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SYMMETRIC NONNEGATIVE REALIZATION OF SPECTRA*

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Abstract. A perturbation result, due to R. Rado and presented by H. Perfect in 1955, shows how to modify r eigenvalues of a matrix of order n, $r \leq n$, via a perturbation of rank r, without changing any of the $n - r$ remaining eigenvalues. This result extended a previous one, due to Brauer, on perturbations of rank $r = 1$. Both results have been exploited in connection with the nonnegative inverse eigenvalue problem. In this paper a symmetric version of Rado's extension is given, which allows us to obtain a new, more general, sufficient condition for the existence of symmetric nonnegative matrices with prescribed spectrum.

Key words. Symmetric nonnegative inverse eigenvalue problem.

AMS subject classifications. 15A18, 15A51.

1. Introduction. The *real nonnegative inverse eigenvalue problem* (hereafter RNIEP) is the problem of characterizing all possible real spectra of entrywise $n \times n$ nonnegative matrices. For $n \geq 5$ the problem remains unsolved. In the general case, when the possible spectrum Λ is a set of complex numbers, the problem has only been solved for $n = 3$ by Loewy and London [11]. The complex cases $n = 4$ and $n = 5$ have been solved for matrices of trace zero by Reams [17] and Laffey and Meehan [10], respectively. Sufficient conditions or realizability criteria for the existence of a nonnegative matrix with a given real spectrum have been obtained in [25, 14, 15, 18, 8, 1, 19, 22] (see [3, §2.1] and references therein for a comprehensive survey). If we additionally require the realizing matrix to be symmetric, we have the *symmetric nonnegative inverse eigenvalue problem* (hereafter SNIEP). Both problems, RNIEP and SNIEP, are equivalent for $n \leq 4$ (see [26]), but are different otherwise [7]. Partial results for the SNIEP have been obtained in [4, 24, 16, 21, 23] (see [3, §2.2] and references therein for more on the SNIEP).

The origin of the present paper is a perturbation result, due to Brauer [2] (Theorem 2.2 below), which shows how to modify one single eigenvalue of a matrix via a rank-one perturbation, without changing any of the remaining eigenvalues. This result was first used by Perfect [14] in connection with the NIEP, and has given rise lately to a number of realizability criteria [19, 20, 22]. Closer to our approach in this paper is Rado's[1] extension (Theorem 2.3 below) of Brauer's result, which was used by

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[1]Perfect points out in [15] that both the extension and its proof are due to Professor R. Rado.

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA

Perfect in [15] to derive a sufficient condition for the RNIEP. Our goal in this paper is twofold: to obtain a symmetric version of Rado's extension and, as a consequence of it, to obtain a new realizability criterion for the SNIEP.

The paper is organized as follows: In section 2 we introduce some notation, basic concepts and results which will be needed throughout the paper, including Rado's Theorem and its new, symmetric version (Theorem 2.6 below). Based on this symmetric version, we give in section 3 a new criterion (Theorem 3.1) for the existence of a symmetric nonnegative matrix with prescribed spectrum, together with an explicit procedure to construct the realizing matrices. Section 4 is devoted to comparing this new criterion with some previous criteria for the SNIEP, and section 5 to illustrate the results with two specific examples.

2. Symmetric rank-r perturbations. Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a set of real numbers. We shall say that Λ is *realizable* (respectively, *symmetrically realizable*) if it exists an entrywise nonnegative (resp., a symmetric entrywise nonnegative) matrix of order n with spectrum Λ.

DEFINITION 2.1. A set K of conditions is said to be a *symmetric realizability criterion* if any set $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ satisfying the conditions K is *symmetrically* realizable.

A real matrix $A = (a_{ij})_{i,j=1}^n$ is said to have *constant row sums* if all its rows sum up to a same constant, say α, i.e.

$$\sum_{j=1}^{n} a_{ij} = \alpha, \qquad i = 1, \ldots, n.$$

The set of all real matrices with constant row sums equal to α is denoted by $\mathcal{CS}_\alpha$. It is clear that any matrix in $\mathcal{CS}_\alpha$ has eigenvector $\mathbf{e} = (1, 1, ...1)^T$ corresponding to the eigenvalue α. Denote by $\mathbf{e}_k$ the vector with one in the k-th position and zeros elsewhere.

The relevance of matrices with constant row sums in the RNIEP is due to the fact [6] that if λ_1 is the dominant element in Λ, then *the problem of finding a nonnegative matrix with spectrum* Λ *is equivalent to the problem of finding a nonnegative matrix in* $\mathcal{CS}_{\lambda_1}$ *with spectrum* Λ.

The following theorem, due to Brauer [2, Thm. 27], is relevant for the study of the nonnegative inverse eigenvalue problem. In particular, Theorem 2.2 plays an important role not only to derive sufficient conditions for realizability, but also to compute a realizing matrix.

THEOREM 2.2. (Brauer [2]) *Let A be an $n \times n$ arbitrary matrix with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. Let $\mathbf{v} = (v_1, v_2, ..., v_n)^T$ be an eigenvector of A associated with the eigenvalue λ_k and let $\mathbf{q}$ be any n-dimensional vector. Then the matrix $A + \mathbf{v}\mathbf{q}^T$ has eigenvalues $\lambda_1, \lambda_2, ..., \lambda_{k-1}, \lambda_k + \mathbf{v}^T\mathbf{q}, \lambda_{k+1}, ..., \lambda_n$.*

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA

The following result, due to R. Rado and presented by Perfect [15] in 1955, is an extension of Theorem 2.2. It shows how to change an arbitrary number r of eigenvalues of an $n \times n$ matrix A (with $n > r$) via a perturbation of rank r, without changing any of the remaining $n - r$ eigenvalues.

THEOREM 2.3. (Rado [15]) *Let A be an $n \times n$ arbitrary matrix with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ and let $\Omega = \text{diag}\{\lambda_1, \lambda_2, ..., \lambda_r\}$ for some $r \leq n$. Let X be an $n \times r$ matrix with rank r such that its columns $\mathbf{x}_1, \ldots, \mathbf{x}_r$ satisfy $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$, $i = 1, 2, ...r,$. Let C be an $r \times n$ arbitrary matrix. Then the matrix $A + XC$ has eigenvalues $\mu_1, \mu_2, ..., \mu_r, \lambda_{r+1}, \lambda_{r+2}, ..., \lambda_n$, where $\mu_1, \mu_2, ..., \mu_r$ are eigenvalues of the matrix $\Omega + CX$.*

Perfect used this extension to derive a realizability criterion for the RNIEP. Although it turns out to be a quite powerful result, inexplicably, this criterion was completely ignored for many years in the literature until it was brought up again in [22]. Perfect's criterion for the RNIEP is the following:

THEOREM 2.4. (Perfect [15]) *Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$ be such that*

$$-\lambda_1 \leq \lambda_k \leq 0, \qquad k = r+1, \ldots, n$$

for a certain $r < n$ and let $\omega_1, \ldots, \omega_r$ be nonnegative real numbers such that there exists an $r \times r$ nonnegative matrix $B \in \mathcal{CS}_{\lambda_1}$ with eigenvalues $\{\lambda_1, \ldots, \lambda_r\}$ and diagonal entries $\{\omega_1, \ldots, \omega_r\}$. If one can partition the set $\{\omega_1, \ldots, \omega_r\} \cup \{\lambda_{r+1}, \ldots, \lambda_n\}$ into r realizable sets $\Gamma_k = \{\omega_k, \lambda_{k2}, ..., \lambda_{kp_k}\}$, $k = 1, \ldots, r$, then Λ is also a realizable set.

Perfect complemented this result with conditions under which $\omega_1, \omega_2, ..., \omega_r$ are the diagonal entries of some $r \times r$ nonnegative matrix $B \in \mathcal{CS}_{\lambda_1}$ with spectrum $\{\lambda_1, \lambda_2, ..., \lambda_r\}$. For the particular case $r = 3$, these conditions, which are necessary and sufficient, are

$$\left.\begin{array}{l} i)\ \ 0 \leq \omega_k \leq \lambda_1, \quad k = 1, 2, 3 \\ ii)\ \ \omega_1 + \omega_2 + \omega_3 = \lambda_1 + \lambda_2 + \lambda_3 \\ iii)\ \ \omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3 \geq \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \\ iv)\ \ \max_k \omega_k \geq \lambda_2 \end{array}\right\}$$

with

$$B = \begin{bmatrix} \omega_1 & 0 & \lambda_1 - \omega_1 \\ \lambda_1 - \omega_2 - p & \omega_2 & p \\ 0 & \lambda_1 - \omega_3 & \omega_3 \end{bmatrix}, \tag{2.1}$$

(see also [22]) where

$$p = \frac{1}{\lambda_1 - \omega_3}\left[\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3 - \lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3\right].$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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To show the power of both Theorem 2.3 and Theorem 2.4, consider the following example; in which, as far as we know, no other realizability criterion is satisfied by the set Λ (except the extended Perfect criterion given in [22]):

EXAMPLE 2.5. Let $\Lambda = \{6, 3, 3, -5, -5\}$. We take the partition

$$\{6, -5\} \cup \{3, -5\} \cup \{3\}$$

with the associated realizable sets

$$\Gamma_1 = \{5, -5\}, \quad \Gamma_2 = \{5, -5\}, \quad \Gamma_3 = \{2\}.$$

Then

$$A = \begin{bmatrix} 0 & 5 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}; \quad X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$, $i = 1, 2, 3$. Now we need to compute a 3×3 matrix $B \in \mathcal{CS}_{\lambda_1}$ with eigenvalues $6, 3, 3$ and diagonal entries $5, 5, 2$. From (2.1), that matrix is

$$B = \begin{bmatrix} 5 & 0 & 1 \\ 1 & 5 & 0 \\ 0 & 4 & 2 \end{bmatrix}$$

and from it we compute the matrix

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{bmatrix}.$$

Then

$$A + XC = \begin{bmatrix} 0 & 5 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 5 & 0 \\ 1 & 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 & 2 \end{bmatrix}$$

is nonnegative with spectrum $\Lambda = \{6, 3, 3, -5, -5\}$.

We finish this section by proving its main result, namely a symmetric version of Rado's Theorem 2.3.

THEOREM 2.6. *Let A be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, and, for some $r \leq n$, let $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_r\}$ be an orthonormal set of eigenvectors of A*

Electronic Journal of Linear Algebra ISSN 1081-3810
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spanning the invariant subspace associated with $\lambda_1, ..., \lambda_r$. *Let* X *be the* $n \times r$ *matrix with* i*-th column* $\mathbf{x}_i$, *let* $\Omega = \mathrm{diag}\{\lambda_1, \ldots, \lambda_r\}$, *and let* C *be any* $r \times r$ *symmetric matrix. Then the symmetric matrix* $A + XCX^T$ *has eigenvalues* $\mu_1, \mu_2, ..., \mu_r, \lambda_{r+1}, ..., \lambda_n$, *where* $\mu_1, \mu_2, ..., \mu_r$ *are the eigenvalues of the matrix* $\Omega + C$.

Proof. Since the columns of X are an orthonormal set, we may complete X to an orthogonal matrix $W = [\, X \; Y \,]$, i.e., $X^TX = I_r,\ Y^TY = I_{n-r},\ X^TY = 0,\ Y^TX = 0$. Then

$$W^{-1}AW = \begin{bmatrix} X^T \\ Y^T \end{bmatrix} A \begin{bmatrix} X & Y \end{bmatrix} = \begin{bmatrix} \Omega & X^TAY \\ 0 & Y^TAY \end{bmatrix}$$
$$W^{-1}(XCX^T)W = \begin{bmatrix} I_r \\ 0 \end{bmatrix} C \begin{bmatrix} I_r & 0 \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore,

$$W^{-1}(A + XCX^T)W = \begin{bmatrix} \Omega + C & X^TAY \\ 0 & Y^TAY \end{bmatrix}$$

and $A + XCX^T$ is a symmetric matrix with eigenvalues $\mu_1, ..., \mu_r, \lambda_{r+1}, ..., \lambda_n$. □

3. A new criterion for symmetric nonnegative realization of spectra. The following result gives a realizability criterion for the SNIEP, that is, if Λ satisfies the criterion then Λ is realizable as the spectrum of a symmetric nonnegative matrix. It is a consequence of Theorem 2.6 in the same way as Theorem 2.4 follows from Theorem 2.3. In section 4 we show that Soto's realizability criterion [19, Theorem 17], which is also sufficient for the symmetric case, is contained in the criterion of Theorem 3.1 below. Example 5.2 in section 5 shows that the inclusion is strict.

THEOREM 3.1. *Let* $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ *be a set of real numbers with* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ *and, for some* $t \leq n$, *let* $\omega_1, \ldots, \omega_t$ *be real numbers satisfying* $0 \leq \omega_k \leq \lambda_1,\ k = 1, \ldots, t$. *Suppose there exist*

i) a partition $\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_t$, *with* $\Lambda_k = \{\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$, $\lambda_{11} = \lambda_1,\ \lambda_{k1} \geq 0;\ \lambda_{k1} \geq \ldots \geq \lambda_{kp_k}$, *such that for each* $k = 1, \ldots, t$ *the set* $\Gamma_k = \{\omega_k, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$ *is realizable by a symmetric nonnegative matrix of order* p_k, *and*

ii) a symmetric nonnegative $t \times t$ *matrix with eigenvalues* $\lambda_{11}, \lambda_{21}, \ldots, \lambda_{t1}$ *and diagonal entries* $\omega_1, \omega_2, \ldots, \omega_t$.

Then Λ *is realizable by a symmetric nonnegative matrix of order* n.

Proof. For each $k = 1, \ldots, t$, denote by A_k the symmetric nonnegative $p_k \times p_k$ matrix realizing Γ_k. We know from i) that the $n \times n$ matrix $A = \mathrm{diag}\{A_1, A_2, \ldots, A_t\}$ is symmetric nonnegative with spectrum $\Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_t$. Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ be an orthonormal set of eigenvectors of A associated, respectively, with $\omega_1, \ldots, \omega_r$. Then, the $n \times r$ matrix X with i-th column $\mathbf{x}_i$ satisfies $AX = X\Omega$ for $\Omega = \mathrm{diag}\{\omega_1, \omega_2, \ldots, \omega_t\}$. Moreover, X is entrywise nonnegative, since each $\mathbf{x}_i$ is a Perron vector of A_i. Now, let

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 16, pp. 1-18, January 2007

B be the symmetric nonnegative $t \times t$ matrix with spectrum $\{\lambda_1, \ldots, \lambda_t\}$ and diagonal entries $\omega_1, \omega_2, \ldots, \omega_t$. If we set $C = B - \Omega$, the matrix C is symmetric nonnegative, and $\Omega + C$ has eigenvalues $\lambda_1, \ldots, \lambda_t$. Therefore, by Theorem 2.6, the symmetric matrix $A + XCX^T$ has spectrum Λ. Moreover, it is nonnegative, since all the entries of A, X and C are nonnegative. □

Theorem 3.1 not only ensures the existence of a realizing matrix, but, as will be shown in the rest of this section, it also allows to construct the realizing matrix. Of course, the key is knowing under which conditions does there exist a symmetric nonnegative matrix B of order t with eigenvalues $\lambda_1, \ldots, \lambda_t$ and diagonal entries $\omega_1, \ldots, \omega_t$.

Necessary and sufficient conditions are known for the existence of a real, not necessarily nonnegative, symmetric matrix. They are due to Horn [5]: *There exists a real symmetric matrix with eigenvalues* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_t$ *and diagonal entries* $\omega_1 \geq \omega_2 \geq \ldots \geq \omega_t$ *if and only if the vector* $(\lambda_1, \ldots, \lambda_t)$ *majorizes the vector* $(\omega_1, \ldots, \omega_t)$, that is, if and only if

$$\left.\begin{array}{l} \sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \omega_i \text{ for } k = 1, 2, ..., t-1 \\ \sum_{i=1}^{t} \lambda_i = \sum_{i=1}^{t} \omega_i. \end{array}\right\} \tag{3.1}$$

From now on, we separate the study in four cases, depending on the number t of subsets in the partition of Λ.

3.1. The case $t = 2$. For $t = 2$ the conditions (3.1) become

$$\left.\begin{array}{c} \lambda_1 \geq \omega_1 \\ \lambda_1 + \lambda_2 = \omega_1 + \omega_2 \end{array}\right\},$$

and they are also sufficient for the existence of a 2×2 *symmetric* nonnegative matrix B with eigenvalues $\lambda_1 \geq \lambda_2$ and diagonal entries $\omega_1 \geq \omega_2 \geq 0$, namely,

$$B = \left[\begin{array}{cc} \omega_1 & \sqrt{(\lambda_1 - \omega_1)(\lambda_1 - \omega_2)} \\ \sqrt{(\lambda_1 - \omega_1)(\lambda_1 - \omega_2)} & \omega_2 \end{array}\right].$$

3.2. The case $t = 3$. There are also necessary and sufficient conditions, obtained by Fiedler [4], for the existence of a 3×3 symmetric nonnegative matrix with prescribed spectrum and diagonal entries:

LEMMA 3.2. (Fiedler [4]) *The conditions*

$$\left.\begin{array}{c} \lambda_1 \geq \omega_1 \\ \lambda_1 + \lambda_2 \geq \omega_1 + \omega_2 \\ \lambda_1 + \lambda_2 + \lambda_3 = \omega_1 + \omega_2 + \omega_3 \\ \lambda_2 \leq \omega_1 \end{array}\right\} \tag{3.2}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 1-18, January 2007

ELA

are necessary and sufficient for the existence of a 3×3 *symmetric nonnegative matrix* B *with eigenvalues* $\lambda_1 \geq \lambda_2 \geq \lambda_3$ *and diagonal entries* $\omega_1 \geq \omega_2 \geq \omega_3 \geq 0$.

REMARK 3.3. **The matrix** B **for** $t = 3$.

Using the proof of Theorem 4.4 in [4], one can write a procedure to construct the symmetric nonnegative matrix B in Lemma 3.2. The procedure is as follows:

1. Define $\mu = \lambda_1 + \lambda_2 - \omega_1$.
2. Construct the 2×2 symmetric nonnegative matrix

$$T = \begin{bmatrix} \omega_2 & \tau \\ \tau & \omega_3 \end{bmatrix}, \qquad \tau = \sqrt{(\mu - \omega_2)(\mu - \omega_3)}$$

   with eigenvalues μ and λ_3. Observe that, using (3.2), we have $\mu = \lambda_1 + \lambda_2 - \omega_1 \geq \omega_1 + \omega_2 - \omega_1 = \omega_2$.
3. Find a normalized Perron vector $\mathbf{u}$ of T

$$T\mathbf{u} = \mu\mathbf{u}, \qquad \mathbf{u}^T\mathbf{u} = 1.$$

4. Construct the 2×2 symmetric nonnegative matrix

$$S = \begin{bmatrix} \mu & s \\ s & \omega_1 \end{bmatrix}, \qquad s = \sqrt{(\lambda_1 - \mu)(\lambda_1 - \omega_1)}$$

   with eigenvalues λ_1 and λ_2. It follows from (3.2) that $\lambda_1 - \mu = \omega_1 - \lambda_2 \geq 0$.
5. By Lemma 2.2 in [4], the matrix

$$\widetilde{B} = \begin{bmatrix} T & s\mathbf{u} \\ s\mathbf{u}^T & \omega_1 \end{bmatrix}$$

   is symmetric nonnegative with the prescribed eigenvalues and diagonal entries. Finally, the matrix

$$B = \begin{bmatrix} \omega_1 & s\mathbf{u}^T \\ s\mathbf{u} & T \end{bmatrix},$$

   similar to $\widetilde{B}$, has the diagonal entries in the order $\omega_1 \geq \omega_2 \geq \omega_3$.

One can easily check that this procedure yields

$$B = \begin{bmatrix} \omega_1 & \sqrt{\frac{\mu-\omega_3}{2\mu-\omega_2-\omega_3}}\, s & \sqrt{\frac{\mu-\omega_2}{2\mu-\omega_2-\omega_3}}\, s \\ \sqrt{\frac{\mu-\omega_3}{2\mu-\omega_2-\omega_3}}\, s & \omega_2 & \sqrt{(\mu-\omega_2)(\mu-\omega_3)} \\ \sqrt{\frac{\mu-\omega_2}{2\mu-\omega_2-\omega_3}}\, s & \sqrt{(\mu-\omega_2)(\mu-\omega_3)} & \omega_3 \end{bmatrix}. \tag{3.3}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 1-18, January 2007

3.3. The case $t = 4$. For $t \geq 4$ we may use the following result:

THEOREM 3.4. (Fiedler [4]) *If $\lambda_1 \geq \ldots \geq \lambda_t$ and $\omega_1 \geq \ldots \geq \omega_t$ are such that*

$$\left.\begin{array}{ll} i) & \sum_{i=1}^{s} \lambda_i \geq \sum_{i=1}^{s} \omega_i,\ 1 \leq s \leq t-1 \\ ii) & \sum_{i=1}^{t} \lambda_i = \sum_{i=1}^{t} \omega_i \\ iii) & \lambda_k \leq \omega_{k-1},\ 2 \leq k \leq t-1 \end{array}\right\} \tag{3.4}$$

then there exists a $t \times t$ symmetric nonnegative matrix B with eigenvalues $\lambda_1, \ldots, \lambda_t$ and diagonal entries $\omega_1, \ldots, \omega_t$.

As before, a procedure to construct the symmetric nonnegative matrix B of Theorem 3.4 can be obtained for the case $t = 4$ following the proofs of Theorem 4.4, Lemma 4.1 and Theorem 4.8 in [4].

REMARK 3.5. **Construction of B for $t = 4$.**
Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ and $\omega_1 \geq \omega_2 \geq \omega_3 \geq \omega_4$ satisfy the conditions (3.4) above. Then, the following procedure leads to a symmetric nonnegative matrix B with eigenvalues λ_i and diagonal entries ω_i.

1. Define $\mu = \lambda_1 + \lambda_2 - \omega_1$.
2. Using the procedure in Remark 3.3, construct a 3×3 symmetric nonnegative matrix T with eigenvalues $\mu,\ \lambda_3,\ \lambda_4$ and diagonal entries $\omega_2,\ \omega_3,\ \omega_4$. Notice that these eigenvalues and diagonal entries satisfy the necessary and sufficient conditions given in Lemma 3.2:

$$\begin{aligned} \mu &= \lambda_1 + \lambda_2 - \omega_1 \geq \omega_1 + \omega_2 - \omega_1 = \omega_2 \\ \mu + \lambda_3 &= \lambda_1 + \lambda_2 + \lambda_3 - \omega_1 \geq \omega_1 + \omega_2 + \omega_3 - \omega_1 = \omega_2 + \omega_3 \\ \mu + \lambda_3 + \lambda_4 &= \lambda_1 + \lambda_2 - \omega_1 + \lambda_3 + \lambda_4 = \omega_1 + \omega_2 - \omega_1 + \omega_3 + \omega_4 \\ &= \omega_2 + \omega_3 + \omega_4 \\ \lambda_3 &\leq \omega_2 \end{aligned}$$

3. Find $\mathbf{u}$ such that

$$T\mathbf{u} = \mu\mathbf{u},\ \mathbf{u}^T\mathbf{u} = 1.$$

4. Construct the 2×2 symmetric nonnegative matrix

$$S = \begin{bmatrix} \mu & s \\ s & \omega_1 \end{bmatrix}, \qquad s = \sqrt{(\lambda_1 - \mu)(\lambda_1 - \omega_1)}$$

with eigenvalues λ_1 and λ_2. It follows from (3.2) that $\lambda_1 - \mu = \omega_1 - \lambda_2 \geq 0$.
5. By Lemma 2.2 in [4],

$$\widetilde{B} = \begin{bmatrix} T & s\mathbf{u} \\ s\mathbf{u}^T & \omega_1 \end{bmatrix}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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is a symmetric nonnegative matrix with the prescribed eigenvalues and diagonal entries. Finally, the matrix

$$B = \begin{bmatrix} \omega_1 & s\mathbf{u}^T \\ s\mathbf{u} & T \end{bmatrix},$$

similar to $\widetilde{B}$, has the diagonal entries in the order $\omega_1 \geq \omega_2 \geq \omega_3 \geq \omega_4$.

REMARK 3.6. *We observe that for $t \geq 4$ the conditions in Theorem 3.4 are only sufficient. The matrix*

$$B = \begin{bmatrix} 5 & 2 & \frac{1}{2} & \frac{1}{2} \\ 2 & 5 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 5 & 2 \\ \frac{1}{2} & \frac{1}{2} & 2 & 5 \end{bmatrix},$$

for instance, has eigenvalues $8, 6, 3, 3$ and its second largest eigenvalue is strictly larger than its largest diagonal entry.

3.4. The case $t \geq 5$. Recursively, the above procedure can be easily extended to $t \geq 5$

REMARK 3.7. **Construction of B for $t \geq 5$.**

Let $\lambda_1 \geq \ldots \geq \lambda_t$ and $\omega_1 \geq \ldots \geq \omega_t$ satisfying the sufficient conditions of Theorem 3.4.

1. Define $\mu = \lambda_1 + \lambda_2 - \omega_1$.
2. Using the above procedure recursively, construct a $(t-1) \times (t-1)$ symmetric nonnegative matrix T with eigenvalues $\mu, \lambda_3, \ldots, \lambda_t$ and diagonal entries $\omega_2, \omega_3, \ldots, \omega_t$. Notice that these eigenvalues and diagonal entries satisfy the conditions of Theorem 3.4:

$$\begin{aligned} \mu &= \lambda_1 + \lambda_2 - \omega_1 \geq \omega_1 + \omega_2 - \omega_1 = \omega_2 \\ \mu + \sum_{j=3}^{l} \lambda_j &= \lambda_1 + \lambda_2 - \omega_1 + \sum_{j=3}^{l} \lambda_j \\ &= \sum_{j=1}^{l} \lambda_j - \omega_1 \geq \sum_{j=2}^{l} \omega_j \text{ for } 3 \leq l \leq t-1. \\ \mu + \sum_{j=3}^{t} \lambda_j &= \lambda_1 + \lambda_2 - \omega_1 + \sum_{j=3}^{t} \lambda_j \\ &= \sum_{j=1}^{t} \lambda_j - \omega_1 = \sum_{j=2}^{t} \omega_j \\ \lambda_k &\leq \omega_{k-1} \text{ for } 3 \leq k \leq t-1 \end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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3. Find $\mathbf{u}$ such that

$$T\mathbf{u} = \mu\mathbf{u},\ \mathbf{u}^T\mathbf{u} = 1.$$

4. Construct the 2×2 symmetric nonnegative matrix

$$S = \begin{bmatrix} \mu & s \\ s & \omega_1 \end{bmatrix}, \qquad s = \sqrt{(\lambda_1 - \mu)(\lambda_1 - \omega_1)}$$

with eigenvalues λ_1 and λ_2. It follows from (3.2) that $\lambda_1 - \mu = \omega_1 - \lambda_2 \geq 0$.

5. By Lemma 2.2 in [4],

$$\widetilde{B} = \begin{bmatrix} T & s\mathbf{u} \\ s\mathbf{u}^T & \omega_1 \end{bmatrix}$$

is a symmetric nonnegative matrix with the prescribed eigenvalues and diagonal entries. Finally, the matrix

$$B = \begin{bmatrix} \omega_1 & s\mathbf{u}^T \\ s\mathbf{u} & T \end{bmatrix},$$

similar to $\widetilde{B}$, has the diagonal entries in the order $\omega_1 \geq \omega_2 \geq .... \geq \omega_t$.

4. Comparison with previous criteria. Several realizability criteria which were first obtained for the RNIEP have later been shown to be realizability criteria for the SNIEP as well. Kellogg's criterion [8], for instance, was shown by Fiedler [4] to imply symmetric realizability. Radwan [16] proved that Borobia's criterion [1] is also a criterion for symmetric realizability , and Soto's criterion for the RNIEP [19] (Theorem 4.2 below), which contains both Kellogg's and Borobia's criteria [20], was shown in [23] to be also a symmetric realizability criterion. In this section we compare the new result in this paper (Theorem 3.1) with some previous realizability criteria for SNIEP. First, we will show that Soto's criterion is actually contained in the new symmetric realizability criterion. Example 5.2 in section 5 shows that the inclusion is strict. Comparisons with results given in [9], [12] and [13] will also be discussed in this section. We begin by recalling Soto's criterion, first in a simplified version which displays the essential ingredients, and then in full generality.

THEOREM 4.1. (Soto [19]) *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a set of real numbers, satisfying $\lambda_1 \geq \ldots \geq \lambda_p \geq 0 > \lambda_{p+1} \geq \ldots \geq \lambda_n$. Let*

$$\begin{aligned} S_j &= \lambda_j + \lambda_{n-j+1},\ j = 2, 3, \ldots, \left[\frac{n}{2}\right] \quad \textit{and} \\ S_{\frac{n+1}{2}} &= \min\{\lambda_{\frac{n+1}{2}}, 0\} \quad \textit{for } n \textit{ odd.} \end{aligned} \tag{4.1}$$

If

$$\lambda_1 \geq -\lambda_n - \sum_{S_j < 0} S_j, \tag{4.2}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 16, pp. 1-18, January 2007

then Λ *is realizable by a nonnegative matrix* $A \in \mathcal{CS}_{\lambda_1}$.

It was shown in [21] that condition (4.2) is also sufficient for the existence of a symmetric nonnegative matrix with prescribed spectrum. In the context of Theorem 4.1 we define

$$T(\Lambda) = \lambda_1 + \lambda_n + \sum_{S_j<0} S_j$$

and observe that (4.2) is equivalent to $T(\Lambda) \geq 0$. If $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ satisfies the sufficient condition (4.2), then

$$\Lambda' = \{-\lambda_n - \sum_{S_j<0} S_j, \lambda_2, \ldots, \lambda_n\}$$

is a *symmetrically* realizable set. The number $-\lambda_n - \sum_{S_j<0} S_j$ is the minimum value that λ_1 may take in order that Λ be *symmetrically* realizable according to Theorem 4.1. Now, suppose that $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is partitioned as $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \ldots \cup \Lambda_t$. Then, according to Theorem 4.1, for each subset $\Lambda_k = \{\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$, $k = 1, 2, \ldots, t$, of the partition,

$$T(\Lambda_k) = T_k = \lambda_{k1} + \lambda_{kp_k} + \sum_{S_{kj}<0} S_{kj}. \tag{4.3}$$

Clearly, Λ_k is symmetrically realizable if and only if $T_k \geq 0$.

The following result is an extension of Theorem 4.1. As mentioned above, it is also a *symmetric* realizability criterion [23].

THEOREM 4.2. (Soto [19]) *Let* $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *as in Theorem 4.1. Let the partition* $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \ldots \cup \Lambda_t$ *with*

$$\Lambda_k = \{\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kp_k}\},\ k = 1, \ldots, t,\ \lambda_{11} = \lambda_1,\ \lambda_{k1} \geq 0,\ \lambda_{k1} \geq \ldots \geq \lambda_{kp_k}.$$

Let T_k *be defined as in (4.3), and let*

$$L = \max\{\lambda_1 - T_1;\ \max_{2\leq k\leq t}\{\lambda_{k1}\}\}. \tag{4.4}$$

If

$$\lambda_1 \geq L - \sum_{T_k<0} T_k, \tag{4.5}$$

then Λ *is realizable by a nonnegative matrix* $A \in \mathcal{CS}_{\lambda_1}$. *Moreover,* Λ *is symmetrically realizable.*

Now we show that the (*symmetric*) realizability criterion of Theorem 4.2 implies Theorem 3.1. Example 5.2 in section 5 shows that the inclusion is strict. We point

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 16, pp. 1-18, January 2007

out that Theorem 4.2 is a constructive criterion in the sense that it allows to compute an explicit realizing matrix. However, to construct a symmetric nonnegative matrix we need a different approach.

THEOREM 4.3. *If the conditions of Theorem 4.2 are satisfied, then the conditions of Theorem 3.1 are satisfied as well.*

Proof. Suppose condition (4.5) of Theorem 4.2 is satisfied. Without loss of generality, we may assume that $\lambda_1 = L - \sum_{T_k<0} T_k$, since increasing the dominant element of a set never leads to a loss of realizability. Consider the partition $\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_t$, $\Lambda_k = \{\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$, $k = 1, \ldots, t$, in Theorem 4.2. We define the sets

$$\Gamma_k = \{\omega_k, \lambda_{k2}, \ldots, \lambda_{kp_k}\}, \quad k = 1, 2, \ldots, t,$$

where

$$\begin{aligned} &\omega_1 = L \\ &\omega_k = \lambda_{k1} - T_k \ \text{ if } \ T_k < 0 \\ &\omega_k = \lambda_{k1} \ \text{ if } \ T_k \geq 0, \quad k = 2, 3, \ldots, t \end{aligned}$$

Then, using the symmetric realizability condition (4.2) given by Theorem 4.1, Γ_k is realizable by a $p_k \times p_k$ symmetric nonnegative matrix A_k. We now show, by checking conditions (3.4), the existence of a symmetric nonnegative matrix B with eigenvalues $\lambda_1, \lambda_{21}, \ldots, \lambda_{t1}$ and diagonal entries $\omega_1, \omega_2, \ldots, \omega_t$: since $\omega_1 = L = \lambda_1 + \sum_{T_k<0} T_k$, we have

$$\begin{aligned} \sum_{k=1}^{s} \lambda_{k1} &= \omega_1 - \sum_{T_k<0} T_k + \sum_{T_k<0,\ k\in\{2,\ldots,s\}} (\omega_k + T_k) + \sum_{T_k\geq 0,\ k\in\{2,\ldots,s\}} \omega_k \\ &= \sum_{k=1}^{s} \omega_k - \sum_{T_k<0,\ k\notin\{2,\ldots,s\}} T_k \\ &\geq \sum_{k=1}^{s} \omega_k, \quad s = 1, \ldots, t-1. \end{aligned}$$

This proves condition $i)$ in (3.4). Next,

$$\begin{aligned} \sum_{k=1}^{t} \lambda_{k1} &= \omega_1 - \sum_{T_k<0} T_k + \sum_{T_k<0,\ k\in\{2,\ldots,t\}} (\omega_k + T_k) + \sum_{T_k\geq 0,\ k\in\{2,\ldots,t\}} \omega_k \\ &= \sum_{k=1}^{t} \omega_k \end{aligned}$$

proves $ii)$. Finally, since

$$\text{if } T_{k-1} < 0 \ \text{ then } \ \lambda_{k1} \leq \lambda_{(k-1)1} = \omega_{k-1} + T_{k-1} \leq \omega_{k-1},$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 1-18, January 2007

or

$$\text{if } \; T_{k-1} \geq 0 \;\; \text{then } \;\; \lambda_{k1} \leq \lambda_{(k-1)1} = \omega_{k-1},$$

condition *iii*) in (3.4) also holds. Thus the conditions of Theorem 3.1 are satisfied. □

In [13], McDonald and Neumann denote as $\mathcal{R}_n$ the set of all points $\sigma = (1, \lambda_2, \ldots, \lambda_n)$, which correspond to spectra realizable by symmetric nonnegative matrices and as $\mathcal{S}_n$ the set of all $\sigma \in \mathcal{R}_n$, which are Soules realizable, that is, there exists an $n \times n$ symmetric nonnegative matrix A and a Soules matrix R such that $R^T AR = diag\{1, \lambda_2, \ldots, \lambda_n\}$. Then they show that $\mathcal{S}_n = \mathcal{R}_n$ for $n = 3$ and $n = 4$. In [9], Knudsen and McDonald establish that $\mathcal{S}_5$ is properly contained in $\mathcal{R}_5$. In particular they show that the point $m = (1, \frac{-1+\sqrt{5}}{4}, \frac{-1+\sqrt{5}}{4}, \frac{-1-\sqrt{5}}{4}, \frac{-1-\sqrt{5}}{4}) \in \mathcal{R}_5$ is not in $\mathcal{S}_5$ and that every point on the line segment from $l = (1, 0, 0, , -\frac{1}{2}, -\frac{1}{2})$ to m, correspond to a set $\{1, \lambda_2, \ldots, \lambda_5\}$, which is realizable by a symmetric nonnegative matrix.

In [3], Egleston et al. study the symmetric realizability of lists of five numbers $\{1, \lambda_2, \ldots, \lambda_5\}$ and point out that there are two cases where SNIEP is unknown. One of these cases is shown to be not realizable as the spectrum of a symmetric nonnegative matrix by using a necessary condition given in ([13], Lemma 4.1). Concerning to the second unresolved case, it is shown in [22] that every point on the line from l to m is also realized by a symmetric nonnegative *circulant* matrix. Conditions

$$\begin{array}{ll} i) & 1 > \lambda_2 \geq \lambda_3 > 0 > \lambda_4 \geq \lambda_5 \\ ii) & 1 + \lambda_2 + \lambda_4 + \lambda_5 < 0 \end{array} \tag{4.6}$$

in the second case (see [3]) imply that Theorem 3.1 gives no information about the realizability of the list $\Lambda = \{1, \lambda_2, \ldots, \lambda_5\}$: from Theorem 3.1 with $t = 3$ we have the partition

$$\begin{array}{c} \Lambda = \{1, \lambda_5\} \cup \{\lambda_2, \lambda_4\} \cup \{\lambda_3\} \quad \text{with} \\ \Gamma_1 = \{-\lambda_5, \lambda_5\}; \;\; \Gamma_2 = \{-\lambda_4, \lambda_4\}; \;\; \Gamma_3 = \{S\} \end{array}$$

where $S = 1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5$. From (4.6), *ii*) and Lemma 3.2 it is easy to see that there is no a 3×3 symmetric nonnegative matrix with eigenvalues and diagonal entries $1, \lambda_2, \lambda_3$ and $-\lambda_4, -\lambda_5, S$, respectively. The same occurs if we take $\omega_1 = -\lambda_5 + S$ or $\omega_1 = -\lambda_5 + \frac{S}{2}$ and $\omega_2 = -\lambda_4 + \frac{S}{2}$, with $\Gamma_3 = \{0\}$. If we take $t = 2$ in Theorem 3.1, that is, if we consider partitions as

$$\begin{array}{c} \Lambda = \{1, \lambda_5\} \cup \{\lambda_2, \lambda_3, \lambda_4\} \quad \text{with} \\ \Gamma_1 = \{-\lambda_5, \lambda_5\}; \;\; \Gamma_2 = \{-\lambda_4, \lambda_3, \lambda_4\}, \end{array}$$

then from (4.6), *ii*) we have $1 + \lambda_2 < -\lambda_4 - \lambda_5$ and therefore there is no a 2×2 symmetric nonnegative matrix with eigenvalues $1, \lambda_2$ and diagonal entries $-\lambda_4, -\lambda_5$.

A recent contribution to the solution of SNIEP for $n = 5$ is due to Loewy and McDonald [12]. They describe the possible patterns $(+, 0)$ for which there exists

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 16, pp. 1-18, January 2007

an extreme symmetric matrix with prescribed spectrum and show that one of these patterns yields to realizable points which have not been known previously. In [12] is presented a graph related with the second unresolved case in [3]. In particular this graph considers points of the form $(1, \lambda_2, \lambda_2, \lambda_3, \lambda_3)$, which are plotted with λ_2 on the horizontal axis and λ_3 on the vertical axis. Then the authors illustrate the boundaries of the following regions of realizable points:

i) The boundary $\lambda_3 = \frac{-1-\lambda_2}{2}$ of the Soules set $\mathcal{S}_5$, of the points that were shown to be realizable in [13]. We observe that every point in that boundary is indeed symmetrically realizable by Theorem 4.1 and consequently by Theorem 3.1.

ii) The boundary of the additional points that were identified as being realizable in [9] is the line segment from m to $a = (1, 1, 1, 1, 1)$. These points have the form $\alpha m + (1-\alpha)a$, that is:

$$(1, 1 + \frac{-5+\sqrt{5}}{4}\alpha, 1 + \frac{-5+\sqrt{5}}{4}\alpha, 1 - \frac{5+\sqrt{5}}{4}\alpha, 1 - \frac{5+\sqrt{5}}{4}\alpha), \tag{4.7}$$

where $0 \leq \alpha \leq 1$. Observe that for $\alpha \leq \frac{4}{5+\sqrt{5}}$, all points in (4.7) have only nonnegative entries. Moreover, positive entries $\lambda_2 = \lambda_3$ dominate $\lambda_4 = \lambda_5$ for $\alpha \leq \frac{4}{5}$ and therefore all these points are trivially realizable by the symmetric nonnegative matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{\lambda_2+\lambda_5}{2} & \frac{\lambda_2-\lambda_5}{2} & 0 & 0 \\ 0 & \frac{\lambda_2-\lambda_5}{2} & \frac{\lambda_2+\lambda_5}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda_3+\lambda_4}{2} & \frac{\lambda_3-\lambda_4}{2} \\ 0 & 0 & 0 & \frac{\lambda_3-\lambda_4}{2} & \frac{\lambda_3+\lambda_4}{2} \end{pmatrix}.$$

So, we consider points in (4.7) for which $\frac{4}{5} < \alpha \leq 1$. In order to see if Theorem 3.1 works here, we look for a 3×3 symmetric nonnegative matrix with eigenvalues and diagonal entries

$$\begin{aligned} \lambda_i : \ & 1, 1 + \frac{-5+\sqrt{5}}{4}\alpha, 1 + \frac{-5+\sqrt{5}}{4}\alpha \\ \omega_i : \ & -(1 - \frac{5+\sqrt{5}}{4}\alpha) + \beta, -(1 - \frac{5+\sqrt{5}}{4}\alpha) + \gamma, \delta, \end{aligned}$$

respectively, where $\beta+\gamma+\delta = 5(1-\alpha)$, the sum of the entries in $\alpha m+(1-\alpha)a$. Then by taking appropriately the numbers β, γ and δ, conditions of Lemma 3.2 are satisfied and the corresponding set $\Lambda = \{1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ is realizable by Theorem 3.1. The points on the line segment from m to a have the form $(1, \lambda_2, \lambda_2, \lambda_3, \lambda_3)$, so they can be written as an even-conjugate vector $(1, \lambda_2, \lambda_3, \lambda_3, \lambda_2)$. Then it is natural to analyze whether they are the spectrum of a symmetric nonnegative circulant matrix. This is the case for all points on the line from m to a.

iii) The boundary of the new additional points identified in [12] as realizable is given by a portion of the curve $\lambda_2\lambda_3 = -\frac{1}{4}$. The corresponding set of points is of the form $(1, \lambda_2, \lambda_2, -\frac{1}{4\lambda_2}, -\frac{1}{4\lambda_2})$ where $\frac{-1+\sqrt{5}}{4} \leq \lambda_2 \leq \frac{\sqrt{3}-1}{2}$. Let us see if Theorem 3.1 is

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 16, pp. 1-18, January 2007

satisfied for this kind of points: For $t = 3$ we have

$$\begin{aligned} \lambda_i &: \ 1, \lambda_2, \lambda_2 \\ \omega_i &: \ \frac{1}{4\lambda_2}, \frac{1}{4\lambda_2}, S, \end{aligned}$$

where $S = 1 + 2\lambda_2 - \frac{1}{2\lambda_2}$. Observe that in this case $S < \frac{1}{4\lambda_2}$. A straight forward calculation shows that $1 + \lambda_2 < \omega_1 + \omega_2$, which contradicts conditions of Lemma 3.2. For $t = 2$, $\lambda_i : 1, \lambda_2$ and $\omega_i : \frac{1}{4\lambda_2}, \frac{1}{4\lambda_2}$. Conditions $1 \geq \omega_1$ and $1 + \lambda_2 = \omega_1 + \omega_2$ are only satisfied for $\lambda_2 = \frac{\sqrt{3}-1}{2}$, that is, for a point on the intersection with the boundary $\lambda_3 = \frac{-1-\lambda_2}{2}$ of the Soules set $\mathcal{S}_5$. Hence, Theorem 3.1 gives no information about realizability of these points.

5. Examples.

EXAMPLE 5.1. Let $\Lambda = \{7, 5, 1, -3, -4, -6\}$. Consider the partition

$$\Lambda_1 = \{7, -6\}, \ \ \Lambda_2 = \{5, -4\}, \ \ \Lambda_3 = \{1, -3\}$$

and the associated realizable sets

$$\Gamma_1 = \{6, -6\}, \ \ \Gamma_2 = \{4, -4\}, \ \ \Gamma_3 = \{3, -3\}.$$

Here $\lambda_1 = 7, \ \lambda_2 = 5, \ \lambda_3 = 1$ and $\omega_1 = 6, \ \omega_2 = 4, \ \omega_3 = 3$. The conditions in Lemma 3.2 are satisfied. Then there exists a symmetric nonnegative matrix B with eigenvalues $7, \ 5, \ 1$ and diagonal entries $6, \ 4, \ 3$. We have $\mu = 7 + 5 - 6 = 6$. From (3.3)

$$B = \begin{bmatrix} 6 & \sqrt{\frac{3}{5}} & \sqrt{\frac{2}{5}} \\ \sqrt{\frac{3}{5}} & 4 & \sqrt{6} \\ \sqrt{\frac{2}{5}} & \sqrt{6} & 3 \end{bmatrix}.$$

Clearly,

$$A_1 = \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}, \ \ A_2 = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}, \ \ A_3 = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$$

are symmetric nonnegative matrices realizing $\Gamma_1, \ \Gamma_2, \ \Gamma_3$, respectively. Let $\Omega = diag\{6, 4, 3\}$. Then the matrix C defined in the proof of Theorem 3.1 is

$$C = \begin{bmatrix} 0 & \sqrt{\frac{3}{5}} & \sqrt{\frac{2}{5}} \\ \sqrt{\frac{3}{5}} & 0 & \sqrt{6} \\ \sqrt{\frac{2}{5}} & \sqrt{6} & 0 \end{bmatrix},$$

while the matrices A and X are

$$A = diag\left\{A_1, A_2, A_3\right\},$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 1-18, January 2007

$$X = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

Then, the symmetric nonnegative matrix

$$A + XCX^T = \begin{bmatrix} 0 & 6 & \frac{1}{2}\sqrt{\frac{3}{5}} & \frac{1}{2}\sqrt{\frac{3}{5}} & \frac{1}{2}\sqrt{\frac{2}{5}} & \frac{1}{2}\sqrt{\frac{2}{5}} \\ 6 & 0 & \frac{1}{2}\sqrt{\frac{3}{5}} & \frac{1}{2}\sqrt{\frac{3}{5}} & \frac{1}{2}\sqrt{\frac{2}{5}} & \frac{1}{2}\sqrt{\frac{2}{5}} \\ \frac{1}{2}\sqrt{\frac{3}{5}} & \frac{1}{2}\sqrt{\frac{3}{5}} & 0 & 4 & \frac{1}{2}\sqrt{6} & \frac{1}{2}\sqrt{6} \\ \frac{1}{2}\sqrt{\frac{3}{5}} & \frac{1}{2}\sqrt{\frac{3}{5}} & 4 & 0 & \frac{1}{2}\sqrt{6} & \frac{1}{2}\sqrt{6} \\ \frac{1}{2}\sqrt{\frac{2}{5}} & \frac{1}{2}\sqrt{\frac{2}{5}} & \frac{1}{2}\sqrt{6} & \frac{1}{2}\sqrt{6} & 0 & 3 \\ \frac{1}{2}\sqrt{\frac{2}{5}} & \frac{1}{2}\sqrt{\frac{2}{5}} & \frac{1}{2}\sqrt{6} & \frac{1}{2}\sqrt{6} & 3 & 0 \end{bmatrix}$$

has the prescribed eigenvalues $7, 5, 1, -3, -4, -6$.

EXAMPLE 5.2. Let $\Lambda = \{7, 5, 1, 1, -4, -4, -6\}$. The conditions of Theorem 4.2 are not satisfied. However, the conditions for the new symmetric realizability criterion, Theorem 3.1, are satisfied: consider the partition $\Lambda = \Lambda_1 \cup \Lambda_2$, where $\Lambda_1 = \{7, -6\}$, $\Lambda_2 = \{5, 1, 1, -4, -4\}$, with associated symmetrically realizable sets

$$\Gamma_1 = \{6, -6\}, \quad \Gamma_2 = \{6, 1, 1, -4, -4\}.$$

It is clear that there exists a 2×2 symmetric nonnegative matrix with diagonal entries $\omega_1 = 6$, $\omega_2 = 6$ and eigenvalues $\lambda_1 = 7$, $\lambda_2 = 5$, namely,

$$B = \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}.$$

By Theorem 3.1, there exists an 7×7 symmetric nonnegative matrix M with the prescribed spectrum Λ. Now we compute the matrix M. Firstly we obtain the symmetric matrices A_1, realizing Γ_1, and A_2, realizing Γ_2, (see [22] for the way in which we obtain A_2):

$$A_1 = \begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} \\ \frac{3+\sqrt{5}}{2} & 0 & \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} & 0 & \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} & 0 & \frac{3+\sqrt{5}}{2} \\ \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} & 0 \end{bmatrix}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 16, pp. 1-18, January 2007

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Then

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$X = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{5}}{5} \\ 0 & \frac{\sqrt{5}}{5} \\ 0 & \frac{\sqrt{5}}{5} \\ 0 & \frac{\sqrt{5}}{5} \\ 0 & \frac{\sqrt{5}}{5} \end{bmatrix}$$

Therefore,

$$M = A + XCX^T = \begin{bmatrix} 0 & 6 & \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{10} \\ 6 & 0 & \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{10} \\ \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{10} & 0 & \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} \\ \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{10} & \frac{3+\sqrt{5}}{2} & 0 & \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{10} & \frac{3-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} & 0 & \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{10} & \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} & 0 & \frac{3+\sqrt{5}}{2} \\ \frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{10} & \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} & 0 \end{bmatrix}$$

is a symmetric nonnegative matrix realizing Λ.

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