

Review

Symmetric Mass Generation

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Abstract: The most well-known mechanism for fermions to acquire a mass is the Nambu–Goldstone–Anderson–Higgs mechanism, i.e., after a spontaneous symmetry breaking, a bosonic field that couples to the fermion mass term condenses, which grants a mass gap for the fermionic excitation. In the last few years, it was gradually understood that there is a new mechanism of mass generation for fermions without involving any symmetry breaking within an anomaly-free symmetry group, also applicable to chiral fermions with anomaly-free chiral symmetries. This new mechanism is generally referred to as the symmetric mass generation (SMG). It is realized that the SMG has deep connections with interacting topological insulator/superconductors, symmetry-protected topological states, perturbative local and non-perturbative global anomaly cancellations, and deconfined quantum criticality. It has strong implications for the lattice regularization of chiral gauge theories. This article defines the SMG, summarizes the current numerical results, introduces an unifying theoretical framework (including the parton-Higgs and the s-confinement mechanisms, as well as the symmetry-extension construction), and presents an overview of various features and applications of SMG.

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1. Introduction

Global symmetry is a central concept in quantum field theories (QFTs). One of the most immediate implications of symmetry is that it restricts the terms that can appear in a field theory action in the path integral formulation, as the partition function Z must remain invariant under the symmetry transformation—the partition function invariance, however, can be up to an invertible complex phase factor $Z \rightarrow e^{i\alpha} Z$ known as a quantum anomaly [1,2]. The quantum anomaly associated with the global symmetry is also known as the 't Hooft anomaly [3], which has profound consequences. The invertible phase $e^{i\alpha}$ that detects the anomaly also specifies a cobordism class of the partition function of one-higher-dimensional invertible topological quantum field theory (invertible TQFT) [4] via the anomaly inflow [5,6].

The anomaly provides a concise and powerful organization principle to classify quantum field theories, and to dictate the influence of the ultraviolet (UV) kinematics on the infrared (IR) dynamics of a field theory because the anomaly does not change under symmetric deformations of the theory, including the renormalization group (RG) flow. The preservation of the anomaly index from UV to IR is known as anomaly matching. If a quantum field theory with a global symmetry G has a non-vanishing 't Hooft anomaly in G , its IR dynamics cannot be trivially gapped while preserving G . As a consequence, either the symmetry G is spontaneously broken in IR [7–11], or the anomaly must be matched by anomalous *gapless* quantum criticality (e.g., the IR limit of the system corresponds to massless free theories or interacting conformal field theories [CFT]) or *gapped* topological order (e.g., the IR limit of the system is described by a topological quantum field theory [TQFT]) [12–25].

Many examples of anomalies involve massless fermions, such as the chiral fermion Adler–Bell–Jackiw anomaly [26,27]. The anomaly causes an obstruction to open a gap in the fermion spectrum (i.e., an obstruction to make the fermion correlation length finite) without breaking the symmetry. The symmetry-breaking mechanism to generate a mass gap is known as the Nambu–Goldstone–Anderson–Higgs mechanism [28–34]. Here, Nambu–Goldstone refers to the spontaneous symmetry breaking, while Anderson–Higgs refers to giving a mass by the elementary or composite boson condensation. However, there are instances where the anomaly vanishes for a collection of massless fermions, yet the symmetry is still restrictive enough to forbid any *fermion bilinear* mass term. In this case, although there is no obstruction toward gapping the fermions symmetrically, the mechanism to achieve the symmetric gapped state must go beyond the free-fermion (perturbatively free or weak-coupling) approach, which potentially leads to a non-perturbative strong-coupling approach in order to generate a finite excitation gap in the fermion many-body spectrum by non-trivial interaction effects. The “strong coupling” here refers to the coupling in the continuum field theory being non-perturbative, or the interaction energy being of the same order as the kinetic energy on the lattice scale (which may also be called the *intermediate-strength* interaction on a lattice) [35]. This phenomenon of gapping out massless fermions by interactions in an anomaly-free system without breaking the anomaly-free symmetry is now called *symmetric mass generation* (SMG) [36,37].

The idea of gapping out massless fermions by interaction dates back to Eichten and Preskill [38] in an attempt to regularize chiral fermions on the lattice. The understanding of SMG has significantly deepened over the past few years, following the development in condensed matter theory regarding symmetry-protected topological (SPT) states [39–43]. SPT states are short-range-entangled quantum many-body states, respecting certain global symmetry G . The bulk of a SPT state is featureless (i.e., gapped, symmetric and non-degenerated). A non-trivial SPT state is most explicitly characterized by its non-trivial boundary features, which are endowed by the non-vanishing ’t Hooft anomaly of G (or mixed G symmetry gauge-gravity anomaly in broader cases) in the boundary effective theory of the SPT state. The one-to-one correspondence between the bulk of a SPT state and its boundary anomaly [44–50] provides the basis to classify distinct SPT states by their distinct boundary anomalies.

The connection between the SPT state and anomaly relates SMG to another topic: the interaction-reduced classification of fermionic SPT states. Fermionic SPT states [4,51–66] are SPT states of fermionic systems with (at least) fermion parity symmetry \mathbb{Z}_2^F (or more generally, invertible topological phases of fermions [4,63]). In the free-fermion limit [67–71], non-trivial fermionic SPT states are characterized by the symmetry-protected gapless fermion boundary modes if the symmetry forbids any fermion bilinear mass on the boundary. However, the free-fermion analysis does not rule out the possibility of gapping out the boundary fermions by interaction through the SMG mechanism. Such a situation can indeed happen when the boundary fermions are actually anomaly free, such that the bulk state should be classified as a trivial gapped state under symmetric interactions, even though it looks like a non-trivial SPT state in the non-interacting limit. This leads to the interaction-reduced classification of fermionic SPT states in the bulk, which is closely related to the SMG for gapless fermions on the boundary.

The first example of interaction-reduced classification was provided by Fidkowski and Kitaev [72,73] in $(1+1)$ D fermion systems, where an explicit interaction was proposed to drive the SMG among fermion zero modes on the $(0+1)$ D boundary. This motivated a sequence of works generalizing the discussion to $(2+1)$ D [74–77], $(3+1)$ D [15–17,78–80], and higher dimensions [81–83]. Studies along this direction reveal families of interactions that could potentially drive the SMG in different dimensions, paving ways for numerical verifications in concrete lattice models. It is also realized that the interaction must be carefully designed to drive the SMG: some symmetric interactions are helpful toward this goal, while other symmetric interactions are not [35].

Currently the most known examples of interaction-reduced classifications all correspond to cancellations of non-perturbative global anomalies. However, there are also cases of SMG that involve cancellations of perturbative local anomalies. Important examples of such are the chiral fermions [84–87] in even dimensional spacetime. Regularizing chiral fermions on a lattice is a long-standing problem in the lattice gauge theory. The Nielsen–Ninomiya no-go theorem [88–90] states that it is not possible to gap the fermion doubler in a non-interacting local lattice model without breaking the chiral symmetry. One possibility to circumvent the no-go theorem is to consider the interaction effects [38], which introduces fermion interaction to gap out the fermion doubler (the mirror fermion) in local lattice fermion models, leaving the normal fermion (the light fermion) untouched. However, the early attempts [87,91–100] were not successful, either because certain anomalies were not carefully canceled or because the appropriate gapping interaction was not found. With a deeper understanding of the SMG mechanism, the problem was revisited by Wen [101] for Spin(10) chiral fermions in $(3 + 1)D$. The idea was further developed by subsequent works in the same dimension [102–106] as well as in lower-dimensional analogs [35,107–109]. More recent numerical works have successfully shown that the SMG indeed provides a feasible solution to regularize chiral fermions [110–113].

The article is organized as follows. We will start by introducing some selective representative models of SMG (one in each spacetime dimension, in $(0 + 1)D$, $(1 + 1)D$, $(2 + 1)D$ and $(3 + 1)D$ respectively) in Section 2. We conclude with Section 2.5 by providing a general definition of SMG in all dimensions. We then review the numerical efforts in Section 3, which is mainly focused on two tasks: (i) to establish the existence of SMG phases in Section 3.1 and (ii) to investigate the nature of SMG transitions in Section 3.2. Based on these backgrounds, we then summarize the recent theoretical progress in Section 4, which aims to (i) understand the SMG phase by the fluctuating bilinear mass picture in Section 4.1 and (ii) describe the SMG transition by fermion fractionalization field theory in Section 4.2. In particular, we unify two currently existing SMG mechanisms (namely the parton-Higgs mechanism and the s-confinement mechanism) under the same theoretical framework of fermion fractionalization. We also make a connection to the further understanding of SMG based on the symmetry extension construction [24,25] in Section 4.4. In Section 5, we discuss other aspects of SMG, including Green’s function zeros in Section 5.1, the connection to deconfined quantum criticality in Section 5.2 and the deformation class of QFTs in Section 5.4, and its application to the standard model regularization in Section 5.3. Finally, we summarize the review in Section 6.

2. Example Models

2.1. $(0 + 1)D$ SMG: Fidkowski-Kitaev Majorana Fermion Model

The simplest example of SMG happens in $(0 + 1)D$ spacetime among a collection of Majorana fermion zero modes, as first shown by Fidkowski and Kitaev [72,73]. The model concerns a system of eight Majorana fermion modes, described by the Majorana fermion operators χ_a ($a = 1, 2, \dots, 8$) satisfying $\{\chi_a, \chi_b\} = 2\delta_{ab}$. Consider an anti-unitary (time-reversal) symmetry $\mathbb{Z}_2^T : \chi_a \rightarrow \chi_a, i \rightarrow -i$ and the fermion parity symmetry $\mathbb{Z}_2^F : \chi_a \rightarrow -\chi_a$. Without involving interactions, the Hamiltonian must take a fermion bilinear form to preserve the fermion parity symmetry \mathbb{Z}_2^F . But any Majorana fermion bilinear term $i\chi_a\chi_b$ (one needs i to keep the operator Hermitian) will break the time-reversal symmetry \mathbb{Z}_2^T , so the free-fermion Hamiltonian has to vanish, e.g., $H = 0$, under both symmetry requirements, and these Majorana modes cannot be gapped on the free-fermion level. Eight Majorana fermion modes form four qubits, and hence the system has a $2^4 = 16$ fold degeneracy, and this degeneracy is protected by the $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ symmetry.

However, it is possible to create a many-body excitation gap by fermion interaction, leaving a unique ground state of this $(0 + 1)D$ system. As introduced by Fidkowski and

Kitaev [72], the following four-fermion interaction suffices to gap out all eight Majorana fermion modes without breaking the time-reversal symmetry \mathbb{Z}_2^T

$$H_{\text{FK}} = - \sum_{a < b < c < d} V_{abcd} \chi_a \chi_b \chi_c \chi_d. \quad (1)$$

The coefficient is specified by $V_{abcd} = \langle 4e | \chi_a \chi_b \chi_c \chi_d | 4e \rangle$, where $|4e\rangle = (|0000\rangle + |1111\rangle) / \sqrt{2}$ is a many-body reference state written in the Fock state basis $|n_1 n_2 n_3 n_4\rangle$, labeled by the fermion occupation numbers $n_i = (1 + i\chi_{2i-1} \chi_{2i}) / 2$ (for $i = 1, 2, 3, 4$). Here the fermion number operator $n_i = c_i^\dagger c_i$ can also be expressed in terms of the complex fermion annihilation operator $c_i = (\chi_{2i-1} + i\chi_{2i}) / 2$, which are constructed by pairing up the Majorana operators. The reference state $|4e\rangle$ describes the quartet condensation of the complex fermions c_i (where four fermions are created or annihilated together in the quantum superposition), which is also known as the charge-4e superconducting state [114–120] in condensed matter physics. It turns out that the state $|4e\rangle$ respects the \mathbb{Z}_2^T symmetry, and is the unique ground state of H_{FK} with a finite excitation gap of 14 (energy units).

The fact that the system has *eight* Majorana zero modes is crucial for the SMG to occur. In contrast, if there are only four Majorana zero modes, the only Hamiltonian allowed by symmetry is

$$H = -g \chi_1 \chi_2 \chi_3 \chi_4 = -g P_1^F P_2^F, \quad (2)$$

where each pair of Majorana fermion operators defines a fermion parity operator $P_i^F = i\chi_{2i-1} \chi_{2i} = 2n_i - 1$ (associated with the i th complex fermion mode). This Hamiltonian always has a two-fold ground state degeneracy, regardless of the sign of g . When $g < 0$, the ground states $|P_1^F = P_2^F = \pm 1\rangle$ are bosonic (as the total fermion parity $P_1^F P_2^F = +1$ is even). When $g > 0$, the ground states $|P_1^F = -P_2^F = \pm 1\rangle$ are fermionic (as $P_1^F P_2^F = -1$). In either cases, the two-fold degenerated ground states transform into each other under \mathbb{Z}_2^T , and form a *Kramers doublet* [121]. The ground state degeneracy implies that the \mathbb{Z}_2^T symmetry is spontaneously broken. Hence, even with interactions, *four* Majorana zero modes still *cannot* be symmetrically gapped. However, by doubling to eight Majorana fermions, it is then possible to couple the two Kramers doublets (originated from the first- and the last-four Majorana fermions under interaction) together via a Heisenberg-type spin–spin interaction as [81,122]

$$H = -P_1^F P_2^F - P_3^F P_4^F + \mathbf{S}_I \cdot \mathbf{S}_{II}, \quad (3)$$

where the first two terms stabilize two Kramers doublets, and $\mathbf{S}_{I/II}$ stands for the effective spin-operator for each Kramers doublet. This will end up with a unique spin-singlet bosonic ground state with a finite gap to all excitations, which successfully gaps out all fermions without breaking the symmetry. Equation (3) can be rewritten as H_{FK} in Equation (1) under some appropriate basis choice.

From the perspective of quantum anomaly, the $(0+1)\text{D}$ Majorana fermions with a $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ internal symmetry (or the Pin^- spacetime-internal symmetry [54,122], or the BDI symmetry class [123,124]) has a \mathbb{Z}_8 class of non-perturbative global anomaly, whose anomaly index corresponds to the number of Majorana modes. With eight Majorana modes, the anomaly vanishes, meaning that the system can be trivially gapped without breaking symmetry. However, the \mathbb{Z}_2^T symmetry is restrictive enough to rule out any fermion bilinear mass, making interaction a necessary ingredient in the fermion mass generation, which corresponds to the SMG mechanism.

The Fidkowski–Kitaev (FK) interaction H_{FK} in Equation (1) has a (unnecessarily high) flavor symmetry of $\text{Spin}(7)$, which rotates seven fermion bilinear operators $\Phi^\alpha := \chi_a \Gamma_{ab}^\alpha \chi_b$ (for $\alpha = 1, \dots, 7$) as a vector representation of $\text{SO}(7)$, with $\Gamma^\alpha = (\sigma^{123}, \sigma^{203}, \sigma^{323}, \sigma^{211}, \sigma^{021},$

$\sigma^{231}, \sigma^{002}$), where $\sigma^{\mu\nu\cdots} = \sigma^\mu \otimes \sigma^\nu \otimes \cdots$ denotes the direct product of Pauli matrices. It can be shown that Equation (1) can be equivalently written as

$$H_{\text{FK}} = -\frac{1}{4!} \sum_{\alpha=1}^7 (\Phi^\alpha \Phi^\alpha - 16), \quad (4)$$

which exhibits the Spin(7) symmetry explicitly. However, it is possible to lower the symmetry to Spin(6) \cong SU(4) for example, without affecting the SMG physics [103],

$$\begin{aligned} H_{\text{SU}(4)} &= -\frac{1}{192} \sum_{\alpha=1}^6 \Phi^\alpha \Phi^\alpha \\ &= -(c_1 c_2 c_3 c_4 + \text{h.c.}) - \frac{1}{3} \sum_{i < j} (n_i - \frac{1}{2})(n_j - \frac{1}{2}), \end{aligned} \quad (5)$$

where c_i (for $i = 1, 2, 3, 4$) are complex fermions in SU(4) fundamental representation and $n_i = c_i^\dagger c_i$ are their number operators. $H_{\text{SU}(4)}$ stabilizes the same symmetric and non-degenerated ground state $|4e\rangle$ as H_{FK} with a finite gap to all excitations. This example illustrates that the interaction that drives SMG is not unique since the SMG is a generic phenomenon in anomaly-free fermion systems.

The decomposition of the four-fermion interaction as a product of two fermion-bilinear operators $\Phi^\alpha \Phi^\alpha$ in both Equations (4) and (5) suggests a common physical picture to understand the SMG as fluctuating bilinear masses. Under the Hubbard–Stratonovich transformation, the SMG interactions in the above examples take the general form of Yukawa–Higgs interaction

$$H_{\text{YH}} = -\phi_\alpha \chi_a \Gamma_{ab}^\alpha \chi_b + \frac{1}{2g} \phi_\alpha \phi_\alpha, \quad (6)$$

where ϕ_α is a bosonic Yukawa field that couples to the fermion bilinear mass, and by integrating out ϕ_α , the desired four fermion interaction is generated. If the Yukawa field condenses, i.e., $\langle \phi_\alpha \rangle \neq 0$, the fermions will be gapped (with a gap size proportional to the amplitude of $\langle \phi_\alpha \rangle$) and the symmetry is also broken spontaneously. However, if it is possible to fluctuate the orientation of the Yukawa field smoothly in the spacetime (which is only the time here for the (0 + 1)D system) without bringing its local amplitude to zero, the fermion could potentially retain the excitation gap while restoring the required symmetry. This intuitive picture leads to fruitful understandings of SMG in higher dimensions [16,108,125], which will be further discussed in Section 4.1.

The SMG in (0 + 1)D is intimately related to the interaction-reduced classification of fermionic SPT states in (1 + 1)D. In the non-interacting limit, the (0 + 1)D Majorana zero modes can be viewed as the topological edge modes of a (1 + 1)D fermionic SPT state protected by the same $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ internal symmetry. A specific lattice model for such SPT root state is Kitaev’s Majorana fermionic chain [126], which supports a dangling Majorana zero mode on each open end. So eight Majorana zero modes can be viewed as the boundary state of eight copies of Majorana chains. Without interaction, the $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ symmetric *free* fermionic SPT states are \mathbb{Z} classified [67–69], where the index corresponds to the number of Majorana edge modes [127]. The fact that eight Majorana zero modes can be trivially gapped out by interaction without breaking the symmetry implies that eight copies of the Majorana chain actually belong to the trivial SPT phase, as their interface with the vacuum state can be made featureless (i.e., gapped and non-degenerated) by the SMG. This indicates that the $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ (or Pin[−]) symmetric *interacting* fermionic SPT states are \mathbb{Z}_8 classified [51,72,73], which is consistent with the formal result $\Omega_2^{\text{Pin}^-}(\text{pt}) = \mathbb{Z}_8$ by the cobordism classification [54,128]. The phenomenon that the fermion SPT classification is reduced from \mathbb{Z} in the non-interacting limit to \mathbb{Z}_8 under interaction is called the *interaction-reduced classification*.

The $\mathbb{Z} \rightarrow \mathbb{Z}_8$ interaction-reduced classification implies that eight copies of the Majorana chain can be smoothly tuned (without closing the bulk gap) to the trivial phase under interaction. The bulk model is defined on a one-dimensional lattice

$$\begin{aligned} H &= H_0 + H_{\text{int}}, \\ H_0 &= \frac{i}{2} \sum_i \sum_{a=1}^8 (u \chi_{2i-1,a} \chi_{2i,a} + v \chi_{2i,a} \chi_{2i-1,a}), \\ H_{\text{int}} &= -g \sum_i \sum_{a < b < c < d} V_{abcd} \chi_{i,a} \chi_{i,b} \chi_{i,c} \chi_{i,d}, \end{aligned} \quad (7)$$

where each site i hosts eight Majorana modes $\chi_{i,a}$ ($a = 1, \dots, 8$). In the free-fermion limit ($g = 0$), the $u < v$ and $u > v$ phases are separated by a quantum phase transition at which the single-particle band gap closes in the bulk. Ref. [72] shows that the quantum critical point can be circumvented by applying the FK interaction on every site with strength g (where V_{abcd} in H_{int} follows the same definition as that in Equation (1)). This can be argued by inspecting the limiting cases ($u = 0, v = 1$) or ($u = 1, v = 0$), where Majorana fermions are fully dimerized along the lattice over even or odd bonds; see Figure 1. The two dimerized states can both be smoothly tuned to the same strong coupling state $\otimes_i |4e\rangle_i$ as $g \rightarrow \infty$ without closing the many-body gap, which can be explicitly verified by diagonalizing the local Hamiltonian across a bond. This establishes a smooth deformation path between the $u < v$ and $u > v$ phases by going through the strong coupling regime as shown in Figure 1, demonstrating that eight copies of the Majorana chain are indeed in the trivial phase.

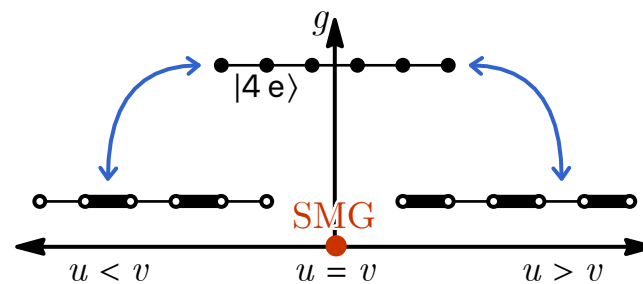


Figure 1. Phase diagram of the interacting Majorana chain model described by Equation (7). Every dot represents a site (not a Majorana mode) on the one-dimensional lattice, which hosts eight Majorana modes (per site) internally. Thick bonds represent inter-site dimerizations of Majorana fermions. Black dots represent on-site charge-4e condensations.

Moreover, the argument also implies that the gapless Majorana fermions (in the $(1+1)$ D bulk) at the free-fermion critical point ($u = v$) can be gapped out by turning on the interaction g (see Figure 1), leading to another example of SMG. As a side note, to pin the bulk at the $u = v$ critical point, an emergent chiral \mathbb{Z}_2 symmetry should be imposed, which leads to a different total symmetry group. This bulk SMG is first analyzed in Ref. [72] using bosonization and $\text{SO}(8)$ triality. Refs. [36,129] provide further understandings from the fermion fractionalization perspective. The bulk SMG of the $(1+1)$ D Majorana chain (at the critical point) is dynamically equivalent to the SMG for the boundary mode of a $(2+1)$ D fermionic topological superconductor (TSC) made of eight copies of the $p_x \pm ip_y$ superconductor [130–132]. In general, each interaction-reduced classification of the fermion SPT state implies the existence of SMG mechanisms for the gapless fermions both on the *boundary* and at the *bulk* critical point between the SPT and the trivial state, where the same local interaction that gaps out the boundary mode can be used to gap out the bulk critical point as well. Using the connection of SMG phenomena between the boundary and the bulk, Ref. [81] obtains a series of interaction-reduced classification of fermionic SPT states

in all spacetime dimensions systematically, thus extending the phenomenon of SMG to higher dimensions.

2.2. (1 + 1)D SMG: U(1) Symmetric Chiral Fermion Model

2.2.1. 3-4-5-0 Chiral Fermion Model

The 3-4-5-0 U(1) chiral fermion model by Wang-Wen [35,109] provides an instructive example of SMG in (1 + 1)D, which involves the cancellation of a perturbative anomaly and is not related to interaction-reduced fermionic SPT classifications. The (1 + 1)D chiral fermion theory is described by the Lagrangian

$$\mathcal{L} = \sum_{a=1}^4 \psi_a^\dagger i(\partial_t - v_a \partial_x) \psi_a, \tag{8}$$

where ψ_a are complex fermion fields (for $a = 1, 2, 3, 4$) with different velocities $v = (1, 1, -1, -1)$. The model can be assigned a U(1) symmetry, under which the fermions transform as $U(1) : \psi_a \rightarrow e^{i\theta q_a} \psi_a$ with $q = (3, 4, 5, 0)$. The two left-moving Weyl fermions ψ_1 and ψ_2 are assigned with charges 3 and 4, and the two right-moving Weyl fermions ψ_3 and ψ_4 are assigned with charges 5 and 0, hence the name of the 3-4-5-0 model.

The seemingly peculiar charge assignment of the chiral fermions is designed to cancel the U(1) 't Hooft anomaly. The (1 + 1)D fermion with U(1) internal symmetry (or $\text{Spin}^c \equiv (\text{Spin} \times U(1))/\mathbb{Z}_2^F$ spacetime-internal symmetry [54,122], or the A symmetry class [123,124]) has a \mathbb{Z} perturbative anomaly, whose anomaly index is given by $\sum_a v_a q_a^2$, which vanishes for the charge assignment of the 3-4-5-0 model. The (1 + 1)D chiral fermions can be also viewed as the chiral edge modes of a (2 + 1)D integer quantum Hall state, also known as the fermionic SPT state in symmetry class A. The integer quantum Hall states are \mathbb{Z} classified by the quantized Hall conductance, and the classification does not further reduce under interaction. The bulk Hall conductance (in unit of e^2/h) matches the boundary anomaly index.

The fact that the U(1) 't Hooft anomaly vanishes for the 3-4-5-0 model indicates that it should be possible to trivially gap out all fermions without breaking the U(1) symmetry. However, the U(1) symmetry is restrictive enough to prevent the gapping to happen on the free-fermion level, because any fermion bilinear term that produces a gap must take the form of $\psi_a^\dagger \psi_b$ or $\psi_a \psi_b$ (with $a \in \{1, 2\}$ and $b \in \{3, 4\}$) that mixes the left- and right-moving fermions. Since the four flavors of fermions all carry distinct U(1) charges that do not add or subtract to zero, any flavor-mixing fermion bilinear term will necessarily break the U(1) symmetry, which makes it impossible to symmetrically gap out these chiral fermions on the free-fermion level.

Nevertheless, the gapping can be achieved with fermion interaction, hence an example of SMG. The interaction to achieve the SMG would involve at least six-fermion terms, which can be derived using the null-vector condition [133–139] for quantum Hall edge states. By bosonization $\psi_a \sim e^{i\varphi_a}$, the (1 + 1)D chiral fermion system can be effectively described by a Luttinger liquid theory

$$\mathcal{L} = \frac{1}{4\pi} (K_{ab} \partial_t \varphi_a \partial_x \varphi_b - V_{ab} \partial_x \varphi_a \partial_x \varphi_b), \tag{9}$$

where $K = \text{diag}(1, 1, -1, -1)$ and $V = \text{diag}(1, 1, 1, 1)$ are diagonal matrices. In the bosonization language, backscattering fermion interactions can be introduced as

$$\mathcal{L}_{\text{int}} = \sum_{\alpha} g_{\alpha} \cos(l_{\alpha,a} \varphi_a), \tag{10}$$

each interaction labeled by a charge vector l_{α} . It is possible to find two charge vectors $l_1 = (1, -2, 1, 2)$ and $l_2 = (2, 1, -2, 1)$ that satisfy the null-vector condition $l_{\alpha}^T K^{-1} l_{\beta} = 0$ (for $\alpha, \beta = 1, 2$). The null-vector condition ensures that the vertex operators $O_{\alpha} = e^{i l_{\alpha} \cdot \varphi}$

are both self-boson and mutual-boson, which can be simultaneously condensed at large coupling g_α . Once these vertex operators condense $\langle O_\alpha \rangle \neq 0$, all fermion excitations will be gapped, since none of the fermion operator braid trivially with any of the vertex operator (as seen from $l_\alpha^\top K^{-1} \neq 0$). Furthermore, the interaction term \mathcal{L}_{int} , as well as the condensate $\langle O_\alpha \rangle$, preserves the U(1) symmetry since $l_\alpha^\top q = 0$ for $\alpha = 1, 2$. Therefore, \mathcal{L}_{int} provides a symmetric way to gap out all chiral fermions in the 3-4-5-0 model, realizing the SMG. In terms of the fermion field, the interaction can be translated from Equation (10) to

$$\begin{aligned} \mathcal{L}_{\text{int}} = & g_1(\psi_1\psi_2^\dagger\partial_x\psi_2^\dagger\psi_3\psi_4\partial_x\psi_4 + \text{h.c.}) \\ & + g_2(\psi_1\partial_x\psi_1\psi_2\psi_3^\dagger\partial_x\psi_3^\dagger\psi_4 + \text{h.c.}), \end{aligned} \quad (11)$$

where the ∂_x operator (that generates infinitesimal translation) is inserted as a point-splitting regularization to avoid identical fermion operators appearing at the same spatial position. In fact, the two g_1 and g_2 interaction terms with $l_1 = (1, -2, 1, 2)$ and $l_2 = (2, 1, -2, 1)$ leave two U(1) internal symmetries unbroken for the four Weyl fermions: one is their charge assignment $q = (3, 4, 5, 0)$, and the other is $q = (0, 5, 4, 3)$ [35,109].

The multi-fermion interaction in Equation (11) can be mediated by two independent Yukawa fields ϕ_1, ϕ_2 via

$$\begin{aligned} \mathcal{L}_{\text{YH}} = & -(\phi_1^2\psi_1\psi_3 + \phi_1^\dagger\psi_2^\dagger\psi_4 + \text{h.c.}) + \frac{1}{\tilde{g}_1}\phi_1^\dagger\phi_1 \\ & -(\phi_2^2\psi_2\psi_4 + \phi_2^\dagger\psi_1\psi_3^\dagger + \text{h.c.}) + \frac{1}{\tilde{g}_2}\phi_2^\dagger\phi_2. \end{aligned} \quad (12)$$

Integrating out the Yukawa fields will generate the interaction in Equation (11) to the leading order of $g_\alpha \sim \tilde{g}_\alpha^2$ (plus additional density–density interactions, such as $\psi_1^\dagger\psi_1\psi_3^\dagger\psi_3$ or $\psi_2^\dagger\psi_2\psi_4^\dagger\psi_4$, whose effect is only to renormalize the V matrix in the Luttinger liquid theory). The Yukawa fields ϕ_1 and ϕ_2 carry the U(1) charges -4 and -2 , respectively. Directly condensing the Yukawa fields would provide Dirac/Majorana masses to all chiral fermions ψ_a at the price of breaking the U(1) symmetry. Nevertheless, the SMG mechanism suggests an alternative scenario that these Yukawa fields are fluctuating in the disordered (i.e., strong-coupling symmetric) phase, such that the U(1) symmetry remains unbroken but the chiral fermions could still acquire a spectral gap via the Yukawa interaction in suitable parameter regimes. A similar idea was numerically explored in Ref. [100] without success, due to the incorrect design of the fermion interaction (which does not satisfy the null-vector condition). The correct design of fermion interaction in Equation (11) or Equation (12) includes only a restricted subset of symmetry-allowed interactions [35], which are helpful for the SMG. A more recent numerical study in Ref. [113] confirmed that the correct interaction indeed leads to the SMG phase.

The SMG transition happens when the interaction is beyond a finite critical strength, because the interaction is perturbatively irrelevant at the free-fermion fixed point, due to its high-order nature. However, a strong enough interaction could lead to non-perturbative effects. Increasing the interaction strength generally tunes the Luttinger parameter (by renormalizing the V matrix) and alters operator scaling dimensions in the Luttinger liquid theory [113]. When the scaling dimension of the interaction term itself is tuned to be marginal, the SMG transition is triggered, which drives the system from the gapless phase to the featureless gapped phase [37,113]. In this case, the SMG transition belongs to the Berezinskii–Kosterlitz–Thouless (BKT) universality class.

2.2.2. Proof on the Equivalence between the Anomaly-Free and SMG Gapping Conditions

Although the earlier discussions focus on the SMG of the (1 + 1)D 3-4-5-0 U(1) chiral fermion model, there is no obstacle to generalize to show that any (1 + 1)D anomaly-free chiral fermion model with multiple U(1) symmetries can allow SMG, following [35]. Ref. [35] proves that the *anomaly-free condition* of the (1 + 1)D multiple U(1) chiral fermion theory is equivalent to the *SMG gapping condition* of the same theory. The proof can be

achieved due to the exact bosonization–fermionization techniques in (1 + 1)D. Here, we recall the proof to complete the discussion of (1 + 1)D SMG.

A generic (1 + 1)D anomaly-free multiple-U(1) chiral fermion theory has equal numbers of left and right moving Weyl fermions, $N_L = N_R = N \in \mathbb{Z}^+$, such that the total Weyl fermion number is $2N = N_L + N_R \in 2\mathbb{Z}^+$, an even positive integer.

1. The SMG gapping condition requires to add N independent compatible gapping terms [133–136] to preserve internal chiral U(1) symmetries. To prove the SMG gapping holds, we bosonize the fermionic theory

$$\mathcal{L} = \sum_{a=1}^{N_L} \psi_a^\dagger i(\partial_t - \partial_x) \psi_a + \sum_{a=1}^{N_R} \psi_a^\dagger i(\partial_t + \partial_x) \psi_a \tag{13}$$

to a multiplet chiral boson theory

$$\mathcal{L} = \frac{1}{4\pi} (K_{ab} \partial_t \varphi_a \partial_x \varphi_b - V_{ab} \partial_x \varphi_a \partial_x \varphi_b), \tag{14}$$

where $K = K_f \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \dots$ and the appropriate rescaled $V = \mathbb{I}_{2N \times 2N}$ are diagonal rank- $2N$ matrices. The K_f is the unimodular symmetric bilinear canonical form for the fermionic system (with $|\det(K)| = 1$). The advantage of Equation (14) is that choosing $K = K_b \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \dots$ works also for the unimodular symmetric bilinear canonical form for the bosonic system (with $|\det(K)| = 1$). The SMG gapping condition requires the following:

- To find a set of N linear-independent of integer-valued $2N$ -component l vectors such that

$$l_\alpha^\top K^{-1} l_\beta = \sum_{a,b} l_{\alpha,a} (K^{-1})_{ab} l_{\beta,b} = 0, \tag{15}$$

for $\alpha, \beta \in \{1, 2, \dots, N\}$ and each l_α vector contains $2N$ components ($l_{\alpha,a}$ with $a = 1, \dots, 2N$). Gapping Equation (14) requires to add the sine-Gordon deformation $\mathcal{L}_{\text{int}} = \sum_{\alpha=1}^N g_\alpha \cos(l_{\alpha,a} \varphi_a)$, which can be fermionized to multi-fermion interactions.

- The massless Weyl fermion theory has, at most, an internal $U(N_f) \times U(N_R)$ symmetry, which contains at most a chiral $U(1)^{2N}$ symmetry. However, for SMG, one can preserve, at most, N -linear independent chiral $U(1)^N$ symmetries, labeled by a set of charge vectors, q_α with $\alpha \in \{1, 2, \dots, N\}$, such that the fermions transform as $\psi_a \rightarrow \psi_a e^{i q_{\alpha,a} \theta}$, and bosonized fields $\varphi_a \rightarrow \varphi_a + q_{\alpha,a} \theta$ with $a = 1, \dots, 2N$ and $\theta \in [0, 2\pi)$. The symmetric sine-Gordon interactions demand

$$l_\alpha^\top q_\beta = \sum_a l_{\alpha,a} q_{\beta,a} = 0 \tag{16}$$

for any $\alpha, \beta \in \{1, 2, \dots, N\}$.

2. Its anomaly-free condition, on the other hand, requires:
 - Gravitational anomaly free (two-point one-loop Feynman diagram of grav² vertices vanish): The left and right chiral central charges $c_L = c_R$, which means $N_L = N_R$.
 - Gauge anomaly free (two-point one-loop Feynman diagram of $U(1)^2$ vertices vanish): For each U(1) symmetry, with left-handed and right-handed Weyl fermion charge vector q_L and q_R , respectively, the anomaly free requires the square sum of each component $\sum q_L^2 - q_R^2 = 0$. In terms of the symmetric bilinear

form K for both bosonic (K_b) and fermionic (K_f) systems, the anomaly-free condition demands that

$$q_\alpha^\top K q_\beta = \sum_{a,b} q_{\alpha,a} K_{ab} q_{\beta,b} = 0, \quad (17)$$

again for $\alpha, \beta \in \{1, 2, \dots, N\}$ and $a, b \in \{1, 2, \dots, 2N\}$.

- The above two anomaly-free conditions are perturbative local anomalies. The $(1+1)$ D nonperturbative global anomalies are classified by cobordism groups (denoted TP_3 or Ω_3 with the special orthogonal SO or Spin group and some internal U(1) symmetries, such as $SO \times U(1)$, $Spin \times U(1)$ and $Spin^c$) which turn out to always vanish [140].
3. To explain that the SMG gapping condition holds implies that the anomaly-free condition also holds, Ref. [35] shows that given the set of l_α satisfying (15), one can find the set of q_α simultaneously satisfying (16) and (17). This is true because given that l_α , we can choose $q_\alpha = K^{-1}l_\alpha$.
 4. To explain that the anomaly-free condition holds implies that the SMG gapping condition also holds; Ref. [35] shows that given the set of q_α satisfying (17), one can find the set of l_α simultaneously satisfying (16) and (15). This is true because given that q_α , we can choose $l_\alpha = Kq_\alpha$.
 5. The above two remarks prove that the *if and only if* (sufficient and necessary) conditions to the equivalence of the anomaly-free condition and the SMG gapping condition. Once the set of q_α and l_α are found, they form $2N$ linear-independent integer-valued vectors spanning completely the $2N$ -dimensional vector space (known as the Narain lattice [141]).
 6. What remains to be explained is why the SMG gapping condition defines a gapped boundary without any topological boundary degeneracy [135]. The idea is viewing the $(1+1)$ D theory (14) as the boundary theory of $(2+1)$ D invertible TQFT with a Chern–Simons action $S_{\text{bulk}} = \frac{K_{ab}}{4\pi} \int_{\mathcal{M}^3} A_a \wedge dA_b$ on a 3-manifold \mathcal{M}^3 . The A_a is a multiplet 1-form gauge field. Hereafter, all repeated indices are summed over. A stable boundary condition requires the variation of S_{bulk} on the boundary 2-manifold $(\partial\mathcal{M})^2$ vanished [134] under the boundary 1-form gauge field $A_\partial \rightarrow A_\partial + \delta A_\partial$ variation: $\delta_{\text{bdry}}(S_{\text{bulk}}) = \frac{K_{ab}}{4\pi} \int_{(\partial\mathcal{M})^2} A_{\partial,a} \wedge \delta A_{\partial,b}$. The differential δ of this variation is a symplectic form $\omega_{\text{Sp}} = \frac{K_{ab}}{4\pi} \int_{(\partial\mathcal{M})^2} \delta A_{\partial,a} \wedge \delta A_{\partial,b}$ on the space of boundary gauge fields. Consistent stable boundary conditions on $(\partial\mathcal{M})^2$ define a Lagrangian submanifold with respect to the symplectic form ω_{Sp} in symplectic geometry.
 - One consistent boundary condition sets one component of A_∂ vanished, such as $(K_{ab}A_{b,t} - V_{ab}A_{b,x})|_{\partial\mathcal{M}} = 0$, which gives a gapless $(1+1)$ D CFT (14).
 - Another boundary condition sets the gauge degrees of freedom $l_{\alpha,a}A_a|_{\partial\mathcal{M}} = 0$ vanish [59]. The boson modes φ_a , originally related by the gauge transformation $A_a \rightarrow A_a + d\lambda_a$ and $\varphi_a \rightarrow \varphi_a - \lambda_a$, now may condense on the boundary with nonzero vacuum expectation values $\langle \exp(i(l_{\alpha,a}\varphi_a)) \rangle|_{\partial\mathcal{M}} \neq 0$, more precisely, indeed $\langle \exp(i(\frac{l_{\alpha,a}}{|\text{gcd}(l_\alpha)|}\varphi_a)) \rangle|_{\partial\mathcal{M}} \neq 0$, where $\text{gcd}(l_\alpha) \equiv \text{gcd}(l_{\alpha,1}, l_{\alpha,2}, \dots, l_{\alpha,2N})$ is the greatest common divisor (gcd) of all components of l_α . This condensation of φ can be triggered by the earlier sine-Gordon cosine term at a strong g coupling. The boundary vertex operator and bulk line operator are connected $\exp(i(l_{\alpha,a}\varphi_a)|_{\partial\mathcal{M}} + i \int l_{\alpha,a}A_a|_{\mathcal{M}})$. The gapped bulk and gapped boundary demand that the partition function Z evaluated on the 3-manifold M with the 2-boundary $\partial\mathcal{M}$ has a finite value (in fact $Z = 1$ when the Z corresponds to counting the dimension of the Hilbert space for the invertible TQFT). This means that arbitrary link configuration of the bulk line operators should give a trivial braiding statistical phase to Z so there are no unwanted quantum fluctuations destabilizing the gapped vacuum—namely, the mutual statistics $\exp(i2\pi l_\alpha^\top K^{-1}l_\beta) = 1$

and the self statistics $\exp(i\pi l_\alpha^T K^{-1} l_\alpha) = 1$ are trivial for all $\alpha, \beta \in \{1, 2, \dots, N\}$. Hence, we derive the correspondence between the N -independent compatible SMG gapping terms and the N null-braiding statistics l_α vectors [133–136]. This completes the proof [35].

2.3. (2 + 1)D SMG: Honeycomb Lattice Model

The honeycomb lattice model is a simple lattice model for SMG in (2 + 1)D. The model is defined on a honeycomb lattice, with each lattice site i hosting four complex fermion modes, denoted as c_{ia} (for $a = 1, 2, 3, 4$). The model is described by the Hamiltonian

$$H = - \sum_{a=1}^4 \sum_{\langle ij \rangle} c_{ia}^\dagger c_{ja} - g \sum_i (c_{i1} c_{i2} c_{i3} c_{i4} + \text{h.c.}), \quad (18)$$

where $\langle ij \rangle$ stands for the bond between nearest neighboring sites i and j on the honeycomb lattice; see Figure 2a. The interaction strength g is the only tuning parameter of this model.

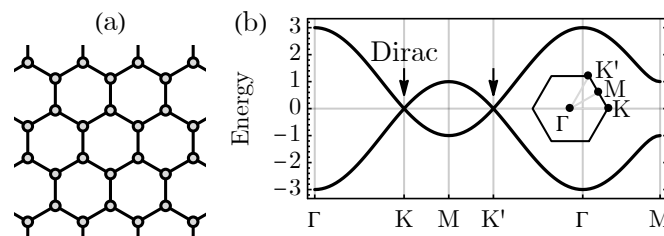


Figure 2. (a) Honeycomb lattice. (b) Graphene band structure (inset defines high symmetry points in the Brillouin zone).

When $g = 0$, the free-fermion hopping on the honeycomb lattice gives rise to the graphene band structure [142], as shown in Figure 2b, that produces $4 \times 2 = 8$ gapless Dirac fermions ψ_{Qa} at low energy, where 4 stands for the four on-site fermion flavors ($a = 1, 2, 3, 4$) and 2 comes of the fermion doubling ($Q = K, K'$) in the Brillouin zone. They can be described by a low-energy effective field theory Lagrangian

$$\mathcal{L} = \sum_{Q=K,K'} \sum_{a=1}^4 \bar{\psi}_{Qa} \gamma^\mu \partial_\mu \psi_{Qa}, \quad (19)$$

where $Q = K, K'$ labels the valley (fermion doubling) freedom, $\gamma^\mu = (\sigma^2, \sigma^1, \sigma^3)$ and ψ_{Qa} are complex Grassmann spinors in (2 + 1)D (with $\bar{\psi}_{Qa} = \psi_{Qa}^\dagger \gamma^0$). These 8 gapless Dirac fermions can be equivalently viewed as 16 gapless Majorana fermions, by decomposing each complex Grassmann field into two real Grassmann fields.

The key physical symmetries that protect these gapless fermions in the non-interacting limit are the lattice translation symmetry and the *anti-unitary* sublattice symmetry $\mathbb{Z}_2^S : c_i \rightarrow (-)^i c_i^\dagger, i \rightarrow -i$ (where $(-)^i$ stands for a sign factor that takes \pm on A/B sublattice of the honeycomb lattice) [67,69,71]. These two symmetries can be combined to create an emergent anti-unitary symmetry at low-energy $\mathbb{Z}_4^{TF} : \psi_{Qa} \rightarrow i\gamma^0 \psi_{Qa}^\dagger, i \rightarrow -i$, whose generator $\mathcal{T} = T_R^{3/4} \mathcal{S}$ consists of a sublattice symmetry \mathbb{Z}_2^S generator \mathcal{S} followed by a 3/4 fraction of the unit-cell translation T_R . Although the 2D lattice translation symmetry \mathbb{Z}^2 (generated by T_R along two linearly independent Bravais lattice vectors \mathbf{R}) cannot be fractionalized on the lattice, yet for the low-energy effective theory Equation (19), the translation symmetry acting on the low-energy fermions ψ_Q becomes an emergent valley $U(1)$, symmetry $T_R : \psi_{K/K'} \rightarrow e^{\pm i2\pi/3} \psi_{K/K'}$ (\pm signs are associated with K and K' valleys, respectively), which can be fractionalized to $T_R^{3/4} : \psi_{K/K'} \rightarrow \pm i \psi_{K/K'}$. Because $\mathcal{T}^2 = -1$ on fermions, \mathcal{T}^2 should correspond to the fermion parity operator, which needs a further square to become an identity, and therefore, \mathcal{T} generates a four-fold cyclic group \mathbb{Z}_4^{TF} .

The anomalous nature of the combined symmetry \mathbb{Z}_4^{TF} is manifested by the fact that \mathbb{Z}_4^{TF} is only an emergent symmetry at low energy and becomes ill-defined on the lattice level.

The (2 + 1)D Majorana fermion with a \mathbb{Z}_4^{TF} internal symmetry (or the Pin^+ spacetime-internal symmetry [54,122], or the DIII symmetry class [123,124]) has a $\Omega_4^{\text{Pin}^+}(\text{pt}) = \mathbb{Z}_{16}$ non-perturbative global anomaly based on the cobordism, whose anomaly index corresponds to the number of gapless Majorana fermions. The honeycomb lattice model precisely has 16 Majorana fermions at low-energy, which is free of the \mathbb{Z}_{16} non-perturbative global anomaly. However, the \mathbb{Z}_4^{TF} symmetry is still restrictive enough to rule out all possible fermion bilinear gapping terms from appearing in Equation (19), which again calls for the SMG mechanism.

The four-fermion interaction g in Equation (18) is one choice of the interaction that drives the desired SMG. On every site, this interaction is the same as the $\text{SU}(4)$ symmetric interaction in Equation (5) (up to unimportant density–density interactions, which do not affect the ground state but only renormalize the gap size). The interaction g explicitly drives a four-fermion condensation, also known as the charge-4e superconducting order [114–120]. In the large g limit, the many-body ground state of the system is simply the product $\otimes_i |4e\rangle_i$ of on-site ground states $|4e\rangle_i = (|0000\rangle_i + |1111\rangle_i) / \sqrt{2}$, which is symmetric, non-degenerated and gapped, realizing the SMG phase. Therefore, by tuning the strength g , one expects to drive an SMG transition at some intermediate $g \sim 1$ (that is comparable with the bandwidth of the lattice fermion). Various numerical simulations of this model (and its variants) [125,143–146] have suggested the existence of such a direct and continuous SMG transition between the gapless and the gapped phases in (2 + 1)D. A field theory description of the SMG quantum critical point was proposed in Ref. [36], which will be further reviewed in Section 4.3.1.

The honeycomb lattice model also has an explicit $\text{SU}(4)$ symmetry $c_{ia} \rightarrow U_{ab}c_{ib}$ that rotates fermions among the four on-site flavors. Although the $\text{SU}(4)$ symmetry does not affect the anomaly analysis in any essential way, it could help to remove all the $\text{SU}(4)$ -breaking relevant perturbations (if there were any) at the critical point, which might help to promote a continuous SMG transition. If the $\text{SU}(4)$ symmetry is also included in the consideration, the total internal symmetry will be $\text{SU}(4) \times_{\mathbb{Z}_2^F} \mathbb{Z}_4^{TF}$ (where $\text{SU}(4)$ and \mathbb{Z}_4^{TF} share the fermion parity \mathbb{Z}_2^F subgroup).

The decomposition of the on-site $\text{SU}(4)$ -symmetric interaction into a product of fermion bilinear operators $\Phi_i^\alpha \Phi_i^\alpha$ as in Equation (5) indicates that one can again introduce a Yukawa field ϕ_α to mediate the fermion interaction, [36]

$$\mathcal{L}_{\text{YH}} = -\phi_\alpha (\psi_{Ka}^\dagger i\gamma^0 \tilde{\Gamma}_{ab}^\alpha \psi_{K'b} + \text{h.c.}) + \frac{1}{2g} \phi_\alpha^* \phi_\alpha, \tag{20}$$

where K, K' label the two valleys (Dirac points) of the low-energy fermions (see Figure 2b), and $\tilde{\Gamma}^\alpha = (\sigma^{12}, \sigma^{20}, \sigma^{32}, i\sigma^{21}, i\sigma^{02}, i\sigma^{23})$ are the flavor–space matrices for the inter-valley flavor-sextet pairing that transform as representation **6** in $\text{SU}(4)$. Importantly, the Yukawa field must transform as $\mathbb{Z}_4^{TF} : \phi_\alpha \rightarrow -\phi_\alpha$ to keep \mathcal{L}_{YH} invariant under \mathbb{Z}_4^{TF} , meaning that directly condensing the Yukawa field $\langle \phi_\alpha \rangle \neq 0$ will necessarily break the protecting symmetry \mathbb{Z}_4^{TF} (as well as breaking the standing by $\text{SU}(4)$ symmetry). The SMG interaction thus provides a mechanism to allow the Yukawa field to condense locally without establishing a long-range order so as to maintain the fermion gap without breaking the symmetry.

2.4. (3 + 1)D SMG: Chiral Fermion Model

The lattice regularization of chiral fermions in (3 + 1)D has always been an important motivation to study SMG in lattice gauge theories. A rich class of SMG transitions was proposed and analyzed in different 4D chiral fermion models recently [37,106]. Here, we will review one of the simplest examples from [37] to illustrate the essential features of these models. The example considers a collection of right-handed massless Weyl fermions

ψ transforming under an internal $SU(N)$ symmetry in the $\square \oplus (N+4)\bar{\square}$ representation, described by the Lagrangian

$$\mathcal{L} = \psi^\dagger i\bar{\sigma}^\mu \partial_\mu \psi, \quad (21)$$

where ψ is the collection of complex Grassmann spinor fields. More explicitly, the fermion field can be split into $\psi = \psi_\lambda \oplus \psi_\psi$, such that ψ_λ denotes the single \square Weyl fermion and ψ_ψ denotes the $(N+4)$ multiples of $\bar{\square}$ Weyl fermions. The goal is to find a path of gapping these fermions without breaking the $SU(N)$ symmetry.

First of all, the symmetric gapping is possible because all fermions together cancel the $SU(N)$ 't Hooft anomaly, as can be verified by the following anomaly index calculation

$$A_{SU(N)}(\square) + (N+4)A_{SU(N)}(\bar{\square}) = (N+4) + (N+4) \times (-1) = 0. \quad (22)$$

Secondly, the $SU(N)$ symmetry is restrictive enough to forbid all possible fermion bilinear masses. For chiral fermions, the only available fermion bilinear mass is the Majorana mass that pairs up fermions, such as $(\psi^\dagger i\sigma^2 \psi + \text{h.c.})$. Since the Majorana mass is already antisymmetric in the spinor subspace, the flavor subspace must be symmetric. However, the fermion representation of the $SU(N)$ symmetry guarantees that its symmetric product does not contain a trivial representation $\mathbf{1}$, as $(\square \oplus (N+4)\bar{\square}) \times_S (\square \oplus (N+4)\bar{\square}) \rightarrow \square \square \oplus (N+4)\square \oplus (N+4)\bar{\square} \square \oplus (N+4)^2 \bar{\square} \bar{\square}$. So any bilinear mass term will transform non-trivially under $SU(N)$ and cannot be condensed in the presence of the $SU(N)$ symmetry. Therefore, one must rely on fermion interactions to achieve symmetric gapping, i.e., the SMG mechanism.

To design the appropriate interaction to realize the SMG transition, one might search for candidate four-fermion interactions by looking for trivial representations in $(\square \oplus (N+4)\bar{\square})^{\times 4}$. However, $(\square \oplus (N+4)\bar{\square})^{\times 4} \not\rightarrow \mathbf{1}$ in general (unless for $N=2,4$), meaning that one might need to look into higher-order fermion interactions, which are even more irrelevant and less interesting to explore. Nevertheless, the three-fermion combination $(\square \oplus (N+4)\bar{\square})^{\times 3}$ contains trivial representations of the form $(\psi_\lambda^\dagger i\sigma^2 \psi_\psi) \psi_\psi$, but this term is fermionic and cannot appear directly in the Lagrangian (otherwise the fermion parity symmetry would be broken). An interesting idea from [37,106] is to bring down $(N+4)(N+3)/2$ additional fermions ψ_χ from the high-energy spectrum which transforms trivially under the $SU(N)$ symmetry, such that a four-fermion interaction

$$\mathcal{L}_{\text{int}} = g((\psi_\lambda^\dagger i\sigma^2 \psi_\psi)(\psi_\chi^\dagger i\sigma^2 \psi_\psi) + \text{h.c.}) \quad (23)$$

can be constructed. In this interaction, the fermion flavors are contracted in the way such that $\psi_\lambda^\dagger i\sigma^2 \psi_\psi$ and $\psi_\chi^\dagger i\sigma^2 \psi_\psi$ transform as \square and $\bar{\square}$ under $SU(N)$, respectively.

It is instructive to note that the chiral fermion model has an additional internal symmetry $SU(N+4)$, under which ψ_λ , ψ_ψ and ψ_χ transform as $\mathbf{1}$, $\bar{\square}$ and \square , respectively, as summarized in Table 1. The representations are so assigned such that the fermions are also free of the $SU(N+4)$ anomaly, as

$$NA_{SU(N+4)}(\bar{\square}) + A_{SU(N+4)}(\square) = N \times (-1) + (N+4-4) = 0. \quad (24)$$

Moreover, there is no mixed anomaly between $SU(N)$ and $SU(N+4)$. The interaction \mathcal{L}_{int} in Equation (23) also respects the larger $SU(N) \times SU(N+4)$ symmetry. Ref. [37] suggests that \mathcal{L}_{int} is a plausible SMG interaction that drives all fermions $\psi = \psi_\lambda \oplus \psi_\psi \oplus \psi_\chi$ to the gapped phase without breaking the $SU(N) \times SU(N+4)$ symmetry. The mechanism will be further reviewed in Section 4.3.2.

Table 1. Representations of the fermion ψ and Yukawa ϕ fields in the chiral fermion model under $SU(N) \times SU(N + 4)$.

		SU(N)	SU(N + 4)	
ψ {	ψ_λ	\square	$\mathbf{1}$	physical fermion
	ψ_ψ	$\bar{\square}$	$\bar{\square}$	
	ψ_χ	$\mathbf{1}$	\square	
ϕ		$\bar{\square}$	\square	Yukawa boson

The interaction \mathcal{L}_{int} in Equation (23) can be decomposed into the following Yukawa couplings

$$\mathcal{L}_{\text{YH}} = -(\phi_1^\dagger (\psi_\lambda^\dagger i\sigma^2 \psi_\psi) + \phi_2 (\psi_\lambda^\dagger i\sigma^2 \psi_\psi) + \text{h.c.}) + \frac{1}{g} (\phi_1^\dagger \phi_2 + \text{h.c.}), \quad (25)$$

where the Yukawa fields ϕ_1 and ϕ_2 are both in the $(\bar{\square}, \square)$ representation of $SU(N) \times SU(N + 4)$. The above Yukawa decomposition again provides an alternative way to understand the SMG transition by driving the Yukawa field into a strong-coupling symmetric phase.

2.5. SMG in General Dimensions

For SMG to happen in a fermion system with a choice of *spacetime-internal symmetry* G (that includes both the spacetime symmetry and internal symmetry) in d -dimensional spacetime, the following two *necessary* conditions must be both satisfied:

- The anomaly index $\nu \in \text{TP}_{d+1}(G)$ of the system must vanish ($\nu = 0$), where $\text{TP}_{d+1}(G)$ denotes the classification of invertible topological phases (with low-energy invertible topological field theories) in $(d + 1)$ -dimensional spacetime [4]. (We may also denote $d + 1 \equiv D$ as the bulk dimension.)
- The single fermion must be in a representation \mathbf{r}_ψ^G of the full spacetime-internal symmetry G , such that the *antisymmetric* product (denoted by $\times_{\mathcal{A}}$) representation $\mathbf{r}_\psi^G \times_{\mathcal{A}} \mathbf{r}_\psi^G$ does not contain the trivial representation $\mathbf{1}^G$ in its direct sum decomposition.

If the first condition is violated ($\nu \neq 0$), trivially and symmetrically gapping out the fermions is impossible with or without interaction, due to the anomaly obstruction. Note that in condensed matter systems, the global symmetry G also includes lattice symmetries and may have the Lieb–Schultz–Mattis (LSM) [147–149] type of anomaly. SMG can not happen if the system is not free from such a LSM anomaly. However, it may be possible to symmetrically gap out the fermions with a symmetric anomalous topological order described by its underlying low-energy topological quantum field theory (TQFT); see Section 4.4. If the second condition is violated, a trivial and symmetric gap can already be achieved at the free-fermion level by a fermion bilinear condensation (through the trivial channel given by $\mathbf{r}_\psi^G \times_{\mathcal{A}} \mathbf{r}_\psi^G \rightarrow \mathbf{1}^G$), such that fermion interaction is not necessary. When both conditions are satisfied, an interacting (strong-coupling) mechanism is needed to generate the fermion mass without breaking the G symmetry, i.e., the SMG mechanism.

These conditions are also sufficient for the existence of an interacting SMG phase (i.e., the featureless gapped phase). However, they do not necessarily imply a direct and continuous SMG transition of fermions from the gapless phase to the gapped phase. A general design of the four-fermion interaction that leads to a single continuous transition between the gapless fermions and the SMG remains a challenging problem. Much of the progress relies on numerical simulations, as to be reviewed in Section 3.

3. Numerical Investigations

3.1. Existence of SMG Phases

Because the SMG is generally a non-perturbative interaction effect (with respect to the gapless free-fermion fixed point), numerical simulations play an important role in the study of SMG. Given an anomaly-free gapless fermion system with a proposed gapping interaction, the numerical study has two major goals:

- Establish the existence of the SMG phase in the strongly interacting limit.
- Investigate the nature of the SMG transition at the critical interaction strength.

The SMG phase refers to the strongly interacting trivial gapped state of fermions without symmetry breaking. It was also called the paramagnetic strong-coupling (PMS) phase [144,150], the strongly coupled symmetric gapped (SCSG) phase [102,103], or the featureless gapped phase [36,151] in the literature. Correspondingly, the gapless Dirac/Weyl/Majorana fermion phase in the weak interaction regime is sometimes also called the paramagnetic weak-coupling (PMW) phase [144,150], or the semi-metal (SM) phase [36,151]. The phase transition separating the SMG phase and the free-fermion phase is called the SMG transition.

Numerical study of the SMG phases and phase transitions has been performed in various spacetime dimensions with different symmetries, as summarized in Table 2. The model Hamiltonian $H = H_0 + H_{\text{int}}$ takes the following general form

$$\begin{aligned}
 H_0 &= - \sum_{i,j} \sum_a t_{ij} \psi_{ia} \psi_{ja} \\
 H_{\text{int}} &= g \sum_i \sum_{a,b,c,d,\dots} V_{abcd\dots} \psi_{ia} \psi_{ib} \psi_{ic} \psi_{id} \dots, \\
 \text{or } H_{\text{int}} &= \sum_i \sum_{a,b,\alpha} \phi_{i\alpha} \psi_{ia} \Gamma_{ab}^\alpha \psi_{ib} + H[\phi_{i\alpha}],
 \end{aligned}
 \tag{26}$$

where ψ_{ia} denotes the fermion of the flavor a on the site i (or in the unit cell i). The free-fermion Hamiltonian H_0 generates gapless fermions at low energy, and the interaction Hamiltonian H_{int} applies properly designed interactions to drive the SMG. The interaction strength will be generally denoted as g in the following discussion. Different models are mainly distinct in the following aspects:

Table 2. Summary of numerical studies of SMG phases. (Dim.—spacetime dimension, Sym.—internal symmetry in terms of Lie algebra).

Dim.	Sym.	Model	Method	Reference
(1 + 1)D	u(1)	ψ^6	DMRG	[113]
	su(2)	YH	disorder average	[108]
	so(4)	ψ^4	QMC	[143]
	su(4)	ψ^4	QMC	[152]
	so(7)	YH	HMC	[112]
(2 + 1)D	so(4)	ψ^4	HMC	[125]
	so(5)	ψ^4	QMC	[143]
	su(4)	ψ^4	FBMC	[144,145]
		ψ^4	QMC	[146]
(3 + 1)D	so(4)	YH	HMC	[150,153]
		QCD	HMC	[154,155]
	su(4)	ψ^4	HMC	[156,157]
		ψ^4	FBMC	[158,159]

- The low-energy fermions can be realized in the lattice model H_0 either as the gapless boundary modes of a fermionic SPT state in one higher spacetime dimension [113],

or as the gapless bulk mode of a semi-metal state in the designated spacetime dimension (such as the honeycomb lattice fermion in $(2 + 1)D$ [143,146] and the stagger fermion in general dimensions [150,153]).

- The interaction H_{int} can (i) either be explicit given by multi-fermion local interaction terms (denoted as ψ^4 for four-fermion interactions or ψ^6 for six-fermion interactions in Table 2) as exemplified in Equations (7), (11) and (23), (ii) or mediated by intermediate Yukawa–Higgs fields (denoted as YH) or non-Abelian gauge fields (denoted as QCD) as exemplified in Equations (12), (20) and (25).
- The $\mathfrak{so}(4)$ (or more precisely $\text{Spin}(3) \times_{\mathbb{Z}_2^F} \text{Spin}(4)$), $\mathfrak{so}(5)$ (or $U(1) \times_{\mathbb{Z}_2^F} \text{Spin}(5)$), $\mathfrak{su}(4)$ (or $SU(4) \cong \text{Spin}(6)$) interactions can all be viewed as lower-symmetry descendants of the $\mathfrak{so}(7)$ (or $\text{Spin}(7)$) Fidkowski–Kitaev interaction, whose relations are discussed in Ref. [103].
- In even spacetime dimensions, the interaction can be restricted to part of the fermions (forming the mirror sector) to study the SMG in chiral fermion systems [112,113].

Many different numerical methods have been used to study these models, which include the density matrix renormalization group (DMRG) [113], auxiliary-field quantum Monte Carlo (QMC) [143], hybrid Monte Carlo (HMC) [155,160], and fermion bag Monte Carlo (FBMC) [161].

The SMG phase was successfully achieved in the studies listed in Table 2, which demonstrate the generality of the SMG mechanism in different spacetime dimensions with various symmetry assignments and under different forms of interaction. The SMG phase has the following defining features that can be checked in numerics:

- Unique ground state with a gap to all excitations (including both fermionic and bosonic excitations);
- Absence of spontaneous symmetry breaking (no fermion bilinear condensation or any higher multi-fermion condensations that break the symmetry);
- Formation of the four-fermion (or higher multi-fermion) condensate that preserves the symmetry.

Directly computing the many-body excitation gap is challenging for most numerical approaches. One way to probe the excitation gap is to measure the correlation function in different channels

$$\begin{aligned} C_{ij}^{(1)} &= \langle \psi_{ia} \psi_{ja} \rangle \sim e^{-|x_i - x_j| / \xi_1}, \\ C_{ij}^{(2)} &= \langle \psi_{ia} \psi_{ib} \psi_{ja} \psi_{jb} \rangle \sim e^{-|x_i - x_j| / \xi_2}. \end{aligned} \quad (27)$$

An exponential decaying correlation function implies a finite excitation gap $\Delta_n \sim 1/\xi_n$ inversely proportional to the correlation length ξ_n . In particular, the single-fermion correlation $C^{(1)}$ probes the single-particle gap Δ_1 for fermionic excitations, and the fermion-bilinear correlation $C^{(2)}$ probes the gap Δ_2 for collective bosonic excitations. Various studies [113,125,143,156] have used the correlation function approach to demonstrate the gap opening (mass generation) in the SMG phase, as illustrated in Figure 3a,b. Ref. [143] observed that the fermion single-particle gap Δ_1 is a bit larger than the bosonic fluctuation gap Δ_2 in the SMG phase, which seems to be consistent with the fluctuating bilinear mass picture (where the bosonic excitations typically have a longer correlation length than the fermionic excitations), as to be discussed in Section 4.1.

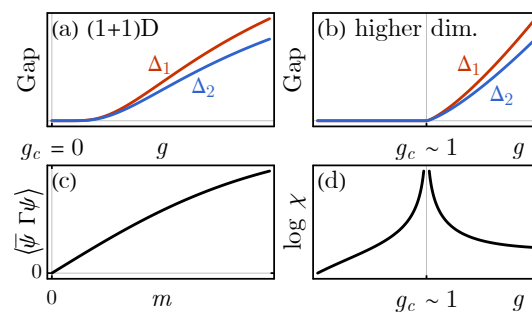


Figure 3. Typical features of SMG observed in numerics. Continuous gap opening in (a) (1 + 1)D SMG driven by four-fermion interactions and (b) higher dimensional SMG in general. (c) Vanishing fermion bilinear expectation $\langle \bar{\psi} \Gamma \psi \rangle$ with its source field m in the SMG phase. (d) Diverging Yukawa field susceptibility χ at the SMG transition.

More explicitly, Ref. [108] investigated the fermion single-particle spectrum of (1 + 1)D chiral gauge theory coupled to a fluctuating Higgs field with finite spacetime correlation. By choosing the Higgs coupling strength and the Higgs field correlation length appropriately, Ref. [108] was able to show that the fermion excitation gap remains open in the SMG phase, despite of the absence of Higgs condensation.

Many numerical works have also confirmed that there is no fermion bilinear condensation in the SMG phase, such that the mass generation is not due to the conventional Anderson–Higgs symmetry-breaking mechanism. To check this statement, one can introduce a small source field m that couples to the fermion bilinear term $\bar{\psi} \Gamma \psi$ of interest (where the vertex matrix Γ can be of any choice of interest that may or may not be the Yukawa vertex in Equation (26)). By showing that in the large system size limit ($L \rightarrow \infty$), the induced fermion bilinear expectation value vanishes as the source field m is turned off

$$\lim_{m \rightarrow 0} \lim_{L \rightarrow \infty} \langle \bar{\psi} \Gamma \psi \rangle = 0, \quad (28)$$

one can rule out the fermion bilinear condensation, as illustrated in Figure 3c. Such a check should be performed for all fermion bilinear terms that transform as different representations under the symmetry. This approach was adopted in various studies [112,146,150,153,156,157]. Another approach is to measure the correlation function of fermion bilinear operators. If the correlation function decays exponentially, then the bilinear operator is not long-range ordered (not condensed). This approach was used in Refs. [113,143].

However, to establish the fact that the SMG phase breaks no symmetry, one also has to check that there is no symmetry-breaking four-fermion or higher multi-fermion condensation developed in the gapped phase. It is generally difficult to check all possible multi-fermion condensations exhaustively in numerics. One possible argument is to show that the multi-fermion condensation that corresponds to the fermion interaction develops in the SMG phase, i.e., $\langle H_{\text{int}} \rangle \neq 0$ [112,125,150,152,153,156,157], which does not break any symmetry (as H_{int} is symmetric by design). Assuming that the SMG phase can be smoothly deformed to the strong interaction limit ($g \rightarrow \infty$) where the Hamiltonian is dominated by H_{int} , and given that the ground state of H_{int} is unique (by design), the possibility of spontaneous symmetry breaking can be ruled out.

3.2. Nature of SMG Transitions

Much effort of the numerical study has been focused on understanding the nature of the SMG transition. Two questions can be asked:

- Is the SMG transition a *direct* transition (i.e., without any intermediate phases setting in)?
- Is the SMG transition *continuous* (i.e., not first order)?

If the answers to both questions are yes, the SMG transition would be a quantum critical point which admits a field-theory description in the continuum limit. Such a case will be highly interesting from the theoretical perspective because the phases on both sides cannot be distinguished by a symmetry-breaking order parameter, and hence a direct and continuous SMG transition is necessarily an exotic quantum phase transition beyond the Landau–Ginzburg–Wilson paradigm.

The nature of the SMG transition depends on the spacetime dimension and the symmetry. Table 3 summarizes the numerical results in the literature. SMG transitions in (1 + 1)D are the most well understood. As four-fermion interactions are marginal perturbations for (1 + 1)D massless fermions, the RG equation for the interaction strength g with respect to the RG scale $\ell = \ln \Lambda$ takes the general form of $dg/d\ell = \alpha g^2$ to the leading order of g . If the coefficient $\alpha > 0$, $g > 0$ is marginally relevant, which flows toward infinity from any finite g . This happens to any SMG transition driven by four-fermion interactions in (1 + 1)D: the system immediately enters the symmetric gapped phase as long as the interaction is turned on (with the correct sign), such that the SMG critical point is at $g_c = 0$. The RG equation predicts that the excitation gap should open as (see Figure 3a)

$$\Delta \sim \exp\left(-\frac{1}{\alpha g}\right), \quad (29)$$

which was verified in Refs. [143,152] with different models.

Table 3. Summary of numerical studies of SMG transitions. (Dir.—direct transition or not; Con.—continuous transition or not).

Dim.	Sym.	Dir.	Con.	Remarks	Reference
(1 + 1)D	u(1)	yes	yes	BKT, $g_c \sim 1$	[113]
	so(4)	yes	yes	$g_c = 0$	[143]
	su(4)	yes	yes	$g_c = 0$	[152]
(2 + 1)D	so(4)	yes	yes		[125]
	so(5)	yes	yes		[143]
	su(4)	yes	yes	$\eta = 1.05, \nu = 1.30$ $\eta = 0.7 \pm 0.1$	[144,145] [146]
(3 + 1)D	so(4)	yes	yes	by frustrating the Yukawa field $\nu \sim 1$	[150,153,154] [155]
	su(4)	no	-	small intermediate SSB phase	[157–159]

However, SMG can also be driven by higher-order fermion interactions. For example, the SMG in (1 + 1)D U(1)-symmetric chiral fermion 3-4-5-0 model is driven by six-fermion interactions, which are irrelevant at the massless free-fermion fixed point. As shown in Ref. [113], the SMG transition in this case happens at a finite interaction strength, and the transition is shown to be in the BKT universality class.

Numerical studies have also found evidence for direct and continuous SMG transitions in higher dimensions, as listed in Table 3. A direct transition can be established by ruling out any intermediate phase between the gapless fermion phase and the SMG phase. The most probable intermediate phase is the spontaneous symmetry breaking (SSB) phase in which the fermion bilinear mass (the Yukawa field) condenses. Such an intermediate SSB phase is often observed in (3 + 1)D systems [157–159]. However, there are also examples showing that it is possible to shrink the intermediate SSB phase by frustrating the Yukawa field (i.e., introducing local couplings of the Yukawa field in conflict with its natural ordering tendency), which could foster a direct SMG transition [150,153,154].

Given a direct SMG transition, one can further check whether the transition is continuous or first-order (discontinuous). The numerical evidence for a continuous transition include the following:

- Continuous gap opening across the SMG transition.
- Universal scaling behavior of physical quantities near the transition.

The excitation gap (or the inverse correlation length) can be extracted from correlation functions (as previously explained around Equation (27)). Refs. [125,143] have demonstrated that the gap opens smoothly across the SMG transition, as depicted in Figure 3b, in support of a continuous transition (i.e., a quantum critical point). As the correlation length diverges, physical quantities should exhibit universal scaling behaviors near the quantum critical point. The quantity that is often studied in numerics is the uniform static susceptibility of the Yukawa field, which is defined as

$$\chi = \frac{1}{L^d} \int d^d x \langle \phi_\alpha(x) \phi_\alpha(0) \rangle, \quad (30)$$

where the integration is over the spacetime, and L^d stands for the spacetime volume (with d being the spacetime dimension). The following universal behavior is observed in Refs. [144,145]

$$\chi = L^{2-\eta} f((g - g_c) L^{1/\nu}), \quad (31)$$

where f is a universal function, and η and ν are critical exponents such that the susceptibility generally diverges as $\chi \sim |g - g_c|^{-\nu(2-\eta)}$ near the SMG critical point, as illustrated in Figure 3d. The exponent η can also be determined from the power-law fitting of the correlation function at the critical point,

$$\langle \phi_{i\alpha} \phi_{j\alpha} \rangle \sim |\mathbf{x}_i - \mathbf{x}_j|^{-(d-2+\eta)}, \quad (32)$$

where d is the spacetime dimension. This approach is used in Ref. [146]. The scaling analysis was performed mainly in $(2+1)$ D systems, where the simulation can achieve a linear system size up to $L = 60$ [145], which enables a rather reliable estimate of critical exponents. As a comparison, for $(3+1)$ D systems, the simulation can only reach $L = 20$ [155]. Numerical studies [144–146] of the $(2+1)$ D $su(4)$ SMG found $\eta \sim 1.05$ and $\nu \sim 1.30$, which are close to the (large- N_f limit) exponents ($\eta = \nu = 1$) of the $(2+1)$ D Gross–Neveu–Yukawa universality class [162,163]. Recent study [155] of the $(3+1)$ D $so(4)$ SMG also favors $\nu \sim 1$, although the result is not conclusive yet. However, the SMG mechanism is physically distinct from the symmetry-breaking mass generation described by the Gross–Neveu–Yukawa model. Their similar critical exponents motivate the idea to view the SMG transition as a hidden Gross–Neveu–Yukawa transition of fermionic partons, as to be discussed in Section 4.2.

4. Theoretical Understandings

4.1. Fluctuating Bilinear Mass Picture

The *fluctuating bilinear mass* (Yukawa field) provides an intuitive physical picture for SMG. It suggests that the SMG can be generally understood in two steps: starting with gapless Weyl/Majorana fermions, first condense a Yukawa field that couples to fermion bilinear mass terms to gap out the fermions at the price of breaking the protecting symmetry, then fluctuate the phase (or direction) of the Yukawa field in the spacetime to restore the symmetry while maintaining the local amplitude of the Yukawa field finite to keep the fermion gap open. The picture can be described by the Yukawa–Higgs model with a symmetry G [108,156]

$$\begin{aligned} Z &= \int \mathcal{D}[\psi, \phi] e^{-S_Y[\psi, \phi] - S_H[\phi]}, \\ S_Y[\psi, \phi] &= \int d^d x (\bar{\psi}_a i \gamma^\mu \partial_\mu \psi_a + \phi_\alpha \bar{\psi}_a \Gamma_{ab}^\alpha \psi_b), \\ S_H[\phi] &= \int d^d x ((\partial_\mu \phi)^2 + V_H(\phi) + \dots), \end{aligned} \quad (33)$$

where ψ_a are the Weyl/Majorana fermions (written as real spinor Grassmann fields with $\bar{\psi}_a = \psi_a^\dagger \gamma^0$) and ϕ_α are the Yukawa bosons (written as real scalar fields). For generality, we assume that all fields are automatically translated into their minimal real embeddings to avoid unnecessary complication of complex conjugations in the discussion. Both the fermion ψ and the boson ϕ fields are in non-trivial representations of the protecting symmetry G , denoted as \mathbf{r}_ψ^G and \mathbf{r}_ϕ^G , respectively. The Yukawa coupling is loosely denoted as $\phi \bar{\psi} \Gamma \psi$ for generality, where the vertex tensor Γ_{ab}^α is set by the Clebsch–Gordan coefficients of the fusion channel $\mathbf{r}_\psi^G \times_{\mathcal{A}} \mathbf{r}_\psi^G \rightarrow \mathbf{r}_\phi^G$. Precise forms of the Yukawa coupling can be found in Equations (6), (12), (20) and (25), which vary from model to model. $S_Y[\psi, \phi]$ describes the gapless fermion ψ coupled to the Yukawa field ϕ , and $S_H[\phi]$ describes the dynamics of the Yukawa field with $V_H(\phi)$ being some Higgs potential.

Integrating out the Yukawa field ϕ leads to a pure fermion model of ψ . Assuming that $S_H[\phi] = \frac{1}{2g} \phi^2$ takes the Gaussian form at the ultraviolet (UV) level, the fermion model will be

$$Z = \int \mathcal{D}[\psi] e^{-S[\psi]}, \quad (34)$$

$$S[\psi] = \int d^d x (\bar{\psi} i \gamma^\mu \partial_\mu \psi + g (\bar{\psi} \Gamma^\alpha \psi) (\bar{\psi} \Gamma^\alpha \psi)).$$

It is assumed that the model is free of G -anomaly, but the symmetry G is still restrictive enough to forbid all possible bilinear mass terms, paving ways for SMG. The fluctuating Yukawa field effectively mediates the fermion interaction g that is responsible to drive the SMG.

The appearance of the SMG in the Yukawa–Higgs model Equation (33) requires some delicate design of $S_H[\phi]$ to achieve the appropriate infrared (IR) dynamics. Assuming the Higgs potential $V_H(\phi)$ pins the Yukawa field ϕ to a finite amplitude (e.g., $|\phi| = 1$) throughout the spacetime (as in a non-linear σ -model), while allows its orientation to fluctuate in its internal flavor space, numerical evidences [108] show that the fluctuating Yukawa field is possible to gap out the fermions ψ without spontaneously breaking the G symmetry if the Yukawa field fluctuation has a finite but large correlation length $\zeta \gg 1$ (s.t. $\langle \phi(x) \phi(0) \rangle \sim e^{-|x|/\zeta}$). The fluctuation must be smooth enough so as not to create sharp domain walls that trap gapless domain-wall fermions and close the fermion gap. However, the fluctuation must also not be too smooth to establish the long-range order of the Yukawa field and break the symmetry. The SMG should be achieved by balancing these two factors.

An alternative way to argue for the SMG is the *topological defect condensation* approach [16,17,102,103,164]. This approach also starts by condensing the Yukawa field with a finite amplitude, but then disordering the Yukawa field orientation by condensing topological point defects (e.g., vortices in 2D space or monopoles in 3D space) of the Yukawa field. Although the point defects could trap fermion zero modes and lead to gapless fermion excitations in the spectrum, these fermion zero modes can be gapped out by local interactions (such as the FK interaction) uniformly applied throughout the system. A non-trivial check in this approach is to show that for anomaly-free fermion systems (that admits SMG), the point defect always traps $8n$ Majorana zero modes that can be trivialized by interaction. The advantage of the topological defect condensation approach is that it does not rely on a delicate tuning of the smoothness of the Yukawa field fluctuation.

Further justification of the fluctuating bilinear mass picture comes from a more explicit trial wave function construction for the SMG state in $(0+1)D$ [36]. Let $|\Psi[\phi]\rangle$ be the quantum many-body ground state of fermions ψ on a background configuration of the Yukawa field ϕ . The fluctuating bilinear mass picture Equation (33) suggests that the SMG state (the featureless gapped state in the strong interacting limit) should be described by

$$|\Psi_{\text{SMG}}\rangle \propto \int \mathcal{D}[\phi] e^{-S_H[\phi]} |\Psi[\phi]\rangle. \quad (35)$$

In the $(0+1)D$ example, the ground state of the Yukawa Hamiltonian $H_Y = -\phi_\alpha \chi_a \Gamma_{ab}^\alpha \chi_b$ is explicitly given by $|\Psi[\phi]\rangle = (1 + \frac{1}{|\phi|} \phi_\alpha \chi_a \Gamma_{ab}^\alpha \chi_b + c_1^\dagger c_2^\dagger c_3^\dagger c_4^\dagger) |0000\rangle$ using notations intro-

duced in Section 2.1. Assuming $S_H[\phi]$ restricts the ϕ vector uniformly on a sphere, the spherical average results in $|\Psi_{\text{SMG}}\rangle \propto (1 + c_1^\dagger c_2^\dagger c_3^\dagger c_4^\dagger)|0000\rangle$ which indeed matches the ground state $|4e\rangle$ of the FK interaction. The construction also applies to $\mathfrak{su}(4)$ symmetric SMGs in all higher dimensions, where the on-site interaction is the same as the $(0+1)\text{D}$ case. In the limit that the Yukawa coupling is strong and the Yukawa field ϕ fluctuates independently on each site, the trial wave function $|\Psi_{\text{SMG}}\rangle$ in Equation (35) approaches to product state $\otimes_i |4e\rangle_i$ —the exact ground state in the strong coupling ($g \rightarrow \infty$) limit. Away from the strong coupling limit, $|\Psi_{\text{SMG}}\rangle$ might still serve as a variational approximation of the SMG state, but this variational ansatz is less well controlled beyond $(0+1)\text{D}$.

Although the fluctuating bilinear mass picture provides a simple understanding for the SMG phase/state, it breaks down near the SMG transition, where the Yukawa field amplitude is no longer large and stable. A major theoretical challenge is to develop field theory descriptions for the continuous SMG transition, which will be the topic of the following Section 4.2.

4.2. Fermion Fractionalization Field Theory

The *fermion fractionalization* [36,151] provides an overarching theoretical framework to understand various continuous SMG transitions in different dimensions. It unifies different theoretical pictures of SMG in literature. Its key proposal is that the physical fermion $\psi \sim \eta^n \psi$ fractionalizes into bosonic η and fermionic ψ partons that deconfine *at and only at* the transition, where the bosonic parton η may appear n multiple times in the fractionalization scheme. Note that upright Greek letters will be used to denote parton fields in the parton theory.

The fermionic parton ψ is generally put in the same gapless phase as the physical fermion ψ . The physical free fermion phase corresponds to the condensed phase of the bosonic parton η , and the SMG phase corresponds to the disordered (gapped) phase of the bosonic parton η . Accompanied with the fractionalization, an additional gauge field a must be introduced to bind the partons together and to remove the redundant degrees of freedom introduced by fractionalization. Let K be the emergent gauge group (typically non-Abelian), then the partons are generally charged under an enlarged symmetry-gauge group $G \times K$ (where G is the symmetry group of the physical fermion), as summarized by Table 4.

Table 4. Representations of different fields under $G \times K$, also under spacetime symmetry $\text{Spin}(d)$. The representation assignment must be consistent with $\psi \sim \eta^n \psi$, $\phi \sim \bar{\psi} \Gamma \psi$, and $\phi \sim \bar{\psi} \Gamma \psi$.

	G	K	Spin	Meaning
ψ	\mathbf{r}_ψ^G	$\mathbf{1}^K$	spinor	physical fermion
ϕ	\mathbf{r}_ϕ^G	$\mathbf{1}^K$	scalar	Yukawa boson
η	\mathbf{r}_η^G	\square^K	scalar	bosonic parton
ψ	\mathbf{r}_ψ^G	\mathbf{r}_ψ^K	spinor	fermionic parton (like ψ)
a	$\mathbf{1}^G$	$\square\square^K$	vector	gauge boson
ϕ	\mathbf{r}_ϕ^G	\mathbf{r}_ϕ^K	scalar	parton-Higgs boson (like ϕ)

The representations are not assigned independently. The choice of representations should satisfy the following general rules:

- Representations of the physical fermion ($\mathbf{r}_\psi^G, \mathbf{1}^K$) and the Yukawa boson ($\mathbf{r}_\phi^G, \mathbf{1}^K$) are given as the starting point by the SMG model in consideration. Both ψ and ϕ fields must be neutral (i.e., as the trivial representation $\mathbf{1}^K$) under the gauge group K by definition.
- The bosonic parton η is always in the (anti)-fundamental representation \square^K of K , such that its condensation can fully Higgs the gauge group K .

- The parton representation must be assigned in consistent with the fermion fractionalization $\psi \sim \eta^n \psi$, such that the following fusion channel must exist

$$(\mathbf{r}_\eta^G, \square^K)^{\times n} \times (\mathbf{r}_\psi^G, \mathbf{r}_\psi^K) \rightarrow (\mathbf{r}_\psi^G, \mathbf{1}^K). \quad (36)$$

- The Yukawa or parton-Higgs field representations must be consistent with their constituting fermions, such that the following fusion channel must exist

$$\mathbf{r}_\psi^G \times_{\mathcal{A}} \mathbf{r}_\psi^G \rightarrow \mathbf{r}_\phi^G \neq \mathbf{1}^G, \quad (37)$$

$$(\mathbf{r}_\psi^G, \mathbf{r}_\psi^K) \times_{\mathcal{A}} (\mathbf{r}_\psi^G, \mathbf{r}_\psi^K) \rightarrow (\mathbf{r}_\phi^G, \mathbf{r}_\phi^K), \quad (38)$$

where $\times_{\mathcal{A}}$ denotes the antisymmetric combination of representations (due to the fermionic nature of ψ and ψ). \mathbf{r}_ϕ^G should not be trivial, otherwise the fermion mass generation can be achieved simply by condensing ϕ without breaking the symmetry, as already explained in Section 2.5.

- The K -gauge field a is always in the trivial representation $\mathbf{1}^G$ of G and the adjoint representation \square^K of K .
- For the SMG to happen in the system, either one of the following two necessary conditions should be satisfied:
 1. To enable the parton-Higgs mechanism, the following branching channel must exist upon $G \times K \rightarrow G \times K'$ breaking:

$$(\mathbf{r}_\phi^G, \mathbf{r}_\phi^K) \rightarrow (\mathbf{1}^G, \mathbf{1}^{K'}); \quad (39)$$

2. To enable the s-confinement mechanism, the following fusion channel must exist within the gauge group K :

$$(\mathbf{r}_\psi^K)^{\times 3} \rightarrow \mathbf{1}^K. \quad (40)$$

The reasoning for these conditions will be explained later in Sections 4.2.1 and 4.2.2.

Given the representation $(\mathbf{r}_\psi^G, \mathbf{1}^K)$ of the physical fermion ψ , there is still the freedom to choose the gauge group K and the representation $(\mathbf{r}_\psi^G, \mathbf{r}_\psi^K)$ of the fermionic parton ψ . Once the choice is made, other representations are (largely) fixed by Equations (36)–(38). There could be multiple parton theories that describe the same SMG transition as long as the condition Equation (39) or Equation (40) is satisfied. Within the present framework, different parton theories are specified by different choices of K and $(\mathbf{r}_\psi^G, \mathbf{r}_\psi^K)$.

The parton theory for the SMG transition generally takes the form of [36,151]

$$\begin{aligned} Z &= \int \mathcal{D}[\eta, \psi, a] e^{-S_B[\eta, a] - S_F[\psi, a]}, \\ S_B[\eta, a] &= \int d^d x (|D_\mu \eta|^2 + r|\eta|^2 + u|\eta|^4), \\ S_F[\psi, a] &= \int d^d x \bar{\psi} i \gamma^\mu D_\mu \psi, \end{aligned} \quad (41)$$

where ψ and η are treated as real fields (with $\bar{\psi} = \psi^\top \gamma^0$) to avoid unnecessary complications of complex conjugation, and $D_\mu = \partial_\mu - i a_\mu$ denotes the covariant derivative (assuming the gauge connection a is automatically represented in the representation of the matter field that it couples to). The parton theory describes the bosonic η and the fermionic ψ partons coupled together by the gauge field a , and is expected to provide an effective description of the low-energy physics around the SMG critical point. The bosonic parton mass r is treated as the only driving parameter in the parton theory, and is responsible for tuning the SMG transition. $r < 0$ corresponds to the gapless phase (the free-fermion fixed point), $r > 0$ corresponds to the trivially gapped phase (the strong-coupling fixed point), and $r = 0$ corresponds to the critical point where the SMG transition happens. The states of different fields in different cases are summarized in Table 5.

Table 5. States of different fields across the SMG transition.

	Gapless Phase ($r < 0$)	SMG ($r = 0$)	Gapped Phase ($r > 0$)
ψ	gapless	fractionalized	gapped
ϕ	gapped ($\xi \sim 1$)	gapped	gapped
η	condensed	critical	gapped
ψ	gapless	gapless	gapped
a	Higgs	deconfined	Higgs/confined
ϕ	gapped	critical	condensed

When $r < 0$, the bosonic parton η condenses (i.e., $\langle \eta \rangle \neq 0$). As η carries the anti-fundamental representation of the gauge group K , the gauge structure is fully Higgs down by the bosonic parton condensation. At the same time, the fermionic parton ψ becomes the physical fermion $\psi \sim \langle \eta \rangle^n \psi$ effectively (as the fields ψ and ψ only differ by a constant $\langle \eta \rangle^n$, which simply serves as a quasi-particle weight). The parton theory then reduces to the physical fermion theory $S[\psi] = \int d^d x \bar{\psi} i \gamma^\mu \partial_\mu \psi$ in the gapless free-fermion phase.

When $r > 0$, the bosonic parton η is gapped and decouples from the gauge field at low energy. Below the gap of η field, the parton theory is effectively a quantum chromodynamics (QCD) theory $S_F[\psi, a]$ involving the fermionic parton ψ coupled to the gauge field a . Then, it relies on an assumption that the QCD dynamics leads to a featureless gapped ground state (i.e., a trivial ground state), such that both the fermionic partons and the gauge bosons are gapped without breaking the G symmetry or generating topological orders. Of course, one necessary condition for the QCD theory to trivialize is that the theory must have vanishing $G \times K$ anomaly. However, even if the QCD theory is anomaly free, it still depends on the dynamical details to achieve a trivially gapped ground state. The major theoretical effort to understand SMG lies in designing an appropriate mechanism to trivialize the QCD theory. Within the fermion fractionalization framework, two mechanisms have been proposed in the literature: (i) parton-Higgs (weak coupling) [36,37,151] and (ii) s-confinement (strong coupling) [37,106]. In some cases, the QCD trivialization may be consistently achieved by both mechanisms.

4.2.1. Parton-Higgs Mechanism

The parton-Higgs mechanism introduces a collection of scalar fields ϕ (parton-Higgs fields) that couple to the fermionic parton ψ via Yukawa interactions, such that the QCD theory $S_F[\psi, a]$ is extended to a QCD–Yukawa–Higgs theory

$$S_F[\psi, \phi, a] = \int d^d x (\bar{\psi} i \gamma^\mu D_\mu \psi + \phi_\alpha \bar{\psi} \Gamma^\alpha \psi + (D_\mu \phi)^2 + V_H(\phi)), \tag{42}$$

where the parton-Higgs field ϕ (as fermionic parton bilinear mass) is in the representation $(\mathbf{r}_\phi^G, \mathbf{r}_\phi^K)$ of $G \times K$. The representation $(\mathbf{r}_\phi^G, \mathbf{r}_\phi^K)$ must be obtained from an antisymmetric product of fermionic parton representations for consistency: $(\mathbf{r}_\psi^G, \mathbf{r}_\psi^K) \times_{\mathcal{A}} (\mathbf{r}_\psi^G, \mathbf{r}_\psi^K) \rightarrow (\mathbf{r}_\phi^G, \mathbf{r}_\phi^K)$, as stated in Equation (38). The Yukawa vertex tensor Γ^α is fixed by this fusion channel. The QCD–Yukawa–Higgs theory in Equation (42) is reminiscent of the Yukawa–Higgs theory in Equation (33) by promoting the physical fermion ψ and physical Yukawa field ϕ to their parton counterparts ψ and ϕ . However, the key difference lies in the different $G \times K$ representations of ψ, ϕ comparing to ψ, ϕ , as listed in Table 4.

Unlike the physical Yukawa field ϕ , which is solely charged under G in a non-trivial representation \mathbf{r}_ϕ^G , such that the condensation of ϕ inevitably breaks the G symmetry; the parton-Higgs field ϕ is in a joint representation $(\mathbf{r}_\phi^G, \mathbf{r}_\phi^K)$ of $G \times K$, which could admit a G -symmetric condensation of ϕ . The sufficient and necessary condition for the existence of a condensed configuration $\langle \phi \rangle$ that preserves the G symmetry is that there exists a

subgroup $G \times K'$ of $G \times K$, such that when $G \times K$ is broken to $G \times K'$, the representation of ϕ admits a trivial branching channel (as summarized in Equation (39)):

$$(\mathbf{r}_\phi^G, \mathbf{r}_\phi^K) \rightarrow (\mathbf{1}^G, \mathbf{1}^{K'}). \quad (43)$$

If the condition is met, a Higgs potential $V_H(\phi)$ can be constructed to drive the condensation of ϕ in the above trivial branching channel. Such a condensation of the parton-Higgs field ϕ will (i) gap out all fermionic partons ψ (that couple to it), (ii) Higgs down the gauge group from H to its subgroup $K' \subseteq K$, and (iii) preserving the symmetry group G . Then the only freedom remaining in the theory is the pure gauge fluctuation of K' . If K' remains a non-Abelian group, its gauge coupling can flow strongly into the confine phase and gap out the remaining gauge bosons. In this way, all freedoms in the QCD–Yukawa–Higgs theory are symmetrically gapped, and the system ends up in the SMG phase. The physical fermion $\psi \sim \eta\psi$ will also be gapped, as its partons are gapped (both η and ψ are gapped).

4.2.2. s-Confinement Mechanism

The s-confinement mechanism (“s” for “smooth”) refers to the strong coupling dynamics of the QCD theory that confines fermionic partons and gauge bosons all together without breaking the G symmetry [165,166]. This is possible if the ’t Hooft anomaly of G is matched up in the theory [3]. One motivation for the s-confinement theory comes from the observation that the four-fermion SMG interaction ψ^4 (for physical fermions ψ) can be broken up into “3+1” as $(\psi^3)\psi$. If there is some strong-coupling mechanism that binds the first three fermions into a composite fermion $\psi_{\text{comp}} \sim \psi^3$ at low-energy, the interaction can then be viewed as a fermion bilinear mass like $\bar{\psi}_{\text{free}}\psi_{\text{comp}}$ (where $\bar{\psi}_{\text{free}} \sim \psi$ stands for the last dangling fermion), which provides a free-fermion understanding for the SMG. This picture is alternative to the “2+2” fluctuation bilinear mass picture, where the interaction is broken up into $(\psi^2)(\psi^2)$ with each ψ^2 being a composite boson (i.e., the Yukawa boson ϕ). However, the Yukawa boson is still interacting, which complicates the analysis of its low-energy dynamics. In contrast, the composite fermion flows to a free fermion critical point, which enables a simple argument for gapping. Nevertheless, this still relies on a strong-coupling mechanism for the composite fermion to form in the first place.

The fermion fractionalization plays an important role in understanding the formation of composite fermions in the s-confinement picture. After fractionalization, physical fermions become fermionic partons that couple to an emergent gauge field, then the emergent gauge force can be employed to confine the fermions into the desired composite. Not all fermions are fractionalized in the s-confinement theory. The physical fermions $\psi = \psi_{\text{frac}} \oplus \psi_{\text{free}}$ are first divided into two sectors, where $\psi_{\text{frac}} \sim \eta^n\psi$ are part of the physical fermions that will be fractionalized to bosonic η and fermionic ψ partons, and ψ_{free} are the remaining physical fermions that do not fractionalize. To drive the SMG transition, the first step is still to gap the bosonic parton η , leaving the low-energy freedom in a QCD theory with the fermionic parton ψ coupled to an emergent (non-Abelian) gauge field a . Under gauge confinement, the fermionic partons ψ are bound together in the IR to form fermion composites, denoted as $\psi_{\text{comp}} \sim (\psi)^3$, just like fundamental quarks forming baryons. The necessary condition for the confinement to happen is the existence of a trivializing fusion channel in the gauge sector (as summarized in Equation (40)):

$$(\mathbf{r}_\psi^K)^{\times 3} \rightarrow \mathbf{1}^K, \quad (44)$$

such that the gauge charges of ψ^3 can be neutralized to produce the physical composite fermion ψ_{comp} . Then the fermion bilinear mass $\bar{\psi}_{\text{free}}\psi_{\text{comp}}$ can be introduced to gap out all fermions together, realizing the SMG. Moreover, the s-confinement mechanism admits the supersymmetric extension [37], where the addition of supersymmetry provides extra control for the strong-coupling confinement dynamics at low energy [167].

4.3. Examples of Fermion Fractionalization

In the following, we will briefly exemplify the fermion fractionalization theory in two SMG models introduced in Section 2.

4.3.1. (2 + 1)D Honeycomb Lattice Model

The SMG in the (2 + 1)D honeycomb lattice model can be understood using the parton-Higgs mechanism within the framework of fermion fractionalization [36]. The internal symmetry in consideration is

$$G = \text{SU}(4) \times_{\mathbb{Z}_2^F} \mathbb{Z}_4^{TF}, \tag{45}$$

where \mathbb{Z}_4^{TF} is the essential protecting symmetry. The physical fermion ψ is in the $\mathbf{4}_1$ representation of G . The proposed fermion fractionalization scheme $\psi \sim \eta\psi$ introduces an emergent gauge group

$$K = \text{SU}(4), \tag{46}$$

and puts the fermionic parton ψ in the $(\mathbf{1}_1, \mathbf{4})$ representation of $G \times K$. This choice fully specifies the parton theory. The physical and fractionalized fields in the parton construction are summarized in Table 6, such that all consistency conditions are met among their symmetry representations.

Table 6. Representations of different fields under $(\text{SU}(4) \times_{\mathbb{Z}_2^F} \mathbb{Z}_4^{TF})_G \times \text{SU}(4)_K$, as well as the Euclidean spacetime symmetry $\text{Spin}(3)$, which are relevant to the SMG in the (2 + 1)D honeycomb lattice model.

	$(\text{SU}(4) \times_{\mathbb{Z}_2^F} \mathbb{Z}_4^{TF})_G$	$\text{SU}(4)_K$	Spin
ψ	$\square_1 = \mathbf{4}_1$	$\mathbf{1}$	spinor (fermion)
ϕ	$\square_2 = \mathbf{6}_2$	$\mathbf{1}$	scalar (boson)
η	$\square_0 = \mathbf{4}_0$	$\bar{\square} = \bar{\mathbf{4}}$	scalar (boson)
ψ	$\mathbf{1}_1$	$\square = \mathbf{4}$	spinor (fermion)
a	$\mathbf{1}_0$	$\square\square = \mathbf{15}$	vector (boson)
ϕ	$\mathbf{1}_2$	$\square = \mathbf{6}$	scalar (boson)

The SMG can be achieved through the parton-Higgs mechanism by gapping out the bosonic parton η and condensing the parton-Yukawa field $\phi \sim \bar{\psi}\Gamma\psi$. This breaks $G \times K$ to $G \times K'$, where the new gauge group is

$$K' = \text{Sp}(2) \subseteq \text{SU}(4) = K. \tag{47}$$

In particular, because the parton-Yukawa field ϕ charges two under both the \mathbb{Z}_4^{TF} symmetry group and the \mathbb{Z}_4^K center of the gauge group $K = \text{SU}(4)$, condensing ϕ necessarily locks the \mathbb{Z}_4^{TF} and \mathbb{Z}_4^K generators together, breaking $\mathbb{Z}_4^{TF} \times \mathbb{Z}_4^K$ down to $\mathbb{Z}_4^{TF} \times \mathbb{Z}_2^{K'}$, which is consistent with the $\mathbb{Z}_2^{K'}$ center of residual gauge group $K' = \text{Sp}(2)$. As $G \times K$ is broken to $G \times K'$, the representation of ϕ splits into

$$(\mathbf{1}_2, \mathbf{6}) \rightarrow (\mathbf{1}_0, \mathbf{1}) \oplus (\mathbf{1}_0, \mathbf{5}), \tag{48}$$

which admits a trivial representation $(\mathbf{1}_0, \mathbf{1})$ in the decomposition, satisfying the trivializing condition in Equation (39) for the parton-Higgs mechanism to work.

4.3.2. (3 + 1)D Chiral Fermion Model

The (3 + 1)D chiral fermion model contains three sectors of physical fermions $\psi = \psi_\lambda \oplus \psi_\psi \oplus \psi_\chi$ transformed under the symmetry group

$$G = \text{SU}(N) \times \text{SU}(N + 4). \tag{49}$$

The SMG in this model can be understood by fermion fractionalization, where only part of the fermions $\psi_{\text{frac}} = \psi_\psi \oplus \psi_\chi$ is fractionalized: $\psi_\psi \sim \eta^\dagger \psi_\psi$ and $\psi_\chi \sim \eta^2 \psi_\chi$, while $\psi_{\text{free}} = \psi_\lambda$ remains untouched. The fractionalization introduces an emergent gauge group

$$K = \text{SU}(N + 4). \tag{50}$$

Table 7 summarizes the representation of fields satisfying all consistency conditions.

To drive the chiral fermions ψ to the SMG phase, the first step is to gap out the bosonic parton η . Then the theory contains a free fermion theory of ψ_λ and a QCD theory of the fermionic partons $\psi_\psi \oplus \psi_\chi$ coupled to the K gauge field a . The trivialization of the QCD theory can be achieved by either the parton-Higgs mechanism or the s-confinement mechanism (they are equivalent).

Table 7. Representations of different fields under $(\text{SU}(N) \times \text{SU}(N + 4))_G \times \text{SU}(N + 4)_K$, as well as the Euclidean spacetime symmetry $\text{Spin}(4)$, which are relevant to the SMG in the (3 + 1)D chiral fermion model.

	$(\text{SU}(N) \times \text{SU}(N + 4))_G$	$\text{SU}(N + 4)_K$	Spin
ψ_λ	$(\square, \mathbf{1})$	$\mathbf{1}$	spinor (fermion)
ψ_ψ	$(\bar{\square}, \bar{\square})$	$\mathbf{1}$	spinor (fermion)
ψ_χ	$(\mathbf{1}, \bar{\square})$	$\mathbf{1}$	spinor (fermion)
ϕ	$(\bar{\square}, \square)$	$\mathbf{1}$	scalar (boson)
η	$(\mathbf{1}, \square)$	$\bar{\square}$	scalar (boson)
ψ_ψ	$(\bar{\square}, \mathbf{1})$	$\bar{\square}$	spinor (fermion)
ψ_χ	$(\mathbf{1}, \mathbf{1})$	$\bar{\square}$	spinor (fermion)
a	$(\mathbf{1}, \mathbf{1})$	$\bar{\square}\square$	vector (boson)
ϕ	$(\bar{\square}, \mathbf{1})$	\square	scalar (boson)

In the parton-Higgs mechanism, two parton-Higgs fields $\phi = \phi_1 \oplus \phi_2$ are introduced and coupled to the fermions as

$$- (\phi_1^\dagger (\psi_\lambda^\dagger i\sigma^2 \psi_\psi) + \phi_2 (\psi_\lambda^\dagger i\sigma^2 \psi_\psi) + \text{h.c.}), \tag{51}$$

which is closely reminiscent of the Yukawa–Higgs decomposition Equation (25) of the SMG interaction in the chiral fermion model. Both parton-Higgs fields $\phi_{1,2}$ are in the representation $((\bar{\square}, \mathbf{1}), \square)$ of $G \times K$. Condensing the parton-Higgs field ϕ in a “color-flavor” locked form, the $\text{SU}(N)$ (flavor) subgroup in G will be identified with the $\text{SU}(N)$ (color) subgroup in $K = \text{SU}(N + 4)$, such that the color-flavor combined $\text{SU}(N)$ rotation will form the diagonal $\text{SU}(N)$ group that leaves ϕ invariant. The color-flavor locking breaks the $G \times K$ group to its subgroup $G \times K'$ with

$$K' = \text{SU}(4) \subseteq \text{SU}(N + 4) = K. \tag{52}$$

Some additional U(1) subgroups are ignored here to simplify the discuss, without hurting the main idea; see [37] for a more rigorous treatment. Under $G \times K \rightarrow G \times K'$, the representation of ϕ decomposes as

$$\phi : ((\bar{\square}, \mathbf{1}), \square) \rightarrow ((\mathbf{1}, \mathbf{1}), \mathbf{1}) \oplus ((\bar{\square}, \mathbf{1}), \mathbf{4}) \oplus ((\bar{\square}\square, \mathbf{1}), \mathbf{1}) \tag{53}$$

which contains the trivial representation $((\mathbf{1}, \mathbf{1}), \mathbf{1})$ that corresponds to the condensed configuration $\langle \phi \rangle$, fulfilling the requirement of Equation (39). Moreover, representations of fermions branch as

$$\begin{aligned} \psi_\lambda & : ((\square\square, \mathbf{1}), \mathbf{1}) \rightarrow ((\square\square, \mathbf{1}), \mathbf{1}), \\ \psi_\psi & : ((\bar{\square}, \mathbf{1}), \bar{\square}) \rightarrow ((\bar{\square}\square, \mathbf{1}), \mathbf{1}) \oplus ((\bar{\square}, \mathbf{1}), \mathbf{1}) \oplus ((\bar{\square}, \mathbf{1}), \bar{\mathbf{4}}), \\ \psi_\chi & : ((\mathbf{1}, \mathbf{1}), \square) \rightarrow ((\square, \mathbf{1}), \mathbf{1}) \oplus ((\square, \mathbf{1}), \mathbf{4}) \oplus ((\mathbf{1}, \mathbf{1}), \mathbf{6}). \end{aligned} \tag{54}$$

The fermion representation has transmuted, morphing from a chiral representation of $G \times K$ into a vector-like representation of the surviving $G \times K'$. The parton-Higgs field condensation $\langle \phi \rangle \neq 0$ will provide mass for almost all fermions via the Yukawa coupling in Equation (51) (by pairing up conjugate fermion representations), with the only exception of the $((\mathbf{1}, \mathbf{1}), \mathbf{6})$ fermions in ψ_χ . However, these fermions are fully neutral under the symmetry group G and only charged under the remaining gauge group K' . The remaining K' gauge bosons and the $((\mathbf{1}, \mathbf{1}), \mathbf{6})$ fermions can be simultaneously removed from the low-energy spectrum once the K' gauge fluctuation flow is strong and confines. Hence, no low-energy freedom is left over, and the system ends up in the SMG phase.

In the s-confinement mechanism, a gauge-neutral combination of fermionic partons must first exist, as required by Equation (40). As the fermionic partons $\psi_\psi \oplus \psi_\chi$ carry the gauge charges $\bar{\square} \oplus \square$, respectively (see Table 7), the gauge neutral three-fermion combination can be found in the fusion channel

$$\square \times \bar{\square} \times \bar{\square} \rightarrow \mathbf{1} \tag{55}$$

So the fermionic partons can be confined into a composite fermion through this confinement channel as

$$\psi_{\text{comp}} \sim (\psi_\chi^\dagger i\sigma^2 \psi_\psi) \psi_\psi, \tag{56}$$

which transforms as $((\bar{\square}\square, \mathbf{1}), \mathbf{1})$ under $G \times K$, precisely conjugate to the representation $((\square\square, \mathbf{1}), \mathbf{1})$ of the remaining physical fermion ψ_λ . Therefore, the composite fermion ψ_{comp} can pair up with the remaining fermion ψ_λ to produce a G-symmetric mass term $\psi_\lambda^\dagger i\sigma^2 \psi_{\text{comp}}$, that gaps out all fermions from the low-energy spectrum, leading to the SMG state.

4.4. Symmetry Extension Construction

4.4.1. Symmetry Breaking vs. Symmetry Extension vs. SMG

From the symmetry perspective, the fermion mass generation generally falls into two categories: (i) the *symmetry breaking* mechanism and (ii) the *symmetry extension* mechanism [24,25] (see Figure 4). At a high level, the SMG is essentially a *symmetry extension* mechanism in the *absence* of the 't Hooft anomaly (strictly speaking, the SMG mechanism does not enlarge the physical symmetry, but rather extends it to a symmetry-gauge structure; nevertheless, we will still call it by symmetry extension in a general sense).

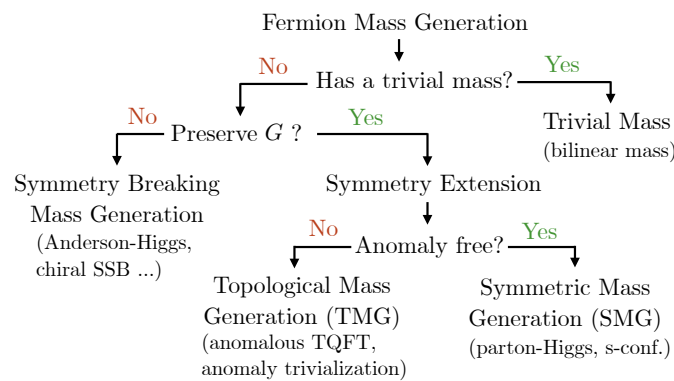


Figure 4. Classification of fermion mass generation mechanisms. Given the spacetime-internal symmetry G and the representation \mathbf{r}_ψ^G of the fermion field under G , the system has a trivial mass if the fusion channel $\mathbf{r}_\psi^G \times_{\mathcal{A}} \mathbf{r}_\psi^G \rightarrow \mathbf{1}^G$ exists; the system is anomaly free if the anomaly index $\nu \in \text{TP}_{d+1}(G)$ vanishes ($\nu = 0$).

1. **Symmetry/Gauge breaking:** Anderson–Higgs mechanism, chiral symmetry breaking, Dirac mass and Majorana mass are induced by the symmetry breaking—breaking either global symmetries or gauge structures, by condensing a Yukawa–Higgs field that couples to a fermion bilinear mass term. More precisely, starting from a symmetry group G (specifically here an internal symmetry, global or gauged), G is broken down to an appropriate subgroup $G' \subseteq G$ to induce quadratic mass term for fermions. Mathematically, it can be described by an *injective* homomorphism ι :

$$G' \xrightarrow{\iota} G. \tag{57}$$

Here are some explicit examples:

- Bardeen–Cooper–Schrieffer (BCS) type \mathbb{Z}_2 -gauged superconductor with a low energy \mathbb{Z}_2 TQFT, we have $G' = \mathbb{Z}_2$ and $G = \text{U}(1)$ electromagnetic gauge group.
 - The standard model electroweak Higgs mechanism breaks $G = G_{\text{SM},q} \equiv (\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)_Y) / \mathbb{Z}_q$ with $q = 1, 2, 3, 6$ and the appropriate greatest common divisor (gcd), down to $G' = (\text{SU}(3) \times \text{U}(1)_{\text{EM}}) / \mathbb{Z}_{\text{gcd}(q,3)}$.
 - (3 + 1)D Dirac mass pairs two Weyl fermions (ψ_L and ψ_R) via the Dirac mass term $m_D(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)$ which breaks the unitary internal $G = \text{U}(2)$ symmetry of two Weyl fermions down to a vector $G' = \text{U}(1)$ symmetry.
 - (1 + 1)D Dirac mass pairs two Weyl fermions (ψ_L and ψ_R) via the Dirac mass term $m_D(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)$ which breaks the unitary internal $G = \text{U}(1)_L \times \text{U}(1)_R$ symmetry of two Weyl fermions down to a vector $\text{U}(1)$ symmetry, so $G' = \text{U}(1)$.
 - (3 + 1)D Majorana mass pairs a single Weyl fermion ψ_L to itself, $\psi_R = i\sigma^2 \psi_L^*$, so the Majorana mass term $m_M \bar{\psi} \psi = m_M(\psi_L^\dagger (i\sigma^2) \psi_L^* + \psi_L^\top (-i\sigma^2) \psi_L)$, which breaks the unitary internal $G = \text{U}(1)$ symmetry of a Weyl fermion down to a fermion parity $G' = \mathbb{Z}_2^F$ symmetry.
2. **Symmetry/Gauge extension:** In contrast, the symmetry extension [25] or gauge enhancement [168] provides a fermion mass generation mechanism that preserves the symmetry. Symmetry extension construction of gapped phases first appears in [24] based on the gauge bundle descriptions, then Ref. [25] refines the idea to the lattice models, group-cohomology cocycle or continuum field theory descriptions. It extends the symmetry group G (that can include the spacetime-internal symmetry, global or gauged) to a larger group \tilde{G} by enlarging the Hilbert space with additional/redundant degrees of freedom, in order to trivialize the 't Hooft anomaly or to lift any other sym-

metry obstruction toward a gapped phase (such as the symmetry-forbidden bilinear mass). Mathematically, it can be described by a *surjective* homomorphism r :

$$\tilde{G} \xrightarrow{r} G, \quad (58)$$

which can be understood as part of the group extension in an exact sequence (see Equation (59)) [25], or can be further generalized as the fibrations of their classifying spaces and higher classifying spaces [59,169,170]. The symmetry extension can induce the *topological mass* in the presence of an anomaly, and the *symmetric mass* in the absence of an anomaly [122].

- (a) **Topological mass generation (TMG)**—if the fermion system has a non-trivial 't Hooft anomaly in G , the anomaly will post an obstruction toward *trivial* gapping, which already rules out SMG, leaving the possibility to TMG. A non-vanishing *perturbative local anomaly* disallows any symmetric gapped phase (even with topological order); also it can never be trivialized by a symmetry extension. So, in order to implement the symmetry extension construction, the non-vanishing G -anomaly must be a *nonperturbative global anomaly* in G . For simplicity, the discussion below focuses on a limited special case of (58). If the global anomaly in G can become anomaly-free in \tilde{G} , by pulling the G group back to the extended \tilde{G} group via a short exact sequence

$$1 \rightarrow K \rightarrow \tilde{G} \xrightarrow{r} G \rightarrow 1, \quad (59)$$

and if the normal subgroup K is a discrete finite group, then the fermion can acquire a topological mass upon gauging K , which gives rise to a discrete K gauge TQFT in appropriate spacetime dimensions and under appropriate criteria detailed in Section 4.4.2. This is also called the group extension of the original quotient group G extended by a normal subgroup K to the total group \tilde{G} . The topological mass refers to the energy gap of a TQFT matching the 't Hooft anomaly of G . Finding the group extension \tilde{G} often requires the essential use of algebraic topology criteria, such as the Lydon–Hochschild–Serre spectral sequence method [25]. Refs. [25,59,63,122,171,172] provide several explicit lattice Hamiltonian or lattice path integral constructions.

- (b) **Symmetric mass generation (SMG)**—In the case of SMG, the fermion system is already anomaly free, but the physical symmetry G is too restrictive to allow any symmetric fermion-bilinear mass (i.e., the symmetry obstruction $\mathbf{r}_\psi^G \times_{\mathcal{A}} \mathbf{r}_\psi^G \not\rightarrow \mathbf{1}^G$). However, with the symmetry extension described by the following short exact sequence

$$1 \rightarrow K \rightarrow \tilde{G} = G \times K \xrightarrow{r} G \rightarrow 1, \quad (60)$$

the fermion-bilinear mass (i.e., the parton-Higgs field ϕ) can be charged under both G and K , which can possibly be condensed, breaking only K to its subgroup K' without breaking G , as long as the condensed fermion-bilinear mass transform trivially under $G \times K' \subseteq G \times K$ (i.e., the fusion channel $(\mathbf{r}_\psi^G, \mathbf{r}_\psi^{K'}) \times_{\mathcal{A}} (\mathbf{r}_\psi^G, \mathbf{r}_\psi^{K'}) \rightarrow (\mathbf{1}^G, \mathbf{1}^{K'})$ exists and the symmetry obstruction is lifted). Moreover, the extended normal subgroup K (and K') can be (and will always be) gauged, such that the actual physical symmetry will not be enlarged by the symmetry extension. This is simply a rephrasing of the parton-Higgs mechanism in the fermion fractionalization framework discussed in Section 4.2.

Figure 5 concludes how the symmetry-gauge group is extended/broken in different phases. A few general requirements for the SMG to happen are summarized as follows:

- The full gauge group K must be large enough to counteract any non-trivial action of the symmetry group G on the parton-Higgs field ϕ , i.e., G must act *projectively* on ϕ , such that the G symmetry can remain unbroken under the condensation of ϕ .
- To achieve SMG in this framework, the deformation path must pass through the strong coupling regime, where neither the bosonic parton η nor the parton-Higgs field ϕ has an expectation value, and the full gauge group K is unbroken.
- After the parton-Higgs field ϕ condenses, the remaining unbroken gauge group K' would better be either trivial or non-Abelian such that either there is no residual K' gauge fluctuation or the residual K' gauge fluctuation can be confined automatically. Otherwise, if K' is Abelian, it becomes possible for the SMG critical point to expand into a critical phase, described by an Abelian K' -gauge theory.

4.4.2. More on Symmetry Extension Construction

The *symmetry extension construction* $\tilde{G} \xrightarrow{r} G$ in (58) based on the pullback G -symmetry to the extended \tilde{G} -symmetry can be interpreted in different languages for different communities [25]:

- For condensed matter, a nontrivial SPT state in the G symmetry cannot be deformed to a trivial tensor product state via a finite-depth of local unitary transformations without breaking the G -symmetry. However, the successful symmetry extension means that we can find an appropriate \tilde{G} such that the SPT state in the extended \tilde{G} symmetric Hilbert space can be deformed to a trivial tensor product state via a finite-depth of local unitary transformations still preserving the \tilde{G} -symmetry.
- For quantum field theory or high-energy physics, the successful symmetry extension means that the 't Hooft anomaly in G -symmetry becomes anomaly-free in \tilde{G} -symmetry.
- For mathematics, the successful symmetry extension means that a nontrivial class of cocycle, cohomology, or cobordism of the G -symmetry becomes a trivial class in the \tilde{G} -symmetry. Suppose the nontrivial class of G -symmetry cocycle, cohomology, or cobordism in the D dimensions is labeled by ω_D^G , then the trivialization means that its pullback (namely $r^*\omega_D^G$) becomes a \tilde{G} -symmetry coboundary in the D dimensions (namely $\omega_D^{\tilde{G}} = \delta\beta_{D-1}^{\tilde{G}}$) which splits to the \tilde{G} cochain (namely $\beta_{D-1}^{\tilde{G}}$) in the $D - 1$ dimensions [25]. In summary, given the ω_D^G of the G -symmetry, the successful symmetry extension requires to find a solution of both the extended \tilde{G} and $\beta_{D-1}^{\tilde{G}}$ to satisfy

$$r^*\omega_D^G = \omega_D^{\tilde{G}} = \delta\beta_{D-1}^{\tilde{G}}. \quad (61)$$

Here, the bulk dimension is $D = d + 1$, while the theory (on the boundary) with 't Hooft anomaly has its dimension $D - 1 = d$. Below, some examples of the symmetry extension construction of gapped phases in various dimensions based on the simplest short exact sequence in (59),

$$1 \rightarrow K \rightarrow \tilde{G} \xrightarrow{r} G \rightarrow 1$$

with a finite group K , are considered. In particular, two issues should be addressed:

1. Given a theory with 't Hooft anomaly in G symmetry (which we shall also call it a dD anomalous boundary theory of a $(d + 1)D$ bulk SPT), can \tilde{G} be found to trivialize the anomaly? If so, what is the minimal \tilde{G} ?
2. If the \tilde{G} is found, there are two implications of the construction based on (59) [25]:
 - \tilde{G} -symmetric extended gapped phase (as a \tilde{G} -symmetric gapped boundary of the bulk G -SPT). In this case, all K, \tilde{G} and G are not dynamically gauged.

- G -symmetric gapped dynamical K -gauge theory with a 't Hooft anomaly in G . When K is dynamically gauged, in some cases, the G is preserved at IR; in other cases, the G becomes spontaneous symmetry breaking (SSB) at IR. One should also be careful to distinguish the two different kinds of dynamics.

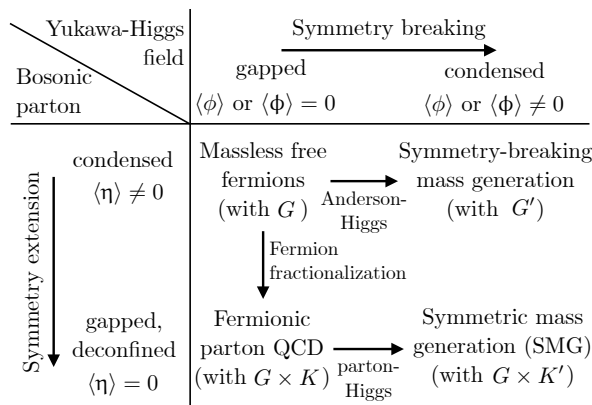


Figure 5. Symmetry-gauge groups in different phases of the Yukawa–Higgs field and the bosonic parton field.

Ref. [25] proves that “given any unitary or anti-unitary finite group G , for $d \geq 1$, there always exists a finite K -extension to a \tilde{G} -symmetric extended gapped phase to trivialize any non-perturbative global anomaly in G (see also the proof given later in [169]).” It is physically meaningful to consider gauging K for $d \geq 2$. Once K is dynamically gauged, Ref. [25] finds that G -symmetric gapped dynamical K -gauge theory with a 't Hooft anomaly in G can only be obtained for $d \geq 3$, but also finds that the G -symmetry is spontaneously broken for $d = 2$.

The above result is established when G is an ordinary 0-symmetry that acts on the 0D point operator. The symmetry extension construction can also be generalized to higher global symmetries [173] such as the 1-symmetry that acts on the 1D line operator (see a recent review [174,175]). Here are some examples (Table 8 for a summary) including ordinary and higher symmetries:

Table 8. Summary of the symmetry-extension construction based on $1 \rightarrow K \rightarrow \tilde{G} \xrightarrow{r} G \rightarrow 1$ in (59). The first and second column show the spacetime dimension d and the G -anomalous theory (as a boundary [bdry] of a bulk $d + 1$ D SPT). The third column shows whether the \tilde{G} -symmetry-extended gapped phase exists. The fourth column shows how the boundary G -anomaly or bulk G -SPT classification is reduced in \tilde{G} , denoted as $TP_{d+1}G \Rightarrow TP_{d+1}\tilde{G}$. The last column shows the dynamics after gauging K : Even if \tilde{G} -symmetry-extended gapped phase exists, the K -gauged dynamics can induce either a \tilde{G} spontaneous symmetry-breaking (SSB) or a G -symmetric K gauge theory.

Dim d	Original System	G	$K \rightarrow \tilde{G} \rightarrow G$	Reduced Class in \tilde{G} of $d + 1$ D G -iTQFT or d D G -Anomaly	Gauge K Dynamics
0+1	bdry of Haldane chain: a doublet or qubit	$SO(3)$ \mathbb{Z}_2^T	$\mathbb{Z}_2 \rightarrow SU(2) \rightarrow G$ $\mathbb{Z}_2 \rightarrow \mathbb{Z}_4^T \rightarrow G$	$\mathbb{Z}_2 \Rightarrow 0$	No
	bdry of 4 Kitaev chains: 4 Majorana modes	$\mathbb{Z}_2^T \times \mathbb{Z}_2^F$	$\mathbb{Z}_2 \rightarrow \mathbb{Z}_4^T \times \mathbb{Z}_2^F \rightarrow G$	$\mathbb{Z}_8 \Rightarrow \mathbb{Z}_4$	
	bdry of 2 Kitaev chains: 2 Majorana modes		$\mathbb{Z}_2 \rightarrow \mathbb{D}_8^{F,T} \rightarrow G$	$\mathbb{Z}_8 \Rightarrow \mathbb{Z}_2$	
	bdry of 2 Kitaev chains: 2 Majorana modes	\mathbb{Z}_4^{TF}	$\mathbb{Z}_4 \rightarrow M_{16}^{F,T} \rightarrow G$	$\mathbb{Z}_2 \Rightarrow 0$	
1+1	bdry of (2 + 1)D CZX model: (1 + 1)D edge modes	\mathbb{Z}_2	$\mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow G$ Symm-extension, but SSB.	$\mathbb{Z}_2 \Rightarrow 0$	G SSB
2+1	bdry of (3 + 1)D $w_1(TM)^4$ SPT: (2 + 1)D surface state	\mathbb{Z}_2^T	$\mathbb{Z}_2 \rightarrow \mathbb{Z}_4^T \rightarrow G$	$\mathbb{Z}_2 \Rightarrow 0$	G-symmetric K-gauge TQFT
	bdry of (3 + 1)D $k\mathcal{P}(B_2)$ higher-SPT: (2 + 1)D surface state	$SO \times \mathbb{Z}_{2,[1]}$	even k : $\mathbb{Z}_2 \rightarrow Spin \times \mathbb{Z}_{2,[1]} \rightarrow G$	$k \in \mathbb{Z}_4 \Rightarrow k \in \mathbb{Z}_2$	G SSB
			but SSB odd k : No symm-extension.		No
bdry of (4 + 1)D A^5 SPT: (3 + 1)D bdry state	\mathbb{Z}_2^T	$\mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow G$	$\mathbb{Z}_2 \Rightarrow 0$		
3+1	bdry of (4 + 1)D $w_3(TM)B_2^e$ higher-SPT	$SO \times \mathbb{Z}_{2,[1]}^e$	$\mathbb{Z}_2 \rightarrow Spin \times \mathbb{Z}_{2,[1]}^e \rightarrow G$	$\mathbb{Z}_2 \Rightarrow 0$	G-symmetric K-gauge TQFT
	bdry of (4 + 1)D $k\mathcal{A}\mathcal{P}(B_2^e)$ higher-SPT	$Spin \times_{\mathbb{Z}_2^F} \mathbb{Z}_8 \times \mathbb{Z}_{2,[1]}^e$	even k : $\mathbb{Z}_2 \rightarrow Spin \times \mathbb{Z}_8 \times \mathbb{Z}_{2,[1]}^e \rightarrow G$	$k \in \mathbb{Z}_4 \Rightarrow k \in \mathbb{Z}_2$	No
			odd k : No symm-extension. No-go obstruction.		
	bdry of (4 + 1)D $\frac{1}{2}\tilde{w}_1(TM)\mathcal{P}(B_2^e)$ higher-SPT: (3 + 1)D $SU(2)_{\theta=\pi}$ YM	$\mathbb{Z}_2^T \times \mathbb{Z}_{2,[1]}^e$	$\mathbb{Z}_2 \rightarrow \mathbb{Z}_2^T \times \mathbb{Z}_{4,[1]}^e \rightarrow G$ Symm-extension, but SSB.	$\mathbb{Z}_2 \Rightarrow 0$	G SSB
bdry of (4 + 1)D $(-N_f)\eta_{4d}(PD(A))$ -SPT: (3 + 1)D $15N_f$ -fermion SM	$Spin \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$	even N_f : $\mathbb{Z}_2 \rightarrow Spin \times \mathbb{Z}_4 \rightarrow G$, $\mathbb{Z}_2 \rightarrow Spin \times \mathbb{Z}_8 \rightarrow Spin \times \mathbb{Z}_4$	$\mathbb{Z}_{16} \Rightarrow \mathbb{Z}_2$	G-symmetric K-gauge TQFT	
		odd N_f : No symm-extension. No-go obstruction.		No	

1. $d = 1$, a (0 + 1)D anomalous theory and a (1 + 1)D bulk:

- Two Kitaev's chains: In Section 2.1, the discussion was limited to gapping (0 + 1)D eight Majorana modes or (1 + 1)D eight Kitaev's chains by the SMG, preserving the $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$, which is free from the \mathbb{Z}_8 global anomaly. Beyond SMG, even for (0 + 1)D two Majorana modes or (1 + 1)D, two Kitaev's chains with 2 mod 8 class of \mathbb{Z}_8 global anomaly, it is still possible to gap the whole system, leaving a single ground state *without breaking* $G = \mathbb{Z}_2^T \times \mathbb{Z}_2^F$, but instead by *extending* the symmetry to a dihedral group of order 8 as $\tilde{G} = \mathbb{D}_8^{F,T} \equiv \mathbb{Z}_4^T \rtimes \mathbb{Z}_2^F$ [122]. Namely, when $T^2 = +1$ is extended to a time-reversal symmetry fictionalization $T^2 = -1$ and $T^4 = +1$, the $\tilde{G} = \mathbb{D}_8^{F,T}$ symmetric interactions can lift up the degenerate Majorana zero modes. Crucially, the time-reversal \mathbb{Z}_2^T generator T does not commute with the fermion parity \mathbb{Z}_2^F generator $(-1)^F$, but $T(-1)^F T^{-1} = -(-1)^F$. This means that T switches a bosonic sector $|B\rangle$ and a fermionic sector $|F\rangle$ in the Hilbert space. This is called *supersymmetry extension* that trivializes this (0 + 1)D fermionic anomaly and also trivializes the (1 + 1)D fermionic SPT state [122]. Another related property is that the preserved symmetry demands that the anomalous boundary theory of ± 2 mod 8 Majorana zero modes must be $\mathcal{N} = 2$ supersymmetric quantum mechanics with two supercharges [176].

Two Kitaev's chains can also allow $G = \mathbb{Z}_4^{TF}$ symmetry with $T^2 = (-1)^F$ [177]. When $T^2 = (-1)^F$ is extended to a time-reversal symmetry fictionalization $T^4 = -1$ and $T^8 = +1$, non-Abelian order-16 finite $\tilde{G} = \mathbb{M}_{16}^{F,T}$ symmetric interactions can lift up the degenerate Majorana zero modes [176].

- Four Kitaev's chains and a Haldane chain: for (0 + 1)D four Majorana modes or (1 + 1)D four Kitaev's chains, its 4 mod 8 class of \mathbb{Z}_8 global anomaly is actually equivalent to a single (1 + 1)D Haldane's chain [81] (tensor product with a trivial gapped fermionic product state) with 1 mod 2 class of \mathbb{Z}_2 global anomaly. The $G = \mathbb{Z}_2^T$ -symmetric Haldane chain can be trivialized in a bosonic $\tilde{G} = \mathbb{Z}_4^T$ [171]. The $G = \text{SO}(3)$ -symmetric Haldane chain can be trivialized in a $\tilde{G} = \text{SU}(2)$ [171].
- Related studies on the fractionalized symmetries on the boundary of the layers of Kitaev chains can also be found in [177–181]. In particular, Refs. [178,180] studied the pure \mathbb{Z}_2 class global gravitational anomaly on the boundary of a single Kitaev chain (which is an invertible fermionic topological order beyond the SPT, known as the mathematical Arf invariant). The pure gravitational anomaly cannot be trivialized by any symmetry extension, but may be "trivialized" by coupling to a gravitational theory [178].

Here, a (0 + 1)D theory has no parity P but only at most time-reversal T , and Majorana fermion has no charge conjugation C symmetry; so only the T fractionalization is found. In higher dimensions, the common theme along the direction of this phenomenon is the C - P - T fractionalization [182].

2. $d = 2$, a (1 + 1)D anomalous theory and a (2 + 1)D bulk:

- The (1 + 1)D edge of a (2 + 1)D CZX model as a \mathbb{Z}_2 -SPT state is known to allow a symmetric gapless or a symmetry-breaking gapped boundary [183]. However, the $G = \mathbb{Z}_2$ can be extended to give a $\tilde{G} = \mathbb{Z}_4$ -symmetry-extended gapped boundary. Unfortunately, gauging the normal subgroup $K = \mathbb{Z}_2$ results in a (1 + 1)D discrete K -gauge theory with $G = \mathbb{Z}_2$ spontaneous symmetry breaking, which is consistent with the standard lore that there is no (1 + 1)D non-invertible intrinsic topological order, at least in the bosonic systems.

Other applications of the symmetry-extension construction on the (1 + 1)D gauge theories and orbifolds can be found in a recent survey [184].

3. $d = 3$, a (2 + 1)D anomalous theory and a (3 + 1)D bulk:

- The (2 + 1)D surface of a (3 + 1)D \mathbb{Z}_2^T -SPT state (topological superconductor) allows a symmetric gapless, symmetry-breaking, or symmetric gapped surface

- topological order boundary [12,185,186]. The $G = \mathbb{Z}_2^T$ can be extended to give a $\tilde{G} = \mathbb{Z}_4^T$ -symmetry-extended gapped boundary [25]. Gauging the normal subgroup $K = \mathbb{Z}_2$ results in a $(2 + 1)$ D discrete K -gauge theory with both electric and magnetic gauge charges that are Kramers doublet with $T^2 = -1$.
- The $(2 + 1)$ D surface state of $(3 + 1)$ D $k\mathcal{P}(B_2)$ higher-SPT state with $k \in \mathbb{Z}_4$: This higher-SPT state is protected by a 1-form symmetry (denoted as $G = \text{SO} \times \mathbb{Z}_{2,[1]}$) which couples to a 2-form background field B_2 . The $\mathcal{P}(B_2)$ means the Pontryagin square of B_2 . The symmetry-extension construction can obtain a gapped phase for an even k via extending to $\tilde{G} = \text{Spin} \times \mathbb{Z}_{2,[1]}$ (although gauging $K = \mathbb{Z}_2$ results in G SSB), but the symmetry-extension trivialization is proven to not exist for an odd k [170]. Later, Ref. [187] proved a no-go theorem that the symmetry-preserving TQFT also does not exist for an odd k . This means the $(2 + 1)$ D surface state must be either symmetric gapless or symmetry-breaking for an odd k .
4. $d = 4$, a $(3 + 1)$ D anomalous theory and a $(4 + 1)$ D bulk:
- The $(3 + 1)$ D boundary of a $(4 + 1)$ D \mathbb{Z}_2 -SPT state allows a symmetric gapless, symmetry-breaking, or symmetric gapped surface topological order boundary. The $G = \mathbb{Z}_2$ can be extended to give a $\tilde{G} = \mathbb{Z}_4$ -symmetry-extended gapped boundary [25]. Gauging the normal subgroup $K = \mathbb{Z}_2$ results in a $(3 + 1)$ D discrete K -gauge theory with both electric and magnetic gauge charges and that carries a fractional G charge.
 - The $(3 + 1)$ D boundary of a $(4 + 1)$ D $w_3(TM)B_2^e$ higher-SPT state is protected by a 1-form electric symmetry (denoted as $G = \mathbb{Z}_{2,[1]}^e$, coupled to a 2-form B_2^e field), while the $w_j(TM)$ is the j th Stiefel–Whitney class of the tangent bundle TM of spacetime manifold M . The corresponding anomaly occurs as a part of the anomaly of $(3 + 1)$ D $\text{SU}(2)$ Yang–Mills theory coupled to two Weyl fermions in the adjoint representation of $\text{SU}(2)$ (below called as the adjoint QCD_4 ; see [170,188–190]). The $G = \mathbb{Z}_{2,[1]}^e$ can be extended to give a $\tilde{G} = \mathbb{Z}_2^F \times \mathbb{Z}_{2,[1]}^e$ -symmetry-extended gapped boundary [170]. Gauging the normal subgroup $K = \mathbb{Z}_2$ results in a $(3 + 1)$ D discrete K -gauge theory such that its electric gauge charge has fermionic statistics.
 - The $(3 + 1)$ D boundary of a $(4 + 1)$ D $kA\mathcal{P}(B_2^e)$ higher-SPT state: again, this is part of the anomaly of the adjoint QCD_4 [189]. The higher SPT is protected by a \mathbb{Z}_8 -axial symmetry (coupled to a 1-form A field, with its fourth power of the symmetry generator equals to $(-1)^F$) and a 1-form electric symmetry $\mathbb{Z}_{2,[1]}^e$ (coupled to a 2-form B_2^e field), which can be denoted as a spacetime-internal symmetry $G = \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_8 \times \mathbb{Z}_{2,[1]}^e$. There is a $k \in \mathbb{Z}_4$ class. The even k class can be trivialized by a \mathbb{Z}_2 -extension to $\tilde{G} = \text{Spin} \times \mathbb{Z}_8 \times \mathbb{Z}_{2,[1]}^e$. The odd k class cannot be trivialized by any symmetry extension [170]. Later, Ref. [187] proved a no-go theorem that the symmetry-preserving TQFT also does not exist. The above results [170,187] turned out to rule out certain UV-IR duality proposal of the adjoint QCD_4 hypothesized in [190].
 - The $(3 + 1)$ D boundary of a $(4 + 1)$ D $\frac{1}{2}\tilde{w}_1(TM)\mathcal{P}(B_2^e)$ higher-SPT state: The boundary turns out to associate with the anomaly of the $(3 + 1)$ D $\text{SU}(2)$ Yang–Mills gauge theory with a $\theta = \pi F \wedge F$ topological term (denoted as $\text{SU}(2)_{\theta=\pi}$ YM) [191–193], while the $\tilde{w}_1(TM)$ is the twisted first Stiefel–Whitney class of TM such that $\frac{1}{2}\tilde{w}_1(TM)\mathcal{P}(B_2^e)$ is a mod 2 class. The $\text{SU}(2)_{\theta=\pi}$ YM kinematically at UV has time-reversal \mathbb{Z}_2^T and 1-form $\mathbb{Z}_{2,[1]}^e$ symmetries. The $G = \mathbb{Z}_2^T \times \mathbb{Z}_{2,[1]}^e$ can be extended to $\tilde{G} = \mathbb{Z}_2^T \times \mathbb{Z}_{4,[1]}^e$ to trivialize the anomaly, thus the \tilde{G} -symmetric extended gapped phase can be constructed [192,193]. However, upon gauging $K = \mathbb{Z}_{2,[1]}$, this induces the G spontaneous symmetry breaking (SSB) [193]. Later, Ref. [187] proved a no-go theorem that the symmetry-preserving TQFT also does not exist. The above results together demand that the IR fate of $\text{SU}(2)_{\theta=\pi}$ YM must be either symmetric gapless or symmetry-breaking only.

- The $(3 + 1)$ D boundary of a $(4 + 1)$ D $k\eta_{4d}(\text{PD}(A))$ SPT state: This is a $k \in \mathbb{Z}_{16}$ class of $(3 + 1)$ D anomaly and $(4 + 1)$ D SPT state, protected by a unitary $\mathbb{Z}_{4,X}$ symmetry such that $X^2 = (-1)^F$; in terms of a spacetime-internal symmetry $G = \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$. The $\eta_{4d}(\text{PD}(A))$ is a 4d Atiyah–Patodi–Singer eta invariant η_{4d} evaluated on the 4-manifold Poincaré dual (PD) to the $(\mathbb{Z}_{4,X}/\mathbb{Z}_2^F)$ -gauge field A . The $\eta_{4d}(\text{PD}(A))$ is a cobordism invariant of the bordism group $\Omega_5^{\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}} = \mathbb{Z}_{16}$, see [63,194,195]. It turns out that $k = (-N_f)$ of such a SPT state captures a global anomaly of $(3 + 1)$ D $15N_f$ -Weyl-fermion standard model (SM), where N_f is the number of families of quarks and leptons [194,196]. If k is odd, Ref. [197] proves an obstruction, so the symmetry-gapped TQFT is not possible to saturate this odd k anomaly. If k is even, Refs. [195,198,199] show that two layers of symmetry extensions can construct the $\tilde{G} = \text{Spin} \times \mathbb{Z}_8$ -symmetry extended gapped phase: the first layer $1 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin} \times \mathbb{Z}_4 \rightarrow \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \rightarrow 1$ and the second layer $1 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin} \times \mathbb{Z}_8 \rightarrow \text{Spin} \times \mathbb{Z}_4 \rightarrow 1$. These constructions may have applications beyond the SM physics [198–200].

5. Features and Applications

5.1. Green's Function Zeros

Given different mass generation mechanisms discussed above, a key question is how to diagnose the SMG mechanism: if a mass gap is observed in an interacting fermionic system, how do we know the mass gap is opened up through the SMG mechanism, rather than a more conventional symmetry-breaking mechanism, say, by condensing a fermion-bilinear mass? Over the years it was gradually realized that one of the characteristic features of the SMG phase is that, the fermion Green's function $\mathcal{G}(i\omega) = -\langle \psi(i\omega)\psi^\dagger(i\omega) \rangle$ in the Matsubara frequency space should have a zero at $\omega \rightarrow 0$, i.e., $\det \mathcal{G}(i\omega \rightarrow 0) = 0$. This applies to the SMG phase either in the bulk or on the boundary of a fermionic system.

There are various ways to argue the necessary existence of this Green's function zero in a system with SMG. For the Green's function $\mathcal{G}(i\omega, k)$ defined in the bulk, one general argument for the existence of zero is based on the topological number associated with the free fermionic SPT state [201,202]. For a fermionic system with translation symmetry, the topological number can be defined in the Matsubara frequency and momentum space using the Euclidean spacetime full fermion Green's function [127,203–208]:

$$n \sim \int d\omega d^{d-1}k \text{tr}[B(\mathcal{G}^{-1}\partial\mathcal{G}) \wedge (\mathcal{G}^{-1}\partial\mathcal{G}) \cdots], \quad (62)$$

where \mathcal{G} is the matrix of the full fermion Green's function, and B is certain matrix in the flavor space. For a Chern insulator at even spatial dimension, B is an identity matrix, and n is equivalent to the Chern number. One can prove mathematically that the number n must be a quantized integer, and it can only change discontinuously when the integrand of Equation (62) becomes singular.

The singularity of $\mathcal{G}^{-1}\partial\mathcal{G}$ happens at two types of "transitions". The first type of transition is a physical transition where $\det \mathcal{G}^{-1}(i\omega = 0) = 0$ vanishing to zero, i.e., the fermions become gapless. This corresponds to the quantum critical line between the fermionic SPT and trivial phases in the weak-coupling regime, as shown in Figure 6. However, note that in the definition of the topological number n in Equation (62), \mathcal{G}^{-1} and \mathcal{G} appear on an equal footing, and hence n can also change when $\det \mathcal{G}(i\omega = 0) = 0$, i.e., when the Green's function has a zero. Hence when the free fermion SPT phase is trivialized by interactions, although there is no unavoidable phase transition between the SPT and the trivial phase, the topological number n still has to change discontinuously somewhere in the phase diagram, as shown in Figure 6. Since there is no real physical transition, the number n has to change through the zero of the full interacting fermion's Green's function. This must hold throughout the SMG phase (line), regardless of the interaction strength g (as long as $g > g_c$).

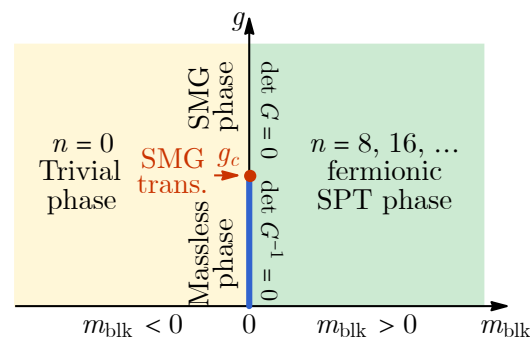


Figure 6. Schematic phase diagram between a trivial phase and a (fake) fermionic SPT phase that can be trivialized by interaction. Massless bulk fermions along the critical line in the weak-coupling limit undergoes the SMG as the interaction g exceeds a critical value g_c . Although the two phases (technically one phase) are smoothly connected through the strong-coupling regime, the topological number n must still jump across the missing “phase boundary”, along which the Green’s function must have a zero.

Computing the full fermion’s Green’s function for a given interacting Hamiltonian is generally a challenging task. To argue that the Green’s function zero is a general feature of the SMG state independent of microscopic details, one relies on the *topological defect condensation* construction for the SMG state, introduced in Section 4.1. Starting with the SMG of eight Majorana zero modes in $(0 + 1)D$, the fermion Green’s function can be explicitly computed [143,209]:

$$\mathcal{G}_{ab}(i\omega) \sim \frac{i\omega\delta_{ab}}{(i\omega)^2 - m^2}, \quad (63)$$

where m is an effective mass proportional to the strength of the fermion interaction. Obviously, in $(0 + 1)D$, $\mathcal{G}_{ab}(i\omega)$ approaches zero as $i\omega \rightarrow 0$. Now one can construct a higher dimensional SMG state by decorating every topological point defect of the Yukawa field with eight interacting Majorana fermions in its SMG state, and then proliferating the point defects to put the Yukawa field in a disordered phase, following the strategy of the decorated domain wall construction [210]. The fermion Green’s function can be evaluated in the spacetime by patching the $(0 + 1)D$ Green’s functions along the world line of the topological defect, and then path integrating all possible world line configurations. Following this approach, Ref. [211] was able to show that the fermion Green’s function takes the general form of

$$\mathcal{G}_{ab}(k) \sim \frac{k_\mu \gamma^\mu \delta_{ab}}{k_\mu k^\mu - m^2}, \quad (64)$$

which universally exhibits a zero $\mathcal{G}(k_\mu = 0) = 0$ in zero momentum-energy limit. Before the non-perturbative prove by Ref. [211], Equation (64) was first obtained in Ref. [209] by a perturbative calculation, and later argued in Refs. [36,107,164] using the fluctuating bilinear mass picture.

5.2. Deconfined Quantum Criticality

While the Green’s function zero provides a key diagnosis of the SMG phase, what about the diagnosis for the SMG transition? The theoretical framework of fermion fractionalization indicates that if the SMG transition is direct and continuous, it should be a deconfined quantum critical point (DQCP) [36], where the physical fermions ψ fractionalize into *deconfined* bosonic η and fermionic ψ partons *at and only at* the critical point.

The concept of DQCP [212–214] was originally introduced to describe the direct continuous transition between the antiferromagnetic phase and the valence bond solid phase in $(2 + 1)D$ quantum spin models. The two phases break distinct symmetries (spin-rotation and lattice-rotation symmetries, respectively) and cannot be connected by a single quantum

critical point without fine-tuning in the Landau–Ginzburg–Wilson paradigm. The DQCP provides an explanation for this exotic quantum critical point by fractionalizing the physical spin into deconfined spinons (partons) at and only at the critical point. The continuous SMG transition is also an exotic quantum critical point that requires the DQCP description. However, unlike the conventional (bosonic) DQCP that fractionalizes bosonic degrees of freedoms, the continuous SMG transition is a *fermionic* version of DQCP, as it fractionalizes fermionic degrees of freedoms.

The SMG critical point can be viewed as the intersection among four phases:

1. The massless fermion phase ($\langle \eta \rangle \neq 0, \langle \phi \rangle = 0$);
2. The SMG phase ($\langle \eta \rangle = 0, \langle \phi \rangle \neq 0$);
3. The spontaneous symmetry breaking (SSB) massive phase ($\langle \eta \rangle \neq 0, \langle \phi \rangle \neq 0$);
4. The fermionic parton QCD phase (if stable) ($\langle \eta \rangle = 0, \langle \phi \rangle = 0$).

The four phases correspond to the four different states of the bosonic parton η and the parton Higgs ϕ fields, as summarized in Figure 5. Figure 7a shows a schematic phase diagram. The DQCP emerges at the origin of the phase diagram, where both η and ϕ are critical. It seems that a direct transition between the massless fermion phase to the SMG phase requires fine tuning through the DQCP. However, if the fermionic parton QCD theory is dynamically unstable, the QCD phase will shrink to a single transition line between the massless fermion and the SMG phases, such that a direct continuous SMG transition can persist without fine tuning.

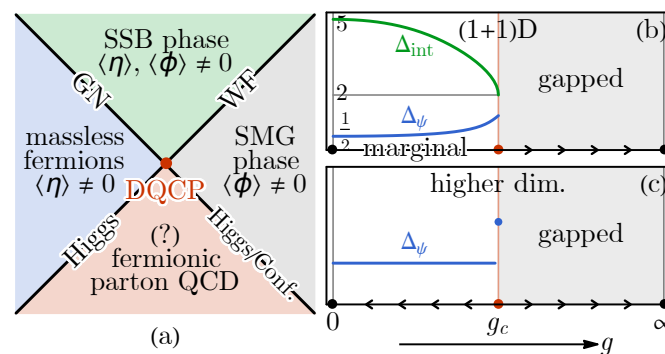


Figure 7. (a) Schematic phase diagram determined by whether η and ϕ are condensed or not. The phase transitions belong to the following universality classes: Gross–Neveu (GN), Wilson–Fisher (WF), Higgs, or confinement (Conf.). The DQCP emerges at the intersection all phases. (b,c) Flow of coupling constant g and the scaling dimensions for physical fermions Δ_ψ and/or the interaction term Δ_{int} in (b) the $(1+1)\text{D}$ 3-4-5-0 model, (c) higher dimensional models.

Strictly speaking, the DQCP picture for the SMG transition is only valid and necessary in $(2+1)\text{D}$ or higher dimensions. The SMG in lower dimensions (i.e., $(0+1)\text{D}$ and $(1+1)\text{D}$) are exceptional and can be understood without involving fermion fractionalization or DQCP.

- In the $(0+1)\text{D}$ Fidkowski–Kitaev model, the four-fermion interaction will immediately open the gap, which can be understood by solving the quantum mechanical problem in Equation (1) exactly. There is no notion of phase transition and quantum criticality in $(0+1)\text{D}$, not to mention DQCP.
- In $(1+1)\text{D}$, take the 3-4-5-0 model for example, the SMG can be understood within the Luttinger liquid framework as a BKT transition. The six-fermion SMG interaction in Equation (11) has a bare scaling dimension $\Delta_{\text{int}}|_{g=0} = 5$ in the free fermion limit, which is perturbatively irrelevant. However, with a non-perturbative (finite) interaction strength, under the RG flow, the Luttinger parameters (as exact marginal parameters) will be modified by the interaction, leading to the decrease in the scaling dimension of the interaction term. When the scaling dimension drops below 2 (which is the spacetime dimension), the SMG interaction will become relevant, driving the system

into the SMG phase [37,113]. So the SMG transition is triggered right at $\Delta_{\text{int}}|_{g=g_c} = 2$; see Figure 7b.

- In higher dimensions ((2 + 1)D and above), interactions are always perturbatively irrelevant at the free fermion fixed point, such that an infinitesimal interaction g will not immediately drive the SMG transition. Therefore, the transition generally requires a finite critical interaction strength g_c . The critical point ($g = g_c$) is expected to be an unstable fixed point under RG, which either flows to the free fermion fixed point ($g = 0$) or the SMG (gapped phase) fixed point ($g = \infty$), as illustrated in Figure 7c.

One implication of fermion fractionalization at the DQCP is that the physical fermion scaling dimension will generally be larger at the SMG transition, compared to the free fermion fixed point [36]. Massless free fermion (Dirac/Weyl/Majorana) in d -dimensional spacetime has the scaling dimension $\Delta_\psi|_{g=0} = (d - 1)/2$. At the SMG transition point ($g = g_c$), the fermion fractionalizes to the critical bosonic parton η and the gapless fermionic parton ψ . In the large- N limit, the scaling dimensions of partons are $\Delta_\eta = d/2 - 1$ and $\Delta_\psi = (d - 1)/2$, such that the leading order estimation of the physical fermion scaling dimension becomes $\Delta_\psi|_{g=g_c} = \Delta_\eta + \Delta_\psi = d - 3/2$. For $d > 2$, the large- N estimation implies

$$\Delta_\psi|_{g=g_c} > \Delta_\psi|_{g=0}, \quad (65)$$

i.e., the fermion scaling dimension will jump to a higher value right at the SMG transition, as shown in Figure 7c. For $d = 2$, the above naive dimension counting seems to indicate $\Delta_\psi|_{g=g_c} = \Delta_\psi|_{g=0}$ (which is not correct, unless $g_c = 0$). More careful Luttinger liquid RG analysis [113] shows that the fermion scaling dimension increases continuously with the interaction until the transition happens, as shown in Figure 7b. Thus, the statement $\Delta_\psi|_{g=g_c} > \Delta_\psi|_{g=0}$ still holds for the (1 + 1)D model.

5.3. Symmetric Mass Generation in the Standard Model

One important motivation to study the SMG is to seek the lattice regularization of chiral gauge theories, in particular the standard model of particle physics. The goal is to gap the fermion doublers (mirror fermions) without affecting the original chiral fermions [38,45,101,103,105,110]. Possible routes to gap the standard model with either $15N_f$ or $16N_f$ Weyl fermions via the SMG are reviewed in the following, where N_f stands for the family (or generation) number and can be taken to be $N_f = 3$.

5.3.1. Symmetry Extension of the $15N_f$ - or $16N_f$ -Weyl-Fermion Standard Model

The (3 + 1)D standard model (SM) has a chiral internal symmetry group of the Lie algebra $\mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1)_Y$, which could correspond to either of the four versions of Lie groups for $q = 1, 2, 3, 6$:

$$G_{\text{SM}_q} \equiv \frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)_Y}{\mathbb{Z}_q}, \quad (66)$$

which are all compatible with all known particle representation data to date. The following discussion is applicable to any of the four versions of SM for $q = 1, 2, 3, 6$. For any specific version, the SM contains $N_f = 3$ families (generations) of matter fermions. In each family, there can be either 15 or 16 left-handed Weyl fermions. The corresponding SM phases will be denoted as $15N_f$ -SM and $16N_f$ -SM, respectively. The first 15 Weyl fermions transform in the following representations of $\mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1)_Y$

$$\begin{aligned} & \bar{d}_R \oplus l_L \oplus q_L \oplus \bar{u}_R \oplus \bar{e}_R \\ & = (\bar{\mathbf{3}}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{2})_{-3} \oplus (\mathbf{3}, \mathbf{2})_1 \oplus (\bar{\mathbf{3}}, \mathbf{1})_{-4} \oplus (\mathbf{1}, \mathbf{1})_6 \end{aligned} \quad (67)$$

in each family. Here u_R and d_R are up and down types of right-handed quarks. The q_L is the $\mathfrak{su}(2)$ doublet of up and down types of left-handed quarks. The e_R is the right-handed electron. The l_L is the $\mathfrak{su}(2)$ doublet of neutrino and electron types of left-handed leptons.

The 16th Weyl fermion corresponds to the sterile neutrino $\bar{\nu}_R = (\mathbf{1}, \mathbf{1})_0$, which can be appended to Equation (67) in any of the families.

For simplicity, all SM fermions are assumed to be massless, without including the electroweak Higgs and its symmetry breaking. To apply SMG to $15N_f$ -SM or $16N_f$ -SM, one shall confirm that the two necessary conditions in Section 2.5 are satisfied. For the second condition in Section 2.5, it is true in the SM. For the first condition, several recent works checked the cobordism group classification of anomalies in the SM [194,196,198,215]. Given the spacetime-internal symmetry $G = \text{Spin} \times G_{\text{SM}_q}$, the anomaly index $\nu \in \text{TP}_5(G)$ belongs to the cobordism group

$$\text{TP}_5(\text{Spin} \times G_{\text{SM}_q}) = \begin{cases} \mathbb{Z}^5 \times \mathbb{Z}_2, & q = 1, 3, \\ \mathbb{Z}^5, & q = 2, 6. \end{cases} \tag{68}$$

As checked in Refs. [196,198,215], The perturbative local anomalies (\mathbb{Z} classes) and nonperturbative global anomalies (\mathbb{Z}_n classes) all vanish ($\nu = 0$) for both the $15N_f$ -SM and $16N_f$ -SM, such that the SMG is possible in either cases. However, if an additional continuous baryon minus lepton symmetry $U(1)_{\mathbf{B-L}}$ is to be preserved, the spacetime-internal symmetry is enlarged to $G = \text{Spin} \times_{\mathbb{Z}_2^f} U(1)_{\mathbf{B-L}} \times G_{\text{SM}_q} = \text{Spin}^c \times G_{\text{SM}_q}$, then the anomaly index $\nu \in \text{TP}_5(G)$ belongs to a different cobordism group

$$\text{TP}_5(\text{Spin}^c \times G_{\text{SM}_q}) = \mathbb{Z}^{11}, \quad q = 1, 2, 3, 6, \tag{69}$$

which only vanishes ($\nu = 0$) for $16N_f$ Weyl fermions [198,215]. Therefore, the SMG preserving an additional $U(1)_{\mathbf{B-L}}$ only works for the $16N_f$ -SM.

Razamat and Tong (Refs. [37,106]) showed that both the $15N_f$ -SM and $16N_f$ -SM can be embedded into a left-right (LR) model with 27 Weyl fermions (denoted as the $27N_f$ -LR) preserving the G_{SM_q} symmetry. The key idea is to bring additional fermions down from high-energy that are vector-like under G_{SM_q} to mix with the low-energy SM chiral fermions in each family:

$(\bar{\mathbf{3}}, \mathbf{1})_2$	$(\mathbf{1}, \mathbf{2})_{-3}$	$(\mathbf{3}, \mathbf{2})_1$	$(\bar{\mathbf{3}}, \mathbf{1})_{-4}$	$(\mathbf{1}, \mathbf{1})_6$	·	
$(\bar{\mathbf{3}}, \mathbf{1})_2$	$(\mathbf{1}, \mathbf{2})_{-3}$					$(\mathbf{1}, \mathbf{1})_0$
$(\mathbf{3}, \mathbf{1})_{-2}$	$(\mathbf{1}, \mathbf{2})_{+3}$					$(\mathbf{1}, \mathbf{1})_0$

(70)

The first row of Equation (70) corresponds to the original 15 left-handed Weyl fermions in Equation (67), and the second and third rows of Equation (70) correspond to 6 left-handed and 6 right-handed additional Weyl fermions in total in a vector-like theory, which add up to 27 Weyl fermions per family. In (70), all right-handed fermions are complex conjugated to be written as their anti-particles, so all fermions are written in the left-handed versions. The sterile neutrino $\bar{\nu}_R$ (the 16th Weyl fermion) corresponds to one of the $(\mathbf{1}, \mathbf{1})_0$ representations in Equation (70).

Hereafter for the left-handed and right-handed notations, we always use the italic fonts L and R to denote spacetime symmetry (Spin group), while we use the text font L and R for internal symmetry.

The $27N_f$ -LR model has an enlarged symmetry $G_{\text{LR}_{q,p}}$ with $q, p \in \{1, 2, 3, 6\}$, in total, 16 versions,

$$\begin{aligned} G_{\text{LR}_{q,p}} &\equiv \frac{G_{\text{SM}_q} \times \text{SU}(2)_R \times \text{U}(1)_R}{\mathbb{Z}_p} \\ &= \frac{\text{SU}(3) \times \text{SU}(2)_L \times \text{SU}(2)_R \times \text{U}(1)_L \times \text{U}(1)_R}{\mathbb{Z}_q \times \mathbb{Z}_p}, \end{aligned} \tag{71}$$

where $\text{SU}(2) \times \text{U}(1)_Y$ in G_{SM_q} is renamed to $\text{SU}(2)_L \times \text{U}(1)_L$ in $G_{\text{LR}_{q,p}}$. The 27 Weyl fermions can be organized by the $\text{SU}(2)_R$ irreducible representations, as indicated by the framed

boxes in Equation (70), or written as the representation of $\mathfrak{su}(3) \times \mathfrak{su}(2)_L \times \mathfrak{su}(2)_R \times \mathfrak{u}(1)_L \times \mathfrak{u}(1)_R$:

$$\begin{aligned}
 &(\bar{d}_R \oplus l_L \oplus q_L \oplus \bar{u}_R \oplus \bar{e}_R \oplus \bar{\nu}_R) \oplus d'_L \oplus \bar{l}'_R = \\
 &((\bar{\mathbf{3}}, \mathbf{1}, \mathbf{2})_{2,-1} \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2})_{-3,3} \oplus (\mathbf{3}, \mathbf{2}, \mathbf{1})_{1,-2} \oplus (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{1})_{-4,2} \\
 &\oplus (\mathbf{1}, \mathbf{1}, \mathbf{1})_{6,-6} \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2})_{0,-3}) \oplus (\mathbf{3}, \mathbf{1}, \mathbf{1})_{-2,4} \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1})_{3,0}.
 \end{aligned}
 \tag{72}$$

This essentially promotes $\bar{d}_R, l_L, \bar{\nu}_R$ to $SU(2)_R$ doublets and introduces new $SU(2)_R$ singlet fermions d'_L, \bar{l}'_R .

The $27N_f$ -LR phase can be viewed as an intermediate quantum phase between the $15N_f$ -SM or $16N_f$ -SM and the SMG phases, as shown in Figure 8. The transition from the $27N_f$ -LR phase to the $16N_f$ -SM phase can be driven by condensing a scalar Higgs field $h_R = (\mathbf{1}, \mathbf{1}, \mathbf{2})_{0,-3}$ that couples to the Weyl fermions by the following Higgs term

$$(\epsilon_R h_R (\bar{d}_R d'_L + \bar{l}'_R l_L) + (h_R^\dagger \bar{\nu}_R) (h_R^\dagger \bar{\nu}_R)) + \text{h.c.},
 \tag{73}$$

where ϵ_R is an anti-symmetric tensor in the $SU(2)_R$ doublet subspace. The ϵ_L and ϵ tensor for the $SU(2)_L$ and Lorentz $\mathfrak{su}(2)$ subspace are omitted. All Lagrangian terms become a scalar in a trivial singlet representation of both spacetime and internal symmetries. For each family, the Higgs condensation $\langle h_R \rangle \neq 0$ (only its upper $SU(2)_R$ doublet component is nonzero and condenses) lifts 11 Weyl fermions with a mass gap (the lower half of both $SU(2)_R$ doublets of \bar{d}_R and l_L obtain mass, while the dimension-5 term $(h_R^\dagger \bar{\nu}_R) (h_R^\dagger \bar{\nu}_R)$ gives Majorana mass to only the upper half of the $SU(2)_R$ doublet $\bar{\nu}_R$), leaves 16 Weyl fermions at low energy, and breaks the symmetry from $G_{LR,q,p}$ down to $G_{SM,q} \times U(1)_{B-L}$.

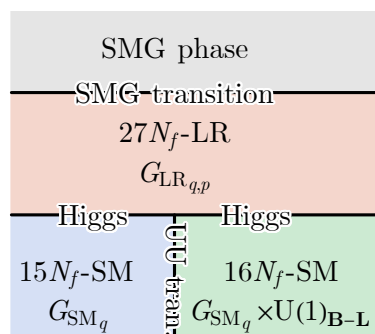


Figure 8. A schematic phase diagram among the SMG, the $27N_f$ -LR, the $15N_f$ -SM or the $16N_f$ -SM phases. The SMG phase is separated from the $27N_f$ -LR phase by the SMG transition. The $15N_f$ -SM and $16N_f$ -SM phases are separated by the ultra unification (UU) transition [198–200], and they can be both obtained from the $27N_f$ -LR model phase by different Higgs transitions.

To further lift the sterile neutrino, an additional scalar Higgs field $h' = (\mathbf{1}, \mathbf{1}, \mathbf{1})_{0,6}$ should be introduced (with $U(1)_{B-L}$ charge -1), such that the Higgs term becomes

$$(\epsilon_R h_R (\bar{d}_R d'_L + \bar{l}'_R l_L) + h' (\epsilon_R \bar{\nu}_R \bar{\nu}_R)) + \text{h.c.}
 \tag{74}$$

In each family, the Higgs condensation $\langle h_R \rangle, \langle h' \rangle \neq 0$ leaves 15 Weyl fermions at low energy (the $h' (\epsilon_R \bar{\nu}_R \bar{\nu}_R)$ gives a Dirac mass to both upper and lower components of the $SU(2)_R$ doublet $\bar{\nu}_R$), breaks the symmetry from $G_{LR,q,p}$ down to $G_{SM,q}$ with no $U(1)_{B-L}$, and drives the transition from the $27N_f$ -LR phase to the $15N_f$ -SM phase.

Embedding the $15N_f$ -SM into the $27N_f$ -LR amounts to the following symmetry extension

$$\frac{SU(2)_R \times U(1)_R}{\mathbb{Z}_{\text{gcd}(p,2)}} \rightarrow \text{Spin} \times G_{LR,q,p} \rightarrow \frac{\text{Spin} \times G_{SM,q}}{\mathbb{Z}_{\text{gcd}(p,3)}},
 \tag{75}$$

and embedding the $16N_f$ -SM into the $27N_f$ -LR amounts to the following symmetry extension

$$SU(2)'_R \rightarrow Spin \times G_{LR,q,p} \rightarrow \frac{Spin^c \times G_{SM_q}}{\mathbb{Z}_{gcd(p,3)}}, \tag{76}$$

where $SU(2)'_R = \frac{SU(2)_R \times U(1)_R}{\mathbb{Z}_{gcd(p,2)} \times U(1)_{B-L}}$.

5.3.2. Symmetric Mass Generation in a $27N_f$ -Weyl-Fermion Left-Right Model

The SMG in the $27N_f$ -LR model can be achieved by both the parton-Higgs mechanism in Section 4.2.1 and the s-confinement mechanism in Section 4.2.2.

1. The s-confinement mechanism: according to Razamat and Tong [106], one first supersymmetrizes the fermions in Equation (70) to their corresponding $\mathcal{N} = 1$ supersymmetric chiral multiplets as $\bar{d}_R \oplus l_L \oplus q_L \oplus \bar{u}_R \oplus \bar{e}_R \oplus \bar{\nu}_R \oplus d'_L \oplus \bar{l}'_R \mapsto D \oplus L \oplus Q \oplus U \oplus E \oplus N \oplus D' \oplus L'$. Then gauge the $SU(2)_R$ symmetry by turning on a dynamical $SU(2)_R$ gauge field that couples to the $su(2)_R$ doublet: D, L, N . A *dangerously irrelevant* superpotential \mathcal{W}_{UV} at UV

$$\mathcal{W}_{UV} = LLE + DDU + LDQ + LNL' + DND' + h.c. \tag{77}$$

becomes a G_{SM_q} -symmetric *relevant* deformation that pairs the 15 mesons ($\tilde{E}, \tilde{U}, \tilde{Q}, \tilde{L}, \tilde{D}$) formed by D, L, N via s-confinement) with the remained 15 superfields (namely Q, U, E, D', L') in a quadratic manner \mathcal{W}_{IR} at IR, consequently gapping out all fields as SMG:

$$\mathcal{W}_{IR} = \tilde{E}E + \tilde{U}U + \tilde{Q}Q + \tilde{L}L' + \tilde{D}D' + h.c. \tag{78}$$

Here, the $su(3)$ color and $su(2)_L$ flavor indices are suppressed, with the understanding that they should be contracted properly to make the Lagrangian a singlet. As \mathcal{W}_{IR} flows strong, all fermions are gapped from low-energy, resulting in the SMG phase. When there are multiple families, independent $SU(2)_R$ gauge fields are introduced in each family, such that the total gauge group is $SU(2)_{R_1} \times SU(2)_{R_2} \times \dots \times SU(2)_{R_{N_f}}$. This guarantees that the s-confinement can induce the fully gapped SMG phase in each family independently.

2. The parton-Higgs mechanism: Tong [37] shows that N_f families of 27 Weyl-fermion model can be fully gapped by preserving *not only* the SM internal symmetry group G_{SM_q} for $q = 1, 2, 3, 6$, *but also* an additional continuous baryon minus lepton symmetry $U(1)_{B-L}$.

The parton-Higgs mechanism introduces the scalar Higgs fields $\phi = (1, 2, 2)_{-3,3}$. Ref. [37] suggests to fully gap (the 27 Weyl fermions per family) to achieve the SMG by adding

$$(\phi(\bar{d}_R q_L + \bar{\nu}_R l'_R + \bar{e}_R l_L) + \phi^{\dagger 2} \bar{u}_R d'_L) + h.c. \tag{79}$$

when the generic condensation $\langle \phi \rangle \neq 0$ occurs. Both $SU(2)_L$ and $SU(2)_R$ doublet components of $\langle \phi \rangle$ can be nonzeros. Here, in Equation (79), the ϵ_L, ϵ_R and ϵ tensors for the $SU(2)_L, SU(2)_R$ and Lorentz $su(2)$ subspaces are omitted, not written explicitly. The internal symmetry breaking pattern of this SMG deformation is

$$\begin{aligned} G_{SM_q} \times \mathbb{Z}_{gcd(p,3)} K &= G_{SM_q} \times \mathbb{Z}_{gcd(p,3)} \frac{SU(2)_R \times U(1)_R}{\mathbb{Z}_{gcd(p,2)}} \\ \mapsto &\frac{SU(3) \times SU(2)_{\text{diagonal}} \times U(1)_{\text{diagonal}}}{\mathbb{Z}_q} \times U(1)_{B-L}, \end{aligned} \tag{80}$$

which leaves a continuous baryon minus lepton number symmetry preserved.

5.4. Deformation Class of Quantum Field Theories

Given the importance of the universal properties of symmetry and its associated 't Hooft anomaly of QFT, Seiberg [216] and others [105] conjectured the following:

- Within the same spacetime-internal symmetry G and the same anomaly $\nu \in \text{TP}_{d+1}(G)$, different QFTs in a d -dimensional spacetime can be deformed to each other by tuning coupling parameters or adding degrees of freedom at short distances that preserve the same symmetry and that maintain the same overall anomaly. (Namely, the whole system allows all symmetric deformations via symmetric interactions between the original QFT (with the anomaly index ν) and any new sectors of symmetric QFTs whose degrees of freedom are brought down from the high energy (anomaly-free in G .)

This organization principle, called the *deformation class of QFTs* [216], enables us to connect different QFTs with the same data $(G, \nu \in \text{TP}_{d+1}(G))$ in the same spacetime via symmetric deformations. The deformation class of QFTs in d -dimensional spacetime is controlled by the *cobordism or deformation class of invertible TQFTs* in $(d + 1)$ -dimensional spacetime [4]. Different QFTs in the same deformation class can be viewed as low-energy effective theories of different quantum phases of the same many-body quantum system, where the symmetric deformation corresponds to tuning the parameters that drive the system among different phases in a phase diagram. As different quantum phases can be separated by quantum phase transitions [217], the same idea applies to different QFTs. For example, it is meaningful to discuss quantum phase transitions between the SM and various grand unified theories (GUTs) [218–221]. The notion of deformation class can be generalized to the context of quantum gravity [222], even in the absence of global symmetries in quantum gravity [223]. In Ref. [222], the cobordism class $\text{TP}_{D-d}(G)$ is obtained by compactifying a quantum gravity theory in a total D -dimensional spacetime down to a d -dimensional QFT, via a $D - d$ -dimensional compact manifold, coupled to gravity. Quantum gravity demands a physical explanation on the trivialization for all of the cobordism classes in $\text{TP}_{D-d}(G)$.

From this deformation class perspective, the SMG in Section 2.5 can be interpreted as follows:

- A symmetric deformation from a gapless free fermion theory (the free limit of a CFT) to a fully gapped trivial theory (i.e., a trivial invertible TQFT with no quantum fields effectively such that the partition function is always $Z = 1$ on any closed manifold), preserving the same symmetry G and the same vanished quantum anomaly $\nu = 0$.

One can further generalize the SMG in Section 2.5 to the following broader definition:

- A symmetric deformation from any gapless theory (including bosonic or fermionic, free or interacting CFT) to a fully gapped trivial theory (i.e., a trivial invertible TQFT), preserving the same symmetry G and the same vanished quantum anomaly $\nu = 0$.

Although the field theories on both sides are quite different, they share the same symmetry group G and quantum anomaly ν (in the case of SMG, the anomaly actually vanishes $\nu = 0$) and hence belong to the same deformation class of QFTs. However, this symmetric deformation might entail non-perturbative effects and strong coupling physics, which is non-trivial. Since stable quantum phases are controlled by attractive RG fixed points, deforming the system from one phase to another always involves introducing what appear to be irrelevant (or sometimes marginally irrelevant) interactions or degrees of freedoms to the low-energy QFT. On the perturbative level, of course, irrelevant deformations can be safely ignored, which is why they are sometimes overlooked in the study of field theories. In contrast, on the non-perturbative level, when the deformation is sufficiently strong, the system might be driven to the phase boundary between two competing QFTs, where the RG flow can render an irrelevant operator dangerously irrelevant. Indeed, within the fermion fractionalization framework, both the parton-Higgs and s-confinement approaches rely on the strong coupling physics somewhere along the deformation path. This is an essential feature of SMG, as advocated in Refs. [36,37,106,151]. Nevertheless,

the challenge is to keep enough control over the strong coupling regime so that one can be sure which operators are dangerously irrelevant and that no symmetry breaking happens along the way.

In a even broader scope beyond the SMG, the concept of the deformation class of QFTs can be further generalized to include also *symmetry-breaking deformations*:

- Within a symmetry G at UV high-energy, and the same anomaly $\nu \in \text{TP}_{d+1}(G)$, different QFTs in a d -dimensional spacetime can be deformed to each other by tuning coupling parameters or adding degrees of freedom at short distances or from high energy that preserve the same symmetry and that maintain the same overall anomaly at some UV energy scale. However, at IR low energy, there could be G -symmetry-breaking down to its subgroup G' ; while there could also be the anomaly matching or anomaly eliminated by G -symmetry-breaking.

Some examples of the deformation class of the standard model (SM) in $d = 4$ are summarized below:

1. Ultra unification transition [198–200,220] and a \mathbb{Z}_{16} deformation class: with $G = \text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_{4,X} \times G_{\text{internal}}$ and a mod 16 anomaly index $\nu \in \text{TP}_{d+1}(G) = \mathbb{Z}_{16} \times \dots$, we can consider the SM with the gauge group as $G_{\text{internal}} = G_{\text{SM}_q}$ or $G_{\text{internal}} = \text{SU}(5)$ for the Georgi–Glashow GUT. The $\mathbb{Z}_{4,X}$ is a discrete version of the baryon minus lepton $\mathbf{B} - \mathbf{L}$ like symmetry. Given the family number N_f , the $16N_f$ -Weyl-fermion SM has the anomaly index $\nu = 0$, while the $15N_f$ -Weyl-fermion SM has the anomaly index $\nu = -N_f \pmod{16}$.
 - (a) If the same symmetry G is preserved to move from the $16N_f$ to $15N_f$ -Weyl-fermion SM, then the $15N_f$ -Weyl-fermion SM requires some new sectors [198–200]: For example, the 4D non-invertible TQFT, 4d CFT, or 5d TQFT (invertible/noninvertible when $\mathbb{Z}_{4,X}$ is treated as a global symmetry/dynamically gauged).
 - (b) If the $\mathbb{Z}_{4,X}$ in G is broken, then the deformation from the $16N_f$ to $15N_f$ models requires no new sectors on the $15N_f$ -Weyl-fermion SM side.
 - (c) Deformation through the SMG phase [37,106]: If $\mathbb{Z}_{4,X}$ in G is broken on the $15N_f$ model side, while the $\mathbb{Z}_{4,X}$ can be either broken, preserved, or enhanced to a $U(1)_X$ on the $16N_f$ model side, the $16N_f$ to $15N_f$ models can be deformed to each other through the SMG phase, shown in Figure 8. The SMG phase in the standard model is discussed in Section 5.3.
2. Deconfined Quantum Criticality (DQC) transition [218–221] and a \mathbb{Z}_2 deformation class: With $G = \text{Spin} \times_{\mathbb{Z}_2} G_{\text{internal}}$ and a mod 2 anomaly index $\nu \in \text{TP}_{d+1}(G) = \mathbb{Z}_2$, one can consider the SM within the gauge group $G_{\text{internal}} = \text{Spin}(10)$ (in general, any $\text{Spin}(n \geq 7)$ works, such as $\text{Spin}(18)$). The $\nu \in \mathbb{Z}_2$ global anomaly in 4d is characterized by a mod 2 class 5d invertible TQFT (as a cup product of Stiefel–Whitney classes).
 - (a) SMG transition: If $\nu = 0$, there can be a SMG phase in the neighborhood to the SM, Georgi–Glashow $su(5)$, the flipped $u(5)$, the Pati–Salam model, and the $so(10)$ GUT phases, etc.
 - (b) DQC transition: If $\nu = 1$, there is a modified $so(10)$ GUT plus an extra Wess–Zumino–Witten-like term such that the nontrivial \mathbb{Z}_2 anomaly can be matched in the G -symmetry preserving phase. The SM, Georgi–Glashow $su(5)$, or the flipped $u(5)$ phases, etc., can be regarded as the anomaly-matching consequences of the symmetry-breaking phases. There can be a gapless DQC region (not a critical point but a stable CFT region) between the deformation from either of the $su(5)$ or $u(5)$ models to the Pati–Salam model. The gapless DQC region can also be replaced by a gapped 4D noninvertible TQFT to match the same \mathbb{Z}_2 anomaly.

In contrast, one can also stay in the same internal or gauge group structure of SM (e.g., see a recent work [224] and references therein), but in that case, the phase transition

there is not a deformation of SM to other neighbor phases of *different* gauge theories, but only within the *same* SM gauge theory.

6. Discussion and Outlook

The symmetric mass generation (SMG) has attracted considerable ongoing interest from both the condensed matter and high energy physics communities in recent years. The SMG transition is beyond the classic Landau–Ginzburg–Wilson paradigm, which potentially involves the fractionalization of physical fermions at and only at the critical point. This article reviewed various aspects related to SMG, including interacting topological insulators (TI), topological superconductors (TSC) [225,226], SPT states [227,228], anomalies, lattice regularization of chiral gauge theories, and the current status of numerical efforts from both the lattice gauge theory and condensed matter communities. Theoretical understandings related to SMG are also discussed, such as the s-confinement mechanism, the connection between SMG and DQCP, etc.

Various numerical works have suggested that the SMG could be a continuous transition, and the critical exponents were measured in some of the numerical works. These works pose a challenge to the further analytical understanding of SMG. Unlike the standard Higgs–Yukawa types of theories, where a large- N or small- ϵ expansion can be applied, theoretically we do not yet have a controlled theory where we can perform a reliable analytical calculation for the critical exponents for SMG, and compare with the numerical simulation. If the SMG is indeed a continuous transition and corresponds to a certain type of conformal field theories, then the conformal bootstrap method [229] can also provide very helpful insights into the nature of the transition. All these require further efforts from different disciplines of theoretical physics.

The problem of the nonperturbative regularization of chiral fermion and chiral gauge theories also potentially bring the high-energy lattice, mathematical physics, quantum information, numerical simulation, and condensed matter communities to work closer together. What else can stimulate different communities to work together other than tackling a profound mysterious problem? Looking back at science history, the phenomena of the anomaly-inflow [5,6] and domain wall fermion [84,86] (as the precursors of the TI/TSC/SPT states) had an old tradition rooted in the high-energy theory and lattice community in the mid 1980s and mid 1990s. Closely related phenomena, such as integer and fractional quantum Hall states, were studied in the condensed matter community already in the early 1980s. Yet the acceleration came much later until the concrete materialization discovery of TI/TSC in 2005 (in both theories and experiments [225,226]), which ignites the serious interest in the classification of the interacting many-body quantum systems of SPT states in the early 2010s [67–69,227,228]. This generates crossing-disincline interests in the classification of anomalies, cobordism classes, and the TQFTs in quantum matter, high-energy string theory, and mathematical physics [4,45–48,54]. Looking back and pondering this history, some curious minds might ask: Why were the discovery of SMG and the classification of TI/TSC/SPT states not made in the high-energy lattice community even earlier, although the lattice gauge theory pioneers, such as Wilson and Kogut already had earlier achievements on the related lattice topics in the 1970s [230]? Perhaps the inputs of the condensed matter material experiments are crucial, or perhaps the crossing-boundary interdisciplinary ideas are much more welcomed in the recent condensed matter developments. As Preskill summarized: combining more ideas together is better than each of the isolated ideas [231]. Hopefully, combining the ideas of (1) domain wall fermions [84], (2) gapping the mirror fermions [38], and (3) the anomaly-free SMG [45,101,103,105,110] already gives the ample insights to fully solve the lattice definition of chiral fermion, chiral gauge theory, and chiral standard model problems.

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