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Symmetric Functions of Generalized Polynomials of Second Order

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Abstract In this paper, we will recover the generating functions of generalized polynomials of second order. The technic used her is based on the theory of the so called symmetric functions.

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1. Introduction

A second order polynomials sequence is of Fibonacci type (Lucas type) if its Binet formula has a structure similar to that for Fibonacci (Lucas) numbers. In the literature, these types of sequences are known as Generalized Fibonacci Polynomial (GFP). They are actually a natural generalization of the sequence, called the Fibonacci polynomial sequence. However, there is no unique generalization of this sequence, one can refer to articles by several authors like André Jeannin [7,10], Bergum et al. [11] and Florez et al. [14,15], to see this. In this paper we use the definition another of our suggested. The generalized polynomials of second order sequence $\{G_n(x)\}_{n\in\mathbb{N}}$ is defined by a recurrence sequence

$$\begin{cases} G_n(x)=(p_0+p_1x)G_{n-1}(x) \\ \quad +(q_0+q_1x)G_{n-2}(x), n\geq 2 \\ G_0(x)=\alpha_0,\ G_1(x)=\beta_0+\beta_1 x \end{cases} \quad (1.1)$$

where p_0, p_1, q_0, q_1, α_0, β_0 and β_1 are complex numbers.

The study of identities for Fibonacci polynomials and Lucas polynomials have received less attention than their counterparts for numerical sequences, even if many of these identities can be proved easily. A natural question to ask is: under what conditions is it possible to extend identities that already exist for Fibonacci and Lucas numbers to the GFP? We observe here that the identities involving Fibonacci and Lucas numbers extend naturally to the GFP that satisfy closed formulas similar to the Binet formulas satisfied by Fibonacci and Lucas numbers.

In fact, the well-known sequences below are special cases of the generalized polynomials sequence

- Putting $p_0=q_1=\beta_0=0$ and $p_1=q_0=\alpha_0=\beta_1=1$ reduces to Fibonacci polynomials.
- Substituting $p_0=q_1=\beta_0=0$ and $\alpha_0=2$, $p_1=q_0=\beta_1=1$ yields Lucas polynomials.
- Taking $p_0=q_1=\alpha_0=\beta_1=0$, $p_1=2$ and $q_0=\beta_0=1$ gives Pell polynomials.
- Taking $p_0=q_1=\beta_0=0, q_0=1$ and $\alpha_0=\beta_1=p_1=2$ gives Pell-Lucas polynomials.
- Taking $p_1=q_0=\alpha_0=\beta_1=0, \beta_0=p_0=1$ and $q_1=2$ gives Jacobsthal polynomials.
- In the case when $p_1=q_0=\beta_1=0$, $\beta_0=p_0=1$ and $\alpha_0=q_1=2$ it reduces to Jacobsthal-Lucas polynomials.
- In the case when $p_0=q_1=\beta_0=0$, $\alpha_0=\beta_1=1$, $p_1=2$ and $q_0=-1$ it reduces to Chebyshev polynomials of first kind.
- In the case when $p_0=q_1=\beta_0=0$, $\alpha_0=1$, $\beta_1=p_1=2$ and $q_0=-1$ it reduces to Chebyshev polynomials of second kind.
- Putting $p_0=q_1=0$, $\alpha_0=1$, $\beta_1=p_1=2$ and $\beta_0=q_0=-1$ we obtain Chebyshev polynomials of third kind.
- Substituting $p_0=q_1=0$, $\alpha_0=\beta_0=1$, $\beta_1=p_1=2$ and $q_0=-1$ yields Chebyshev polynomials of fourth kind.
- Taking $p_0=q_1=\beta_1=0$, $\beta_0=q_0=1$, $p_1=2$ and $\alpha_0=i$ we get Gaussian Pellpolynomials.
- Putting $p_1=q_0=\beta_1=0$, $\beta_0=p_0=1$, $q_1=2$ and $\alpha_0=\frac{i}{2}$ we obtain Gaussian Jacobsthal polynomials.

- Putting $p_1 = q_0 = 0,\ \beta_0 = p_0 = 1,\ q_1 = 2$ and $\alpha_0 = 2 - \frac{i}{2},\ \beta_1 = 2i$ we obtain Gaussian Jacobsthal-Lucas polynomials.

In order to determine generating functions of generalized polynomials sequence, we use analytical means and series manipulation methods. In the sequel, we derive new symmetric functions and some new properties. We also give some more useful definitions which are used in the subsequent sections. From these definitions, we prove our main results given in Section 3.

2. Definitions and Some Properties

In order to render the work self-contained we give the necessary preliminaries tools; we recall some definitions and results.

Proposition 1. *(Favard's Theorem [1]).* Let $\{P_n\}_{n\geq 0}$ be a monic polynomial sequence. Then $\{P_n\}_{n\geq 0}$ is orthogonal if and only if there exist two sequences of complex numbers $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$, such that $\gamma_n \neq 0,\ n \geq 1$ and satisfies the three-term recurrence relation

$$\begin{cases} P_{-1}(x) = 0,\ P_0(x) = 1, \\ P_{n+1}(x) = (x - \beta_n) P_n(x) \\ \quad -\gamma_n P_{n-1}(x),\ n \geq 0. \end{cases} \tag{2.1}$$

Remark 2. If $\beta_n = 0$ so $\{P_n\}_{n\geq 0}$ is called symmetric when $P_n(-x) = (-1)^n P_n(x)$.

Definition 3. Let k and n be two positive integers and $\{a_1, a_2, ..., a_n\}$ are set of given variables the k-th elementary symmetric function $e_k(a_1, a_2, ..., a_n)$ is defined by

$$e_k(a_1, a_2, ..., a_n) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n}\ (0 \leq k \leq n),$$

with $i_1,\ i_2,\ ...,\ i_n = 0$ or 1.

Definition 4. Let k and n be two positive integers and $\{a_1, a_2, ..., a_n\}$ are set of given variables the k-th complete homogeneous symmetric function $h_k(a_1, a_2, ..., a_n)$ is defined by

$$h_k(a_1, a_2, ..., a_n) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n}\ (0 \leq k \leq n),$$

with $i_1,\ i_2,\ ...,\ i_n \geq 0$.

Remark 5. Set $e_0(a_1, a_2, ..., a_n) = 1$ and $h_0(a_1, a_2, ..., a_n) = 1$, by usual convention. For $k < 0$, we set $e_k(a_1, a_2, ..., a_n) = 0$ and $h_k(a_1, a_2, ..., a_n) = 0$.

Definition 6. [2] Let A and P be any two alphabets. We define $S_n(A - P)$ by the following form

$$\Pi_{p\in P}(1 - pt) / \Pi_{a\in A}(1 - at) = \sum_{n=0}^{\infty} S_n(A - P) t^n, \tag{2.2}$$

with the condition $S_n(A - P) = 0$ for $n < 0$.

Equation (2.2) can be rewritten in the following form

$$\sum_{n=0}^{\infty} S_n(A-P)t^n = \left(\sum_{n=0}^{\infty} S_n(A)t^n\right) \times \left(\sum_{n=0}^{\infty} S_n(-P)t^n\right), \tag{2.3}$$

where

$$S_n(A - P) = \sum_{j=0}^{n} S_{n-j}(-P) S_j(A).$$

Definition 7. [3] Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows

$$\partial_{p_i p_{i+1}}(f) = \frac{\begin{Bmatrix} f(p_1, \cdots, p_i, p_{i+1}, \cdots p_n) \\ -f(p_1, \cdots p_{i-1}, p_{i+1}, p_i, p_{i+2} \cdots p_n) \end{Bmatrix}}{p_i - p_{i+1}}.$$

Definition 8. [4] the symmetrizing operator $\delta_{e_1 e_2}^k$ is defined by

$$\delta_{p_1 p_2}^k(g) = \frac{p_1^k g(p_1) - p_2^k g(p_2)}{p_1 - p_2}$$

for all $k \in \mathbb{N}$.

3. Generating Function of Generalized Polynomials of Second Order

The following theorem is one of the key tools of the proof of our main result which is already given its proof in [5].

Theorem 9. *Given an alphabet* $E = \{e_1, e_2\}$ *two sequences* $\sum_{n=0}^{\infty} a_n t^n$, $\sum_{n=0}^{\infty} b_n t^n$ *such that* $\left(\sum_{n=0}^{\infty} a_n t^n\right) \left(\sum_{n=0}^{\infty} b_n t^n\right) = 1$, *then*

$$\frac{\sum_{n=0}^{\infty} b_n \delta_{e_1 e_2}^k (e_1^n) t^n}{\left(\sum_{n=0}^{\infty} b_n e_1^n t^n\right)\left(\sum_{n=0}^{\infty} b_n e_2^n t^n\right)}$$

$$= \sum_{n=0}^{\infty} a_n e_1^n e_2^n \delta_{e_1 e_2}^{k-1}(e_1) t^n - e_1^k e_2^k t^{k+1} \sum_{n=0}^{\infty} a_{n+k+1} \delta_{e_1 e_2}^n (e_1) t^n.$$

If $k = 1$ for the case $\frac{1}{1-t} = \sum_n t^n$. The following lemmas allow us to obtain many generating functions of generalized polynomials and some well-known polynomials cited above, using a technique symmetric functions.

Lemma 10. Given an alphabet $E = \{e_1, e_2\}$, we have

$$\sum_{n=0}^{+\infty} S_n(e_1 + e_2) t^n = 1 \Big/ \prod_{e\in E} (1 - et). \tag{3.1}$$

Lemma 11. Given an alphabet $E=\{e_1,e_2\}$, we have

$$\sum_{n=0}^{+\infty} S_{n-1}(e_1+e_2)t^n=\frac{t}{\prod_{e\in E}(1-et)}. \tag{3.2}$$

- Replacing e_2 by $[-e_2]$ in the formulas (3.1) and (3.2), we have

$$\sum_{n=0}^{\infty} S_n(e_1+[-e_2])t^n=\frac{1}{1-(e_1-e_2)t-e_1e_2t^2}, \tag{3.3}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(e_1+[-e_2])t^n=\frac{t}{1-(e_1-e_2)t-e_1e_2t^2}. \tag{3.4}$$

Multiplying the equation (3.3) by α_0 and adding it from (3.4) multiplying by $((\beta_0-\alpha_0p_0)+(\beta_1-\alpha_0p_1)x)$ and setting $\begin{cases} e_1-e_2=p_0+p_1x, \\ e_1e_2=q_0+q_1x, \end{cases}$ we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty}\begin{bmatrix}\alpha_0S_n(a_1+[-a_2])+((\beta_0-\alpha_0p_0)\\ +(\beta_1-\alpha_0p_1)x)S_{n-1}(a_1+[-a_2])\end{bmatrix}t^n \\ &=\frac{\alpha_0+[(\beta_0-\alpha_0p_0)+(\beta_1-\alpha_0p_1)x]t}{1-(p_0+p_1x)t-(q_0+q_1x)t^2}, \end{aligned} \tag{3.5}$$

and we have the following Proposition.

Proposition 12. For $n\in\mathbb{N}$, the new generating function of generalized polynomials is given by

$$\sum_{n=0}^{+\infty} G_n(x)t^n=\frac{\alpha_0+[(\beta_0-\alpha_0p_0)+(\beta_1-\alpha_0p_1)]t}{1-(p_0+p_1x)t-(q_0+q_1x)t^2},$$

with

$$G_n(x)=\begin{pmatrix}\alpha_0S_n(a_1+[-a_2])+((\beta_0-\alpha_0p_0)\\ +(\beta_1-\alpha_0p_1)x)S_{n-1}(a_1+[-a_2])\end{pmatrix}. \tag{3.6}$$

Proof. Theordinary generating function associated is defined by

$$G(G_n,t)=\sum_{n=0}^{+\infty} G_n(x)t^n.$$

Using the initial conditions, we get

$$\begin{aligned}\sum_{n=0}^{+\infty} G_n(x)t^n &= G_0(x)+G_1(x)t+\sum_{n=2}^{+\infty} G_n(x)t^n \\ &=\alpha_0+(\beta_0+\beta_1x)t+\sum_{n=2}^{+\infty}((p_0+p_1x)G_{n-1}(x) \\ &\quad +(q_0+q_1x)G_{n-2}(x))t^n \\ &=\alpha_0+(\beta_0+\beta_1x)t+(p_0+p_1x)t\sum_{n=1}^{+\infty} G_n(x)t^n \\ &\quad +(q_0+q_1x)t^2\sum_{n=0}^{+\infty} G_n(x)t^n \\ &=\alpha_0+(\beta_0+\beta_1x)t-\alpha_0(p_0+p_1x)t \\ &\quad +(p_0+p_1x)t\sum_{n=0}^{+\infty} G_n(x)t^n+(q_0+q_1x)t^2\sum_{n=0}^{+\infty} G_n(x)t^n \\ &=\alpha_0+[(\beta_0-\alpha_0p_0)+(\beta_1-\alpha_0p_1)x]t \\ &\quad +(p_0+p_1x)t\sum_{n=0}^{+\infty} G_n(x)t^n+(q_0+q_1x)t^2\sum_{n\geq 0} G_n(x)t^n,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\sum_{n=0}^{+\infty} G_n(x)t^n(1-(p_0+p_1x)t-(q_0+q_1x)t^2) \\ &=\alpha_0+[(\beta_0-\alpha_0p_0)+(\beta_1-\alpha_0p_1)x]t,\end{aligned}$$

therefore

$$\sum_{n=0}^{+\infty} G_n(x)t^n=\frac{\alpha_0+[(\beta_0-\alpha_0p_0)+(\beta_1-\alpha_0p_1)x]t}{1-(p_0+p_1x)t-(q_0+q_1x)t^2}.$$

Accordingly, we conclude the following Corollaries.

Corollary 13. For $n\in\mathbb{N}$, the generating function of Fibonacci polynomials is given by

$$\begin{aligned}&\sum_{n=0}^{\infty} F_n(x)t^n=\frac{1}{1-xt-t^2}, \\ &\text{with } F_n(x)=S_n(e_1+[-e_2]).\end{aligned} \tag{3.7}$$

Replacing t by $(-t)$ in the formula (3.7), we have

$$\begin{aligned}&\sum_{n=0}^{\infty} F_n(-x)t^n=\frac{1}{1+xt-t^2}, \\ &\text{with } F_n(-x)=(-1)^nS_n(e_1+[-e_2]).\end{aligned}$$

Corollary 14. For $n\in\mathbb{N}$, the generating function of Lucas polynomials is given by

$$\begin{aligned}&\sum_{n=0}^{\infty} L_n(x)t^n=\frac{2-xt}{1-xt-t^2}, \\ &\text{with } L_n(x)=2S_n(e_1+[-e_2]) \\ &-xS_{n-1}(e_1+[-e_2]).\end{aligned} \tag{3.8}$$

Replacing t by $(-t)$ in the formula (3.8), we have

$$\begin{aligned}&\sum_{n=0}^{\infty} L_n(-x)t^n=\frac{2+xt}{1+xt-t^2}, \\ &\text{with } L_n(-x)=(-1)^n\begin{pmatrix}2S_n(e_1+[-e_2])\\ -xS_{n-1}(e_1+[-e_2])\end{pmatrix}.\end{aligned}$$

Corollary 15. For $n\in\mathbb{N}$, the generating function of Pell polynomials is given by

$$\begin{aligned}&\sum_{n=0}^{\infty} P_n(x)t^n=\frac{t}{1-2xt-t^2}, \\ &\text{with } P_n(x)=S_{n-1}(e_1+[-e_2]).\end{aligned}$$

Corollary 16. For $n \in \mathbb{N}$, the generating function of Pell-Lucas polynomials is given by

$$\sum_{n=0}^{\infty} Q_n(x)t^n = \frac{2-2xt}{1-2xt-t^2},$$
$$\text{with } Q_n(x) = 2S_n(e_1+[-e_2]) - 2xS_{n-1}(e_1+[-e_2]).$$

Corollary 17. For $n \in \mathbb{N}$, the generating function of Jacobsthal polynomials is given by

$$\sum_{n=0}^{\infty} J_n(x)t^n = \frac{t}{1-t-2xt^2}, \quad \text{with } J_n(x) = S_{n-1}(e_1+[-e_2]). \tag{3.9}$$

Replacing t by $(-t)$ in the formula (3.9), we have

$$\sum_{n=0}^{\infty} J_n(-x)t^n = \frac{-t}{1+t-2xt^2},$$
$$\text{with } J_n(-x) = (-1)^n S_{n-1}(e_1+[-e_2]).$$

Corollary 18. For $n \in \mathbb{N}$, the generating function of Jacobsthal- Lucas polynomials is given by

$$\sum_{n=0}^{\infty} j_n(x)t^n = \frac{2+(1-2x)t}{1-t-2xt^2}, \quad \text{with } j_n(x) = \begin{pmatrix} 2S_n(e_1+[-e_2]) \\ +(1-2x)S_{n-1}(e_1+[-e_2]) \end{pmatrix}. \tag{3.10}$$

Replacing t by $(-t)$ in the formula (3.10), we have

$$\sum_{n=0}^{\infty} j_n(-x)t^n = \frac{2-(1-2x)t}{1+t-2xt^2},$$
$$\text{with } j_n(-x) = (-1)^n \begin{pmatrix} 2S_n(e_1+[-e_2]) \\ +(1-2x)S_{n-1}(e_1+[-e_2]) \end{pmatrix}.$$

Corollary 19. For $n \in \mathbb{N}$, the generating function of Gaussian- Pell polynomials is given by

$$\sum_{n=0}^{\infty} GP_n(x)t^n = \frac{i+(1-2ix)t}{1-2xt-t^2}, \quad \text{with } GP_n(x) = \begin{pmatrix} iS_n(e_1+[-e_2]) \\ +(1-2ix)S_{n-1}(e_1+[-e_2]) \end{pmatrix}. \tag{3.11}$$

Replacing t by $(-t)$ in the formula (3.11), we have

$$\sum_{n=0}^{\infty} GP_n(-x)t^n = \frac{i-(1-2ix)t}{1+2xt-t^2},$$
$$\text{with } GP_n(-x) = (-1)^n \begin{pmatrix} iS_n(e_1+[-e_2]) \\ +(1-2ix)S_{n-1}(e_1+[-e_2]) \end{pmatrix}.$$

Corollary 20. For $n \in \mathbb{N}$, the generating function of Gaussian -Jacobsthal polynomials is given by

$$\sum_{n=0}^{\infty} GJ_n(x)t^n = \frac{\frac{i}{2}+\left(1-\frac{i}{2}\right)t}{1-t-2xt^2},$$
$$\text{with } GJ_n(x) = \begin{pmatrix} \frac{i}{2}S_n(e_1+[-e_2]) \\ +\left(1-\frac{i}{2}\right)S_{n-1}(e_1+[-e_2]) \end{pmatrix}. \tag{3.12}$$

Replacing t by $(-t)$ in the formula (3.12), we have

$$\sum_{n=0}^{\infty} GJ_n(-x)t^n = \frac{\frac{i}{2}-\left(1-\frac{i}{2}\right)t}{1+t-2xt^2},$$
$$\text{with } GJ_n(-x) = (-1)^n \begin{pmatrix} \frac{i}{2}S_n(e_1+[-e_2]) \\ +\left(1-\frac{i}{2}\right)S_{n-1}(e_1+[-e_2]) \end{pmatrix}.$$

Corollary 21. For $n \in \mathbb{N}$, the generating function of Gaussian –Jacobsthal- Lucas polynomials is given by

$$\sum_{n=0}^{\infty} Gj_n(x)t^n = \frac{\left(2-\frac{i}{2}\right)-\left(1-\frac{i}{2}-2ix\right)t}{1-t-2xt^2},$$
$$\text{with } Gj_n(x) = \begin{pmatrix} \left(2-\frac{i}{2}\right)S_n(e_1+[-e_2]) \\ -\left(1-\frac{i}{2}-2ix\right)S_{n-1}(e_1+[-e_2]) \end{pmatrix}. \tag{3.13}$$

Replacing t by $(-t)$ in the formula (3.13), we have

$$\sum_{n=0}^{\infty} Gj_n(-x)t^n = \frac{\left(2-\frac{i}{2}\right)+\left(1-\frac{i}{2}-2ix\right)t}{1+t-2xt^2},$$
$$\text{with } Gj_n(-x) = (-1)^n \begin{pmatrix} \left(2-\frac{i}{2}\right)S_n(e_1+[-e_2]) \\ -\left(1-\frac{i}{2}-2ix\right)S_{n-1}(e_1+[-e_2]) \end{pmatrix}.$$

- Replacing e_1 by $[2e_2]$ and e_2 by $[2e_2]$ in the formulas (3.1) and (3.2), we have

$$\sum_{n=0}^{\infty} S_n(2e_1+[-2e_2])t^n = \frac{1}{1-2(e_1-e_2)t-4e_1e_2t^2} \tag{3.14}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(2e_1+[-2e_2])t^n = \frac{t}{1-2(e_1-e_2)t-4e_1e_2t^2} \tag{3.15}$$

Multiplying the equation (3.14) by α_0 and adding it from (3.15) multiplying by $((\beta_0-\alpha_0 p_0)+(\beta_1-\alpha_0 p_1)x)$ and setting $\begin{cases} e_1-e_2 = p_0+p_1x, \\ 4e_1e_2 = -q_0+q_1x, \end{cases}$ we obtain

$$\sum_{n=0}^{\infty} \begin{bmatrix} \alpha_0 S_n(2a_1+[-2a_2]) + ((\beta_0-\alpha_0 p_0) \\ +(\beta_1-\alpha_0 p_1)x)S_{n-1}(2a_1+[-2a_2]) \end{bmatrix} t^n = \frac{\alpha_0+[(\beta_0-\alpha_0 p_0)+(\beta_1-\alpha_0 p_1)x]t}{1-2(p_0+p_1x)t+(q_0+q_1x)t^2}, \tag{3.16}$$

and we have the following Proposition.

Proposition 22. For $n \in \mathbb{N}$, the new generating function of generalized polynomials is given by

$$\sum_{n=0}^{+\infty} G_n(x)t^n = \frac{\alpha_0 + \left[(\beta_0 - \alpha_0 p_0) + (\beta_1 - \alpha_0 p_1)x\right]t}{1 - 2(p_0 + p_1 x)t + (q_0 + q_1 x)t^2},$$

$$with\ G_n(x) = \begin{pmatrix} \alpha_0 S_n(2a_1 + [-2a_2]) + ((\beta_0 - \alpha_0 p_0) \\ +(\beta_1 - \alpha_0 p_1)x)S_{n-1}(2a_1 + [-2a_2] \end{pmatrix}. \quad (3.17)$$

Corollary 23. For $n \in \mathbb{N}$, the generating function of Chebyshev polynomials of first kind is given by

$$\sum_{n=0}^{\infty} T_n(x)t^n = \frac{1 - xt}{1 - 2xt + t^2},$$

$$with\ T_n(x) = S_n(2e_1 + [-2e_2]) - xS_{n-1}(2e_1 + [-2e_2]).$$

Corollary 24. For $n \in \mathbb{N}$, the generating function of Chebyshev polynomials of second kind is given by

$$\sum_{n=0}^{\infty} U_n(x)t^n = \frac{1}{1 - 2xt + t^2},$$

$$with\ U_n(x) = S_n(2e_1 + [-2e_2]).$$

Corollary 25. For $n \in \mathbb{N}$, the generating function of Chebyshev polynomials of third kind is given by

$$\sum_{n=0}^{\infty} V_n(x)t^n = \frac{1 - t}{1 - 2xt + t^2},$$

$$with\ V_n(x) = S_n(2a_1 + [-2a_2]) - S_{n-1}(2a_1 + [-2a_2]).$$

Corollary 26. For $n \in \mathbb{N}$, the generating function of Chebyshev polynomials of fourth kind is given by

$$\sum_{n=0}^{\infty} W_n(x)t^n = \frac{1 + t}{1 - 2xt + t^2},$$

$$with\ W_n(x) = S_n(2e_1 + [-2e_2]) + S_{n-1}(2e_1 + [-2e_2]).$$

4. Conclusion

In this paper, by making use of equations (3.1) and (3.2), we have derived some new generating functions for generalized polynomials of second order. It would be interesting to apply the methods shown in the paper to families of other special polynomials.

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