Symmetric Non-Negative Forms and Sums of Squares

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Abstract

We study symmetric non-negative forms and their relationship with symmetric sums of squares. For a fixed number of variables n and degree $2d$, symmetric non-negative forms and symmetric sums of squares form closed, convex cones in the vector space of n-variate symmetric forms of degree $2d$. Using representation theory of the symmetric group we characterize both cones in a uniform way. Further, we investigate the asymptotic behavior when the degree $2d$ is fixed and the number of variables n grows. Here, we show that, in sharp contrast to the general case, the difference between symmetric non-negative forms and sums of squares does not grow arbitrarily large for any fixed degree $2d$. We consider the case of symmetric quartic forms in more detail and give a complete characterization of quartic symmetric sums of squares. Furthermore, we show that in degree 4 the cones of non-negative symmetric forms and symmetric sums of squares approach the same limit, thus these two cones asymptotically become closer as the number of variables grows. We conjecture that this is true in arbitrary degree $2d$.

Keywords Non-Negative polynomials · Sums of squares · Symmetric polynomials · Symmetric inequalities · Symmetric group

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1 Introduction

Throughout the paper let $\mathbb{R}[X_1, \ldots, X_n]$ denote the ring of polynomials in n real variables and $H_{n,k}$ the set of homogeneous polynomials (forms) of degree k in $\mathbb{R}[X_1, \ldots, X_n]$. Certifying that a form $f \in H_{n,2d}$ assumes only non-negative values is one of the fundamental questions of real algebra. One such possible certificate is a decomposition of f as a sum of squares, i.e., one finds forms $p_1, \ldots, p_m \in H_{n,d}$ such that $f = p_1^2 + \cdots + p_m^2$. In 1888 Hilbert [16] gave a beautiful proof showing that in general not all non-negative forms can be written as a sum of squares. In fact, he showed that the sum of squares property only characterizes non-negativity in the cases of binary forms, of quadratic forms, and of ternary quartics. In all other cases there exist forms that are non-negative but do not allow a decomposition as a sum of squares. Despite its elegance, Hilbert's proof was not constructive. A constructive approach to Hilbert's proof appeared in an article by Terpstra [37] in 1939, but the first explicit example was found by Motzkin in 1965 [22] and an explicit example based on Hilbert's method was constructed by Robinson in 1969 [29]. We refer the interested reader to [24,33] for more background on this topic.

The sum of squares decomposition of non-negative polynomials has been the cornerstone of recent developments in polynomial optimization. Following ideas of Lasserre and Parrilo, polynomial optimization problems, i.e., the task of finding $f^* = \min f(x)$ for a polynomial f, can be relaxed and transferred into semidefinite optimization problems. If $f - f^*$ can be written as a sum of squares, these semidefinite relaxations are in fact exact. Hence a better understanding of the difference of sums of squares and non-negative polynomials is highly desirable.

We study the case of forms in n variables of degree $2d$ that are symmetric, i.e., invariant under the action of the symmetric group $\mathcal{S}_n$ that permutes the variables. Let $\mathbb{R}[X_1, \ldots, X_n]^S$ denote the ring of symmetric polynomials and $H_{n,2d}^S$ denote the real vector space of symmetric forms of degree $2d$ in n variables. Let $\Sigma_{n,2d}^S$ be the cone of forms in $H_{n,2d}^S$ that can be decomposed as sums of squares and $\mathcal{P}_{n,2d}^S$ be the cone of non-negative symmetric forms. Choi and Lam [7] showed that the following symmetric form of degree 4 in 4 variables is non-negative but cannot be written as a sum of squares:

$$\sum X_i^2 X_j^2 + \sum X_i^2 X_j X_k - 4 X_1 X_2 X_3 X_4.$$

Thus one can conclude that $\Sigma_{4,4}^S \neq \mathcal{P}_{4,4}^S$ and therefore even in the case of symmetric polynomials the sum of squares property already fails to characterize non-negativity in the first case covered by Hilbert's classical result. These results have been recently extended by Goel et al. [13] into a full characterization of equality cases between $\Sigma_{n,2d}^S$ and $\mathcal{P}_{n,2d}^S$. Unfortunately, there are no other interesting cases of equality beyond those covered by Hilbert's Theorem.

The case of even symmetric forms has also received some attention. Choi et al. [8] fully described the cones of even symmetric sextics in any number of variables, and showed that under some normalization these cones have the same limit as the number of variables grows. Harris [15] showed that even symmetric ternary octics are non-

negative, only if they are sums of squares, providing a new interesting case of equality between non-negative polynomials and sums of squares. Goel et al. [14] showed that there are no other interesting cases of equality beyond Harris' and Hilbert's results for even symmetric forms.

In addition to the qualitative statement of Hilbert's characterization, a quantitative understanding of the gap between sums of squares and non-negative forms has been studied by several authors. In particular, in [3] the first author added to the work of Hilbert by showing that the gap between sum of squares and non-negative forms of fixed degree grows infinitely large with the number of variables if the degree is at least 4. This result has been recently refined by Ergur to the multihomogeneous case [10]. In this article we study the relationship between symmetric sums of squares and symmetric non-negative forms. In particular, we are interested in the asymptotic behavior of the cones, which we can realize for example as symmetric mean inequalities naturally associated to a symmetric polynomial. The study of such symmetric inequalities has a long history (see for example [9]) and it is an interesting question to ask when one can use sum of squares certificates to verify such an inequality. For instance, Hurwitz [17] showed that a sum of squares decomposition can be used to verify the arithmetic mean–geometric mean inequality. Recently, Frenkel and Horváth [11] studied the connection of Minkowski's inequality to sums of squares. Our results imply that a positive fraction of such inequalities come from sums of squares symmetric polynomials. Furthermore, in degree 4 we show that a family of symmetric power mean inequalities is valid for all n if and only if each member can be written as a sum of squares. We conjecture that this holds for all degrees.

2 Overview and Main Results

2.1 Symmetric Sums of Squares

Symmetric polynomials are classical objects in algebra. In order to represent symmetric polynomials, we will make use of the power sum polynomials.

Definition 2.1 For $i \in \mathbb{N}$ define

$$P_i^{(n)} := X_1^i + \cdots + X_n^i$$

to be the ith power sum polynomial. We will also work with the power means:

$$p_i^{(n)} := \frac{1}{n} P_i^{(n)}.$$

It is known (for example [20, 2.11]) that $\mathbb{R}[X_1, \ldots, X_n]^S$ is freely generated by the algebraically independent polynomials $P_1^{(n)}, \ldots, P_n^{(n)}$. Hence it follows that every symmetric polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]^S$ of degree $2d \leq n$ can uniquely be written as

$$f = g\big(P_1^{(n)}, \ldots, P_{2d}^{(n)}\big)$$

for some polynomial $g \in \mathbb{R}[z_1, \ldots, z_{2d}]$, with $\deg_w g = \deg f$, where $\deg_w$ denotes the weighted degree corresponding to the weight $(1, \ldots, 2d)$. Recall that for a natural number k a partition λ of k (written $\lambda \vdash k$) is a sequence of weakly decreasing positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ with $\sum_{i=1}^{l} \lambda_i = k$. For $n \geq k$ and to a partition $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash k$ we associate polynomials

$$P_\lambda^{(n)} := P_{\lambda_1}^{(n)} \cdot P_{\lambda_2}^{(n)} \cdots P_{\lambda_l}^{(n)} \quad \text{and} \quad p_\lambda^{(n)} := p_{\lambda_1}^{(n)} \cdot p_{\lambda_2}^{(n)} \cdots p_{\lambda_l}^{(n)}.$$

It now follows that for every $n \geq k$ the families of polynomials $\{P_\lambda : \lambda \vdash k\}$ as well as $\{p_\lambda^{(n)} : \lambda \vdash k\}$ form a basis of $H_{n,k}^S$. In particular, if $n \geq k$ then the dimension of $H_{n,k}^S$ is equal to $\pi(k)$, the number of partitions of k. Thus the dimension of $H_{n,k}^S$ is constant for fixed k and all sufficiently large n.

Using representation theory of the symmetric group, and in particular so-called higher Specht polynomials, we are able to give a uniform representation of the cone of symmetric sums of squares of fixed degree $2d$ in terms of matrix polynomials, with coefficients that are rational functions in n (see Theorem 4.15) and similarly a uniform representation of the sequence of dual cones in terms of linear matrix polynomials whose coefficients are "symmetrizations" of sums of squares in $2d$ variables. This gives us in particular a better understanding of the faces of $\Sigma_{n,2d}^S$ that are not faces of $\mathcal{P}_{n,2d}^S$. We make these findings more concrete in the case of quartic symmetric forms, where we completely characterize the cone $\Sigma_{n,4}$ and its boundary. This in particular allows us to easily compute a family of symmetric sums of square polynomials that are on the boundary of $\Sigma_{n,4}^S$ without having a real zero, thus certifying the difference between symmetric sums of squares and symmetric non-negative forms (see Theorem 5.5).

2.2 Asymptotic Behavior of Sums of Squares and Non-Negative Forms

Our characterization allows us to study the asymptotic relationship between symmetric sums of squares and symmetric non-negative forms of fixed degree in a growing number of variables. Even though vector spaces $H_{n,2d}^S$ have the same dimension $\pi(2d)$ for all $n \geq 2d$, there is no canonical way to identify vector spaces $H_{n,2d}^S$ for different n. In fact there are several natural ways to define transition maps identifying vector spaces of symmetric forms in different numbers of variables (see for example [2]), and different transition maps will lead to different limits as n goes to infinity. The system of vector spaces $H_{n,2d}^S$ together with transition maps will define a *directed system* of vector spaces, and we can define the *direct limit* $H_{\infty,2d}^S$ of vector space $H_{n,2d}^S$ [30, Sect. 7.6]. One way of defining these transitions is by symmetrization:

Definition 2.2 For $f \in \mathbb{R}[X]$ we define the symmetrization of f as

$$\text{sym}_n f := \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \sigma(f).$$

The composition of the natural inclusion $i_{n,n+1} \colon H_{n,2d} \to H_{n+1,2d}$ with sym_{n+1} defines injective maps $\varphi_{n,n+1} \colon H^S_{n,2d} \to H^S_{n+1,2d}$. Therefore, we have the following.

Proposition 2.3 *For $n, m \in \mathbb{N}$ with $n > m$ consider the maps $\varphi_{m,n} \colon H^S_{m,2d} \to H^S_{n,2d}$ defined by*

$$\varphi_{m,n}(p) = \mathrm{sym}_n \, p.$$

Then the system of vector spaces $H^S_{n,2d}$ together with the maps $\varphi_{m,n}$ defines a directed system and for $m \geq 2d$ the maps $\varphi_{m,n}$ are isomorphisms.

We consider the direct limit $H^\varphi_{\infty,k}$ of the directed system above. Since the maps $\varphi_{m,n}$ are isomorphisms for $m \geq 2d$, it follows that $H^\varphi_{\infty,k}$ is also a real vector space of dimension $\pi(2d)$. Therefore we have natural isomorphisms $\varphi_n \colon H^\varphi_{\infty,2d} \to H^S_{n,2d}$ for $n \geq 2d$, which allow us to view the cones $\Sigma^S_{n,2d}$ and $\mathcal{P}^S_{n,2d}$ as subsets of $H^\varphi_{\infty,2d}$. Note that we have $\varphi_{m,n}(\Sigma^S_{m,2d}) \subseteq \Sigma^S_{n,2d}$ and $\varphi_{m,n}(\mathcal{P}^S_{m,2d}) \subseteq \mathcal{P}^S_{n,2d}$. It follows that with transition maps $\varphi_{m,n}$ the cones of sums of squares and the cones of non-negative polynomials form nested increasing sequences in $H^\varphi_{\infty,2d}$. We define the following cones of non-negative elements and sums of squares in $H^\varphi_{\infty,k}$:

$$\mathcal{P}^\varphi_{\infty,2d} := \left\{ \mathrm{f} \in H^\varphi_{\infty,2d} \; : \; \varphi_n(\mathrm{f}) \in \mathcal{P}^S_{n,2d} \text{ for all } n \geq 2d \right\},$$
$$\Sigma^\varphi_{\infty,2d} := \left\{ \mathrm{f} \in H^\varphi_{\infty,2d} \; : \; \varphi_n(\mathrm{f}) \in \Sigma^S_{n,2d} \text{ for all } n \geq 2d \right\}.$$

The following theorem is immediate from the above discussion.

Theorem 2.4 *The cones $\mathcal{P}^\varphi_{\infty,2d}$ and $\Sigma^\varphi_{\infty,2d}$ are full-dimensional convex cones in $H^\varphi_{\infty,2d} \simeq \mathbb{R}^{\pi(2d)}$.*

Forms in fixed degree make up a vanishingly small portion of non-negative forms as the number of variables grows [3]. More precisely, (non-symmetric) non-negative forms and sums of squares in n variables of degree $2d$ with average 1 on the unit sphere form compact convex sets $\bar{P}_{n,2d}$ and $\bar{\Sigma}_{n,2d}$ of dimension $D = \binom{n+d-1}{d} - 1$. It was shown in [3] that the ratio of volumes

$$\left(\frac{\mathrm{vol}\, \bar{\Sigma}_{n,2d}}{\mathrm{vol}\, \bar{P}_{n,2d}} \right)^{1/D}$$

converges to 0 for all $2d \geq 4$ as n goes to infinity. The ratio of volumes is raised to the power $1/D$ to take into account the effects of large dimension on volumes as the volume of $(1+\varepsilon)\Sigma_{n,2d}$ is equal to $(1+\varepsilon)^D \mathrm{vol}\, \Sigma_{n,2d}$.

By contrast, the cones of symmetric non-negative forms and sums of squares of fixed degree live in the vector space $H^S_{n,2d}$ which has fixed dimension $\pi(2d)$ for a sufficiently large number of variables n. Therefore, to prove that asymptotically symmetric sums of squares make up a non-trivial portion of symmetric non-negative forms (with respect to some transition maps) it suffices to show that both limits are full-dimensional in $H^\varphi_{\infty,2d} \simeq \mathbb{R}^{\pi(2d)}$, which is done in Theorem 2.4.

Besides the direct limit we also study *symmetric power mean inequalities*. We can express a symmetric form f in $H_{n,2d}^S$ in the power mean basis $p_\lambda^{(n)}$ with $\lambda \vdash 2d$:

$$f = \sum_{\lambda \vdash 2d} c_\lambda p_\lambda^{(n)}.$$

Using the power mean basis we can define transition maps $\rho_{m,n}$ by identifying

$$\sum_{\lambda \vdash 2d} c_\lambda p_\lambda^{(m)} \quad \text{with} \quad \sum_{\lambda \vdash 2d} c_\lambda p_\lambda^{(n)}.$$

As before the system of vector spaces $H_{n,2d}^S$ together with the maps $\rho_{m,n}$ defines a directed system, and for $m \geq 2d$ the maps $\rho_{m,n}$ are isomorphisms. We consider the direct limit $H_{\infty,k}^\rho$. Since the maps $\rho_{m,n}$ are isomorphisms for $m \geq 2d$, it follows that $H_{\infty,k}^\rho$ is again a real vector space of dimension $\pi(2d)$. The natural isomorphisms $\rho_n : H_{\infty,2d}^\rho \to H_{n,2d}^S$ for $n \geq 2d$, allow us to view the cones $\Sigma_{n,2d}^S$ and $\mathcal{P}_{n,2d}^S$ as subsets of $H_{\infty,2d}^\rho$. We will denote these images by $\Sigma_{n,2d}^\rho$ and $\mathcal{P}_{n,2d}^\rho$ and consider the limit cones:

Definition 2.5

$$\mathfrak{P}_{2d} := \left\{ f \in H_{\infty,2d}^\rho \ : \ \rho_n(f) \in \mathcal{P}_{n,2d}^S \text{ for all } n \geq 2d \right\} \quad \text{and}$$
$$\mathfrak{S}_{2d} := \left\{ f \in H_{\infty,2d}^\rho \ : \ \rho_n(f) \in \Sigma_{n,2d}^S \text{ for all } n \geq 2d \right\}.$$

The sequences $\mathcal{P}_{n,2d}^\rho$ and $\Sigma_{n,2d}^\rho$ are not nested in general. Let $x = (X_1, \ldots, X_n)$ be a point in $\mathbb{R}^n$ and let $\tilde{x}$ be the point in $\mathbb{R}^{k \cdot n}$ with each X_i repeated k times. Then

$$p_i^{(k \cdot n)}(\tilde{x}) = \frac{1}{k \cdot n}(kX_1^i + \cdots + kX_n^i) = p_i^{(n)}(x).$$

It follows that $f^{(k \cdot n)} \in \mathcal{P}_{k \cdot n,d}^P$ implies $f^{(n)} \in \mathcal{P}_{n,d}^P$ and hence we get the following.

Proposition 2.6 *Consider the cones $\mathcal{P}_{n,2d}^P$ as convex subsets of $\mathbb{R}^{\pi(d)}$ using the coefficients c_λ of p_λ. Then for every $n \geq 2d$ and $k \in \mathbb{N}$ we have*

$$\mathcal{P}_{k \cdot n,2d}^\rho \subseteq \mathcal{P}_{n,2d}^\rho \subset H_{\infty,2d}^\rho \simeq \mathbb{R}^{\pi(2d)}.$$

Remark 2.7 We note that the same proof also yields that $\Sigma_{k \cdot n,2d}^\rho \subseteq \Sigma_{n,2d}^\rho$.

It is not directly clear from Proposition 2.6 that the sequences $\mathcal{P}_{n,2d}^\rho$ and $\Sigma_{n,2d}^\rho$ have limits, which we show separately:

Theorem 2.8 (a) *The cones $\mathfrak{S}_{2d}$ and $\mathfrak{P}_{2d}$ are full-dimensional cones.*

$$\mathfrak{P}_{2d} = \lim_{n \to \infty} \mathcal{P}_{n,2d}^\rho \quad \text{and} \quad \mathfrak{S}_{2d} = \lim_{n \to \infty} \Sigma_{n,2d}^\rho. \tag{b}$$

Although the cone of symmetric non-negative quartics is strictly bigger than the cone of symmetric quartic sums of squares for any number of variables $n \geq 4$, we show that in the limit the two cones coincide:

Theorem 2.9 $\mathfrak{P}_4 = \mathfrak{S}_4$.

In particular, this result applies in the situation of power mean inequalities studied in [23], and hence it is possible to verify any such inequality using sums of squares. We conjecture that this happens in arbitrary degree $2d$, i.e., we suggest the following.

Conjecture 1 $\mathfrak{P}_{2d} = \mathfrak{S}_{2d}$ for all $d \in \mathbb{N}$.

2.3 Structure of the Article and Guide for the Reader

This article is structured as follows: We provide a characterization of symmetric non-negative forms and the limit cone in Sect. 3. Section 4 provides a detailed study of symmetric sums of squares. To this end we present the general framework of how to use representation theory to study invariant sums of squares in Sect. 4.1. In Sect. 4.2 we outline the basic notions of the representation theory of the symmetric group. These results are then used in Sect. 4.3 to represent the cone of symmetric sums of squares (without restrictions on the degree) in terms of matrix polynomials in Theorems 4.11 and 4.12. The subsequent Sect. 4.4 then discusses how restricting degree allows for a uniform description of the cones $\Sigma_{n,2d}$ in terms of the power mean bases $p_\lambda^{(n)}$ (Theorem 4.15). The final subsection of Sect. 4 discusses some results on the dual cone which are needed in the sequel. The subsequent Sect. 5 makes these results more concrete as we give a description of the cone of symmetric quartic sums of squares (Theorem 5.1). Furthermore, we describe the elements of the boundary of $\Sigma_{n,4}$ that are strictly positive in Theorem 5.3 and give an explicit example of such a polynomial for every $n \geq 4$ in Example 5.4. From this example it follows in particular that besides the cases where Hilbert showed the equality of sums of squares and non-negative forms there always exist symmetric positive definite forms which are not sums of squares (see Theorem 5.5). In Sect. 6 we explore the two notions of limits and prove Theorem 2.8. We also discuss the connection with the power mean inequalities. These power mean inequalities are then again studied in more detail in the final Sect. 7, where we show in particular that all valid power mean inequalities of degree 4 are sums of squares (Theorem 2.9).

The order of sections was chosen to present the more general statements in Sects. 3, 4, and 6 and then apply them in the quartic case in Sects. 5 and 7. Depending on reader's preferences, one can also begin by reading Sect. 5 first before actually diving into Sect. 4, and similarly Sect. 7 before Sect. 6, while taking the necessary results from previous sections for granted.

3 Symmetric PSD Forms

We begin by characterizing the cone $\mathfrak{P}_{2d}$. One key result needed to describe the non-negative symmetric forms is the so-called half degree principle (see [26,27,38]): For

a natural number $k \in \mathbb{N}$ we define A_k to be the set of all points in $\mathbb{R}^n$ with at most k distinct components, i.e.,

$$A_k := \{x \in \mathbb{R}^n : |\{X_1, \ldots, X_n\}| \leq k\}.$$

The half degree principle says that a symmetric form of degree $2d > 2$ is non-negative, if and only if it is non-negative on A_d:

Proposition 3.1 (Half degree principle) *Let $f \in H^S_{n,2d}$ and set $k := \max\{2, d\}$. Then f is non-negative if and only if*

$$f(y) \geq 0 \quad \text{for all } y \in A_k.$$

Remark 3.2 By considering $f - \epsilon(X_1^2 + \cdots + X_n^2)^d$ for a sufficiently small $\epsilon > 0$ we see that we can also replace *non-negative* by *positive* in the above theorem, thus characterizing strict positivity of symmetric forms.

A non-increasing sequence of k natural numbers $\vartheta := (\vartheta_1, \ldots, \vartheta_k)$ such that $\vartheta_1 + \cdots + \vartheta_k = n$ is called a k-partition of n (written $\vartheta \vdash_k n$). Given a symmetric form $f \in H^S_{n,2d}$ and ϑ a k-partition of n we define $f^\vartheta \in \mathbb{R}[t_1, \ldots, t_k]$ via

$$f^\vartheta(t_1, \ldots, t_k) := f(\underbrace{t_1, \ldots, t_1}_{\vartheta_1}, \underbrace{t_2, \ldots, t_2}_{\vartheta_2}, \ldots, \underbrace{t_k, \ldots, t_k}_{\vartheta_k}).$$

From now on assume that $2d > 2$. Then the half-degree principle implies that non-negativity of $f = \sum_{\lambda \vdash 2d} c_\lambda p_\lambda$ is equivalent to non-negativity of

$$f^\vartheta := \sum_{\lambda \vdash 2d} c_\lambda p_\lambda^\vartheta(t_1, \ldots t_k)$$

for all $\vartheta \vdash_d n$, since the polynomials f^ϑ give the values of f at all points with at most d parts. We note that for all $i \in \mathbb{N}$ we have

$$p_i^\vartheta = \frac{1}{n}(\vartheta_1 t_1^i + \vartheta_2 t_2^i + \cdots + \vartheta_d t_d^i).$$

For a partition $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash 2d$ we define a $2d$-variate form Φ_λ in the variables $s_1, \ldots, s_d$ and $t_1, \ldots, t_d$ by

$$\Phi_\lambda(s_1, \ldots, s_d, t_1, \ldots, t_d) = \prod_{i=1}^l (s_1 t_1^{\lambda_i} + s_2 t_2^{\lambda_i} + \cdots + s_d t_d^{\lambda_i})$$

and use it to associate to any form $f \in H^S_{n,2d}$, $f = \sum_{\lambda \vdash 2d} c_\lambda p_\lambda$, the form

$$\Phi_f := \sum_{\lambda \vdash 2d} c_\lambda \Phi_\lambda.$$

Note that

$$\Phi_\lambda \left(\frac{\vartheta_1}{n}, \ldots, \frac{\vartheta_d}{n}, t_1, \ldots t_d \right) = p_\lambda^\vartheta (t_1, \ldots, t_d).$$

We define the set

$$W_n = \left\{ w = (w_1, \ldots w_d) \in \mathbb{R}^d : n \cdot w_i \in \mathbb{N} \cup \{0\} \text{ and } w_1 + \cdots + w_d = 1 \right\}.$$

It follows from the arguments above that $f \in H_{n,2d}^S$ is non-negative if and only if the forms $\Phi_f(s, t)$ are non-negative forms in t for all $w \in W_n$. This is summarized in the following corollary.

Corollary 3.3 *Let $f = \sum_{\lambda \vdash 2d} c_\lambda p_\lambda$ be a form in $H_{n,2d}^S$. Then f is non-negative (positive) if and only if for all $w \in W_n$ the d-variate forms $\Phi_f(w, t)$ are non-negative (positive).*

This result enables us to characterize the elements of $\mathfrak{P}_{2d}$. We expand the sets W_n to the standard simplex Δ in $\mathbb{R}^d$:

$$\Delta := \left\{ \alpha = (\alpha_1, \ldots, \alpha_d) \in [0, 1]^d : \alpha_1 + \cdots + \alpha_d = 1 \right\}.$$

Then we have the following theorem characterizing $\mathfrak{P}_{2d}$.

Theorem 3.4 *Let $\mathfrak{f} \in H_{\infty,2d}^\rho$ be the sequence defined by $f^{(n)} = \sum_{\lambda \vdash 2d} c_\lambda p_\lambda^{(n)}$. Then $\mathfrak{f} \in \mathfrak{P}_{2d}$ if and only if the $2d$-variate polynomial $\Phi_f(s, t)$ is non-negative on $\Delta \times \mathbb{R}^d$.*

Proof Suppose that $\Phi_f(s, t)$ is non-negative on $\Delta \times \mathbb{R}^d$. Let $f^{(n)} = \sum c_\lambda p_\lambda^{(n)}$. Since $W_n \subset \Delta$ for all n, we see from Corollary 3.3 that $f^{(n)}$ is a non-negative form for all n and thus $\mathfrak{f} \in \mathfrak{P}_{2d}$.

On the other hand, suppose there exists $\alpha_0 \in \Delta$ such that $\Phi_f(\alpha_0, t) < 0$ for some $t \in \mathbb{R}^d$. Then we can find a rational point $\alpha \in \Delta$ with all positive coordinates and sufficiently close to α_0 so that $\Phi_f(\alpha, t) < 0$. Let h be the least common multiple of the denominators of α. Then we have $\alpha \in W_{ah}$ for all $a \in \mathbb{N}$. Choose a such that $ah \geq 2d$. Then $f^{(ah)}$ is negative at the corresponding point and we have $\mathfrak{f} \notin \mathfrak{P}_{2d}$. $\square$

4 Symmetric Sums of Squares

We now consider symmetric sums of squares. It was already observed in [12] that invariance under a group action allows us to demand sum of squares decompositions which put strong restrictions on the underlying squares. First, we explain the general approach, which uses representation theory and can be used for other groups as well. Our presentation follows the ideas of [12] which we present in a slightly different way. The interested reader is advised to consult there for more details.

4.1 Invariant Sums of Squares

Let G be a finite group acting linearly on $\mathbb{R}^n$. As G acts linearly on $\mathbb{R}^n$ also the $\mathbb{R}$-vector space $\mathbb{R}[X]$ can be viewed as a G-module and by Maschke's theorem (the reader may consult for example [34] for basics in linear representation theory) there exists a decomposition of the form

$$\mathbb{R}[X] = V^{(1)} \oplus V^{(2)} \oplus \cdots \oplus V^{(h)} \tag{4.1}$$

with $V^{(j)} = W_1^{(j)} \oplus \cdots \oplus W_{\eta_j}^{(j)}$ and $\nu_j := \dim W_i^{(j)}$. Here, the $W_i^{(j)}$ are the *irreducible components* and the $V^{(j)}$ are the *isotypic components*, i.e., the direct sums of isomorphic irreducible components. The component with respect to the trivial irreducible representation is the invariant ring $\mathbb{R}[X]^G$. The elements of the other isotypic components are called *semi-invariants*. It is classically known that each isotypic component is a finitely generated $\mathbb{R}[X]^G$-module (see [36, Theorem 1.3]). To any element $f \in H_{n,d}$ we can associate a *symmetrization* by which we mean its image under the following linear map:

Definition 4.1 For a finite group G the linear map $\mathcal{R}_G \colon H_{n,d} \to H_{n,d}^G$ defined by

$$\mathcal{R}_G(f) := \frac{1}{|G|} \sum_{\sigma \in G} \sigma(f)$$

is called the Reynolds operator of G. In the case of $G = \mathcal{S}_n$ we say that $\mathcal{R}_{\mathcal{S}_n}(f)$ is a symmetrization of f and we write sym f in this case.

For a set of polynomials $f_1, \ldots, f_l$ we will write $\sum \mathbb{R}\{f_1, \ldots, f_l\}^2$ to refer to the sums of squares of elements in the linear span of the polynomials $f_1, \ldots, f_l$. It has already been observed by Gaterman and Parrilo [12] that invariant sums of squares can be written as sums of squares of semi-invariants using Schur's Lemma. However, a closer inspection of the situation allows in many cases—as for example in the case of $\mathcal{S}_n$—a finer analysis of the decomposition into sums of squares. Consider a set of forms $\{f_{1,1}, \ldots, f_{1,\eta_1}, f_{2,1}, \ldots, f_{h,\eta_h}\}$ such that for fixed j the forms $f_{j,i}$ generate irreducible components of $V^{(j)}$. Further assume that they are chosen in such a way, that for each j and each pair (l, k) there exists a G-isomorphism $\rho_{l,k}^{(j)} \colon V^j \to V^j$ which maps $f_{j,l}$ to $f_{j,k}$. Now for every j we consider the set $\{f_{j,1}, \ldots, f_{j,\eta_j}\}$ which contains only one polynomial per irreducible module. However, since every irreducible module is generated by the G-orbit of only one element, every such set uniquely describes the chosen decomposition. We call such a set a *symmetry basis* and show that invariant sums of squares are in fact symmetrizations of sums of squares of a symmetry basis. The following theorem, which we state in a slightly more general setup, highlights the use of a symmetry basis.

Theorem 4.2 *Let G be a finite group and assume that all real irreducible representations $V \subset H_{n,d}$ are also irreducible over their complexification. Let p be a form of degree $2d$ that is invariant with respect to G. If p is a sum of squares, then p can be*

written in the form

$$p = \sum_{j}^{h} q_j, \quad \text{where each} \quad q_j \in \sum \mathbb{R}\{f_{j,1}, \dots f_{j,\eta_j}\}^2.$$

The main tool for the proof is Schur's Lemma, and we remark that a dual version of this theorem can be found in [28, Thm. 3.4] and [25].

Proof Let $p \in H_{n,2d}$ be a G-invariant sum of squares. Then there exists a symmetric positive semidefinite bilinear form

$$B \colon H_{n,d} \times H_{n,d} \to \mathbb{R}$$

which is a Gram matrix for p, i.e., for every $x \in \mathbb{R}^n$ we can write $p(x) = B(X^d, X^d)$, where X^d stands for the d-th power of x in the symmetric algebra of $\mathbb{R}^n$. Since p is G-invariant, we have $p = \mathcal{R}_G(p)$ and by linearity we may assume that B is a G-invariant bilinear form. Now decompose $H_{n,2d}$ as in (4.1) and consider the restriction of B to

$$B^{ij} \colon V^{(i)} \times V^{(j)} \to \mathbb{R} \quad \text{with} \quad i \neq j.$$

For every $v \in V^{(i)}$ the quadratic form B^{ij} defines a linear map $\phi_v \colon V^{(j)} \to \mathbb{R}$ via $\phi_v(w) := B^{ij}(v, w)$ and so the form B^{ij} naturally can be seen as an element of $\operatorname{Hom}^G(V^{(i)*}, V^{(j)})$. Since real representations are self-dual we have that $V^{(i)*}$ and $V^{(j)}$ are not isomorphic and thus by Schur's Lemma we find that $B^{ij}(v, w) = 0$ for all $v \in V^{(i)}$ and $w \in V^{(j)}$. So the isotypic components are orthogonal with respect to B and hence it suffices to look at

$$B^{jj} \colon V^{(j)} \times V^{(j)} \to \mathbb{R}$$

individually. We have $V^{(j)} := \bigoplus_{k=1}^{l} W_k^{(j)}$, where each $W_k^{(j)}$ is generated by a semi-invariant $f_{j,k}$, i.e., there is a basis $f_{j,k,1}, \dots, f_{j,k,\nu_j}$ for every $W_k^{(j)}$ such that the basis elements $f_{j,k,i}$ are taken from the orbit of $f_{j,k}$ under G. To again use Schur's Lemma we identify B_j with its complexification $B_j^{\mathbb{C}}$, which is possible since we assumed that all representations are irreducible also over $\mathbb{C}$. Consider a pair $W_{k_1}^{(j)}, W_{k_2}^{(j)}$, where we allow $k_1 = k_2$. To apply Schur's Lemma we relate the quadratic from B^{jj} to a linear map $\psi_{k_1,k_2}^{(j)} \colon W_{k_1}^{(j)} \to W_{k_2}^{(j)}$ defined on the generating set $f_{j,k_1,1}, \dots, f_{j,k_1,\nu_j}$ by

$$\psi_{k_1,k_2}^{(j)}(f_{j,k_1,u}) := \sum_{v} B^{jj}(f_{j,k_1,u}, f_{j,k_2,v}) f_{j,k_2,v}.$$

Since we assumed that $W_k^{(j)}$ are absolutely irreducible we have by Schur's Lemma

$$\dim \operatorname{Hom}^G\big(W_{k_1}^{(j)}, W_{k_2}^{(j)}\big) = 1$$

and we can conclude that this map is unique up to scalar multiplication. Therefore it can be represented in the form $\psi_{k_1,k_2}^{(j)} = c_{k_1,k_2}\rho_{k_1,k_2}$, where ρ_{k_1,k_2} is the G-isomorphism with $\rho_{k_1,k_2}(f_{j,k_1}) = f_{j,k_2}$ as above. It therefore follows that

$$B^{jj}(f_{j,k_1,u}, f_{j,k_2,v}) = \delta_{u,v}c_{k_1,k_2},$$

where $\delta_{u,v}$ denotes the Kronecker delta. By considering the matrix of B with respect to the basis $f_{j,k,l}$ of $H_{n,d}$ we see that p has the desired decomposition. □

Remark 4.3 The above statement also holds true in the situation where one looks at sums of squares of elements of an arbitrary G-closed submodule $T \subset \mathbb{R}[X]$.

In some situations it is convenient to formulate the above Theorem 4.2 in terms of matrix polynomials, i.e., matrices with polynomial entries. Given two $k \times k$ symmetric matrices A and B define their inner product as $\langle A, B \rangle = \text{trace}(AB)$. Define a block-diagonal symmetric matrix A with h blocks $A^{(1)}, \ldots, A^{(h)}$ with the entries of each block given by:

$$A_{ik}^{(j)} = g_{ik}^{(j)} = \mathcal{R}_G(f_{j,i} \cdot f_{j,k}).$$

Then Theorem 4.2 is equivalent to the following statement:

Corollary 4.4 *With the conditions as in Theorem 4.2 let $p \in \mathbb{R}[X]^G$. Then p is a sum of squares of polynomials in T if and only if p can be written as $p = \langle A, B \rangle$, where B is a positive semidefinite matrix with real entries.*

We now aim to apply Theorem 4.2 to a symmetric form $p \in H_{n,2d}^S$. In order to do this we need to identify an explicit representative in every irreducible $\mathcal{S}_n$-submodule of $H_{n,d}$. We first recall some useful facts from the representation theory of $\mathcal{S}_n$. The irreducible representations in this case are the so-called *Specht modules*, which we will define in the following section. We refer to [18,31] for more details.

4.2 Specht Modules as Polynomials

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$ be a partition of n. A *Young tableau* of shape λ consists of l rows, with λ_i entries in the i-th row. Each entry is an element in $\{1, \ldots, n\}$, and each of these numbers occurs exactly once. A *standard Young tableau* is a Young tableau in which all rows and columns are increasing. An element $\sigma \in \mathcal{S}_n$ acts on a Young tableau by replacing each entry by its image under σ. Two Young tableaux T_1 and T_2 are called *row-equivalent* if the corresponding rows of the two tableaux contain the same numbers. The classes of row-equivalent Young tableaux are called *tabloids*, and the equivalence class of a tableau T is denoted by $\{T\}$. The stabilizer of a row-equivalence class is called the row-stabilizer, denoted by RStab_T. If $R_1, \ldots, R_l$ are the rows or a given Young tableau T this group can be written as

$$\text{RStab}_T = \mathcal{S}_{R_1} \times \mathcal{S}_{R_2} \times \cdots \times \mathcal{S}_{R_l},$$

where $\mathcal{S}_{R_i}$ is the symmetric group on the elements of row i. The action of $\mathcal{S}_n$ on the equivalence classes of row-equivalent Young tableaux gives rise to the *permutation module M^λ corresponding to λ* which is the $\mathcal{S}_n$-module defined by

$$M^\lambda = \mathbb{R}\{\{T_1\}, \ldots, \{T_s\}\},$$

where $\{T_1\}, \ldots, \{T_s\}$ is a complete list of λ-tabloids and $\mathbb{R}\{\{T_1\}, \ldots, \{T_s\}\}$ denotes their $\mathbb{R}$-linear span.

Let T be a Young tableau for $\lambda \vdash n$, and let C_i be the entries in the i-th column of t. The group

$$\mathrm{CStab}_T = \mathcal{S}_{C_1} \times \mathcal{S}_{C_2} \times \cdots \times \mathcal{S}_{C_v},$$

where $\mathcal{S}_{C_i}$ is the symmetric group on the elements of columns i, is called the *column stabilizer* of T. The irreducible representations of the symmetric group $\mathcal{S}_n$ are in 1-1-correspondence with the partitions of n, and they are given by the Specht modules, as explained below. For $\lambda \vdash n$, the *polytabloid associated with T* is defined by

$$e_T = \sum_{\sigma \in \mathrm{CStab}_t} \mathrm{sgn}(\sigma) \sigma \{t\}.$$

Then for a partition $\lambda \vdash n$, the *Specht module S^λ* is the submodule of the permutation module M^λ spanned by the polytabloids e_T. The dimension of S^λ is given by the number of standard Young tableaux for $\lambda \vdash n$, which we will denote by s_λ.

A classical construction of Specht realizes Specht modules as submodules of the polynomial ring (see [35]): For $\lambda \vdash n$ let T_λ be a standard Young tableau of shape λ and $\mathcal{C}_1, \ldots, \mathcal{C}_v$ be the columns of T_λ. To T_λ we associate the monomial $X^{t_\lambda} := \prod_{i=1}^n X_i^{m(i)-1}$, where $m(i)$ is the index of the row of T_λ containing i. Note that for any λ-tabloid $\{T_\lambda\}$ the monomial X^{T_λ} is well defined, and the mapping $\{T_\lambda\} \mapsto X^{T_\lambda}$ is an $\mathcal{S}_n$-isomorphism. For any column $\mathcal{C}_i$ of T_λ we denote by $\mathcal{C}_i(j)$ the element in the j-th row and we associate to it a Vandermonde determinant:

$$\mathrm{Van}_{\mathcal{C}_i} := \det \begin{pmatrix} X_{\mathcal{C}_i(1)}^0 & \cdots & X_{\mathcal{C}_i(k)}^0 \\ \vdots & \ddots & \vdots \\ X_{\mathcal{C}_i(1)}^{k-1} & \cdots & X_{\mathcal{C}_i(k)}^{k-1} \end{pmatrix} = \prod_{j<l} (X_{\mathcal{C}_i(j)} - X_{\mathcal{C}_i(l)}).$$

The *Specht polynomial sp_{T_λ}* associated to T_λ is defined as

$$sp_{T_\lambda} := \prod_{i=1}^v \mathrm{Van}_{\mathcal{C}_i} = \sum_{\sigma \in \mathrm{CStab}_{T_\lambda}} \mathrm{sgn}(\sigma) \sigma (X^{T_\lambda}),$$

where $\mathrm{CStab}_{T_\lambda}$ is the column stabilizer of T_λ. By the $\mathcal{S}_n$-isomorphism $\{T_\lambda\} \mapsto X^{t_\lambda}$, $\mathcal{S}_n$ acts on sp_{T_λ} in the same way as on the polytabloid e_{T_λ}. If $T_{\lambda,1}, \ldots, T_{\lambda,k}$ denote all standard Young tableaux associated to λ, then the polynomials $sp_{T_{\lambda,1}}, \ldots, sp_{T_{\lambda,k}}$ are

called the *Specht polynomials* associated to λ. We then have the following proposition; see [35].

Proposition 4.5 *The Specht polynomials $sp_{T_{\lambda,1}}, \ldots, sp_{T_{\lambda,k}}$ span an $\mathcal{S}_n$-submodule of $\mathbb{R}[X]$ which is isomorphic to the Specht module S^λ.*

The Specht polynomials identify a submodule of $\mathbb{R}[X]$ isomorphic to $\mathcal{S}^\lambda$. In order to get a decomposition of the entire ring $\mathbb{R}[X]$ we will use a generalization of this construction which is described in the next section.

4.3 Higher Specht Polynomials and the Decomposition of $\mathbb{R}[X]$

In what follows we will need to understand the decomposition of the polynomial ring $\mathbb{R}[X]$ and $\mathcal{S}_n$-module $H_{n,d}$ in terms of $\mathcal{S}_n$-irreducible representations. Notice that such a decomposition is not unique. It is classically known that the ring $\mathbb{R}[X]$ is a free module of dimension $n!$ over the ring of symmetric polynomials. Similarly, every isotypic component is a free $\mathbb{R}[X]^{\mathcal{S}_n}$-module. Therefore, one general strategy in order to get a symmetry basis of $\mathbb{R}[X]$ consists in building a free module basis for $\mathbb{R}[X]$ over $\mathbb{R}[X]^{\mathcal{S}_n}$ which additionally is symmetry adapted, i.e., which respects a decomposition into irreducible $\mathcal{S}_n$-modules. One such construction, which generalizes Specht's original construction presented above, is due to Ariki et al. [1].

Definition 4.6 Let $n \in \mathbb{N}$.

  (i) A finite sequence $w = (w_1, \ldots, w_n)$ of non-negative integers is called a word of length n. A word w of length n is called a permutation if the set of non-negative integers forming a word of length n is $\{1, \ldots, n\}$.
 (ii) Given a word w and a permutation u we define the monomial associated to the pair as $X_u^w := X_{u_1}^{w_1} \cdots X_{u_n}^{w_n}$.
(iii) Given a permutation w we associate to w its index, denoted by $i(w)$, by constructing the following word of length n. The word $i(w)$ contains 0 exactly at the same position where 1 occurs in w and the other entries we define recursively with the following rule: Suppose that the entry in $i(w)$ at a given position is c and that k occurs in w at the same position, then $i(w)$ should be also c if it lies to the right of k and it should be $c + 1$ if it lies to the left of k.
 (iv) For $\lambda \vdash n$ and T being a standard Young tableau of shape λ, we define the *word of T*, denoted by $w(T)$, by collecting the entries of T from the bottom to the top in consecutive columns starting from the left.
  (v) For a pair (T, V) of standard λ-tableaux we define the monomial associated to this pair as $X_{w(V)}^{i(w(T))}$.

Example 4.7 Consider the tableau

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}.$$

The resulting word is given by $w(T) = 31524$, with $i(w(T)) = 10001$. Taking

$$V = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}$$

we obtain $X_{w(V)}^{i(w(T))} = X_1^0 X_2^1 X_3^0 X_4^2 X_5^1$.

Definition 4.8 Let $\lambda \vdash n$ and T be a λ-tableau. Then the *Young symmetrizer* associated to T is the element in the group algebra $\mathbb{R}[\mathcal{S}_n]$ defined to be

$$\varepsilon_T = \sum_{\sigma \in \mathrm{RStab}_T} \sum_{\tau \in \mathrm{CStab}_T} \mathrm{sgn}(\tau) \tau \sigma.$$

Now let T be a standard Young tableau, and define the *higher Specht polynomial* associated with the pair (T, V) to be

$$F_V^T(X_1, \ldots, X_n) := \varepsilon_V\big(X_{w(V)}^{i(w(T))}\big).$$

For $\lambda \vdash n$ we will denote by

$$\mathcal{F}_\lambda = \big\{F_V^T : T, V \text{ run over all standard } \lambda\text{-tableaux}\big\}$$

the set of all standard higher Specht polynomials corresponding to λ and by

$$\mathcal{F} = \bigcup_{\lambda \vdash n} \mathcal{F}_\lambda$$

the set of all standard higher Specht polynomials.

Remark 4.9 Let s_λ denote the number of standard Young tableaux of shape λ. It follows from the so-called Robinson–Schensted correspondence (see [31]) that

$$\sum_{\lambda \vdash n} s_\lambda^2 = n!$$

Therefore the cardinality of $\mathcal{F}$ is exactly $n!$

The importance of the higher Specht polynomials now is summarized in the following theorem which can be found in [1, Thm. 1].

Theorem 4.10 *The following holds for the set of higher Specht polynomials.*

(i) *The set $\mathcal{F}$ is a free basis of the ring $\mathbb{R}[X]$ over the invariant ring $\mathbb{R}[X]^{\mathcal{S}_n}$.*
(ii) *For any $\lambda \vdash n$ and standard λ-tableau T, the space spanned by the polynomials in*

$$\mathcal{F}_\lambda^T := \big\{F_V^T : V \text{ runs over all standard } \lambda\text{-tableaux}\big\}$$

is an irreducible $\mathcal{S}_n$-module isomorphic to the Specht module S^λ.

For every $\lambda \vdash n$ we denote by V_0^λ the standard λ-tableau with entries $\{1, \ldots, \lambda_1\}$ in the first row, $\{\lambda_1 + 1, \ldots, \lambda_2\}$ in the second row, and so on. Consider the set

$$\mathcal{Q}_\lambda := \big\{F_{V_0^\lambda}^T : T \text{ runs over all standard } \lambda\text{-tableaux}\big\},$$

which is of cardinality s_λ. The set $\mathcal{Q}_\lambda$ is a symmetry basis of the vector space spanned by $\mathcal{F}$. Using these polynomials we define $s_\lambda \times s_\lambda$ matrix polynomials Q^λ by:

$$Q^\lambda(T, T') := \operatorname{sym} F_{V_0^\lambda}^T F_{V_0^\lambda}^{T'}, \tag{4.2}$$

where T, T' run over all standard λ-tableaux. Since by (i) in Theorem 4.10 we know that every polynomial $h \in \mathbb{R}[X]$ can be uniquely written as a linear combination of elements in $\mathcal{F}$ with coefficients in $\mathbb{R}[X]^{\mathcal{S}_n}$, the following theorem can be thought of as a generalization of Corollary 4.4 to sums of squares from an $\mathcal{S}_n$-module with coefficients in an $\mathcal{S}_n$-invariant ring (see also [12, Thm. 6.2]):

Theorem 4.11 *Let $p \in \mathbb{R}[X]^{\mathcal{S}_n}$ be a symmetric polynomial. Then p is a sum of squares if and only if it can be written in the form*

$$p = \sum_{\lambda \vdash n} \langle B^\lambda, Q^\lambda \rangle,$$

where Q^λ is defined in (4.2) and each $B^\lambda \in \mathbb{R}[X]^{s_\lambda \times s_\lambda}$ is a sum of symmetric squares matrix polynomial, i.e., $B^\lambda(x) = L^t(x)L(x)$ for some matrix polynomial $L(x)$ whose entries are symmetric polynomials.

Each entry of the matrix Q^λ is a symmetric polynomial and thus can be represented as a polynomial in any set of generators of the ring of symmetric polynomials. We will use the power means $p_1, \ldots, p_n$ to phrase the next theorem. However, any other choice works similarly. With this choice of basis it follows that there exists a matrix polynomial $\tilde{Q}^\lambda(z_1, \ldots, z_n)$ in n variables $z_1, \ldots, z_n$ such that

$$\tilde{Q}^\lambda(p_1(x), \ldots, p_n(x)) = Q^\lambda(x). \tag{4.3}$$

With this notation one can restate Theorem 4.11 in the following way:

Theorem 4.12 *Let $f \in \mathbb{R}[X]^{\mathcal{S}_n}$ be a symmetric polynomial and $g \in \mathbb{R}[z_1, \ldots, z_n]$ such that $f = g(p_1, \ldots, p_n)$. Then f is a sum of squares if and only if g can be written in the form*

$$g = \sum_{\lambda \vdash n} \langle B^\lambda, \tilde{Q}^\lambda \rangle,$$

where $\tilde{Q}^\lambda$ is defined in (4.3) and each $B^\lambda \in \mathbb{R}[z]^{s_\lambda \times s_\lambda}$ is a sum of squares matrix polynomial, i.e., $B^\lambda := L(z)^t L(z)$ for some matrix polynomial L.

While Theorems 4.11 and 4.12 give a characterization of symmetric sums of squares in a given number of variables, we need to understand the behavior of the $\mathcal{S}_n$-module $H_{n,d}$ for polynomials of a fixed degree d in a growing number of variables n. This will be done in the next section.

4.4 The Cone $\Sigma_{n,2d}^S$

A symmetric sum of squares $f \in \Sigma_{n,2d}^S$ has to be a sum of squares from $H_{n,d}$. Therefore we now consider restricting the degree of the squares in the underlying sum of squares representation. With a little abuse of notation we denote by $\mathcal{F}_{n,d}$ the vector space spanned by higher Specht polynomials for the group $\mathcal{S}_n$ of degree at most d. Further, for a partition $\lambda \vdash n$ let $\mathcal{F}_{\lambda,d}$ denote the span of the higher Specht polynomials of degree at most d corresponding to the Specht module $\mathcal{S}^\lambda$, i.e., $\mathcal{F}_{\lambda,d}$ is exactly the isotypic component of $\mathcal{F}_{n,d}$ corresponding to $\mathcal{S}^\lambda$. In order to describe this isotypic component combinatorially, recall that the degree of the higher Specht polynomial F_T^S is given by the charge $c(S)$ of S. Thus, it follows from the above construction that

$$\mathcal{F}_{\lambda,d} = \operatorname{span}\left\{F_T^S : S, T \text{ are standard } \lambda\text{-tableaux and } c(S) \leq d\right\}.$$

We now show that sums of squares of degree $2d$ in n variables can be constructed by symmetrizing sums of squares in $2d$ variables. So we first consider the case $n = 2d$. Let

$$\mathcal{F}_{2d,d} = \bigoplus_{\lambda \vdash 2d} m_\lambda \mathcal{S}^\lambda$$

be the decomposition of $\mathcal{F}_{2d,d}$ as an $\mathcal{S}_{2d}$-module. The following proposition gives the multiplicities of the different $\mathcal{S}_n$-modules appearing in the vector space of homogeneous polynomials of degree d.

Proposition 4.13 *The multiplicities m_λ of the $\mathcal{S}_n$-modules $\mathcal{S}^\lambda$ which appear in an isotypic decomposition $H_{n,d}$ coincide with the number of standard λ-tableaux S with the charge of S at most d: $c(S) \leq d$.*

For a partition $\lambda \vdash 2d$ and $n \geq 2d$ define a new partition $\lambda^{(n)} \vdash n$ by simply increasing the first part of λ by $n - 2d$: $\lambda_1^{(n)} = \lambda_1 + n - 2d$ and $\lambda_i^{(n)} = \lambda_i$ for $i \geq 2$. Then the decomposition Theorem 4.10 in combination with [28, Thm. 4.7] yields that

$$\mathcal{F}_{n,d} = \bigoplus_{\lambda \vdash 2d} m_\lambda \mathcal{S}^{\lambda^{(n)}}.$$

For every $\lambda \vdash 2d$ we choose m_λ many higher Specht polynomials $q_1^\lambda, \ldots, q_{m_\lambda}^\lambda$ that form a symmetry basis of the λ-isotypic component of $\mathcal{F}_{2d,d}$. Let $q_\lambda = (q_1^\lambda, \ldots, q_{m_\lambda}^\lambda)$ be the vector with entries q_i^λ. As before we construct the matrix Q_{2d}^λ by

$$Q_{2d}^\lambda = \operatorname{sym}_{2d} q_\lambda^t q_\lambda, \qquad Q_{2d}^\lambda(i,j) = \operatorname{sym}_{2d} q_i^\lambda q_j^\lambda.$$

Further, we define the matrix Q_n^λ by

$$Q_n = \operatorname{sym}_n q_\lambda^t q_\lambda, \qquad Q_n^\lambda(i,j) = \operatorname{sym}_n q_i^\lambda q_j^\lambda.$$

By construction we have the following:

Proposition 4.14 *The matrix Q_n^λ is the $\mathcal{S}_n$-symmetrization of the matrix Q_{2d}^λ:*

$$Q_n^\lambda = \mathrm{sym}_n\, Q_{2d}^\lambda.$$

We now give a parametric description of the family of cones $\Sigma_{n,2d}^S$. Note again that this statement is given in terms of a particular basis, but similarly can be stated with any set of generators.

Theorem 4.15 *Let $f := \sum_{\lambda \vdash 2d} c_\lambda p_\lambda^{(n)} \in H_{n,2d}^S$. Then f is a sum of squares if and only if it can be written in the form*

$$f = \sum_{\lambda \vdash 2d} \langle B^\lambda, Q_n^\lambda \rangle,$$

where each $B^\lambda \in \mathbb{R}\big[p_1^{(n)}, \ldots, p_d^{(n)}\big]^{m_\lambda \times m_\lambda}$ is a sum of squares matrix of power sum polynomials, i.e., $B^\lambda = L_\lambda^t L_\lambda$ for some matrix polynomial $L_\lambda\big(p_1^{(n)}, \ldots, p_d^{(n)}\big)$ whose entries are weighted homogeneous forms. Additionally, we have for every column k of L_λ,

$$\deg_w Q_n^\lambda(i,k) + 2\deg_w L_\lambda(k,i) = 2d$$

or, equivalently, every entry $B^\lambda(i,j)$ of B^λ is a weighted homogeneous form such that

$$\deg_w Q_n^\lambda(i,j) + \deg_w B^\lambda(i,j) = 2d.$$

Proof In order to apply Theorem 4.11 to our fixed degree situation we have to show that the forms $\{q_1^\lambda, \ldots, q_{m_\lambda}^\lambda\}$ when viewed as functions in n variables also form a symmetry basis of the $\lambda^{(n)}$-isotypic component of $\mathcal{F}_{n,d}$ for all $n \geq 2d$. Indeed, consider a standard Young tableau t_λ of shape λ and construct a standard Young tableau $t_{\lambda^{(n)}}$ of shape $\lambda^{(n)}$ by adding numbers $2d+1, \ldots, n$ as rightmost entries of the top row of $t_{\lambda^{(n)}}$, while keeping the rest of the filling of $t_{\lambda^{(n)}}$ the same as for t_λ. It follows by construction of the Specht polynomials that

$$sp_{t_\lambda} = sp_{t_{\lambda^{(n)}}}.$$

We may assume, the $q_k^{(\lambda)}$ were chosen so that they map to sp_{t_λ} by an $\mathcal{S}_{2d}$-isomorphism. We observe that sp_{t_λ} (and therefore $sp_{t_{\lambda^{(n)}}}$) and q_k^λ do not involve any of the variables X_j, $j > 2d$. Therefore both are stabilized by $\mathcal{S}_{n-2d}$ (operating on the last $n - 2d$ variables), and further the action on the first $2d$ variables is exactly the same. Thus there is an $\mathcal{S}_n$-isomorphism mapping q_k^λ to $sp_{t_\lambda^{(n)}}$ and the $\mathcal{S}_n$-modules generated by the two polynomials are isomorphic. Therefore it follows that $q_k^{(\lambda)}$ also form a symmetry basis of the $\lambda^{(n)}$-isotypic component of $\mathcal{F}_{n,d}$. □

Remark 4.16 We remark that the sum of squares decomposition of $f = \sum_{\lambda \vdash 2d} \langle B^\lambda, Q_n^\lambda \rangle$, with $B^\lambda = L_\lambda^t L_\lambda$ can be read off as follows:

$$f = \sum_{\lambda \vdash 2d} \mathrm{sym}_n \, q_\lambda^t B^\lambda q_\lambda = \sum_{\lambda \vdash 2d} \mathrm{sym}_n (L_\lambda q_\lambda)^t L_\lambda q_\lambda. \qquad (4.4)$$

In particular, if for a fixed $\lambda \vdash n$ and for every $1 \leq i \leq m_\lambda$ we denote $\delta_i := d - \deg q_i^\lambda$, then the set of polynomials

$$\bigcup_{i=1}^{m_\lambda} \bigcup_{\nu \vdash \delta_i} \{q_i^\lambda p_\nu\} \qquad (4.5)$$

is a symmetry basis of the isotypic component $H_{n,d}$ corresponding to λ.

4.5 The Dual Cone of Symmetric Sums of Squares

Recall, that for a convex cone $K \subset \mathbb{R}^n$ the dual cone K^* is defined as

$$K^* := \{l \in \mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R}) : \ell(x) \geq 0 \text{ for all } x \in K\}.$$

Our analysis of the dual cone $(\Sigma_{n,2d}^S)^*$ proceeds similarly to the analysis of the dual cone in the non-symmetric situation given in [4,6].

Let $S_{n,d}$ be the vector space of real quadratic forms on $H_{n,d}$. Let $S_{n,d}^+$ be the cone of positive semidefinite quadratic forms in $S_{n,d}$. An element $\mathcal{Q} \in S_{n,d}$ is said to be $\mathcal{S}_n$-invariant if $\mathcal{Q}(f) = \mathcal{Q}(\sigma(f))$ for all $\sigma \in \mathcal{S}_n$, $f \in H_{n,d}$. We will denote by $\bar{S}_{n,d}$ the space of $\mathcal{S}_n$-invariant quadratic forms on $H_{n,d}$. Further we can identify a linear functional $l \in (H_{n,2d}^S)^*$ with a quadratic form $\mathcal{Q}_l$ defined by

$$\mathcal{Q}_\ell(f) = \ell(\mathrm{sym}\, f^2).$$

Let $\bar{S}_{n,d}^+$ be the cone of positive semidefinite forms in $\bar{S}_{n,d}$, i.e.,

$$\bar{S}_{n,d}^+ := \{\mathcal{Q} \in \bar{S}_{n,d} : \mathcal{Q}(f) \geq 0 \text{ for all } f \in H_{n,d}\}.$$

The following lemma is straightforward, but very important, as it allows us to identify the elements of dual cone $l \in (\Sigma_{n,2d}^S)^*$ with quadratic forms $\mathcal{Q}_\ell$ in $\bar{S}_{n,d}^+$.

Lemma 4.17 *A linear functional $\ell \in (H_{n,2d}^S)^*$ belongs to the dual cone $(\Sigma_{n,2d}^S)^*$ if and only if the quadratic form $\mathcal{Q}_\ell$ is positive semidefinite.*

Since for $\ell \in (H_{n,2d}^S)^*$ we have $\mathcal{Q}_\ell \in \bar{S}_{n,d}$, Schur's Lemma again applies and we can use the symmetry basis constructed above to simplify the condition that $\mathcal{Q}_\ell$ is positive semidefinite. In order to arrive at a dual statement of Corollary 4.15 we construct the following matrices:

Definition 4.18 For a partition $\lambda \vdash 2d$ consider the block-matrix $M_{n,\lambda}$ defined by

$$M_{n,\lambda}^{(i,j)}(\alpha, \beta) := \ell\big(p_\alpha \cdot p_\beta \cdot Q_n^\lambda(i, j)\big),$$

where in each block i, j the indices (α, β) run through all pairs of weakly decreasing sequences $\alpha = (\alpha_1, \ldots, \alpha_a)$ and $\beta = (\beta_1, \ldots, \beta_b)$ such that

$$2d - \deg_w Q_n^\lambda(i, j) = \alpha_1 + \cdots + \alpha_a + \beta_1 + \cdots + \beta_b.$$

With this notation the following lemma is just the dual version of Corollary 4.15 and is established by expressing Lemma 4.17 in the basis given in (4.5):

Lemma 4.19 *Let $\ell \in H_{n,2d}^*$ be a linear functional. Then $\ell \in (\Sigma_{n,2d})^*$ if and only if for all $\lambda \vdash 2d$ the above matrices $M_{n,\lambda}$ are positive semidefinite.*

In order to examine the kernels of quadratic forms we use the following construction. Let $W \subset H_{n,d}$ be any linear subspace. We define $W^{<2>}$ to be the symmetrization of the degree $2d$ part of the ideal generated by W:

$$W^{<2>} := \left\{ h \in H_{n,2d}^S : h = \mathrm{sym} \sum f_i g_i \text{ with } f_i \in W, \ g_i \in H_{n,d} \right\}.$$

In Lemma 4.17 we identified the dual cone $(\Sigma_{n,2d}^S)^*$ with a linear section of the cone of positive semidefinite quadratic forms $S_{n,d}^+$ with the subspace $\bar{S}_{n,d}$ of symmetric quadratic forms. By a slight abuse of terminology we think of positive semidefinite forms Q_ℓ as elements of the dual cone $(\Sigma_{n,2d})^*$. The following important proposition is a straightforward adaptation of the equivalent result in the non-symmetric case [6, Proposition 2.1]:

Proposition 4.20 *Let $\ell \in (\Sigma_{n,2d}^S)^*$ be a linear functional non-negative on squares and let $W_\ell \subset H_{n,d}$ be the kernel of the quadratic form Q_ℓ. The linear functional ℓ spans an extreme ray of $(\Sigma_{n,2d}^S)^*$ if and only if $W_\ell^{<2>}$ is a hyperplane in $H_{n,2d}^S$. Equivalently, the kernel of Q_ℓ is maximal, i.e., if $\ker Q_\ell \subseteq \ker Q_m$ for some $m \in H_{n,2d}^*$ then $m = \lambda\ell$ for some $\lambda \in \mathbb{R}$.*

The dual correspondence yields that any facet F of a cone K, i.e., any maximal face of K, is given by an extreme ray of the dual cone K^*. More precisely, for any maximal face F of K there exists an extreme ray of K^* spanned by a linear functional $\ell \in K^*$ such that

$$F = \{x \in K : \ell(x) = 0\}.$$

We now aim to characterize the extreme rays of $(\Sigma_{n,2d}^S)^*$ that are not extreme rays of the cone $(\mathcal{P}_{n,2d}^S)^*$. For $v \in \mathbb{R}^n$ define a linear functional

$$\ell_v : H_{n,2d}^S \to R, \quad \ell_v(f) = f(v).$$

We say that the linear functional ℓ_v corresponds to point evaluation at v. It is easy to show with the same proof as in the non-symmetric case that the extreme rays of the cone $(\mathcal{P}^S_{n,2d})^*$ are precisely the point evaluations ℓ_v (see [5, Chap. 4] for the non-symmetric case). Therefore we need to identify extreme rays of $(\Sigma^S_{n,2d})^*$ which are not point evaluations. We now examine the case of degree 4 in detail, and give an explicit construction of an element of $(\Sigma^S_{n,4})^*$ which does not belong to $(\mathcal{P}^S_{n,2d})^*$.

5 Symmetric Quartic Sums of Squares

We now look at the decomposition of $H_{n,2}$ as an $\mathcal{S}_n$-module in order to apply Theorem 4.2 and characterize all symmetric sums of squares of degree 4.

Theorem 5.1 *Let $f^{(n)} \in H_{n,4}$ be symmetric and $n \geq 4$. If $f^{(n)}$ is a sum of squares then it can be written in the form*

$$
\begin{aligned}
f^{(n)} =\ & \alpha_{11} p_{(1^4)} + 2\alpha_{12} p_{(2,1^2)} + \alpha_{22} p_{(2^2)} \\
& + \beta_{11}\big(p_{(2,1^2)} - p_{(1^4)}\big) + 2\beta_{12}\big(p_{(3,1)} - p_{(2,1^2)}\big) + \beta_{22}\big(p_{(4)} - p_{(2^2)}\big) \\
& + \gamma\left(\frac{1}{2} p_{(1^4)} - p_{(2,1^2)} + \frac{n^2 - 3n + 3}{2n^2} p_{(2^2)} + \frac{2n-2}{n^2} p_{(31)} + \frac{1-n}{2n^2} p_{(4)}\right),
\end{aligned}
$$

where $\gamma \geq 0$ and the matrices $(\alpha_{ij})_{2\times 2}$ and $(\beta_{ij})_{2\times 2}$ are positive semidefinite.

Proof The statement follows directly from the arguments presented in Sect. 4.4. Following Theorem 4.15 we get that $f^{(n)}$ has a decomposition in the form

$$
f^{(n)} = B^{(n)} + \big\langle B^{(n-1,1)}, Q_n^{(n-1,1)}\big\rangle + B^{(n-2,2)} \cdot Q_n^{(n-2,2)},
$$

where $B^{(n)}$ is a sum of symmetric squares, $B^{(n-1,1)}$ is a 2×2 sum of symmetric squares matrix polynomial and due to the degree restrictions $B^{(n-2,2)}$ is a non-negative scalar. It remains to calculate the matrices $Q_n^{(n-1,1)}$ and $Q_n^{(n-2,2)}$ appearing in the statement decomposition. These are defined as the symmetrization of the pairwise products of those Specht polynomials which generate the corresponding Specht modules in degree 2. In degree 2 the Specht polynomials $X_n - X_1$ and $X_n^2 - X_1^2$ generate two disjoint irreducible $\mathcal{S}_n$-modules isomorphic to the $S^{(n-1,1)}$ part and the Specht polynomial $(X_{n-1} - X_1)(X_n - X_2)$ generates a module isomorphic to $S^{(n-2,2)}$. Thus we have:

$$
\begin{aligned}
Q_n^{(n-1,1)} &= \begin{pmatrix} \mathrm{sym}_n(X_n - X_1)^2 & \mathrm{sym}_n(X_n - X_1)(X_n^2 - X_1^2) \\ \mathrm{sym}_n(X_n - X_1)(X_n^2 - X_1^2) & \mathrm{sym}_n(X_n^2 - X_1^2)^2 \end{pmatrix}, \\
Q_n^{(n-2,2)} &= \mathrm{sym}_n(X_{n-1} - X_1)^2(X_n - X_2)^2.
\end{aligned}
$$

Then the symmetrization can be calculated quite directly, since every product involves at most 4 variables. These calculations then yield

$$Q_n^{(n-1,1)} = \frac{2n}{n-1} \begin{pmatrix} p_{(2)} - p_{(1^2)} & p_{(3)} - p_{(2,1)} \\ p_{(3)} - p_{(2,1)} & p_{(4)} - p_{(2^2)} \end{pmatrix},$$

$$Q_n^{(n-2,2)} = \frac{8n^3}{n^3 - 6n^2 + 11n - 6} \tag{5.1}$$
$$\times \left(\frac{1}{2} p_{(1^4)} - p_{(2,1^2)} + \frac{n^2 - 3n + 3}{2n^2} p_{(2^2)} + \frac{2n-2}{n^2} p_{(3,1)} + \frac{1-n}{2n^2} p_{(4)} \right),$$

which gives exactly the statement in the theorem.   □

5.1 The Boundary of $\Sigma_{n,4}^S$

We now apply Proposition 4.20 to the case of degree 4 and examine the possible kernels of an extreme ray of $(\Sigma_{n,4}^S)^*$ which do not come from a point evaluation.

Lemma 5.2 *Suppose a linear functional ℓ spans an extreme ray of $(\Sigma_{n,4}^S)^*$ that is not an extreme ray of $(\mathcal{P}_{n,4}^S)^*$. Let Q be quadratic form corresponding to ℓ. Then*

$$\text{Ker } Q \simeq \mathcal{S}^{(n)} \oplus \mathcal{S}^{(n-1,1)} \quad or \quad \ell \left(\sum_{\lambda \vdash 4} c_\lambda p_\lambda \right) = c_{(4)} + c_{(2^2)}$$

and n is odd.

Proof Since Q is an $\mathcal{S}_n$-invariant quadratic form, its kernel Ker $Q \subseteq H_{n,2}$ is an $\mathcal{S}_n$-module. It follows from the arguments in the proof of Theorem 5.1 that Ker Q decomposes as

$$\text{Ker } Q \simeq \alpha \mathcal{S}^{(n)} \oplus \beta \mathcal{S}^{(n-1,1)} \oplus \gamma \mathcal{S}^{(n-2,2)},$$

where $\alpha, \beta \in \{0, 1, 2\}$ and $\gamma \in \{0, 1\}$. We now examine the possible combinations of α, β, and γ.

As above let W denote the kernel of Q. We first observe that $\alpha = 2$ is not possible: if $\alpha = 2$ then we have $p_2 \in W$ and so $p_2^2 \in W_S^{<2>}$, which is a contradiction since p_2^2 is not on the boundary of $\Sigma_{n,4}^{\mathcal{S}_n}$.

By Proposition 4.20 the kernel W of Q must be maximal. Let $w \in \mathbb{R}^n$ be the all 1 vector: $w = (1, \ldots, 1)$. We now observe that $\alpha = 0$ is also impossible: if $\alpha = 0$ then all forms in the kernel W of Q are 0 at w. Therefore ker $Q \subseteq$ ker Q_{ℓ_w} and by Proposition 4.20 we have $Q = \lambda Q_{\ell_w}$, which is a contradiction, since Q does not correspond to point evaluation. Thus we must have $\alpha = 1$.

Since we have dim $H_{n,4}^S = 5$, from Corollary 4.20 we see that dim $W^{<2>} = 4$. This excludes the case $\beta = 0$, since even with $\alpha = 1$ and $\gamma = 1$ the dimension of $W^{<2>}$ is at most 3. Now suppose that $\beta = 2$, i.e., the $\mathcal{S}_n$-module generated by $(X_1 - X_2)p_1$ and $X_1^2 - X_2^2$ lies in W as well as a polynomial $q = ap_1^2 + bp_2$. We

consider the symmetrizations of the five pairwise products and express these in the basis $\{p_{(4)}, p_{(3,1)}, p_{(2^2)}, p_{(2,1^2)}, p_{(1^4)}\}$.

Now the condition $\dim W^{<2>} = 4$ implies that these five products cannot be linearly independent and an explicit calculation of the determinant of the corresponding matrix M yields $\det M = b(a + b)$. We now examine the possible roots of this determinant. In the case when $a = -b$ all polynomials in W (even if $\gamma = 1$) will be zero at $(1, \ldots, 1)$, which is excluded. Therefore the only possible case is $b = 0$. In that case, by calculating the kernel of M we see that the unique (up to a constant multiple) linear functional ℓ vanishing on $W^{<2>}$ is given by

$$\ell\left(\sum_{\lambda \vdash 4} c_\lambda p_\lambda\right) = c_{(4)} + c_{(2^2)}.$$

We observe using (5.1) that we must have $\gamma = 0$ since $\ell\left(\mathrm{sym}_n(X_1 - X_2)^2(X_3 - X_4)^2\right) > 0$ for $n \geq 4$. Now suppose that n is even and let $w \in \mathbb{R}^n$ be given by $w = (1, \ldots, 1, -1, \ldots, -1)$ where 1 and -1 occur $n/2$ times each. It is easy to verify that for all $f \in W$ we have $f(w) = 0$. Therefore it follows that $W \subseteq \ker Q_{\ell_w}$, which is a contradiction, since W is a kernel of an extreme ray that does not come from point evaluation.

When n is odd the forms in W have no common zeros and therefore ℓ is not a positive combination of point evaluations. It is not hard to verify that ℓ is non-negative on squares and the kernel W is maximal. Therefore by Proposition 4.20 we know that ℓ spans an extreme ray of $(\Sigma_{n,2d}^S)^*$

Finally we need to deal with the case $\alpha = \beta = \gamma = 1$. Suppose that the $\mathcal{S}_n$-module W is generated by three polynomials:

$$q_1 := ap_1^2 + bp_2, \quad q_2 := c(X_1 - X_2)p_1 + d(X_1^2 - X_2^2), \quad q_3 := (X_1 - X_2)(X_3 - X_4).$$

Again we consider the symmetrizations of the five pairwise products and represent these in a matrix M. Explicit calculations now show that

$$\det M = -(a + b)(ad^2n^2 - 4ad^2n + 4ad^2 + bd^2n^2 + 4bcdn + bc^2n - 4bcd - bc^2).$$

As $\alpha = \beta = \gamma = 1$ we must have $\mathrm{rank}\, M = 4$ since the rows of M generate $W^{<2>}$. Again we cannot have $a = -b$, and thus

$$ad^2n^2 - 4ad^2n + 4ad^2 + bd^2n^2 + 4bcdn + bc^2n - 4bcd - bc^2 = 0. \quad (5.2)$$

Therefore there exists a unique linear functional ℓ, which vanishes on $W^{<2>}$ and comes from the kernel of M.

Let $w \in \mathbb{R}^n$ be a point with coordinates $w = (s, \ldots, s, t)$ with $s, t \in \mathbb{R}$ such that

$$cn(s + t) + d((n - 1)s + t) = 0.$$

We see that $q_3(w) = 0$ and from the above equation it also follows that for all f in the $\mathcal{S}_n$-module generated by q_2 we have $f(w) = 0$. Direct calculation shows that (5.2) also implies $q_1(w) = 0$. Thus we have $W \subseteq Q_{\ell_w}$, which is a contradiction by Proposition 4.20, since W is a kernel of an extreme ray that does not come from point evaluation. We remark that it is possible to show that the functional ℓ vanishing on $W^{<2>}$ and giving rise to W is in fact a multiple of ℓ_w, but this is not necessary for us to finish the proof. □

The above description allows us to explicitly characterize degree 4 symmetric sums of squares that are positive and on the boundary of $\Sigma^S_{n,4}$.

Theorem 5.3 *Let $n \geq 4$ and $f^{(n)} \in H_{n,4}$ be symmetric and positive on the boundary of $\Sigma^S_{4,n}$. Then*

(i) *either $f^{(n)}$ can be written as*

$$
\begin{aligned}
f^{(n)} &= a^2 p^{(n)}_{(4)} + 2abp^{(n)}_{(31)} + (c^2 - a^2)p^{(n)}_{2^2} \\
&\quad + (2cd + b^2 - 2ab)p^{(n)}_{(2,1^2)} + (d^2 - b^2)p^{(n)}_{(1^4)},
\end{aligned}
$$

with non-zero coefficients $a, b, c, d \in \mathbb{R} \setminus \{0\}$ which additionally satisfy

$$
\begin{aligned}
&0 \leq \frac{a(c-d) + b(d+c)}{ac}, \quad 0 \leq \frac{a(c+d)(bc-ad)}{ac}, \\
&0 \leq -\frac{c+d}{a^2c^2}\big(a^2(c-d) + b(ac+bc)\big), \\
&0 \leq -\frac{c+d}{a^2c^2}\big((abc + b^2c - a^2d)a^2c - (-a^2d)^2\big), \\
&0 \leq (c+d)\big((ca^2 + cab)n^2 + (b^2c - 3cab + 3a^2d)n - b^2c + 3cab - 3a^2d\big),
\end{aligned}
\tag{5.3}
$$

(ii) *or, if n is odd, then $f^{(n)}$ may have the form*

$$
\begin{aligned}
f^{(n)} &= a^2 p_{(1^4)} + b_{11}\big(p_{(2,1^2)} - p_{(1^4)}\big) + 2b_{12}\big(p_{(3,1)} - p_{(2,1^2)}\big) \\
&\quad + b_{22}\big(p_{(4)} - p_{(2^2)}\big),
\end{aligned}
$$

with coefficients $a, b_{11}, b_{12}, b_{22} \in \mathbb{R}$ which additionally satisfy

$$
a \neq 0, \quad b_{11} + b_{22} \geq 0, \quad b_{11}b_{22} - b^2_{12} \geq 0.
\tag{5.4}
$$

Proof Suppose that $f^{(n)}$ is a strictly positive form on the boundary of $\Sigma^S_{4,n}$. Then there exists a non-trivial functional l spanning an extreme ray of the dual cone $(\Sigma^S_{4,n})^*$ such that $\ell(f^{(n)}) = 0$. Let $W_l \subset H_{n,2}$ denote the kernel of Q_ℓ. In view of Lemma 5.2 we see that there are two possible situations that we need to take into consideration.

(i) We first assume that

$$
W_\ell \simeq \mathcal{S}^{(n)} \oplus \mathcal{S}^{(n-1,1)}.
\tag{5.5}
$$

In view of (5.5) we may assume that the $\mathcal{S}_n$-module W_ℓ is generated by two polynomials:

$$q_1 := cp_1^2 + dp_2 \quad \text{and} \quad q_2 := \frac{n-1}{2n}\left(a(X_1^2 - X_2^2) + b(X_1 - X_2)p_1\right),$$

where $a, b, c, d \in \mathbb{R}$ are chosen such that $(0, 0) \neq (a, b)$ and $(0, 0) \neq (c, d)$.

Let $q \in H_{n,2}$. By Proposition 4.20 we have

$$q \in W_l \quad \text{if and only if} \quad \text{the } \mathcal{S}_n\text{-linear map } p \mapsto \ell(pq) \text{ is the zero map on } H_{n,2}.$$

The dimension of the vector space of $\mathcal{S}_n$-invariant quadratic maps from $H_{n,2}$ to $\mathbb{R}$ is 5. However, since $q \in W_\ell$, Schur's lemma implies $\ell(qr) = 0$ for all r in the isotypic component of the type $(n-2, 2)$. Let $y_\lambda = \ell(p_\lambda)$. Using explicit calculations we find that the coefficients y_λ are characterized by the following system of four linear equations:

$$0 = \ell(\operatorname{sym} q_1 p_2) = cy_{(2^2)} + dy_{(2,1^2)},$$
$$0 = \ell(\operatorname{sym} q_1 p_1^2) = cy_{(2,1^2)} + dy_{(1^4)},$$
$$0 = \ell(\operatorname{sym} q_2(X_1^2 - X_2^2)) = ay_{(4)} - ay_{(2^2)} + by_{(3,1)} - by_{(2,1^2)},$$
$$0 = \ell(\operatorname{sym} q_2(X_1 - X_2)p_1) = ay_{(3,1)} - ay_{(2,1^2)} + by_{(2,1^2)} - by_{(1^4)}.$$

Since in addition we want that the form $l \in (\Sigma_{n,d}^S)^*$ we must also take into account that the corresponding quadratic form Q_ℓ has to be positive semidefinite. By Lemma 4.17 this is equivalent to checking that each of the two matrices

$$M_{(n)} := \begin{pmatrix} y_{(2^2)} & y_{(2,1^2)} \\ y_{(2,1^2)} & y_{(1^4)} \end{pmatrix}, \quad M_{(n-1,1)} := \begin{pmatrix} y_{(4)} - y_{(2^2)} & y_{(3,1)} - y_{(2,1^2)} \\ y_{(3,1)} - y_{(2,1^2)} & y_{(2,1^2)} - y_{(1^4)} \end{pmatrix}$$

is positive semidefinite and

$$M_{(n-2,2)} := \frac{n^2}{2}y_{(1^4)} - n^2 y_{(21^2)} + (2n-2)y_{(31)}$$
$$+ \frac{n^2 - 3n + 3}{2}y_{(2^2)} + \frac{1-n}{2}y_{(4)} \geq 0.$$

Now assuming $a = 0$ we find that either $b = 0$, which is excluded, or any solution of the above linear system will have

$$y_{(2^2)} = y_{(3,1)} = y_{(2,1^2)} = y_{(1^4)}.$$

By substituting this into $M_{(n-2,2)}$ we find that

$$\frac{1-n}{2}(y_{(4)} - y_{(2^2)}) \geq 0,$$

while from $M_{(n-1,1)}$ we have $y_{(4)} - y_{(2)^2} \geq 0$. It follows that

$$y_{(3,1)} = y_{(2,1^2)} = y_{(1^4)} = y_{(4)} = y_{(2^2)}.$$

But then we find that ℓ is proportional to the functional that simply evaluates at the point $(1, 1, 1, \ldots, 1)$, which is a contradiction since $f^{(n)}$ is strictly positive. Thus $a \neq 0$.

Now suppose that $c = 0$. Then we find that

$$y_{(3,1)} = y_{(1^4)} = y_{(2,1^2)} = 0 \quad \text{and} \quad a(y_{(4)} - y_{(2^2)}) = 0.$$

Since $a \neq 0$ we find that the linear functional ℓ is given by

$$\ell\left(\sum_{\lambda \vdash 4} c_\lambda p_\lambda\right) = c_{(4)} - c_{(2^2)}.$$

By Lemma 5.2 we must have n odd in order for ℓ not to be a point evaluation and this sends us to case (ii) discussed below. Meanwhile with $a, c \neq 0$ the solution of the linear system (up to a common multiple) is given by

$$y_{(4)} = \frac{-b^2cd - b^2c^2 + a^2d^2}{c^2a^2}, \quad y_{(2^2)} = -\frac{da - db - bc}{ca},$$

$$y_{(3,1)} = \frac{d^2}{c^2}, \quad y_{(2,1^2)} = -\frac{d}{c}, \quad y_{(1^4)} = 1,$$

which then yields the conditions in (5.3).

(ii) If n is odd we know from Lemma 5.2 that there is one additional case: $f^{(n)}$ is a sum of the square $(ap_{(11)})^2$, and a sum of squares of elements from the isotypic component of $H_{n,2}$ which corresponds to the representation $S^{(n-1,1)}$. Since $f^{(n)}$ is strictly positive, we must have $a \neq 0$ (otherwise $f^{(n)}$ has a zero at $(1, \ldots, 1)$) and it also follows that the matrix $(b_{ij})_{2 \times 2}$ must be strictly positive definite. Therefore we get the announced decomposition from Theorem 5.1.   □

Note that although the first symmetric counterexample by Choi and Lam in four variables gives $\Sigma_{4,4}^S \subsetneq \mathcal{P}_{4,4}^S$ it does not immediately imply that we have strict containment for all n. However, using our methods, one can produce a sequence of strictly positive symmetric quartics that lie on the boundary of $\Sigma_{n,4}^S$ for all n as a witness for the strict inclusion.

Example 5.4 For $n \geq 4$ consider the family of polynomials

$$f^{(n)} := a^2 p_{(4)}^{(n)} + 2ab p_{(31)}^{(n)} + (c^2 - a^2) p_{(2^2)}^{(n)}$$
$$+ (2cd + b^2 - 2ab) p_{(2,1^2)}^{(n)} + (d^2 - b^2) p_{(1^4)}^{(n)},$$

where we set $a = 1$, $b = -13/10$, $c = 1$, and $d = -5/4$. Further consider the linear functional $\ell \in H_{n,4}^*$ with

$$\ell\left(p_{(4)}^{(n)}\right) = \frac{397}{200}, \quad \ell\left(p_{(2^2)}^{(n)}\right) = \frac{63}{40}, \quad \ell\left(p_{(3,1)}^{(n)}\right) = \frac{25}{16}, \quad \ell\left(p_{(2,1^2)}^{(n)}\right) = \frac{5}{4}, \quad \ell\left(p_{(1^4)}^{(n)}\right) = 1.$$

Then we have $\ell(f^{(n)}) = 0$. In addition the corresponding matrices become

$$M_{(n)} := \begin{pmatrix} 63/40 & 5/4 \\ 5/4 & 1 \end{pmatrix}, \quad M_{(n-1,1)} := \begin{pmatrix} 41/100 & 5/16 \\ 5/16 & 1/4 \end{pmatrix},$$

$$M_{(n-2,2)} := \frac{3n^2}{80} + \frac{21n}{80} - \frac{21}{80}.$$

These matrices are all positive semidefinite for $n \geq 4$ and therefore we have $\ell \in (\Sigma_{n,4}^S)^*$. This implies that $f^{(n)} \in \partial\Sigma_{n,4}^S$.

Now we argue that for any $n \in \mathbb{N}$ the forms $f^{(n)}$ are strictly positive. By Corollary 3.3 it follows that $f^{(n)}$ has a zero, if and only if there exists $k \in \{1/n, \ldots, (n-1)/n\}$ such that the bivariate form

$$\begin{aligned}
h_k(x, y) &= \Phi_f(k, 1 - k, x, y) \\
&= kx^4 + (1 - k)y^4 - \frac{13}{5}\left(kx^3 + (1 - k)y^3\right)(kx + (1 - k)y) \\
&\quad + \frac{179}{100}\left(kx^2 + (1 - k)y^2\right)(kx + (1 - k)y)^2 - \frac{51}{400}(kx + (1 - k)y)^4
\end{aligned}$$

has a real projective zero (x, y). Since $f^{(n)}$ is a sum of squares and therefore non-negative, we also know that $h_k(x, y)$ is non-negative for all $k \in \{1/n, \ldots, (n-1)/n\}$. Therefore the real projective roots of $h_k(x, y)$ must have even multiplicity. This implies that $h_k(x, y)$ has a real root only if its discriminant $\delta(h_k)$—viewed as polynomial in the parameter k—has a root in the admissible range for k. We calculate

$$\delta(h_k) := -10^{-8}(10000 - 37399k + 37399k^2)(149k^2 - 149k + 25)^2(k - 1)^3k^3.$$

We see that $\delta(h_k)$ is zero only for

$$k \in \left\{0, 1, \frac{1}{2} \pm \frac{7\sqrt{149}}{298}, \frac{1}{2} \pm \frac{51\sqrt{37399}}{74798}i\right\}.$$

Thus we see that for all natural numbers n there is no $k \in \{1/n, \ldots, (n-1)/n\}$ such that $h_k(x, y)$ has a real projective zero. Therefore we can conclude that for any $n \in \mathbb{N}$ the form $f^{(n)}$ will be strictly positive.

From the above example the following characterization, which recently had been independently given by Goel et al. [13], is an immediate consequence.

Theorem 5.5 *The inclusion $\Sigma_{n,2d}^S \subset \mathcal{P}_{n,2d}^S$ is strict except in the cases of symmetric bivariate forms, or symmetric quadratic forms, or symmetric ternary quartics.*

Proof The well-known Robinson form

$$X_1^6 + X_2^6 + X_3^6 - X_1^4 X_2^2 - X_1^2 X_2^4 - X_1^4 X_3^2 - X_2^4 X_3^2 - X_1^2 X_3^4 - X_2^2 X_3^4 + 3 X_1^2 X_2^2 X_3^2$$

is a non-negative form which is not a sum of squares. Furthermore, for the case $2d = 4$ and $n \geq 4$, Example 5.4 above gives for every n a positive polynomial $f^{(n)}$ which lies on the boundary of $\Sigma_{n,4}^S$ and therefore guarantees the existence of $h_{n,4} \in P_{n,2d}^S$ which is positive definite but not a sum of squares. The result now follows by observing that for any positive definite form $h \in H_{n,2d}$ that is not a sum of squares, the form $(X_1 + \cdots + X_n)^2 h \in H_{n,2d+2}$ is also positive definite and not a sum of squares. Indeed, if $(X_1 + \cdots + X_n)^2 h = f_1^2 + \ldots + f_m^2$ then $(X_1 + \cdots + X_n)^2$ will divide f_i^2 which yields that h is a sum of squares. $\qquad\square$

6 Asymptotic Behavior

In this section we study the relationship of sums of squares and non-negative forms when the number of variables tends to infinity.

6.1 Full-Dimensionality

We now consider the power mean inequalities and their limits. In order to talk about limits of our sequences of cones we use the following notion of limit of a sequence of sets, which is due to Kuratowski [19], and we refer the reader to [21,32] for details in the context of sequences of convex sets.

Definition 6.1 Let $\{K_n\}_{n \in \mathbb{N}}$ be a sequence of subsets of $\mathbb{R}^k$. Then a set $K \subset \mathbb{R}^k$ is called the limit of the sequence, denoted by $K = \lim\limits_{n \to \infty} K_n$, if we have

$$\limsup_{n \to \infty} K_n \subset K \subset \liminf_{n \to \infty} K_n,$$

where

$$\liminf_{n \to \infty} K_n = \left\{ x \in \mathbb{R}^k : x = \lim_{n \in \mathbb{N}} x_n, \ x_n \in K_n \right\},$$

$$\limsup_{n \to \infty} K_n = \left\{ x \in \mathbb{R}^k : x = \lim_{m \in M \subset \mathbb{N}} x_m, \ x_m \in K_m \right\}, \quad \text{for some infinite } M \subset \mathbb{N}.$$

Remark 6.2 Note that the limit defined above is a closed set.

It will be convenient for the proof of Theorem 2.8 to relate the power mean inequalities to the sequences formed by the Reynolds operator. Let $\mu = (\mu_1, \ldots, \mu_r)$ be a partition of $2d$. Associate to μ the monomial $X_1^{\mu_1} \cdots X_r^{\mu_r}$ and define a symmetric form $m_\mu^{(n)}$ by:

$$m_\mu^{(n)} = \mathrm{sym}_n \, X_1^{\mu_1} \cdots X_r^{\mu_r}.$$

This is a monomial mean basis of $H_{n,2d}^S$. We observe that with this choice of basis of $H_{n,2d}^S$ the transition maps $\varphi_{m,n}$ are given by the identity matrices. Since the stabilizer of the monomial $X_1^{\mu_1} \cdots X_r^{\mu_r}$ is isomorphic to $\mathcal{S}_{s_1} \times \ldots \times \mathcal{S}_{s_t} \times \mathcal{S}_{n-r}$, it follows that

$$m_\mu^{(n)} = \frac{s_1! \cdots s_k!(n-r)!}{n!} \bar{m}_\mu^{(n)} = \binom{n}{s_1 \ldots s_k}^{-1} \bar{m}_\mu^{(n)},$$

where $\bar{m}_\mu^{(n)}$ is the monomial symmetric polynomial associated to μ.

Proposition 6.3 *Consider the sequences $\Sigma_{n,2d}^\varphi$ and $\mathcal{P}_{n,2d}^\varphi$ embedded into $\mathbb{R}^{\pi(2d)}$ via the monomial mean basis. Then the limits of the resulting sequences of convex cones in $\mathbb{R}^{\pi(2d)}$ have limits, which we will denote by $\mathfrak{S}_{2d}^\varphi$ and $\mathfrak{P}_{2d}^\varphi$. Both of these limits are closed and full-dimensional.*

Proof Since we have $\varphi_{n,n+1}(\Sigma_{n,2d}) \subseteq \Sigma_{n+1,2d}^S$ and $\varphi_{n,n+1}(P_{n,2d}) \subseteq \mathcal{P}_{n+1,2d}^S$ the resulting sequences of cones are increasing. Thus by [32, Prop. 1] the limits exist and are given by

$$\mathfrak{S}_{2d}^\varphi = \overline{\left\{ f^{(n)} := \sum_{\lambda \vdash 2d} c_\lambda m_\lambda^{(n)} \text{ with } f^{(m)} \in \Sigma_{m,2d} \text{ for one } m \in \mathbb{N} \right\}},$$

$$\mathfrak{P}_{2d}^\varphi = \overline{\left\{ f^{(n)} := \sum_{\lambda \vdash 2d} c_\lambda m_\lambda^{(n)} \text{ with } f^{(m)} \in \mathcal{P}_{m,2d} \text{ for one } m \in \mathbb{N} \right\}}.$$

Clearly, both cones are full-dimensional. $\square$

In order to establish the result for the power mean basis, we first have to study the relationship between these two bases:

Proposition 6.4 *Let M_n be the matrix acting between the monomial mean and power mean bases of $H_{n,2d}^S$. Then M_n converges entry-wise to a full rank matrix M^* as n grows to infinity.*

Proof The transition matrix between power sum symmetric polynomials and monomial symmetric polynomials is well understood [2]. Converting to our mean bases we have the following: let $\nu := (\nu_1, \ldots, \nu_l) \vdash 2d$, $\mu = (\mu_1, \ldots, \mu_r) \vdash 2d$, then

$$m_\mu^{(n)} = \sum_{\nu \vdash 2d} (-1)^{r-l} \frac{(n-r)! |\mathcal{BL}(\mu)^\nu|}{n!} n^l p_\nu^{(n)},$$

where $|\mathcal{BL}(\mu)^\nu|$ is the number of μ-brick permutations of shape ν [2]. We observe that the unique highest order of growth in n for a coefficient of $p_\nu^{(n)}$ occurs when the number of parts of ν is maximized. The unique ν with the largest number of parts and non-zero $|\mathcal{BL}(\mu)^\nu|$ is μ. Thus we have $\nu = \mu$, $r = l$, and

$$|\mathcal{BL}(\nu)^\nu| = \nu_1! \cdots \nu_l! \quad \text{and} \quad \lim_{n \to \infty} \frac{n^r (n-r)!}{n!} = 1.$$

Therefore we see that asymptotically

$$m_\mu^{(n)} = p_\mu^{(n)} + \sum_{v \vdash 2d,\, v \neq \mu} a_{\mu,v}(n)\, p_v^{(n)},$$

where the coefficients $a_{\mu,v}(n)$ tend to 0 as $n \to \infty$. The proposition now follows. □

Now with these preparations the proof of Theorem 2.8 will be immediate after the following two lemmas.

Lemma 6.5 *Let V be a finite-dimensional vector space. Let A_i be a sequence of subsets of V converging to A. Let M_i be a sequence of linear maps from V to itself converging to identity. Let $B_i = M_i(A_i)$. Then*

$$A \subseteq \liminf_{i \to \infty} B_i \quad and \quad \limsup_{i \to \infty} B_i \subseteq A,$$

so A is the limit of B_i.

Proof For the first inclusion, let $a \in A$. Since A is the limit of A_i there exists N such that for all $i \geq N$, a is contained in A_i. Let $b_i = M_i a$. Then $b_i \in B_i$ for all $i \geq N$ and moreover, since the linear maps M_i converge to identity, we have that b_i converges to a. This in turn implies that $a \in \liminf_{i \to \infty} B_i$. For the second inclusion, we remark that $\left(\limsup_{i \to \infty} B_i\right)^c = \liminf_{i \to \infty} B_i^c$. Hence, one can argue in an analogous way by considering the complement of A. □

From the above lemma we can easily obtain the following generalization, which shows that the conclusions also hold if the limit of the linear maps M_i is any full-rank map.

Lemma 6.6 *Let V be a finite-dimensional vector space. Let A_i be a sequence of subsets of V converging to A. Let M_i be a sequence of linear maps from V to itself, converging to a full-rank linear map M. Let $B_i = M_i(A_i)$. Then*

$$M(A) \subseteq \liminf_{i \to \infty} B_i \quad and \quad \limsup_{i \to \infty} B_i \subseteq M(A),$$

so $M(A)$ is the limit of B_i.

Proof We can apply Lemma 6.5 to the sequence $C_i = M^{-1} M_i(A_i)$. Since $B_i = M(C_i)$, and M is a full-rank linear map, the desired conclusions follow for the sequence B_i as well. □

The existence of the limits of sequences and their full-dimensionality now can be established by translating from Proposition 6.3.

Proof of Theorem 2.8 We only give the proof for $\mathfrak{S}_{2d}$ since the statement for $\mathfrak{P}_{2d}$ follows in an analogous manner. We first observe that the sequence $\Sigma_{n,2d}^\rho$ is semi-nested; it follows that $\liminf \Sigma_{n,2d}^\rho = \bigcap_{n \geq 2d} \Sigma_{n,2d}^\rho$. We now apply Lemma 6.5 to the sequence $\Sigma_{n,2d}^\rho$, with $A_i = \Sigma_{n,2d}^\varphi$ and M_i being transition maps between monomial

mean and power mean bases. From Proposition 6.4 we know that the maps M_i converge to identity. Therefore, we see that

$$\mathfrak{S}_{2d} \subseteq \bigcap_{n \geq 2d} \Sigma^{\rho}_{n,2d} \quad \text{and} \quad \limsup \Sigma^{\rho}_{n,2d} \subseteq \mathfrak{S}_{2d}.$$

The theorem now follows, since the full-dimensionality is a direct consequence of Proposition 6.3.   □

7 Symmetric Mean Inequalities of Degree Four

In this last section we characterize quartic symmetric mean inequalities that are valid for all values of n. Recall from Section 2 that $\mathfrak{P}_4$ denotes the cone of all sequences $\mathfrak{f} = (f^{(4)}, f^{(5)}, \ldots)$ of degree 4 power means that are non-negative for all n and $\mathfrak{S}_4$ the cone of such sequences that can be written as sums of squares.

In the case of quartic forms the elements of $\mathfrak{P}_4$ can be characterized by a family of univariate polynomials as Theorem 3.4 specializes to the following

Proposition 7.1 *Let*

$$\mathfrak{f} := \sum_{\lambda \vdash 4} c_\lambda \mathfrak{p}_\lambda$$

be a linear combination of quartic symmetric power means. Then $\mathfrak{f} \in \mathfrak{P}_4$ if any only if for all $\alpha \in [0, 1]$ the bivariate form

$$\Phi^{\alpha}_{\mathfrak{f}}(x, y) = \Phi_{\mathfrak{f}}(\alpha, 1-\alpha, x, y) := \sum_{\lambda \vdash 4} c_\lambda \Phi_\lambda(\alpha, 1-\alpha, x, y)$$

is non-negative.

Now we turn to the characterization of the elements on the boundary of $\mathfrak{P}_4$.

Lemma 7.2 *Let $0 \neq \mathfrak{f} \in \mathfrak{P}_4$. Then $\mathfrak{f}$ is on the boundary $\partial \mathfrak{P}_4$ if and only if there exists $\alpha \in (0, 1)$ such that the bivariate form $\Phi^{\alpha}_{\mathfrak{f}}(x, y)$ has a double real root.*

Proof Let $\mathfrak{f} \in \partial \mathfrak{P}_4$. Suppose that for all $\alpha \in (0, 1)$ the bivariate form $\Phi^{\alpha}_{\mathfrak{f}}$ has no double real roots. From Proposition 7.1 we know that $\Phi^{\alpha}_{\mathfrak{f}}$ is a non-negative form for all $\alpha \in [0, 1]$ and thus $\Phi^{\alpha}_{\mathfrak{f}}$ is strictly positive for all $\alpha \in (0, 1)$. Thus for a sufficiently small perturbation $\tilde{\mathfrak{f}}$ of the coefficients c_λ of $\mathfrak{f}$ the form $\Phi^{\alpha}_{\tilde{\mathfrak{f}}}$ will remain positive for all $\alpha \in (0, 1)$. Now we deal with the cases $\alpha = 0, 1$.

We observe that for all $\mathfrak{g} \in H^{\rho}_{\infty,4}$,

$$\Phi^{0}_{\mathfrak{g}}(x, y) = \Phi^{1/2}_{\mathfrak{g}}(y, y) = \Phi^{1/2}_{\mathfrak{g}}(1, 1) y^4 \quad \text{and} \quad \Phi^{1}_{\mathfrak{g}}(x, y) = \Phi^{1/2}_{\mathfrak{g}}(1, 1) X^4.$$

By the above we must have $\Phi^{1/2}_{\mathfrak{g}}(1, 1) > 0$ and the same will be true for a sufficiently small perturbation $\tilde{\mathfrak{f}}$ of $\mathfrak{f}$. But then it follows by Proposition 7.1 that a neighborhood

of $\mathfrak{f}$ is in $\mathfrak{P}_4$, which contradicts the assumption that $\mathfrak{f} \in \partial\mathfrak{P}_4$. Therefore there exists $\alpha \in (0, 1)$ such that $\Phi_{\mathfrak{f}}^{\alpha}(x, y)$ has a double real root.

Now suppose $\mathfrak{f} \in \mathfrak{P}_4$ and $\Phi_{\mathfrak{f}}^{\alpha}(x, y)$ has a double real root for some $\alpha \in (0, 1)$. Let $\mathfrak{f}_{\epsilon} = f - \epsilon \mathfrak{p}_{2^2}$. It follows that for all $\epsilon > 0$ we have $\mathfrak{f}_{\epsilon} \notin \mathfrak{P}_4$, since $\Phi_{\mathfrak{f}_{\epsilon}}^{\alpha}$ is strictly negative at the double zero of $\Phi_{\mathfrak{f}}^{\alpha}$. Thus $\mathfrak{f}$ is on the boundary of $\mathfrak{P}_4$. □

We now deduce a corollary from Theorem 5.1, completely describing the polynomials in $\mathfrak{S}_4$.

Corollary 7.3 *We have $\mathfrak{f} \in \mathfrak{S}_4$ if and only if*

$$
\begin{aligned}
\mathfrak{f} &= \alpha_{11}\mathfrak{p}_{(1^4)} + 2\alpha_{12}\mathfrak{p}_{(2,1^2)} + \alpha_{22}\mathfrak{p}_{(2^2)} + \beta_{11}\big(\mathfrak{p}_{(2,1^2)} - \mathfrak{p}_{(1^4)}\big) \\
&\quad + 2\beta_{12}\big(\mathfrak{p}_{(3,1)} - \mathfrak{p}_{(2,1^2)}\big) + \beta_{22}\big(\mathfrak{p}_{(4)} - \mathfrak{p}_{(2^2)}\big),
\end{aligned}
$$

where the matrices $(\alpha_{ij})_{2\times 2}$ and $(\beta_{ij})_{2\times 2}$ are positive semidefinite.

Proof We observe from Theorem 5.1 that the coefficients of the squares of symmetric polynomials and of $(n-1, 1)$ semi-invariants do not depend on n. Thus the cone generated by these sums of squares is the same for any n, and it corresponds precisely to the cone given in the statement of the corollary. Now observe that the limit of the square of the $(n-2, 2)$ component is equal to $\mathfrak{p}_{(1^4)}/2 - \mathfrak{p}_{(21^2)} + \mathfrak{p}_{(2^2)}/2$, which is a sum of symmetric squares. Thus the squares from the $(n-2, 2)$ component do not contribute anything in the limit. □

In order to algebraically characterize the elements on the boundary recall that the discriminant disc f of a bivariate form f is a homogeneous polynomial in the coefficients of f, which vanishes exactly on the set of forms with multiple projective roots. However, note that disc $f = 0$ alone does not guarantee that f has a double real root, since the double root may be complex.

Proposition 7.4 *Let $\mathfrak{f} \in H_{\infty,4}^{\rho}$ be of the form*

$$
\mathfrak{f} = a^2\mathfrak{p}_{(1^4)} + b_{11}\big(\mathfrak{p}_{(2,1^2)} - \mathfrak{p}_{(1^4)}\big) + 2b_{12}\big(\mathfrak{p}_{(3,1)} - \mathfrak{p}_{(2,1^2)}\big) + b_{22}\big(\mathfrak{p}_{(4)} - \mathfrak{p}_{(2^2)}\big),
$$

such that the coefficients meet the conditions in (5.4). Then for $\alpha = 1/2$ the associated form $\Phi_{\mathfrak{f}}^{\alpha}(X, Y)$ has a double root at $(x, y) = (1, -1)$.

Lemma 7.5 *Let $\mathfrak{f} \in H_{\infty,4}^{\rho}$ be of the form*

$$
\mathfrak{f} = a^2\mathfrak{p}_4 + 2ab\mathfrak{p}_{31} + (c^2 - a^2)\mathfrak{p}_{22} + (2cd + b^2 - 2ab)\mathfrak{p}_{211} + (d^2 - b^2)\mathfrak{p}_{1111},
$$

such that the coefficients a, b, c, d meet the conditions in (5.3). Consider the associated form $\Phi_{\mathfrak{f}}^{\alpha}$. Then there is a value α, $0 < \alpha < 1$, such that $\Phi_{\mathfrak{f}}^{\alpha}$ has a real double root.

Proof We first show that there is a (possibly complex) double root by examining the discriminant. To this end, we find that this discriminant $\delta_{\mathfrak{f}}(\alpha)$ factors as

$$
\delta_{\mathfrak{f}}(\alpha) = 16(\alpha - 1)^3(c + d)^2\alpha^3\delta_1(\alpha)\delta_2(\alpha)^2,
$$

where δ_1 and δ_2 are quadratic polynomials in α. We examine these factors δ_1 and δ_2 now assuming the conditions on a, b, c, d imposed by (5.3).

One easily checks that $\delta_1(\alpha) = \delta_1(1 - \alpha)$ and $\delta_1(0) = \delta_1(1) = -16a^2(c+d)^2 < 0$. Further,

$$\delta_1\left(\frac{1}{2}\right) = -\frac{1}{4}(4a^2 + 4ab + 4cd + 4c^2 + b^2)^2 < 0.$$

This clearly implies that the quadratic polynomial δ_1 is strictly negative on $(0, 1)$. Moreover, the conditions in (5.3) yield $\delta_2(0) = \delta_2(1) = a^2(c+d) > 0$ and $\delta_2(1/2) = c(2a+b)^2/4 < 0$ since c is supposed to be negative.

It follows now that $\delta_2(\alpha)$ has two real roots $\alpha_1, 1 - \alpha_1 \in (0, 1)$ and hence the polynomial $\Phi_\mathfrak{f}(\alpha_{1/2}, 1 - \alpha_{1/2}, x, 1)$ has a double root. However, it remains to verify that this double root is indeed real. In order to establish this we examine the polynomial $\delta_2(\alpha, a, b, d, c)$ more carefully. We have

$$\delta_2(\alpha, a, b, c, d) := a^2 d + a^2 c + 4\alpha^2 a^2 d - 4a^2 \alpha d$$
$$- 4ab\alpha^2 c + 4ab\alpha c - \alpha^2 b^2 c + b^2 \alpha c,$$

and for $\alpha \neq 1/2$, one can solve for d to find

$$d = -c(a^2 - 4ab\alpha^2 + 4ab\alpha - \alpha^2 b^2 + b^2\alpha)(2a\alpha - a)^{-2}.$$

This yields that $\Phi_\mathfrak{f}(\alpha^*, 1 - \alpha^*, x, 1)$ contains the factor $(ax + a + b\alpha x - b\alpha + b)^2$ and hence in this case $\Phi_\mathfrak{f}(\alpha^*, 1 - \alpha^*, x, 1)$ has real double root.

In the case $\alpha = 1/2$ it follows from the observations made above, that at a root of $\delta_2(\alpha, a, b, c, d)$ also all second partial derivatives have to vanish. By explicit calculations one finds that this can happen only if $a = -1/2$ in which case the polynomial δ_2 specializes to $c(b-1)^2/4$. Since $c < 0$ it follows that only for $b = 1$ the discriminant can vanish. In this situation one gets

$$\Phi_\mathfrak{f}\left(\frac{1}{2}, \frac{1}{2}, x, 1\right) = \frac{1}{16}(d + 2dx + 2c + 2cx^2 + dx^2)^2$$

and it follows that $X_{1/2} := \pm\left(-d + 2\sqrt{c(d-c)}(2c+d) - 1\right)$ are the two double roots in this case. The conditions imposed on c, d by Theorem 5.3 ensure that $c(d-c) \geq 0$, and hence these roots will also be real. Therefore we have shown that in all cases the roots are indeed real.   □

We are now in the position to show that $\mathfrak{P}_4 = \mathfrak{S}_4$.

Proof of Theorem 2.9 Since $\mathfrak{S}_4 \subset \mathfrak{P}_4$ and both sets are closed convex cones, it suffices to show that every $\mathfrak{f}$ on the boundary of $\mathfrak{S}_4$ also lies in the boundary of $\mathfrak{P}_4$. It follows from Theorem 5.3 that a sequence $\mathfrak{f} := (f^{(4)}, f^{(5)}, \ldots)$ in the boundary of $\mathfrak{S}_4$ that is not in the boundary of $\mathfrak{P}_4$ has to be a form as considered in Lemma 7.5, but by combining Lemmas 7.5 and 7.2 we find that $\mathfrak{f} \in \partial\mathfrak{S}_4$ implies $\mathfrak{f} \in \partial\mathfrak{P}_4$, and we can conclude that $\mathfrak{S}_4 = \mathfrak{P}_4$.   □

8 Conclusion and Open Questions

Besides Conjecture 1 there is another important question left open in our work. Corollary 7.3 gave a description of the asymptotic symmetric sums of squares cone in terms of the squares involved. In this description of the limit not all semi-invariant polynomials were necessary. It is natural to investigate the situation also in arbitrary degree:

Question 1 *Let $\mathfrak{f} \in \mathfrak{S}_{2d}$. What semi-invariant polynomials are necessary for a description of $\mathfrak{f}$ as a sum of squares?*

The general setup of our work focused on the case of a fixed degree. Examples like the difference of the geometric and the arithmetic mean show however, that it would be very interesting to understand also the situation where the degree is not fixed.

Question 2 *What can be said about the quantitative relationship between the cones $\Sigma_{n,2d}^{S}$ and $\mathcal{P}_{n,2d}^{S}$ in asymptotic regimes other than fixed degree $2d$?*

References

1. Ariki, S., Terasoma, T., Yamada, H.-F.: Higher Specht polynomials. Hiroshima Math. J. **27**(1), 177–188 (1997)
2. Beck, D.A., Remmel, J.B., Whitehead, T.: The combinatorics of transition matrices between the bases of the symmetric functions and the B_n analogues. Discrete Math. **153**(1–3), 3–27 (1996)
3. Blekherman, G.: There are significantly more nonnegative polynomials than sums of squares. Israel J. Math. **153**, 355–380 (2006)
4. Blekherman, G.: Nonnegative polynomials and sums of squares. J. Am. Math. Soc. **25**(3), 617–635 (2012)
5. Blekherman, G., Parrilo, P.A., Thomas, R.R. (eds.): Semidefinite Optimization and Convex Algebraic Geometry. MOS-SIAM Series on Optimization, vol. 13. Society for Industrial and Applied Mathematics & Mathematical Optimization Society, Philadelphia (2013)
6. Blekherman, G., Sinn, R.: Extreme rays of Hankel spectrahedra for ternary forms. J. Symb. Comput. **79**(1), 23–42 (2017)
7. Choi, M.-D., Lam, T.-Y.: Extremal positive semidefinite forms. Math. Ann. **231**(1), 1–18 (1977)
8. Choi, M.D., Lam, T.Y., Reznick, B.: Even symmetric sextics. Math. Z. **195**(4), 559–580 (1987)
9. Cuttler, A., Greene, C., Skandera, M.: Inequalities for symmetric means. Eur. J. Comb. **32**(6), 745–761 (2011)

10. Ergür, A.A.: Multihomogeneous nonnegative polynomials and sums of squares. Discrete Comput. Geom. **60**(2), 318–344 (2018)
11. Frenkel, P.E., Horváth, P.: Minkowski's inequality and sums of squares. Cent. Eur. J. Math. **12**(3), 510–516 (2014)
12. Gatermann, K., Parrilo, P.A.: Symmetry groups, semidefinite programs, and sums of squares. J. Pure Appl. Algebra **192**(1–3), 95–128 (2004)
13. Goel, C., Kuhlmann, S., Reznick, B.: On the Choi–Lam analogue of Hilbert's 1888 theorem for symmetric forms. Linear Algebra Appl. **496**, 114–120 (2016)
14. Goel, C., Kuhlmann, S., Reznick, B.: The analogue of Hilbert's 1888 theorem for even symmetric forms. J. Pure Appl. Algebra **221**(6), 1438–1448 (2017)
15. Harris, W.R.: Real even symmetric ternary forms. J. Algebra **222**(1), 204–245 (1999)
16. Hilbert, D.: Ueber die Darstellung definiter Formen als Summe von Formenquadraten. Math. Ann. **32**(3), 342–350 (1888)
17. Hurwitz, A.: Ueber den Vergleich des arithmetischen und des geometrischen Mittels. J. Reine Angew. Math. **108**, 266–268 (1891)
18. James, G., Kerber, A.: Representation Theory of the Symmetric Group. Encyclopedia of Mathematics and its Applications, vol. 16. Addison-Wesley, Reading (1981)
19. Kuratowski, C.: Topologie. Vol. I. Monografie Matematyczne, vol. 20. Państwowe Wydawnictwo Naukowe, Warsaw (1958)
20. Macdonald, I.G.: Symmetric Functions and Hall Polynomials. Oxford Mathematical Monographs. Oxford University Press, New York (1995)
21. Mosco, U.: Convergence of convex sets and of solutions of variational inequalities. Adv. Math. **3**(4), 510–585 (1969)
22. Motzkin, T.S.: The arithmetic-geometric inequality. In: 1967 Inequalities (Proc. Sympos. Wright–Patterson Air Force Base, Ohio 1965), pp. 205–224. Academic Press, New York (1967)
23. Reznick, B.: Some inequalities for products of power sums. Pac. J. Math. **104**(2), 443–463 (1983)
24. Reznick, B.: Some concrete aspects of Hilbert's 17th Problem. In: Real Algebraic Geometry and Ordered Structures (Baton Rouge 1996). Contemp. Math., vol. 253, pp. 251–272. American Mathematical Society, Providence (2000)
25. Riener, C.: Symmetries in Semidefinite and Polynomial Optimization. PhD thesis, Johann Wolfgang Goethe-Universität, Frankfurt am Main (2011)
26. Riener, C.: On the degree and half-degree principle for symmetric polynomials. J. Pure Appl. Algebra **216**(4), 850–856 (2012)
27. Riener, C.: Symmetric semi-algebraic sets and non-negativity of symmetric polynomials. J. Pure Appl. Algebra **220**(8), 2809–2815 (2016)
28. Riener, C., Theobald, T., Andrén, L.J., Lasserre, J.B.: Exploiting symmetries in SDP-relaxations for polynomial optimization. Math. Oper. Res. **38**(1), 122–141 (2013)
29. Robinson, R.M.: Some definite polynomials which are not sums of squares of real polynomials. In: Selected Questions of Algebra and Logic, pp. 264–282. Izdat. "Nauka" Sibirsk. Otdel., Novosibirsk (1973). (in Russian)
30. Rotman, J.J.: Advanced Modern Algebra. Graduate Studies in Mathematics, vol. 114. American Mathematical Society, Providence (2010)
31. Sagan, B.E.: The Symmetric Group Representations, Combinatorial Algorithms, and Symmetric Functions. Graduate Texts in Mathematics, vol. 203. Springer, New York (2001)
32. Salinetti, G., Wets, R.J.-B.: On the convergence of sequences of convex sets in finite dimensions. SIAM Rev. **21**(1), 18–33 (1979)
33. Scheiderer, C.: Positivity and sums of squares: a guide to recent results. In: Emerging Applications of Algebraic Geometry. IMA Vol. Math. Appl., vol. 149, pp. 271–324. Springer, New York (2009)
34. Serre, J.-P.: Linear Representations of Finite Groups. Graduate Texts in Mathematics, vol. 42. Springer, New York (1977)
35. Specht, W.: Zur Darstellungstheorie der symmetrischen Gruppe. Math. Z. **42**(1), 774–779 (1937)
36. Stanley, R.P.: Invariants of finite groups and their applications to combinatorics. Bull. Am. Math. Soc. **1**(3), 475–511 (1979)
37. Terpstra, F.J.: Die Darstellung biquadratischer Formen als Summen von Quadraten mit Anwendung auf die Variationsrechnung. Math. Ann. **116**(1), 166–180 (1939)

38. Timofte, V.: On the positivity of symmetric polynomial functions. Part I: general results. J. Math. Anal. Appl. **284**(1), 174–190 (2003)

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