Tribonacci and the Tribonacci Lucas numbers. In factTribonacci numbers have been firstly defined by M. Feinberg in 1963 and then some important properties over this numbers have been created by [5-9]. On the other hand, Tribonacci Lucas numbers have been introduced and investigated by author in [10].

For $n \geq 3$, it is known that while the Tribonacci Lucas sequence $\{K_n\}_{n\in\mathbb{N}}$ is defined by

$$\begin{cases} K_n = K_{n-1} + K_{n-2} + K_{n-3} \\ K_0 = 3, K_1 = 1, K_2 = 3 \end{cases}. \tag{1.1}$$

Consider the characteristic polynomial $x^3 - x^2 - x - 1 = 0$ associated to recursive relation (1.1) with having the roots

$$\alpha = \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3},$$
$$\beta = \frac{1+w\sqrt[3]{19+3\sqrt{33}}+w^2\sqrt[3]{19-3\sqrt{33}}}{3},$$
$$\gamma = \frac{1+w^2\sqrt[3]{19+3\sqrt{33}}+w\sqrt[3]{19-3\sqrt{33}}}{3},$$

where $w = \frac{-1+i\sqrt{3}}{2}$.

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The Binet formula for Tribonacci Lucas numbers is:

$$K_n = \alpha^n + \beta^n + \gamma^n.$$

In this paper, we will recover the new generating functions of some products of Tribonacci Lucas numbers and orthogonal polynomials.

The further contents of this paper are as follows: Section 1 gives introduction, in section 2, we introduce symmetric functions and some of its properties. We also give some more useful definitions which are used in the following sections. In Section 3 we prove our main result which relates the generating functions of binary products of Tribonacci Lucas numbers and orthogonal polynomials.

2. PRELIMINARIES AND SOME PROPERTIES

Definition 2.1. [11] *Let* $\{P_n(x)\}_{n\in\mathbb{N}}$ *polynomials sequence of second ordre defined by*

$$\begin{cases} P_n(x) = \left(p_0 + p_1 x\right) P_{n-1}(x) + \left(q_0 + q_1 x\right) P_{n-2}(x), \ n \geq 2 \\ P_0(x) = \alpha, P_1(x) = \beta_0 + \beta_1 x \end{cases},$$

where $p_0, p_1, q_0, q_1, \alpha, \beta_0$ and β_1 are real numbers.

In fact, the well-known sequences below are special cases of the generalized polynomials sequence $\{P_n(x)\}_{n\in\mathbb{N}}$

- Putting $p_0 = q_1 = \beta_0 = 0$ and $p_1 = q_0 = \alpha = \beta_1 = 1$ reduces to **Fibonacci polynomials**.
- Substituting $p_0 = q_1 = \beta_0 = 0$ and $\alpha = 2$, $p_1 = q_0 = \beta_1 = 1$ yields **Lucas polynomials.**
- Taking $p_0 = q_1 = \alpha = \beta_1 = 0$, $p_1 = 2$ and $q_0 = \beta_0 = 1$ gives **Pell polynomials**.
- Taking $p_0 = q_1 = \beta_0 = 0$, $q_0 = 1$ and $\alpha = \beta_1 = p_1 = 2$ gives **Pell Lucas polynomials**.
- Taking $p_1 = q_0 = \alpha = \beta_1 = 0$, $\beta_0 = p_0 = 1$ and $q_1 = 2$ gives **Jacobsthal polynomials**.
- In the case when $p_1 = q_0 = \beta_1 = 0$, $\beta_0 = p_0 = 1$ and $\alpha = q_1 = 2$ it reduces to **Jacobsthal Lucas polynomials**.
- In the case when $p_0 = q_1 = \beta_0 = 0$, $\alpha = \beta_1 = 1,\ p_1 = 2$ and $q_0 = -1$ it reduces to **Chebyshev polynomials of first kind**.
- In the case when $p_0 = q_1 = \beta_0 = 0,\ \alpha = 1,\ \beta_1 = p_1 = 2$ and $q_0 = -1$ it reduces to **Chebyshev polynomials of second kind**.
- Putting $p_0 = q_1 = 0,\ \alpha = 1,\ \beta_1 = p_1 = 2$ and $\beta_0 = q_0 = -1$ we obtain **Chebyshev polynomials of third kind**.
- Substituting $p_0 = q_1 = 0,\ \alpha = \beta_0 = 1,\ \beta_1 = p_1 = 2$ and $q_0 = -1$ yield **Chebyshev polynomials of fourth kind**.

Corollary 2.1. [12] *For* $n \in \mathbb{N}$*, Tribonacci Lucas numbers, can be defined by*

$$K_n = 3S_n(e_1+e_2+e_3) - 2S_{n-1}(e_1+e_2+e_3) - S_{n-2}(e_1+e_2+e_3),$$

where e_1*,* e_2 *and* e_3 *are the roots of characteristic polynomial associated to recursive relation (1.1).*

Proposition 2.1. [11] *For* $n \in \mathbb{N}$*, the generating function of generalized polynomials of second ordre is given by*

$$\sum_{n=0}^{+\infty} P_n(x)t^n = \frac{\alpha + \left[\left(\beta_0 - \alpha p_0\right) + \left(\beta_1 - \alpha p_1\right)\right]t}{1 - \left(p_0 + p_1 x\right)t - \left(q_0 + q_1 x\right)t^2},$$

with $P_n(x) = \alpha S_n\left(a_1 + [-a_2]\right) + \left[\left(\beta_0 - \alpha p_0\right) + \left(\beta_1 - \alpha p_1\right)x\right] S_{n-1}\left(a_1 + [-a_2]\right).$

Definition 2.2. *[17] Let* A *and* B *be any two alphabets, then we give* $S_n(A-B)$ *by the following form:*

$$\frac{\Pi_{b\in B}(1-bt)}{\Pi_{a\in A}(1-at)} = \sum_{n=0}^{+\infty} S_n(A-B)t^n, \tag{2.1}$$

with the condition $S_n(A-B) = 0$ *for* $n < 0.$

Remark 2.2. *Taking* $A = 0$ *in (2.1) gives*

$$\Pi_{b\in B}(1-bt) = \sum_{n=0}^{+\infty} S_n(-B)t^n.$$

Further, in the case $A = 0$ or $B = 0$,we have

$$\sum_{n=0}^{+\infty} S_n(A-B)t^n = \left(\sum_{n=0}^{+\infty} S_n(A)t^n\right)\left(\sum_{n=0}^{+\infty} S_n(-B)t^n\right),$$

thus,

$$S_n(A-B) = \sum_{k=0}^{n} S_{n-k}(A)S_k(-B).$$

Definition 2.3. *Let* g *be any function on* $\mathbb{R}^n$*, then we consider the divided difference operator as the following form*

$$\partial_{x_i x_{i+1}}(g) = \frac{g(x_1,\cdots,x_i,x_{i+1},\cdots x_n) - g(x_1,\cdots x_{i-1},x_{i+1},x_i,x_{i+2}\cdots x_n)}{x_i - x_{i+1}}, \text{ (see } [13])$$

Definition 2.4. [14] *Given an alphabet* $E = \{e_1, e_2\}$*, the symmetrizing operator* $\delta^k_{e_1e_2}$ *is defined by*

$$\delta^k_{e_1e_2}(e_1^n) = \frac{e_1^{k+n} - e_2^{k+n}}{e_1 - e_2} = S_{k+n-1}(e_1+e_2), \text{ for all } k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

3. MAIN RESULTS AND LEMMAS

The following Theorem is one of the key tools of the proof of our main result. It has been proved in [18].

Theorem 3.1. *Given two alphabets* $A=\{a_1,a_2\}$ *and* $E=\{e_1,e_2,e_3\}$, *we have*

$$\sum_{n=0}^{+\infty} S_n(E)\delta_{a_1a_2}^k(a_1^n)t^n = \frac{\sum_{n=0}^{+\infty} S_n(-E)\delta_{a_1a_2}^k\left(a_2^n\right)t^n}{\left(\sum_{n=0}^{+\infty} S_n(-E)a_2^n t^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E)a_1^n t^n\right)}, k\in\mathbb{N}_0.$$

Proof : Applying the operator $\delta_{a_1a_2}^k$ to the series $f\left(a_1t\right)=\sum_{n=0}^{+\infty} S_n(E)a_1^n t^n$, we have

$$\begin{aligned}\delta_{a_1a_2}^k f\left(a_1t\right) &= \delta_{a_1a_2}^k\left(\sum_{n=0}^{+\infty} S_n(E)a_1^n t^n\right)\\ &= \frac{\sum_{n=0}^{+\infty} S_n(E)a_1^{n+k}t^n - \sum_{n=0}^{+\infty} S_n(E)a_2^{n+k}t^n}{a_1-a_2}\\ &= \sum_{n=0}^{+\infty} S_n(E)\frac{a_1^{n+k}-a_2^{n+k}}{a_1-a_2}t^n = \sum_{n=0}^{+\infty} S_n\left(E\right)\delta_{a_1a_2}^k(a_1^n)t^n.\end{aligned}$$

On the other part, since $\sum_{n=0}^{+\infty} S_n(E)a_1^n t^n = \frac{1}{\prod_{e\in E}(1-ea_1t)}$ we have

$$\begin{aligned}\delta_{a_1a_2}^k f\left(a_1t\right) &= \delta_{a_1a_2}^k\left(\frac{1}{\prod_{e\in E}\left(1-ea_1t\right)}\right) = \frac{\frac{a_1^k}{\prod_{e\in E}(1-ea_1t)} - \frac{a_2^k}{\prod_{e\in E}(1-ea_2t)}}{a_1-a_2}\\ &= \frac{a_1^k\prod_{e\in E}\left(1-ea_2t\right) - a_2^k\prod_{e\in E}\left(1-ea_1t\right)}{(a_1-a_2)\prod_{e\in E}\left(1-ea_1t\right)\prod_{e\in E}\left(1-ea_2t\right)}\end{aligned}$$

Using the fact that : $\sum_{n=0}^{+\infty} S_n(-E)a_1^n t^n = \prod_{e\in E}\left(1-ea_1t\right)$, then

$$\begin{aligned}\delta_{a_1a_2}^k f\left(a_1t\right) &= \frac{a_1^k\sum_{n=0}^{+\infty} S_n(-E)a_2^n t^n - a_2^k\sum_{n=0}^{+\infty} S_n(-E)a_1^n t^n}{(a_1-a_2)\left(\sum_{n=0}^{+\infty} S_n(-E)a_2^n t^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E)a_1^n t^n\right)}\\ &= \frac{\sum_{n=0}^{+\infty} S_n(-E)\frac{a_1^k a_2^n - a_2^k a_1^n}{a_1-a_2}t^n}{\left(\sum_{n=0}^{+\infty} S_n(-E)a_2^n t^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E)a_1^n t^n\right)} = \frac{\sum_{n=0}^{+\infty} S_n(-E)\delta_{a_1a_2}^k\left(a_2^n\right)t^n}{\left(\sum_{n=0}^{+\infty} S_n(-E)a_2^n t^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E)a_1^n t^n\right)}.\end{aligned}$$

This completes the proof.

If $k=0$, $k=1$, $k=2$ and $k=3$ in Theorem 3.1 we deduce the following lemmas

Lemma 3.1. *Let* $A=\{a_1,a_2\}$ *and* $E=\{e_1,e_2,e_3\}$ *two alphabets, we have*

$$\sum_{n=0}^{+\infty} S_n(E)S_{n-1}(A)t^n = \frac{-S_1(-E)t-(a_1+a_2)S_2(-E)t^2-((a_1+a_2)^2-a_1a_2)S_3(-E)t^3}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1-ea_2t)}. \tag{3.2}$$

Lemma 3.2. *Let* $A=\{a_1,a_2\}$ *and* $E=\{e_1,e_2,e_3\}$ *two alphabets, we have*

$$\sum_{n=0}^{+\infty} S_n(E)S_n(A)t^n = \frac{1-a_1a_2S_2(-E)t^2-a_1a_2(a_1+a_2)S_3(-E)t^3}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1-ea_2t)}, \tag{3.3}$$

from (3.3) we deduce

$$\sum_{n=0}^{+\infty} S_{n-1}(E)S_{n-1}(A)t^n = \frac{t-a_1a_2S_2(-E)t^3-a_1a_2(a_1+a_2)S_3(-E)t^4}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1-ea_2t)}. \tag{3.4}$$

Lemma 3.3. *Given two alphabets* $E=\{e_1,e_2,e_3\}$ *and* $A=\{a_1,a_2\}$, *we have*

$$\sum_{n=0}^{+\infty} S_n(E)S_{n+1}(\mathrm{A})t^n = \frac{(a_1+a_2)+S_1(-E)a_1a_2t-S_3(-E)a_1^2a_2^2t^3}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1-ea_2t)}, \tag{3.5}$$

from (3.5) we deduce

$$\sum_{n=0}^{+\infty} S_{n-1}(E)S_n(\mathrm{A})t^n = \frac{(a_1+a_2)t+S_1(-E)a_1a_2t^2-S_3(-E)a_1^2a_2^2t^4}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1-ea_2t)}, \tag{3.6}$$

from (3.6) we deduce

$$\sum_{n=0}^{+\infty} S_{n-2}(E)S_{n-1}(\mathrm{A})t^n = \frac{(a_1+a_2)t^2+S_1(-E)a_1a_2t^3-S_3(-E)a_1^2a_2^2t^5}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1-ea_2t)}. \tag{3.7}$$

Lemma 3.4. *Given two alphabets* $E=\{e_1,e_2,e_3\}$ *and* $A=\{a_1,a_2\}$, *we have*

$$\sum_{n=0}^{+\infty} S_{n-2}(E)S_n(\mathrm{A})t^n = \frac{\left((a_1+a_2)^2-a_1a_2\right)t^2+a_1a_2(a_1+a_2)S_1(-E)t^3+S_2(-E)a_1^2a_2^2t^4}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1-ea_2t)}. \tag{3.8}$$

In this part, we now derive the new generating functions of the products of Tribonacci Lucas numbers with Fibonacci, Lucas, Pell, Pell Lucas, Jacobsthal and Jacobsthal Lucas polynomials.

- For the case $A=\{a_1,-a_2\}$ and $E=\{e_1,e_2,e_3\}$ with replacing a_2 by $(-a_2)$ in (3.2), (3.3), (3.4), (3.6), (3.7) and (3.8), we have

$$\sum_{n=0}^{+\infty} S_n(E)S_{n-1}\left(a_1+[-a_2]\right)t^n=\frac{-S_1(-E)t-(a_1-a_2)S_2(-E)t^2-((a_1-a_2)^2+a_1a_2)S_3(-E)t^3}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1+ea_2t)}. \quad (3.9)$$

$$\sum_{n=0}^{+\infty} S_n(E)S_n\left(a_1+[-a_2]\right)t^n=\frac{1+a_1a_2S_2(-E)t^2+a_1a_2(a_1-a_2)S_3(-E)t^3}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1+ea_2t)}. \quad (3.10)$$

$$\sum_{n=0}^{+\infty} S_{n-1}(E)S_{n-1}\left(a_1+[-a_2]\right)t^n=\frac{t+a_1a_2S_2(-E)t^3+a_1a_2(a_1-a_2)S_3(-E)t^4}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1+ea_2t)}. \quad (3.11)$$

$$\sum_{n=0}^{+\infty} S_{n-1}(E)S_n\left(a_1+[-a_2]\right)t^n=\frac{(a_1-a_2)t-S_1(-E)a_1a_2t^2-S_3(-E)a_1^2a_2^2t^4}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1+ea_2t)}, \quad (3.12)$$

$$\sum_{n=0}^{+\infty} S_{n-2}(E)S_{n-1}\left(a_1+[-a_2]\right)t^n=\frac{(a_1-a_2)t^2-S_1(-E)a_1a_2t^3-S_3(-E)a_1^2a_2^2t^5}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1+ea_2t)}, \quad (3.13)$$

$$\sum_{n=0}^{+\infty} S_{n-2}(E)S_n\left(a_1+[-a_2]\right)t^n=\frac{\left((a_1-a_2)^2+a_1a_2\right)t^2-a_1a_2(a_1-a_2)S_1(-E)t^3+a_1^2a_2^2S_2(-E)t^4}{\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1+ea_2t)}, \quad (3.14)$$

with

$$\begin{aligned}\prod_{e\in E}(1-ea_1t)\prod_{e\in E}(1+ea_2t)=&1+(a_1-a_2)S_1(-E)t+\left[(a_1-a_2)^2S_2(-E)-a_1a_2(S_1^2(-E)-2S_2(-E))\right]t^2\\&+\left[(a_1-a_2)^3S_3(-E)-a_1a_2(a_1-a_2)(S_1(-E)S_2(-E)-3S_3(-E))\right]t^3\\&-\left[a_1a_2(a_1-a_2)^2S_3(-E)S_1(-E)-a_1^2a_2^2(S_2^2(-E)-2S_3(-E)S_1(-E))\right]t^4\\&+a_1^2a_2^2(a_1-a_2)S_3(-E)S_2(-E)t^5-a_1^3a_2^3S_3^2(-E)t^6.\end{aligned}$$

This case consists of three related parts.

First, the substitutions $\begin{cases}a_1-a_2=x\\a_1a_2=1\end{cases}$ and $\begin{cases}S_1(-E)=-1\\S_2(-E)=-1,\\S_3(-E)=-1\end{cases}$ in (3.10), (3.12) and (3.14) give

$$\sum_{n=0}^{+\infty} S_n(E)S_n\left(a_1+\left[-a_2\right]\right)t^n = \frac{1-t^2-xt^3}{1-xt-(x^2+3)t^2-(x^3+4x)t^3-(x^2+1)t^4+xt^5-t^6}. \quad (3.15)$$

$$\sum_{n=0}^{+\infty} S_{n-1}(E)S_n\left(a_1+\left[-a_2\right]\right)t^n = \frac{xt+t^2+t^4}{1-xt-(x^2+3)t^2-(x^3+4x)t^3-(x^2+1)t^4+xt^5-t^6}. \quad (3.16)$$

$$\sum_{n=0}^{+\infty} S_{n-2}(E)S_n\left(a_1+\left[-a_2\right]\right)t^n = \frac{\left(x^2+1\right)t^2+xt^3-t^4}{1-xt-(x^2+3)t^2-(x^3+4x)t^3-(x^2+1)t^4+xt^5-t^6}. \quad (3.17)$$

Multiplying the equation (3.15) by 3 and substracting it from (3.16) multiplying by 2 and substracting it from (3.17), we have

$$\begin{aligned}\sum_{n=0}^{+\infty}\left[3S_n(E)-2S_{n-1}(E)-S_{n-2}(E)\right]S_n\left(a_1+\left[-a_2\right]\right)t^n &= \frac{3-2xt-(6+x^2)t^2-4xt^3-t^4}{1-xt-(x^2+3)t^2-(x^3+4x)t^3-(x^2+1)t^4+xt^5-t^6}\\ &= \sum_{n=0}^{+\infty} K_nF_n(x)t^n, \quad (3.18)\end{aligned}$$

which represents the new generating function for the combined Tribonacci Lucas numbers and Fibonacci polynomials, with $K_nF_n(x)=\left[3S_n(E)-2S_{n-1}(E)-S_{n-2}(E)\right]S_n\left(a_1+\left[-a_2\right]\right)$.

We have the following theorem.

Theorem 3.2. *For* $n\in\mathbb{N}$, *the new generating function for the combined Tribonacci Lucas numbers and Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} K_nL_n(x)t^n = \frac{6-5xt-(12+4x^2)t^2-(12x+3x^3)t^3-(2+2x^2)t^4+xt^5}{1-xt-(x^2+3)t^2-(x^3+4x)t^3-(x^2+1)t^4+xt^5-t^6}.$$

Proof: Recall that, we have [11]

$$L_n(x)=2S_n\left(a_1+\left[-a_2\right]\right)-xS_{n-1}\left(a_1+\left[-a_2\right]\right).$$

we see that

$$\begin{aligned}\sum_{n=0}^{+\infty} K_nL_n(x)t^n &= \sum_{n=0}^{+\infty}\left[3S_n(E)-2S_{n-1}(E)-S_{n-2}(E)\right]\left[2S_n\left(a_1+\left[-a_2\right]\right)-xS_{n-1}\left(a_1+\left[-a_2\right]\right)\right]t^n\\
&= 2\sum_{n=0}^{+\infty}\left[3S_n(E)-2S_{n-1}(E)-S_{n-2}(E)\right]S_n\left(a_1+\left[-a_2\right]\right)t^n\\
&\quad -3x\sum_{n=0}^{+\infty} S_n(E)S_{n-1}\left(a_1+\left[-a_2\right]\right)t^n+2x\sum_{n=0}^{+\infty} S_{n-1}(E)S_{n-1}\left(a_1+\left[-a_2\right]\right)t^n\\
&\quad +x\sum_{n=0}^{+\infty} S_{n-2}(E)S_{n-1}\left(a_1+\left[-a_2\right]\right)t^n\\
&= 2\sum_{n=0}^{+\infty} K_nF_n(x)t^n-3x\sum_{n=0}^{+\infty} S_n(E)S_{n-1}\left(a_1+\left[-a_2\right]\right)t^n+2x\sum_{n=0}^{+\infty} S_{n-1}(E)S_{n-1}\left(a_1+\left[-a_2\right]\right)t^n\\
&\quad +x\sum_{n=0}^{+\infty} S_{n-2}(E)S_{n-1}\left(a_1+\left[-a_2\right]\right)t^n,\end{aligned}$$

according the relations (3.9), (3.11) and (3.13), and reduce to same denominator, we obtain the following result:

$$\sum_{n=0}^{+\infty} K_n L_n(x)t^n = 2\sum_{n=0}^{+\infty} K_n F_n(x)t^n - \frac{p_1(x)t + p_2(x)t^2 + p_3(x)t^3 + p_4(x)t^4 + p_5(x)t^5}{1+q_1(x)t+q_2(x)t^2+q_3(x)t^3+q_4(x)t^4+q_5(x)t^5+q_6(x)t^6},$$

where

$$p_1(x) = x,$$
$$p_2(x) = 2x(a_1 - a_2),$$
$$p_3(x) = (3x(a_1-a_2)^2 + 4xa_1a_2),$$
$$p_4(x) = 2xa_1a_2(a_1-a_2),$$
$$p_5(x) = -a_1^2a_2^2x,$$

and

$$q_1(x) = -(a_1 - a_2),$$
$$q_2(x) = -((a_1-a_2)^2 + 3a_1a_2),$$
$$q_3(x) = -((a_1-a_2)^3 + 4a_1a_2(a_1-a_2)),$$
$$q_4(x) = -a_1a_2((a_1-a_2)^2 + a_1a_2),$$
$$q_5(x) = a_1^2a_2^2(a_1-a_2),$$
$$q_6(x) = -a_1^3a_2^3.$$

After a simple calculation, of $p_i(x)$ and $q_i(x)$ we obtain

$$\sum_{n=0}^{+\infty} K_n L_n(x)t^n = \frac{6-5xt-(12+4x^2)t^2-(12x+3x^3)t^3-(2+2x^2)t^4+xt^5}{1-xt-(x^2+3)t^2-(x^3+4x)t^3-(x^2+1)t^4+xt^5-t^6}.$$

This completes the proof.

Second, the substitutions $\begin{cases} a_1 - a_2 = 2x \\ a_1a_2 = 1 \end{cases}$ and $\begin{cases} S_1(-E) = -1 \\ S_2(-E) = -1, \\ S_3(-E) = -1 \end{cases}$ in (3.9), (3.11) and (3.13) give

$$\sum_{n=0}^{+\infty} S_n(E)S_{n-1}\left(a_1 + [-a_2]\right)t^n = \frac{t+2xt^2+(4x^2+1)t^3}{1-2xt-(4x^2+3)t^2-(8x^3+8x)t^3-(4x^2+1)t^4+2xt^5-t^6}. \quad (3.19)$$

$$\sum_{n=0}^{+\infty} S_{n-1}(E)S_{n-1}\left(a_1 + [-a_2]\right)t^n = \frac{t-t^3-2xt^4}{1-2xt-(4x^2+3)t^2-(8x^3+8x)t^3-(4x^2+1)t^4+2xt^5-t^6}. \quad (3.20)$$

$$\sum_{n=0}^{+\infty} S_{n-2}(E)S_{n-1}\left(a_1 + [-a_2]\right)t^n = \frac{2xt^2+t^3+t^5}{1-2xt-(4x^2+3)t^2-(8x^3+8x)t^3-(4x^2+1)t^4+2xt^5-t^6}. \quad (3.21)$$

Multiplying the equation (3.19) by 3 and substracting it from (3.20) multiplying by 2 and substracting it from (3.21) we have

$$\sum_{n=0}^{+\infty}\left[3S_n(E)-2S_{n-1}(E)-S_{n-2}(E)\right]S_{n-1}\left(a_1+\left[-a_2\right]\right)t^n = \frac{t+4xt^2+\left(4+12x^2\right)t^3+4xt^4-t^5}{1-2xt-(4x^2+3)t^2-(8x^3+8x)t^3-(4x^2+1)t^4+2xt^5-t^6}=\sum_{n=0}^{+\infty}K_nP_n(x)t^n, \quad (3.22)$$

which represents a new generating function for the combined Tribonacci Lucas numbers and Pell polynomials, with $K_nP_n(x)=\left[3S_n(E)-2S_{n-1}(E)-S_{n-2}(E)\right]S_{n-1}\left(a_1+\left[-a_2\right]\right)$.

We have the following theorem.

Theorem 3.3. *For* $n\in\mathbb{N}$, *the new generating function for the combined Tribonacci Lucas numbers and Pell Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty}K_nQ_n(x)t^n=\frac{6-10xt-\left(12+16x^2\right)t^2-\left(24x+24x^3\right)t^3-\left(2+8x^2\right)t^4+2xt^5}{1-2xt-(4x^2+3)t^2-(8x^3+8x)t^3-(4x^2+1)t^4+2xt^5-t^6}.$$

Proof: By referred to [11], we have

$$Q_n(x)=2S_n\left(a_1+\left[-a_2\right]\right)-2xS_{n-1}\left(a_1+\left[-a_2\right]\right).$$

We see that

$$\begin{aligned}\sum_{n=0}^{+\infty}K_nQ_n(x)t^n &= \sum_{n=0}^{+\infty}\left[3S_n(E)-2S_{n-1}(E)-S_{n-2}(E)\right]\left[2S_n\left(a_1+\left[-a_2\right]\right)-2xS_{n-1}\left(a_1+\left[-a_2\right]\right)\right]t^n\\ &=6\sum_{n=0}^{+\infty}S_n(E)S_n\left(a_1+\left[-a_2\right]\right)t^n-4\sum_{n=0}^{+\infty}S_{n-1}(E)S_n\left(a_1+\left[-a_2\right]\right)t^n\\ &\quad -2\sum_{n=0}^{+\infty}S_{n-2}(E)S_n\left(a_1+\left[-a_2\right]\right)t^n\\ &\quad -2x\sum_{n=0}^{+\infty}\left[3S_n(E)-2S_{n-1}(E)-S_{n-2}(E)\right]S_{n-1}\left(a_1+\left[-a_2\right]\right)t^n\\ &=6\sum_{n=0}^{+\infty}S_n(E)S_n\left(a_1+\left[-a_2\right]\right)t^n-4\sum_{n=0}^{+\infty}S_{n-1}(E)S_n\left(a_1+\left[-a_2\right]\right)t^n\\ &\quad -2\sum_{n=0}^{+\infty}S_{n-2}(E)S_n\left(a_1+\left[-a_2\right]\right)t^n-2x\sum_{n=0}^{+\infty}K_nP_n(x)t^n,\end{aligned}$$

by using the relationphis (3.10), (3.12) and (3.14) after some calculations we find

$$\sum_{n=0}^{+\infty}K_nQ_n(x)t^n=\frac{6+f_1(x)t+f_2(x)t^2+f_3(x)t^3+f_4(x)t^4}{1+h_1(x)t+h_2(x)t^2+h_3(x)t^3+h_4(x)t^4+h_5(x)t^5+h_6(x)t^6}-2x\sum_{n=0}^{+\infty}K_nP_n(x)t^n,$$

where

$$f_1(x) = -4(a_1 - a_2),$$
$$f_2(x) = -(12a_1a_2 + 2(a_1 - a_2)^2),$$
$$f_3(x) = -8a_1a_2(a_1 - a_2),$$
$$f_4(x) = -2a_1^2a_2^2,$$

and

$$h_1(x) = -(a_1 - a_2),$$
$$h_2(x) = -((a_1 - a_2)^2 + 3a_1a_2),$$
$$h_3(x) = -((a_1 - a_2)^3 + 4a_1a_2(a_1 - a_2)),$$
$$h_4(x) = -a_1a_2((a_1 - a_2)^2 + a_1a_2),$$
$$h_5(x) = a_1^2a_2^2(a_1 - a_2),$$
$$h_6(x) = -a_1^3a_2^3.$$

After a simple calculation, of $f_i(x)$ and $h_i(x)$ we obtain

$$\sum_{n=0}^{+\infty} K_n Q_n(x)t^n = \frac{6 - 10xt - \left(12 + 16x^2\right)t^2 - \left(24x + 24x^3\right)t^3 - \left(2 + 8x^2\right)t^4 + 2xt^5}{1 - 2xt - (4x^2 + 3)t^2 - (8x^3 + 8x)t^3 - (4x^2 + 1)t^4 + 2xt^5 - t^6}.$$

This completes the proof.

Third, the substitutions $\begin{cases} a_1 - a_2 = 1 \\ a_1a_2 = 2x \end{cases}$ and $\begin{cases} S_1(-E) = -1 \\ S_2(-E) = -1, \\ S_3(-E) = -1 \end{cases}$ in (3.9), (3.11) and (3.13) give

$$\sum_{n=0}^{+\infty} S_n(E)S_{n-1}\left(a_1 + [-a_2]\right)t^n = \frac{t + t^2 + (2x+1)t^3}{1 - t - (1+6x)t^2 - (1+8x)t^3 - (2x + 4x^2)t^4 + 4x^2t^5 - 8x^3t^6}. \quad (3.23)$$

$$\sum_{n=0}^{+\infty} S_{n-1}(E)S_{n-1}\left(a_1 + [-a_2]\right)t^n = \frac{t - 2xt^3 - 2xt^4}{1 - t - (1+6x)t^2 - (1+8x)t^3 - (2x + 4x^2)t^4 + 4x^2t^5 - 8x^3t^6}. \quad (3.24)$$

$$\sum_{n=0}^{+\infty} S_{n-2}(E)S_{n-1}\left(a_1 + [-a_2]\right)t^n = \frac{t^2 + 2xt^3 + 4x^2t^5}{1 - t - (1+6x)t^2 - (1+8x)t^3 - (2x + 4x^2)t^4 + 4x^2t^5 - 8x^3t^6}. \quad (3.25)$$

Multiplying the equation (3.23) by 3 and substracting it from (3.24) multiplying by 2 and substracting it from (3.25) we have

$$\begin{aligned} &\sum_{n=0}^{+\infty} \left[3S_n(E) - 2S_{n-1}(E) - S_{n-2}(E)\right]S_{n-1}\left(a_1 + [-a_2]\right)t^n \\ &= \frac{t + 2t^2 + \left(3 + 8x\right)t^3 + 4xt^4 - 4x^2t^5}{1 - t - (1+6x)t^2 - (1+8x)t^3 - (2x + 4x^2)t^4 + 4x^2t^5 - 8x^3t^6} = \sum_{n=0}^{+\infty} K_n J_n(x)t^n, \end{aligned} \quad (3.26)$$

which represents the new generating function for the combined Tribonacci Lucas numbers

and Jacobsthal polynomials, with $K_n J_n(x) = \left[3S_n(E) - 2S_{n-1}(E) - S_{n-2}(E)\right] S_{n-1}\left(a_1 + [-a_2]\right)$.
We have the following theorem.

Theorem 3.4. *For* $n \in \mathbb{N}$, *the new generating function for the combined Tribonacci Lucas numbers and Jacobsthal Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} K_n j_n(x) t^n = \frac{6x + (1-6x)t + (2-6x-24x^2)t^2 + \left(3+2x-32x^2\right)t^3 + (4x-8x^2-8x^3)t^4 - (4x^2-8x^3)t^5}{1 - t - (1+6x)t^2 - (1+8x)t^3 - (2x+4x^2)t^4 + 4x^2t^5 - 8x^3t^6}.$$

Proof: By [11], we have $j_n(x) = 2xS_n\left(a_1 + [-a_2]\right) - (2x-1)S_{n-1}\left(a_1 + [-a_2]\right)$. Then, we can see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} K_n j_n(x) t^n &= \sum_{n=0}^{+\infty} \left[3S_n(E) - 2S_{n-1}(E) - S_{n-2}(E)\right]\left[2xS_n\left(a_1 + [-a_2]\right) - (2x-1)S_{n-1}\left(a_1 + [-a_2]\right)\right] t^n \\
&= 6x\sum_{n=0}^{+\infty} S_n(E) S_n\left(a_1 + [-a_2]\right) t^n - 4x\sum_{n=0}^{+\infty} S_{n-1}(E) S_n\left(a_1 + [-a_2]\right) t^n \\
&\quad -2x\sum_{n=0}^{+\infty} S_{n-2}(E) S_n\left(a_1 + [-a_2]\right) t^n \\
&\quad -(2x-1)\sum_{n=0}^{+\infty} \left[3S_n(E) - 2S_{n-1}(E) - S_{n-2}(E)\right] S_{n-1}\left(a_1 + [-a_2]\right) t^n \\
&= 6x\sum_{n=0}^{+\infty} S_n(E) S_n\left(a_1 + [-a_2]\right) t^n - 4x\sum_{n=0}^{+\infty} S_{n-1}(E) S_n\left(a_1 + [-a_2]\right) t^n \\
&\quad -2x\sum_{n=0}^{+\infty} S_{n-2}(E) S_n\left(a_1 + [-a_2]\right) t^n - (2x-1)\sum_{n=0}^{+\infty} K_n J_n(x) t^n,
\end{aligned}$$

by using the relations (3.10), (3.12), (3.14) and make some calculations, we get

$$\sum_{n=0}^{+\infty} K_n j_n(x) t^n = \frac{6x + l_1(x)t + l_2(x)t^2 + l_3(x)t^3 + l_4(x)t^4}{1 + m_1(x)t + m_2(x)t^2 + m_3(x)t^3 + m_4(x)t^4 + m_5(x)t^5 + m_6(x)t^6} - (2x-1)\sum_{n=0}^{+\infty} K_n J_n(x) t^n,$$

with

$$\begin{aligned}
l_1(x) &= -4x(a_1 - a_2), \\
l_2(x) &= -(2x(a_1 - a_2)^2 + 12xa_1a_2), \\
l_3(x) &= -8xa_1a_2(a_1 - a_2), \\
l_4(x) &= -2xa_1^2a_2^2,
\end{aligned}$$

and

$$m_1(x) = -(a_1 - a_2),$$
$$m_2(x) = -((a_1 - a_2)^2 + 3a_1a_2),$$
$$m_3(x) = -((a_1 - a_2)^3 + 4a_1a_2(a_1 - a_2)),$$
$$m_4(x) = -a_1a_2((a_1 - a_2)^2 + a_1a_2),$$
$$m_5(x) = a_1^2a_2^2(a_1 - a_2),$$
$$m_6(x) = -a_1^3a_2^3.$$

Therefore

$$\sum_{n=0}^{+\infty} K_n j_n(x)t^n = \frac{\begin{array}{c}6x + (1-6x)t + (2-6x-24x^2)t^2 + \left(3+2x-32x^2\right)t^3 \\ +(4x-8x^2-8x^3)t^4 - (4x^2-8x^3)t^5\end{array}}{1 - t - (1+6x)t^2 - (1+8x)t^3 - (2x+4x^2)t^4 + 4x^2t^5 - 8x^3t^6}.$$

This completes proof.

4. GENERATRING FUNCTIONS OF BINARY PRODUCTS OF TRIBONACCI LUCAS NUMBERS AND CHEBYSHEV POLYNOMIALS

In this section, we now derive the new generating functions of the products of Tribonacci Lucas numbers with Chebyshev polynomials of first, second, third and fourth kind.

- For the case $A = \{2a_1, -2a_2\}$ with replacing a_1 by $2a_1$ and a_2 by $(-2a_2)$ in (3.2), (3.3), (3.4), (3.6), (3.7) and (3.8) we have

$$\sum_{n=0}^{+\infty} S_n(E)S_{n-1}\left(2a_1 + [-2a_2]\right)t^n = \frac{-S_1(-E)t - 2(a_1-a_2)S_2(-E)t^2 - 4((a_1-a_2)^2 + a_1a_2)S_3(-E)t^3}{\prod_{e\in E}(1-2ea_1t)\prod_{e\in E}(1+2ea_2t)}. \tag{4.1}$$

$$\sum_{n=0}^{+\infty} S_n(E)S_n\left(2a_1 + [-2a_2]\right)t^n = \frac{1 + 4a_1a_2S_2(-E)t^2 + 8a_1a_2(a_1-a_2)S_3(-E)t^3}{\prod_{e\in E}(1-2ea_1t)\prod_{e\in E}(1+2ea_2t)}. \tag{4.2}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(E)S_{n-1}\left(2a_1 + [-2a_2]\right)t^n = \frac{t + 4a_1a_2S_2(-E)t^3 + 8a_1a_2(a_1-a_2)S_3(-E)t^4}{\prod_{e\in E}(1-2ea_1t)\prod_{e\in E}(1+2ea_2t)}. \tag{4.3}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(E)S_n\left(2a_1 + [-2a_2]\right)t^n = \frac{2(a_1-a_2)t - 4S_1(-E)a_1a_2t^2 - 16S_3(-E)a_1^2a_2^2t^4}{\prod_{e\in E}(1-2ea_1t)\prod_{e\in E}(1+2ea_2t)}. \tag{4.4}$$

$$\sum_{n=0}^{+\infty} S_{n-2}(E)S_{n-1}\left(2a_1 + [-2a_2]\right)t^n = \frac{2(a_1-a_2)t^2 - 4S_1(-E)a_1a_2t^3 - 16S_3(-E)a_1^2a_2^2t^5}{\prod_{e\in E}(1-2ea_1t)\prod_{e\in E}(1+2ea_2t)}. \tag{4.5}$$

$$\sum_{n=0}^{+\infty} S_{n-2}(E)S_n\left(2a_1+\left[-2a_2\right]\right)t^n = \frac{4\left((a_1-a_2)^2+a_1a_2\right)t^2-8a_1a_2(a_1-a_2)S_1\left(-E\right)t^3+16a_1^2a_2^2S_2\left(-E\right)t^4}{\prod_{e\in E}(1-2ea_1t)\prod_{e\in E}(1+2ea_2t)}, \quad (4.6)$$

with

$$\begin{aligned}\prod_{e\in E}(1-2ea_1t)\prod_{e\in E}(1+2ea_2t) = {} & 1+2(a_1-a_2)S_1\left(-E\right)t+4\left[(a_1-a_2)^2S_2\left(-E\right)-a_1a_2(S_1^2\left(-E\right)-2S_2\left(-E\right))\right]t^2 \\ & +8\left[(a_1-a_2)^3S_3\left(-E\right)-a_1a_2(a_1-a_2)(S_1\left(-E\right)S_2\left(-E\right)-3S_3\left(-E\right))\right]t^3 \\ & -16\left[a_1a_2(a_1-a_2)^2S_3\left(-E\right)S_1\left(-E\right)-a_1^2a_2^2(S_2^2\left(-E\right)-2S_3\left(-E\right)S_1\left(-E\right))\right]t^4 \\ & +32a_1^2a_2^2(a_1-a_2)S_3\left(-E\right)S_2\left(-E\right)t^5-64a_1^3a_2^3S_3^2\left(-E\right)t^6.\end{aligned}$$

Theorem 4.1. *For* $n\in\mathbb{N}$, *the new generating function for the combined Tribonacci Lucas numbers and Chebyshev polynomials of second kind is given by*

$$\sum_{n=0}^{+\infty} K_nU_n(x)t^n = \frac{3-4xt+\left(6-4x^2\right)t^2+8xt^3-t^4}{1-2xt-(4x^2-3)t^2-(8x^3-8x)t^3+(4x^2-1)t^4+2xt^5+t^6}.$$

Proof: Recall that, we have [13]

$$U_n\left(x\right)=S_n(2a_1+[-2a_2]).$$

We see that

$$\begin{aligned}\sum_{n=0}^{+\infty} K_nU_n\left(x\right)t^n &= \sum_{n=0}^{+\infty}\left[3S_n(E)-2S_{n-1}(E)-S_{n-2}(E)\right]S_n(2a_1+[-2a_2])t^n \\ &= 3\sum_{n=0}^{+\infty} S_n(E)S_n(2a_1+[-2a_2])t^n-2\sum_{n=0}^{+\infty} S_{n-1}(E)S_n(2a_1+[-2a_2])t^n \\ &\quad -\sum_{n=0}^{+\infty} S_{n-2}(E)S_n(2a_1+[-2a_2])t^n,\end{aligned}$$

according the relations (4.2), (4.4), (4.6) and reduce to same denominator, we obtain the following result

$$\sum_{n=0}^{+\infty} K_nU_n(x)t^n = \frac{3+u_1(x)t+u_2(x)t^2+u_3(x)t^3+u_4(x)t^4}{1+v_1(x)t+v_2(x)t^2+v_3(x)t^3+v_4(x)t^4+v_5(x)t^5+v_6(x)t^6},$$

where

$$\begin{aligned} u_1(x) &= -4(a_1-a_2), \\ u_1(x) &= -(24a_1a_2+4(a_1-a_2)^2), \\ u_1(x) &= -32a_1a_2(a_1-a_2), \\ u_1(x) &= -16a_1^2a_2^2, \end{aligned}$$

and

$$v_1(x) = -2(a_1 - a_2),$$
$$v_2(x) = -(12a_1a_2 + 4(a_1 - a_2)),$$
$$v_3(x) = -(8(a_1 - a_2)^3 + 32a_1a_2(a_1 - a_2)),$$
$$v_4(x) = -\left(16a_1a_2(a_1 - a_2)^2 + 16a_1^2a_2^2\right),$$
$$v_5(x) = 32a_1^2a_2^2(a_1 - a_2),$$
$$v_6(x) = -64a_1^3a_2^3.$$

Therefore

$$\sum_{n=0}^{+\infty} K_n U_n(x)t^n = \frac{3 - 4xt + \left(6 - 4x^2\right)t^2 + 8xt^3 - t^4}{1 - 2xt - (4x^2 - 3)t^2 - (8x^3 - 8x)t^3 + (4x^2 - 1)t^4 + 2xt^5 + t^6}.$$

This completes proof.

Theorem 4.2. *For* $n \in \mathbb{N}$, *the new generating function for the combined Tribonacci Lucas numbers and Chebyshev polynomials of first kind is given by*

$$\sum_{n=0}^{+\infty} K_n T_n(x)t^n = \frac{3 - 5xt + \left(6 - 8x^2\right)t^2 + \left(12x - 12x^2\right)t^3 - \left(1 - 4x^2\right)t^4 + xt^5}{1 - 2xt - (4x^2 - 3)t^2 - 8x(x^2 - 1)t^3 + (4x^2 - 1)t^4 + 2xt^5 + t^6}.$$

Proof: By referred to [13], we have $T_n(x) = S_n\left(2a_1 + [-2a_2]\right) - xS_{n-1}\left(2a_1 + [-2a_2]\right)$. Then, we see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} K_n T_n(x)t^n &= \sum_{n=0}^{+\infty}\left[3S_n(E) - 2S_{n-1}(E) - S_{n-2}(E)\right]\left[S_n\left(2a_1 + [-2a_2]\right) - xS_{n-1}\left(2a_1 + [-2a_2]\right)\right]t^n \\
&= \sum_{n=0}^{+\infty}\left[3S_n(E) - 2S_{n-1}(E) - S_{n-2}(E)\right]S_n\left(2a_1 + [-2a_2]\right)t^n \\
&\quad -3x\sum_{n=0}^{+\infty} S_n(E)S_{n-1}\left(2a_1 + [-2a_2]\right)t^n + 2x\sum_{n=0}^{+\infty} S_{n-1}(E)S_{n-1}\left(2a_1 + [-2a_2]\right)t^n \\
&\quad +x\sum_{n=0}^{+\infty} S_{n-2}(E)S_{n-1}\left(2a_1 + [-2a_2]\right)t^n \\
&= \sum_{n=0}^{+\infty} K_n U_n(x)t^n - 3x\sum_{n=0}^{+\infty} S_n(E)S_{n-1}\left(2a_1 + [-2a_2]\right)t^n \\
&\quad +2x\sum_{n=0}^{+\infty} S_{n-1}(E)S_{n-1}\left(2a_1 + [-2a_2]\right)t^n + x\sum_{n=0}^{+\infty} S_{n-2}(E)S_{n-1}\left(2a_1 + [-2a_2]\right)t^n,
\end{aligned}$$

using the relations (4.1), (4.3) and (4.5), we obtain

$$\sum_{n=0}^{+\infty} K_n T_n(x)t^n = \sum_{n=0}^{+\infty} K_n U_n(x)t^n - \frac{u_1(x)t + u_2(x)t^2 + u_3(x)t^3 + u_4(x)t^4 + u_5(x)t^5}{1 + v_1(x)t + v_2(x)t^2 + v_3(x)t^3 + v_4(x)t^4 + v_5(x)t^5 + v_6(x)t^6},$$

where

$$u_1(x) = x,$$
$$u_2(x) = 4(a_1 - a_2)x,$$
$$u_3(x) = (16a_1a_2 + 12(a_1 - a_2)^2)x,$$
$$u_4(x) = 16xa_1a_2(a_1 - a_2),$$
$$u_5(x) = -16a_1^2a_2^2x,$$

and

$$v_1(x) = -2(a_1 - a_2),$$
$$v_2(x) = -(12a_1a_2 + 4(a_1 - a_2)),$$
$$v_3(x) = -(8(a_1 - a_2)^3 + 32a_1a_2(a_1 - a_2)),$$
$$v_4(x) = -\left(16a_1a_2(a_1 - a_2)^2 + 16a_1^2a_2^2\right),$$
$$v_5(x) = 32a_1^2a_2^2(a_1 - a_2),$$
$$v_6(x) = -64a_1^3a_2^3.$$

Therefore

$$\sum_{n=0}^{+\infty} K_n T_n(x)t^n = \frac{3 - 5xt + \left(6 - 8x^2\right)t^2 + \left(12x - 12x^2\right)t^3 - \left(1 - 4x^2\right)t^4 + xt^5}{1 - 2xt - (4x^2 - 3)t^2 - 8x(x^2 - 1)t^3 + (4x^2 - 1)t^4 + 2xt^5 + t^6}.$$

This completes proof.

Theorem 4.3. *For* $n \in \mathbb{N}$*, the new generating function for the combined Tribonacci Lucas numbers and Chebyshev polynomials of third kind is given by*

$$\sum_{n=0}^{+\infty} K_n V_n\left(x\right)t^n = \frac{3 - (4x+1)t + (6 - 4x - 4x^2)t^2 + \left(4 + 8x - 12x^2\right)t^3 - \left(1 - 4x\right)t^4 + t^5}{1 - 2xt - (4x^2 - 3)t^2 - 8x(x^2 - 1)t^3 + (4x^2 - 1)t^4 + 2xt^5 + t^6}.$$

Proof: We have [15]

$$V_n\left(x\right) = S_n\left(2a_1 + [-2a_2]\right) - S_{n-1}\left(2a_1 + [-2a_2]\right).$$

We see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} K_n V_n\left(x\right)t^n &= \sum_{n=0}^{+\infty}\left[3S_n(E) - 2S_{n-1}(E) - S_{n-2}(E)\right]\left[S_n\left(2a_1 + [-2a_2]\right) - S_{n-1}\left(2a_1 + [-2a_2]\right)\right]t^n \\
&= \sum_{n=0}^{+\infty}\left[3S_n(E) - 2S_{n-1}(E) - S_{n-2}(E)\right]S_n\left(2a_1 + [-2a_2]\right)t^n \\
&\quad -3\sum_{n=0}^{+\infty} S_n(E)S_{n-1}\left(2a_1 + [-2a_2]\right)t^n + 2\sum_{n=0}^{+\infty} S_{n-1}(E)S_{n-1}\left(2a_1 + [-2a_2]\right)t^n \\
&\quad +\sum_{n=0}^{+\infty} S_{n-2}(E)S_{n-1}\left(2a_1 + [-2a_2]\right)t^n \\
&= \sum_{n=0}^{+\infty} K_n U_n(x)t^n - 3\sum_{n=0}^{+\infty} S_n(E)S_{n-1}\left(2a_1 + [-2a_2]\right)t^n + 2\sum_{n=0}^{+\infty} S_{n-1}(E)S_{n-1}\left(2a_1 + [-2a_2]\right)t^n \\
&\quad +\sum_{n=0}^{+\infty} S_{n-2}(E)S_{n-1}\left(2a_1 + [-2a_2]\right)t^n,
\end{aligned}$$

according to relatioshpis (4.1), (4.3) and (4.5) this give the following equality

$$\sum_{n=0}^{+\infty} K_n V_n(x) t^n = \sum_{n=0}^{+\infty} K_n U_n(x) t^n - \frac{t + b_2(x)t^2 + b_2(x)t^3 + b_3(x)t^4 + b_4(x)t^5}{1 + v_1(x)t + v_2(x)t^2 + v_3(x)t^3 + v_4(x)t^4 + v_5(x)t^5 + v_6(x)t^6},$$

where

$$\begin{aligned} b_1(x) &= 4(a_1 - a_2), \\ b_2(x) &= (16a_1a_2 + 12(a_1 - a_2)^2), \\ b_3(x) &= 16a_1a_2(a_1 - a_2), \\ b_4(x) &= -16a_1^2a_2^2, \end{aligned}$$

and

$$\begin{aligned} v_1(x) &= -2(a_1 - a_2), \\ v_2(x) &= -(12a_1a_2 + 4(a_1 - a_2)), \\ v_3(x) &= -(8(a_1 - a_2)^3 + 32a_1a_2(a_1 - a_2)), \\ v_4(x) &= -\left(16a_1a_2(a_1 - a_2)^2 + 16a_1^2a_2^2\right), \\ v_5(x) &= 32a_1^2a_2^2(a_1 - a_2), \\ v_6(x) &= -64a_1^3a_2^3. \end{aligned}$$

Therefore

$$\sum_{n=0}^{+\infty} K_n V_n(x) t^n = \frac{3 - (4x+1)t + (6 - 4x - 4x^2)t^2 + \left(4 + 8x - 12x^2\right)t^3 - \left(1 - 4x\right)t^4 + t^5}{1 - 2xt - (4x^2 - 3)t^2 - 8x(x^2 - 1)t^3 + (4x^2 - 1)t^4 + 2xt^5 + t^6}.$$

This completes the proof.

Theorem 4.4. *For $n \in \mathbb{N}$, the new generating function for the combined Tribonacci Lucas numbers and Chebyshev polynomials of fourth kind is given by*

$$\sum_{n=0}^{+\infty} K_n W_n(x) t^n = \frac{3 - (4x-1)t + (6 + 4x - 4x^2)t^2 - \left(4 - 8x - 12x^2\right)t^3 - \left(1 + 4x\right)t^4 - t^5}{1 - 2xt - (4x^2 - 3)t^2 - 8x(x^2 - 1)t^3 + (4x^2 - 1)t^4 + 2xt^5 + t^6}.$$

Proof: By [15], we have $W_n(x) = S_n\left(2a_1 + [-2a_2]\right) + S_{n-1}\left(2a_1 + [-2a_2]\right)$. Then, we can see that

$$\sum_{n=0}^{+\infty} K_n W_n(x) t^n = \sum_{n=0}^{+\infty} \left[3S_n(E) - 2S_{n-1}(E) - S_{n-2}(E)\right]\left[S_n\left(2a_1 + [-2a_2]\right) + S_{n-1}\left(2a_1 + [-2a_2]\right)\right] t^n$$
$$= \sum_{n=0}^{+\infty} \left[3S_n(E) - 2S_{n-1}(E) - S_{n-2}(E)\right] S_n\left(2a_1 + [-2a_2]\right) t^n$$
$$+3\sum_{n=0}^{+\infty} S_n(E) S_{n-1}\left(2a_1 + [-2a_2]\right) t^n - 2\sum_{n=0}^{+\infty} S_{n-1}(E) S_{n-1}\left(2a_1 + [-2a_2]\right) t^n$$
$$-\sum_{n=0}^{+\infty} S_{n-2}(E) S_{n-1}\left(2a_1 + [-2a_2]\right) t^n$$
$$= \sum_{n=0}^{+\infty} K_n U_n(x) t^n - 3\sum_{n=0}^{+\infty} S_n(E) S_{n-1}\left(2a_1 + [-2a_2]\right) t^n$$
$$-2\sum_{n=0}^{+\infty} S_{n-1}(E) S_{n-1}\left(2a_1 + [-2a_2]\right) t^n - \sum_{n=0}^{+\infty} S_{n-2}(E) S_{n-1}\left(2a_1 + [-2a_2]\right) t^n,$$

according to relationshpis (4.1), (4.3) and (4.5) this give the following equality

$$\sum_{n=0}^{+\infty} K_n V_n(x) t^n = \sum_{n=0}^{+\infty} K_n U_n(x) t^n + \frac{t + r_1(x)t^2 + r_2(x)t^3 + r_3(x)t^4 + r_4(x)t^5}{1 + v_1(x)t + v_2(x)t^2 + v_3(x)t^3 + v_4(x)t^4 + v_5(x)t^5 + v_6(x)t^6},$$

where

$$r_1(x) = 4(a_1 - a_2),$$
$$r_2(x) = (16a_1a_2 + 12(a_1 - a_2)^2),$$
$$r_3(x) = 16a_1a_2(a_1 - a_2),$$
$$r_4(x) = -16a_1^2a_2^2,$$

and

$$v_1(x) = -2(a_1 - a_2),$$
$$v_2(x) = -(12a_1a_2 + 4(a_1 - a_2)),$$
$$v_3(x) = -(8(a_1 - a_2)^3 + 32a_1a_2(a_1 - a_2)),$$
$$v_4(x) = -\left(16a_1a_2(a_1 - a_2)^2 + 16a_1^2a_2^2\right),$$
$$v_5(x) = 32a_1^2a_2^2(a_1 - a_2),$$
$$v_6(x) = -64a_1^3a_2^3,$$

Therefore

$$\sum_{n=0}^{+\infty} K_n W_n(x) t^n = \frac{3 - (4x-1)t + (6 + 4x - 4x^2)t^2 - \left(4 - 8x - 12x^2\right)t^3 - \left(1 + 4x\right)t^4 - t^5}{1 - 2xt - (4x^2 - 3)t^2 - 8x(x^2 - 1)t^3 + (4x^2 - 1)t^4 + 2xt^5 + t^6}.$$

This completes the proof.

5. CONCLUSION

In this paper, we have derived new theorems in order to determine new generating functions of product of Tribonnacci Lucas numbers and orthogonal polynomials. The derived theorems are based on symmetric functions and products of these numbers and polynomials.

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