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SYMMETRIC POSITIVE SYSTEMS WITH BOUNDARY CHARACTERISTIC OF CONSTANT MULTIPLICITY

BY

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ABSTRACT. The theory of maximal positive boundary value problems for symmetric positive systems is developed assuming that the boundary is characteristic of constant multiplicity. No such hypothesis is needed on a neighborhood of the boundary. Both regularity theorems and mixed initial boundary value problems are discussed. Many classical ideas are sharpened in the process.

1. Introduction. Suppose that $\Omega \subset \mathbf{R}^\nu$ is a bounded open set lying on one side of its C^1 boundary $\partial\Omega$. In $\overline{\Omega}$ suppose that

$$L = \sum_{j=1}^{\nu} A_j(x)\partial_j + B(x) \tag{1}$$

is a first order system of differential operators with

$$A_j \in \operatorname{Lip}(\overline{\Omega}: \operatorname{Hom}(\mathbf{C}^k)), \tag{2}$$

$$B \in L^\infty(\overline{\Omega}: \operatorname{Hom}(\mathbf{C}^k)). \tag{3}$$

We are interested in boundary value problems for the system

$$Lu = f \in \mathscr{L}^2(\Omega). \tag{4}$$

Our first result is concerned with Green's identity,

$$\int_\Omega \langle Lu, v\rangle = \int_\Omega \langle u, L^*v\rangle + \int_{\partial\Omega} \langle A_n u, v\rangle d\sigma \tag{5}$$

when L^* is the formal adjoint of L, $n = (n_1, n_2, \dots, n_\nu)$ is the unit outward normal to $\partial\Omega$ and $A_n \equiv \Sigma_j n_j A_j$. It has long been recognized that if $u \in \mathscr{L}^2(\Omega)$ and $Lu \in \mathscr{L}^2(\Omega)$, then $A_n u|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ and Green's identity holds for $v \in H^1(\Omega)$. In fact, less is needed. Let

$$\mathscr{K}_L = \{ u \in \mathscr{L}^2(\Omega) | Lu \in H^1(\Omega)' \}, \quad \|u\|^2_{\mathscr{K}_L} = \|u\|^2_{\mathscr{L}^2(\Omega)} + \|Lu\|^2_{H^1(\Omega)'},$$

$$\mathscr{H}_L = \{ u \in \mathscr{L}^2(\Omega) | Lu \in \mathscr{L}^2(\Omega) \}, \quad \|u\|^2_{\mathscr{H}_L} = \|u\|^2_{\mathscr{L}^2(\Omega)} + \|Lu\|^2_{\mathscr{L}^2(\Omega)}.$$

The space $\mathscr{H}_{L^*}$ is defined similarly.

PROPOSITION 1. *$\mathscr{K}_L$ and $\mathscr{H}_L$ are Hilbert spaces and $C^1(\overline{\Omega})$ is dense in each of them.*

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The proof of this and other results is given in later sections. If $u \in \mathscr{L}^2(\Omega)$, then automatically $Lu \in \mathring{H}^1(\Omega)'$ so $\mathscr{H}_L$ is only slightly smaller than $\mathscr{L}^2(\Omega)$, the restriction coming near $\partial\Omega$.

THEOREM 1. *The map*

$$C^1(\bar{\Omega}) \ni u \mapsto A_n u|_{\partial\Omega}$$

extends uniquely to a continuous linear map $\mathscr{H}_L \to H^{-1/2}(\partial\Omega)$ *and Green's identity* (5) *holds for* $u \in \mathscr{H}_L$, $v \in H^1(\Omega)$.

In the proofs of energy inequalities one wants to take $u = v$ and for that purpose this theorem is not sufficient. For $X \subset \mathbf{R}^\nu$ we denote by $\mathrm{Lip}(X)$ the set of uniformly Lipshitzean functions on X normed by

$$\|u\|_{\mathrm{Lip}(X)} = \sup_{x \in X} \|u(x)\| + \sup_{\substack{x, y \in X \times X \\ x \neq y}} \frac{\|u(x) - u(y)\|}{\|x - y\|}.$$

THEOREM 2. *The map*

$$C^1(\bar{\Omega}) \times C^1(\bar{\Omega}) \ni (u, v) \mapsto \langle A_n u, v\rangle|_{\partial\Omega}$$

extends uniquely to a continuous bilinear map $\mathscr{H}_L \times \mathscr{H}_{L^*} \to \mathrm{Lip}(\partial\Omega)'$ *and Green's identity* (5) *holds for* $(u, v) \in \mathscr{H}_L \times \mathscr{H}_{L^*}$.

REMARK. The space $\mathrm{Lip}(\partial\Omega)' \subset \mathscr{D}'(\partial\Omega)$ does not have a useful elementary description.

REMARK. The boundary integral in Green's identity is interpreted as the action of $\langle A_n u, v\rangle$ on the Lipshitz continuous function 1.

The next result expresses the idea that traces on nearby surfaces are close. Note that $u \in \mathscr{H}_L$ (resp. $u \in \mathscr{H}_L$) implies that $\phi u \in \mathscr{H}_L$ (resp. $\mathscr{H}_L$) for $\phi \in C^1(\bar{\Omega})$. Thus it suffices to consider functions supported in a small neighborhood of a point $p \in \partial\Omega$. Introduce local coordinates (x_1, x') near p so that Ω becomes $\{|x| < 1 \text{ and } x_1 > 0\}$. Theorem 2 then implies that for $\varepsilon \geqslant 0$, and $u \in \mathscr{H}_L$, $A_1 u|_{x_1 = s} \in H^{-1/2}(\mathbf{R}^{\nu-1}) \cap \mathscr{E}'(\mathbf{R}^{\nu-1})$, where A_1 comes from the expression for L in the new coordinates.

THEOREM 3. *If* $u \in \mathscr{H}_L$ *is supported in the coordinate patch above, then the map*

$$\overline{\mathbf{R}}_+ \ni s \mapsto A_1\, u|_{x_1 = s} \in H^{-1/2}(\mathbf{R}^{\nu-1})$$

is continuous. Similarly if $u \in \mathscr{H}_L$, $v \in \mathscr{H}_{L^*}$ *are supported in the patch, then the map*

$$\overline{\mathbf{R}}_+ \ni s \mapsto \langle A_1 u, v\rangle|_{x_1 = s} \in \mathrm{Lip}(\mathbf{R}^{\nu-1})'$$

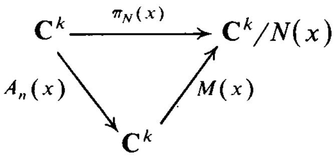
is continuous.

We are interested in boundary value problems for the system (4). For simplicity we consider homogeneous linear conditions $u(x) \in N(x)$ for $x \in \partial\Omega$ when $N(x)$ is a linear subspace of $\mathbf{C}^k$ for each $x \in \partial\Omega$. We suppose that

(6) $\quad N(x)$ depends Lipshitz continuously on x,

(7) $\quad N(x) \supset \ker A_n(x)$ for all $x \in \partial\Omega$.

Roughly speaking, since $A_n(x)u(x)$ is meaningful in $x \in \partial\Omega$ we expect that $u(x)$ is determined modulo $\ker A_n(x)$ and since $N(x) \supset \ker A_n(x)$ the equivalence class $u(x) \bmod N(x)$ is determined. To make this precise let $\pi_N(x)$: $\mathbf{C}^k \to \mathbf{C}^k/N(x)$ be the canonical projection. Since $N(x) \supset \ker A_n(x)$ there is a unique $M(x)$ so that



is a commutative diagram. Since $N(x)$ and $M(x)$ are Lipshitzean, $\mathbf{C}^k/N(x)$ is a Lipshitz continuous vector bundle over $\partial\Omega$ and M is a Lipshitzean bundle map. The Sobolev space of sections $H^s(\partial\Omega : \mathbf{C}^k/N(x))$ is well defined for $|s| \leqslant 1$. These remarks and Proposition 1 yield the following result.

PROPOSITION 2. *The map*

$$C^1(\overline{\Omega}) \ni u \mapsto u \bmod N(x) \in \mathrm{Lip}\big(\partial\Omega : \mathbf{C}^k/N(x)\big)$$

extends uniquely to a continuous map from $\mathscr{X}_L$ *to* $H^{-1/2}(\partial\Omega : \mathbf{C}^k/N(x))$. *For* u *in* $\mathscr{X}_L$ *the image is equal to* $M(A_n u|_{\partial\Omega})$.

DEFINITION 1. For $u \in \mathscr{X}_L$ we say that $u \in N$ at $\partial\Omega$ if the image of u in

$$H^{-1/2}\big(\partial\Omega : \mathbf{C}^k/N(x)\big)$$

vanishes.

Inhomogeneous boundary conditions, $u \equiv g \bmod N$ at $\partial\Omega$, can be reduced to the homogeneous case when $g \in H^{1/2}(\partial\Omega)$ by merely subtracting an element of H^1 which achieves these boundary data. The adjoint boundary space $N^*(x)$ is defined by

$$N^*(x) \equiv \left[A_n(x)(N(x))\right]^{\perp}.$$

Since $N(x) \supset \ker A_n(x)$ we see that

$$\dim A_n(N) = \dim N(x) - \dim\ker A_n(x).$$

Thus N^* has locally constant dimension if and only if the nullity of $A_n(x)$ is locally constant.

DEFINITION 2. The boundary of Ω is characteristic of constant multiplicity if $\dim\ker A_n(x)$ is constant on each component of $\partial\Omega$.

In this case, N^* is Lipshitz continuous. *We will assume from here on that* $\partial\Omega$ *is characteristic of constant multiplicity*. The next result is fundamental.

THEOREM 4 (WEAK = STRONG). *If* $u \in \mathscr{X}_L$ *(resp.* $\mathscr{H}_L$*) and* $u \in N$ *at* $\partial\Omega$, *then there is a sequence* $u_k \in C^1(\overline{\Omega})$ *with* $u_k(x) \in N(x)$ *for* $x \in \partial\Omega$ *and* $u_k \to u$ *in* $\mathscr{X}_L$ *(resp.* $\mathscr{H}_L$*).*

As a consequence, it is not difficult to prove.

PROPOSITION 3. *If $\partial\Omega$ is characteristic of constant multiplicity, $u \in \mathcal{H}_L$ and $f = Lu$, then $u \in N$ at $\partial\Omega$ if and only if for all $v \in \mathrm{Lip}(\overline{\Omega})$ with $v(x) \in N^*(x)$ for all $x \in \partial\Omega$,*

$$\int_{\Omega} \langle u, L^* v\rangle = f(v). \tag{8}$$

Here the right-hand side is the value of $f \in H^1(\Omega)'$ at v.

REMARK. In case $f \in \mathcal{L}^2(\Omega)$, the right-hand side is an integral and the formula (8) was used by Friedrichs as the definition of a weak solution to the boundary value problem.

REMARK. Using Theorem 2 we see that for $u \in \mathcal{H}_L$, the equality (8) extends to all $v \in \mathcal{H}_L$ satisfying $v \in N^*$ at $\partial\Omega$ in the sense of Definition 1.

Theorems 1–4 provide the basic calculus on which the theory of boundary value problems for (4) is built. Most earlier work on the subject assumed a stronger hypothesis than in Definition 2. They assumed that there was an extension of $n(x)$ to a C^1 function on $\overline{\Omega}$ so that $\dim\ker A_n(x)$ was constant on a *neighborhood* of each component of the boundary. In problems involving the flow of fluids it is quite common for this stronger hypothesis to fail (see [**1**, **3**, **13**, **22**]). This paper was written to provide a theory which was sufficiently strong to handle these problems and, secondly, refines and simplifies the standard results, even when the stronger hypothesis is valid. Another class of problems arises when $\dim\ker A_n(x)$ is not locally constant on $\partial\Omega$. Here examples are known when weak is not equal to strong (see [**12**, **14**, **17**]). Some positive results can be found in [**17**, **20**].

The symmetric positive problems we study have an elementary a priori estimate thanks to two positivity assumptions. First *we suppose that L is symmetric positive*, that is $A_j = A_j^*$ for all $x \in \Omega$, and there is a constant $a > 0$ so that

$$Z(x) \equiv \frac{B + B^*}{2} - \sum_j \partial_j A_j \geqslant aI \tag{9}$$

for all $x \in \Omega$. Second, *we suppose that N is maximal positive* in the sense that

$$\langle A_n(x) v, v\rangle \geqslant 0 \quad \forall x \in \partial\Omega, v \in N(x). \tag{10}$$

$$\dim N = \#\ \text{nonnegative eigenvalues of } A_n \text{ counting multiplicity.} \tag{11}$$

The maximality condition (11) implies that N cannot be enlarged while preserving (10), in particular it implies that $N \supset \ker A_n$. If $u \in H^1(\Omega)$, then Green's identity (5) with $u = v$ yields the energy identity

$$\mathrm{Re}(u, f)_{\Omega} = (Zu, u)_{\Omega} + \int_{\partial\Omega} \langle A_n u, u\rangle\, d\sigma. \tag{12}$$

The positivity hypotheses (9) and (10) yield the L^2 a priori estimate

$$a\|u\|_{\mathcal{L}^2(\Omega)} \leqslant \|Lu\|_{\mathcal{L}^2(\Omega)} \tag{13}$$

for $u \in H^1(\Omega)$ with $u(x) \in N(x)$ for almost all $x \in \partial\Omega$. Using Theorems 2 and 4 it is easy to prove the following.

THEOREM 5. *For any $f \in \mathcal{L}^2(\Omega)$ there is a unique $u \in \mathcal{L}^2(\Omega)$ satisfying $Lu = f$ in Ω and $u \in N$ at $\partial\Omega$. In addition the distribution $\langle A_n u, u\rangle|_{\partial\Omega}$ is nonnegative, and the estimate (13) holds.*

REMARK. For A_n invertible this was proved by Friedrichs [**6**] and a direct proof valid under the more restrictive constant multiplicity hypothesis was given by Lax and Phillips [**9**].

For problems with characteristic boundary, one does not expect full regularity of u even if $f \in C^\infty(\overline{\Omega})$. However, there is a good tangential regularity theorem.

DEFINITION 3. A smooth vector field γ on $\overline{\Omega}$ is called *tangential* if and only if, for every $x \in \partial\Omega$, $\langle \gamma(x), n(x) \rangle = 0$. For $s \in \mathbf{Z}_+$, the space $H^s_{\tan}(\Omega)$ consists of those $u \in \mathscr{L}^2(\Omega)$ with the property that for any $l \leqslant s$ and tangential fields $\{\gamma_i\}_{i=1}^l$, $\gamma_1\gamma_2 \cdots \gamma_l u \in \mathscr{L}^2(\Omega)$. Clearly elements of $H^s_{\tan}$ lie in $H^s_{\mathrm{loc}}(\Omega)$. Near $p \in \partial\Omega$ one may localize to ϕu then introduce coordinates (x_1, x') so that $\operatorname{supp} \phi u \subset \{|x| < 1$ and $x_1 \geqslant 0\}$. The elements ϕu are characterized by

$$\sum_{|\alpha| \leqslant s} \left\| (x_1\partial_1, \partial_2, \dots, \partial_\nu)^\alpha \phi u \right\|^2_{\mathscr{L}^2(\mathbf{R}^\nu_+)} < \infty .$$

This yields a natural Hilbert space structure for $H^s_{\tan}(\Omega)$ (see [**2**]). Assuming that $\partial\Omega$ and the coefficients of L are sufficiently regular, one has tangental regularity as follows.

THEOREM 6. *Suppose $s \in \mathbf{Z}_+$, A, N and $\partial\Omega$ are of class $C^{s,1}$ and B is of class $C^{s-1,1}$. Then there are real numbers λ_s and C_s so that $\lambda_0 \leqslant \lambda_1 \leqslant \cdots$ and if $u \in H^s_{\tan}$, $u \in N$ at $\partial\Omega$, and $Lu \in H^s_{\tan}$, then for all $\lambda \in \mathbf{C}$*

$$\operatorname{Re}(\lambda - \lambda_s)\|u\|_{H^s_{\tan}} \leqslant C_s\Big(\|(L + \lambda)u\|_{H^s_{\tan}} + |\lambda| \, \|u\|_{H^{s-1}_{\tan}} \Big). \tag{14}$$

Conversely if λ_s, C_s are as above, $\operatorname{Re}\lambda > \lambda_s$ *and $f \in H^s_{\tan}(\Omega)$, then the unique solution u to $(L + \lambda)u = f$, $u \in N$ at $\partial\Omega$, lies in $H^s_{\tan}(\Omega)$.*

REMARK 1. In case $\partial\Omega$ is noncharacteristic it follows that $u \in H^s(\Omega)$. In the characteristic case, one cannot expect full regularity even if $f \in H^s(\Omega)$ (see [**11**, **24**]). However, for some important problems of mathematical physics one does get full regularity (see [**11**, **13**, **22**]).

REMARK 2. An example of Friedrichs [**6**] shows that without a condition that λ be sufficiently large ($\lambda > 0$ does not always suffice) one gets regularity no better than $\mathscr{L}^2$.

REMARK 3. In the noncharacteristic case with $s = 1$ this result was proved by Friedrichs. Higher s was studied by many authors [7, **15**, **16**, **23**]. Problems characteristic of constant multiplicity on a neighborhood of the boundary were studied in [**11**, **16**, **24**], where partial results can be found.

Results analogous to Theorems 1–6 are valid for time dependent problems $(\partial_t - L(t))u = f$ in cylindrical domains $[0, T] \times \Omega$. These results are described and proved in §4. §2 is devoted to the proofs of Theorems 1–5 while §3 contains the proof of Theorem 6.

Studying these problems for the last decade the author has benefitted from correspondence and conversations with J. Ralston, D. Tartakoff and L. Sarason. Thanks!

2. The $\mathscr{L}^2$ theory.

PROOF OF PROPOSITION 1. Only the density requires comment. Cover $\overline{\Omega}$ by finitely many coordinate patches C^1 diffeomorphic to $\{|x| < 1\}$ or to $\{|x| < 1$ and $x_1 > 0\}$ by diffeomorphisms χ_i. Choose a finite smooth partition of unity ϕ_i subordinate to this cover. Choose $j \in C_0^\infty$ ($|x| < 1$ and $x_1 < 0$), $\int j = 1$ and let $j_\varepsilon(x) = \varepsilon^{-n} j(\varepsilon^{-1}x)$. Let

$$u_\varepsilon = \sum_i \left(j_\varepsilon * \left(\phi_i u \circ \chi_i^{-1} \right) \right) \circ \chi_i.$$

Then $u_\varepsilon \in C^1(\overline{\Omega})$ and $u_\varepsilon \to u$ in $\mathscr{L}^2(\Omega)$. The classical lemma of Friedrichs **[4]** implies that $(Lu)_\varepsilon - L(u_\varepsilon) \to 0$ in $\mathscr{L}^2(\Omega)$, so $L(u_\varepsilon) \to Lu$ in $H^1(\Omega)'$. Thus $u_\varepsilon \to u$ in $\mathscr{H}_L$. □

PROOF OF THEOREM 1. Given $\psi \in H^{1/2}(\partial\Omega)$ choose $\Psi \in H^1(\Omega)$ such that $\Psi|_{\partial\Omega} = \psi$, $\|\Psi\|_{H^1} < c\|\psi\|_{H^{1/2}}$ with c independent of ψ. Then for $u \in C^1(\overline{\Omega})$,

$$\int_{\partial\Omega} \langle A_n u, \psi \rangle \, d\sigma = \int_\Omega \langle u, L^*\Psi \rangle + \langle Lu, \Psi \rangle \, dx.$$

Thus

$$\left| \int_{\partial\Omega} \langle A_n u, \psi \rangle \, d\sigma \right| \leqslant \|u\|_{\mathscr{L}^2(\Omega)} \|L^*\Psi\|_{\mathscr{L}^2(\Omega)} + \|Lu\|_{H^1(\Omega)'} \|\Psi\|_{H^1(\Omega)}$$
$$\leqslant c\|\psi\|_{H^{1/2}(\partial\Omega)} \|u\|_{\mathscr{H}_L}.$$

Thus, $\|A_n u\|_{H^{-1/2}(\partial\Omega)} \leqslant c\|u\|_{\mathscr{H}_L}$ which proves the existence of a continuous extension. By Proposition 1, $C^1(\overline{\Omega})$ is dense so the extension is unique. □

PROOF OF THEOREM 2. Given $\psi \in \mathrm{Lip}(\partial\Omega)$ choose $\Psi \in \mathrm{Lip}(\overline{\Omega})$ so that $\Psi|_{\partial\Omega} = \psi$ and $\|\Psi\|_{\mathrm{Lip}(\overline{\Omega})} \leqslant c\|\psi\|_{\mathrm{Lip}(\partial\Omega)}$ with c independent of ψ. Then for $u, v \in C^1(\overline{\Omega})$, Green's identity yields

$$\int_{\partial\Omega} \psi \langle A_n u, v \rangle \, d\sigma = \int_\Omega \langle Lu, \Psi v \rangle + \langle u, L^*(\Psi v) \rangle \, dx.$$

Since the commutator $[L^*, \Psi]$ is of order zero we see that with c independent of ψ

$$\left| \int \psi \langle A_n u, v \rangle \, d\sigma \right| \leqslant c\|\psi\|_{\mathrm{Lip}(\partial\Omega)} \|u\|_{\mathscr{H}_L} \|v\|_{\mathscr{H}_{L^*}}.$$

Thus $\|\langle A_n u, v \rangle\|_{\mathrm{Lip}(\partial\Omega)'} \leqslant c\|u\|_{\mathscr{H}_L} \|v\|_{\mathscr{H}_{L^*}}$. Since $C^1(\overline{\Omega})$ is dense, Theorem 2 follows. □

PROOF OF THEOREM 3. The proof of Theorem 2 shows that for $u \in C^1$

$$\left\| A_1 u|_{x_1 = s} \right\|_{H^{-1/2}(\mathbf{R}^{\nu-1})} \leqslant c\|u\|_{\mathscr{H}_L} \tag{15}$$

with c independent of u *and* s. For $u \in \mathscr{H}_L$, choose $u_n \in C^1$ with support in the coordinate patch and $u_n \to u$ in $\mathscr{H}_L$. By (15) u_n is a Cauchy sequence in $C(\overline{\mathbf{R}}_+ : H^{-1/2}(\mathbf{R}^{\nu-1}))$, so there is $\tilde{u}$ such that $u_n \to \tilde{u}$ in $C(\overline{\mathbf{R}}_+ : H^{-1/2}(\mathbf{R}^{\nu-1}))$. Thus $u_n \to \tilde{u}$ in $\mathscr{D}'(\mathbf{R}^\nu_+)$ so $u = \tilde{u}$. Thus $u \in C(\mathbf{R}_+ : H^{-1/2}(\mathbf{R}^{\nu-1}))$ and (15) holds for u.

Similarly, for $u, v \in C^1$, the proof of Theorem 2 yields

$$\left\| \langle A, u, v \rangle|_{x_1 = s} \right\|_{\mathrm{Lip}(\mathbf{R}^{\nu-1})'} \leqslant c\|u\|_{\mathscr{H}_L} \|v\|_{\mathscr{H}_{L^*}}.$$

with c independent of u, v *and* s. Using this in the same fashion as (15), the second part of Theorem 3 follows. □

PROOF OF THEOREM 4. With the aid of a partition of unity write $u = \Sigma\phi_i u \equiv \Sigma u^i$. Then

$$Lu^i = \phi_i f + \sum c_{ij}(x)u^j \equiv f_i.$$

If we can find u^i_ε in $C^1(\overline{\Omega})$ supported near $\operatorname{supp}\phi_i$, $u^i_\varepsilon \in N$ at $\partial\Omega$, and $u^i_\varepsilon \to u^i$ in $\mathscr{K}_L$ (resp. $\mathscr{H}_L$), then letting $u_\varepsilon = \Sigma u^i_\varepsilon$ gives the desired approximation for u. Thus, it suffices to consider u supported in a small coordinate patch. The interesting patches are at $\partial\Omega$. Performing a change of independent variable we are reduced to the case $\Omega = \mathbf{R}^n_+$, $\operatorname{supp} u \subset \{|x| < 1 \text{ and } x_1 \geqslant 0\}$.

Under a change of dependent variable, $\tilde{u} = M(x)^{-1}u$, the differential equation is transformed to $\tilde{L}\tilde{u} = M^*f$ where

$$\tilde{L}v = \sum M^*A_j\partial_j(Mw) + M^*BMw.$$

By hypothesis, $A_1(0, x')$ has rank independent of x' so by a Lipshitzean change, $M(x')$, we can transform $A_1(0, x')$ to

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix}.$$

Multiplying on the left by

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix}$$

transforms to an equivalent, but nonsymmetric, system with

$$A_1(0, x') = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{16}$$

Since $N \supset \ker A_1$ we may choose a Lipshitz continuous unitary $U(x')$ such that U leaves $\ker A_1$ invariant and $U^*(x')(N(x'))$ is equal to

$$\left\{u \in \mathbf{C}^k | u_{l+1} = \cdots = u_k = 0\right\} \equiv N_0. \tag{17}$$

The change of dependent variable $\tilde{\tilde{u}} = U^*\tilde{u}$ transforms to an equivalent system with A_1 unchanged since $U^*A_1U = A_1$ and with the boundary space N replaced by N_0. These changes have simultaneously transformed $A_1(0, x')$ to the form (16) and N to N_0 in (17), both independent of x'. We now drop the tildes and work with the transformed boundary value problem.

Suppose $u \in \mathscr{K}_L$ and $u \in N_0$ on $x_1 = 0$. The approximation u_ε is made in three steps. First we construct $u_\varepsilon \in H^1_{\tan} \cap \mathscr{K}_L$, $u_\varepsilon \in N_0$, $u_\varepsilon \to u$ in $\mathscr{K}_L$. The construction uses a variant of Friedrichs's mollifiers. Choose $j \in C_0^\infty(\{|x| < 1 \text{ and } x_1 > 0\})$, $j \geqslant 0$ and $\int j = 1$. Let

$$u_\varepsilon \equiv J_\varepsilon u \equiv \int u\left(x_1 e^{\varepsilon y_1}, x' + \varepsilon y'\right) j(y)\, dy.$$

The novelty here is that instead of $x_1 + \varepsilon y_1$ we have $x_1 e^{\varepsilon y_1}$ which is the point ε units of time along the integral curve of $x_1\partial/\partial x_1$ with initial point x_1. The gain is that $u_\varepsilon|_{x_1=0}$ is determined by $u|_{x_1=0}$ and the loss is that the mollifier is not completely smoothing; one gains $(x_1\partial_1, \partial')$ derivatives but not ∂_1. A related idea, convoluting only in the x' variables, was used by Lax and Phillips [9]. For that method it appears necessary to suppose that $\dim\ker A_1$ is constant on a *neigborhood* of $x_1 = 0$.

LEMMA. (1) *With* $X = H^1(\mathbf{R}^\nu_+)$ *or* $X = H^s_{\tan}(\mathbf{R}^\nu_+)$ *and* $\phi \in X$ *with compact support,* $\{J_\varepsilon\phi\}_{0<\varepsilon<1}$ *is a bounded subset of* X. *As* $\varepsilon \to 0$, $J_\varepsilon\phi$ *converges to* ϕ *in* X.

(2) *If* $Z = (x_1\partial_1, \partial_2, \dots, \partial_\nu)$ *and* ϕ *is as above, then* $Z^\alpha J_\varepsilon \phi \in X$ *for all* $\alpha \in \mathbf{Z}^\nu$.

(3) *Suppose* $s \geqslant 0$ *is an integer,* $\phi \in H^s_{\tan}(\mathbf{R}^\nu_+)$ *with compact support and* $A \in C^{s,1}((\overline{\mathbf{R}^\nu_+}) : \mathrm{Hom}(\mathbf{C}^k))$. *Then if* $|\alpha| = 1$ *the family* $[AZ^\alpha, J_\varepsilon]\phi$, $0 < \varepsilon < 1$, *is bounded in* $H^s_{\tan}$ *and as* $\varepsilon \to 0$, $[AZ^\alpha, J_\varepsilon]\phi \to 0$ *in* $H^s_{\tan}$.

PROOF OF LEMMA. (1) For $\phi \in X$ with compact support we have

$$\phi\left(x_1 e^{\varepsilon y_1}, x' + \varepsilon y'\right) \in X \quad \text{for } |y| \leqslant 1, 0 \leqslant \varepsilon \leqslant 1.$$

In fact they lie in a bounded subset of X. As $J_\varepsilon\phi$ is a convex combination it is bounded in X uniformly in $\varepsilon \leqslant 1$, with bound depending only on $\|\phi\|_X$ and $\operatorname{supp}\phi$. Approximating ϕ in X by elements of $C^1_{(0)}(\overline{\mathbf{R}^\nu})$ with uniformly bounded supports, part (1) follows since $J_\varepsilon\psi \to \psi$ in X for $\psi \in C^\infty_{(0)}(\overline{\mathbf{R}^\nu_+})$.

(2) Consider first $x_1\partial_1$. For $\phi \in C^\infty_{(0)}(\overline{\mathbf{R}^\nu_+})$, $J_\varepsilon\phi \in C^\infty_{(0)}(\overline{\mathbf{R}^\nu_+})$ and differentiating under the integral sign

$$x_1\partial_1 J_\varepsilon\phi = \int x_1 e^{\varepsilon y_1}\frac{\partial\phi}{\partial x_1}\left(x_1 e^{\varepsilon y_1}, x' + \varepsilon y'\right) j(y)\, dy$$

$$= \int j(y)\frac{1}{\varepsilon}\frac{\partial}{\partial y_1}\left(\phi\left(x_1 e^{\varepsilon y_1}, x' + \varepsilon y'\right)\right) dy.$$

Integrating by parts using the fact that $j = 0$ when $y_1 = 0$ yields

$$= -\frac{1}{\varepsilon}\int \phi\left(x_1 e^{\varepsilon y_1}, x' + \varepsilon y'\right)\frac{\partial j}{\partial y_1}(y)\, dy.$$

More generally we have

$$Z^\alpha J\varepsilon\phi = \left(\frac{-1}{\varepsilon}\right)^{|\alpha|}\int \phi\left(x_1 e^{\varepsilon y_1}, x' + \varepsilon y'\right)\left(\partial_y^\alpha j\right)(y)\, dy.$$

Thus for ε and α fixed and $K \subset \overline{\mathbf{R}^\nu_+}$ compact there is $C = C(\varepsilon, \alpha, K)$ so that $\|Z^\alpha J_\varepsilon\phi\|_X \leqslant c\|\phi\|_X$. Approximating $\phi \in X$ with compact support by a sequence $\phi_k \in C^\infty_{(0)}$, (2) follows.

(3) We treat $\alpha = (1, 0, \dots, 0)$ and $s = 0$. The other cases are similar. For $\phi \in C^\infty_{(0)}(\overline{\mathbf{R}^\nu_+})$ and $\Gamma \equiv Z^\alpha$ we have

$$[A\Gamma, J_\varepsilon]\phi = \int \left(A(x) - A\left(x_1 e^{\varepsilon y_1}, x' + \varepsilon y'\right)\right) j(y)\frac{1}{\varepsilon}\frac{\partial}{\partial y_1}\left(\phi\left(x_1 e^{\varepsilon y_1}, x' + \varepsilon y'\right)\right) dy.$$

Now integrate by parts. When the y derivative falls on the A term we find

$$\int x_1 e^{\varepsilon y_1}\frac{\partial A}{\partial x_1}\left(x_1 e^{\varepsilon y_1}, x' + \varepsilon y'\right) j(y)\phi\left(x_1 e^{\varepsilon y_1}, x' + \varepsilon y'\right) dy$$

whose $\mathscr{L}^2(\mathbf{R}^\nu_+)$ norm is bounded independent of $\varepsilon \in (0,1]$ since $x_1 \partial A/\partial x_1$ is bounded. When the y derivative falls on j we find

$$-\int \frac{A(x) - A\left(x_1 e^{\varepsilon y_1}, x' + \varepsilon y'\right)}{\varepsilon} \frac{\partial j}{\partial y_1}(y) \phi\left(x_1 e^{\varepsilon y_1}, x' + \varepsilon y'\right) dy$$

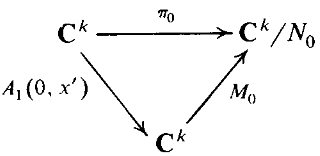
whose $\mathscr{L}^2(\mathbf{R}^n_+)$ norm is bounded independent of $\varepsilon \in (0,1]$ because the difference quotient in the integrand is bounded since $A \in \text{Lip}$. Thus $[A\Gamma, J_\varepsilon]$ maps $\mathscr{L}^2(\mathbf{R}^\nu_+)$ to itself with norm independent of $\varepsilon \in (0,1]$. Since $\mathscr{L}^2\text{-lim}[A\Gamma, J_\varepsilon]\phi = 0$ for $\phi \in C^\infty_{(0)}(\overline{\mathbf{R}^\nu_+})$, a dense subset of $\mathscr{L}^2$, part (3) follows. □

Applying part (2) of the lemma with $X = \mathscr{L}^2(\mathbf{R}^\nu_+)$ we see that $u_\varepsilon \equiv J_\varepsilon u \in H^1_{\tan}(\mathbf{R}^\nu_+)$. Part (1) with $X = H^1(\mathbf{R}^\nu_+)'$ shows $J_\varepsilon(Lu) \to Lu$ in $H^1(\mathbf{R}^\nu_+)'$. Finally, part (3) shows that $J_\varepsilon(Lu) - L(u_\varepsilon) \to 0$ in $\mathscr{L}^2(\mathbf{R}^\nu_+)$. Putting this together we see that for $u \in \mathscr{K}_L$, $u_\varepsilon \to u$ in $\mathscr{K}_L$.

Since N_0 is independent of x', it is clear on a formal level that $u_\varepsilon \in N_0$ at $x_1 = 0$. To prove this, first observe that for $\phi \in C^\infty_{(0)}(\overline{\mathbf{R}^\nu_+})$,

$$J_\varepsilon\phi|_{x_1=0} = \gamma_\varepsilon * \left(\phi|_{x_1=0}\right), \quad \gamma_\varepsilon(x') \equiv \varepsilon^{-(\nu-1)}\gamma(x'/\varepsilon), \quad \gamma(x') \equiv \int j(x_1, x')\, dx_1,$$

the convolution in x' variables only. Define M_0: $\mathbf{C}^k \to \mathbf{C}^k/N_0$ by the commutative diagram:



Since M_0 and A_1 are independent of x' we have

$$M_0 A_1 J_\varepsilon \phi|_{x_1=0} = \gamma_\varepsilon * \left(M_0 A_1 \phi|_{x_1=0}\right),$$

the convolution in x' variables. Using Theorem 1 we see that this identity extends by continuity to $\phi \in \mathscr{K}_L$, the equality expressed in $H^{-1/2}(\mathbf{R}^{\nu-1})$. For $u \in \mathscr{K}_L$ with $u \in N$ at $\partial\Omega$, $M_0 A_1 u|_{x_1=0} = 0$ so we find that $M_0 A_1 u_\varepsilon = 0$, that is $u_\varepsilon \in N_0$ at $x_1 = 0$.

Thus, replacing u by u_ε we may suppose without loss of generality that $u \in H^1_{\tan}(\mathbf{R}^\nu_+)$.

The differential equation and $u \in H^1_{\tan}$ imply that $\partial_1(A_1 u) \in \mathscr{L}^2(\mathbf{R}^\nu_+)$, so $A_1 u \in H^1(\mathbf{R}^\nu_+)$. Let $u^{II} = (0, 0, \ldots, 0, u_{l+1}, \ldots, u_k)$ be the projection of u orthogonal to N_0. Since $N_0 \supset \ker A_1$ we see that $u^{II} \in H^1(\mathbf{R}^\nu_+)$. The boundary condition, though expressed weakly, implies $u^{II} \in \mathring{H}^1(\mathbf{R}^\nu_+)$. Extend u^{II} to be zero for $x_1 < 0$, so $u^{II} \in H^1(\mathbf{R}^\nu)$. Let $u^I = u - u^{II}$ and for $\eta \in (0,1]$,

$$u_\eta(x) = u^I(x) + u^{II}\left(x_1 - \eta, x'\right).$$

Then as $\eta \to 0$, $u_\eta - u \to 0$ in $H^1(\mathbf{R})^\nu_+)$, so $Lu_\eta \to Lu$ in $\mathscr{L}^2(\mathbf{R}^\nu_+)$. Thus $u_\eta \to u$ in $\mathscr{K}_L$. Replacing u by u_η we may suppose without loss of generality that $u^{II} = 0$ for $0 < x_1 < \eta$.

With j as before and $j_\varepsilon(x) \equiv \varepsilon^{-\nu} j(-x_1/\varepsilon, x/\varepsilon)$ and $\varepsilon < \eta$ let $u_\varepsilon = j_\varepsilon * u \in C^\infty_{(0)}(\overline{\mathbf{R}^\nu_+})$, $(u_\varepsilon)^{II} = 0$ for $x_1 < \eta - \varepsilon$. Thus $u_\varepsilon \in N_0$ at the boundary. That $u_\varepsilon \to u$ in $\mathcal{L}^2(\mathbf{R}^\nu_+)$ and $j_\varepsilon * Lu \to Lu$ in $H^1(\mathbf{R}^\nu_+)'$ are routine. Friedrichs' classical lemma asserts that $j_\varepsilon * Lu - Lu_\varepsilon \to 0$ in $\mathcal{L}^2(\mathbf{R}^\nu_+)$. In total, $u_\varepsilon \to u$ in $\mathcal{K}_L$ and the proof of Theorem 4 for $\mathcal{K}_L$ is complete. For $\mathcal{H}_L$ one merely repeats the proof replacing $\mathcal{K}$ by $\mathcal{H}$ and $H^1(\mathbf{R}^\nu_+)'$ by $\mathcal{L}^2(\mathbf{R}^\nu_+)$. □

Proof of Proposition 3. If $u \in \mathcal{K}_L$ with $u \in N$ at $\partial\Omega$ we may choose $u_\varepsilon \in \mathcal{K}_L \cap C^1(\overline{\Omega})$, $u_\varepsilon \in N$ at $\partial\Omega$ and $u_\varepsilon \to u$ in $\mathcal{K}_L$. Green's identity for u_ε yields

$$\int_\Omega \langle u_\varepsilon, L^*v\rangle\, dx = (Lu_\varepsilon)(v) + \int_{\partial\Omega} \langle A_n u_\varepsilon, v\rangle\, d\sigma.$$

For $v \in \operatorname{Lip}(\partial\Omega)$ with $v \in N^*$ the boundary term vanishes. Passing to the limit $\varepsilon \to 0$ yields (8).

Conversely, suppose (8) holds for $u \in \mathcal{K}_L$. Then for any $\psi \in \operatorname{Lip}(\partial\Omega)$ with $\psi \in N^*$ we may choose $v \in \operatorname{Lip}(\overline{\Omega})$, $v|_{\partial\Omega} = \psi$. Then identity (8) implies that $(A_n u|_{\partial\Omega})(\psi) = 0$. Thus if $\pi_{N^*(x)}$ is the orthogonal projection in $\mathbf{C}^k$ onto N^* we see that $A_n u|_{\partial\Omega}$ annihilates $\pi_{N^*}v$ for any $v \in \operatorname{Lip}(\partial\Omega)$. Since π_{N^*} is selfadjoint, this is equivalent to $\pi_{N^*}(A_n u|_{\partial\Omega}) = 0$ in $H^{-1/2}(\partial\Omega)$. Here we have used the fact that π_{N^*} multiplies $H^s(\partial\Omega)$ to itself for all $|s| \leqslant 1$. Since $N \supset \ker A_n$, we have $\ker \pi_{N^*}A_n = N$ for all $x \in \partial\Omega$. An argument like that in Proposition 2 then shows that for $u \in \mathcal{K}_L$, $u \in N$ at $\partial\Omega \Leftrightarrow \pi_{N^*}(A_n u|_{\partial\Omega}) = 0$ in $H^{-1/2}(\partial\Omega)$. We conclude that if (8) holds then $u \in N$ at $\partial\Omega$ and Proposition 3 is proved. □

Proof of Theorem 5. If $u \in \mathcal{H}_L$ and $u \in N$ at $\partial\Omega$ we may choose $u_\varepsilon \in C^1(\overline{\Omega})$ with $u_\varepsilon \in N$ at $\partial\Omega$ and $u_\varepsilon \to u$ in $\mathcal{H}_L$. Then $\langle A_n u_\varepsilon, u_\varepsilon\rangle|_{\partial\Omega} \geqslant 0$ and $\langle A_n u_\varepsilon, u_\varepsilon\rangle \to \langle A_n u, u\rangle$ in $\operatorname{Lip}(\partial\Omega)'$. This implies that $\langle A_n u, u\rangle|_{\partial\Omega}$ is a positive distribution. Green's identity (5) with $v = u$ yields estimate (13), and, in particular, uniqueness.

To prove existence let $\mathcal{B}$ be the set of $v \in \operatorname{Lip}(\overline{\Omega})$ with $v \in N^*$ at $\partial\Omega$. Since N^* is maximal positive we have $a\|v\|_{\mathcal{L}^2(\Omega)} \leqslant \|L^*v\|_{\mathcal{L}^2(\Omega)}$ for all $v \in \mathcal{B}$, in particular L^* is a bijection from $\mathcal{B}$ to $\mathcal{R} \equiv L^*(\mathcal{B})$. Define $l : \mathcal{R} \to \mathbf{C}$ by

$$L^*v \mapsto \int_\Omega v\bar{f}\, dx.$$

The estimate for L^* yields $a|l(v)| \leqslant \|f\|_{\mathcal{L}^2(\Omega)}$. Riesz's theorem implies that there is a $u \in \mathcal{L}^2(\Omega)$ so that $l(w) = (w, u)_{\mathcal{L}^2(\Omega)}$ for all $w \in \mathcal{R}$. This is exactly identity (8) of Proposition 3. Considering $w = L^*v$, $v \in C_0^\infty(\Omega)$ we find $Lu = f$. Proposition 3 shows $u \in N$ at $\partial\Omega$ so u is the desired solution. □

3. Tangential regularity.

Proof of Theorem 6. We begin with the derivation of the a priori estimate (14). Let $f = (L + \lambda)u$. Cover $\overline{\Omega}$ by a finite family of open sets $\mathcal{U}_i$ so that either $\mathcal{U}_i \subset\subset \Omega$ or $\mathcal{U}_i \cap \overline{\Omega}$ is $C^{s,1}$ diffeomorphic to $\{x_1 \geqslant 0\} \cap \{|x| < 1\}$ with $\partial\Omega$ mapping to $\{x_1 = 0\}$. Choose a finite partition of unity $\{\phi_i\}$ subordinate to this cover and let $u = \Sigma\phi_i u \equiv \Sigma u_i$. Changing coordinates in the boundary patches yields functions $u_i \circ \chi_i$ defined on $\{x_1 \geqslant 0\} \cap \{|x| < 1\}$ which we continue to call u_i. Each function u_i satisfies an equation of the form

$$(18) \qquad (L_i + \lambda)u_i = \phi_i f \circ \chi_i + \sum c_{i,j}(x) u_j,$$

where L_i is the operator expressed in the new coordinates and the matrices c_{ij} are of class $C^{s-1,1}$. Taking the $\mathscr{L}^2$ scalar product with u_i yields

$$|(u_i, (L_i + \lambda)u_i)| \leqslant c\|u\|_{\mathscr{L}^2}(\|(L + \lambda)u\|_{\mathscr{L}^2} + \|u\|_{\mathscr{L}^2}).$$

However, as u_i satisfies the maximal positive boundary condition $u_i \in N \circ \chi_i$ on $x_1 = 0$, Green's identity implies that

$$\operatorname{Re}(u_i, (L_i + \lambda)u_i) \geqslant (\operatorname{Re}\lambda - c)\|u_i\|^2.$$

Summing on i, we find constants ω_0, c_0 so that

$$(\operatorname{Re}\lambda - \omega_0)\|u\|_{\mathscr{L}^2(\Omega)} \leqslant c_0\|(L + \lambda)u\|_{\mathscr{L}^2(\Omega)}.$$

We want such an estimate for the tangential derivatives of u. The basic idea is to apply the $\mathscr{L}^2$ estimate to the tangential derivatives of u. There are two problems, first the tangential derivatives γu need not satisfy the boundary conditions and, second, $L\gamma u$ need not have $\mathscr{L}^2$ norm dominated by $\|u\|_{H^1_{\tan}}$. To overcome these difficulties the problem is transformed to a convenient form.

Since $\ker A_1(0, x') \subset N(x')$ we may choose a unitary matrix valued function U_i of class $C^{s,1}$ so that

$$U_i^*(N) = \{u \in \mathbf{C}^k \colon u_{l+1} = \cdots = u_k = 0\} \equiv N_0,$$
$$U_i^*(\ker A_1) = \{u \in \mathbf{C}^k \colon u_{\alpha+1} = \cdots = u_k = 0\}$$

with $\alpha \leqslant l$. Then $\tilde{u}_i \equiv U_i^* u_i$ satisfies the boundary condition $\tilde{u}_i \in N_0$ and $L_i + \lambda$ is transformed to $\tilde{L}_i + \lambda$, where

$$\tilde{L}_i = \sum_j U_i^* A_j U_i \partial_j + \text{l.o.t.} \equiv \sum \tilde{A}_j \partial_j + \tilde{B}.$$

The symmetry of $\tilde{A}_1$ shows that

$$A_1(0, x') = \begin{bmatrix} 0 & 0 \quad 0 \\ 0 & \\ 0 & \left[\text{NONSING}\right] \end{bmatrix}, \tag{19}$$

where the $(k - \alpha) \times (k - \alpha)$ lower right-hand block is invertible.

What we have done is to transform the problem so that N and $\ker A_1(0, x')$ are independent of x'. It is worth noting that one cannot, in general, arrange that N and $A_1(0, x')$ be constant. To see this consider A_1 constant and $N(x')$ which varies from dissipative to conservative with x'.

For notational simplicity we drop the tildes. Next we examine the commutator of L_i with tangential derivatives. In local coordinates, the tangential vector fields are generated by the Z_j, where

$$(Z_1, Z_2, \ldots, Z_\nu) \equiv (x_1\partial_1, \partial_2, \ldots, \partial_\nu).$$

The critical observation is that for each Z_j and L_i there are matrices Γ_β, Ψ so that

$$[L_i, Z_j] = \sum_{|\beta| \leqslant 1} \Gamma_\beta Z^\beta + \Psi L_i, \tag{20}$$

where Γ_β and Ψ are of class $C^{s-2,1}$ or L^∞ depending on whether s is greater than or equal to one. The crucial commutators are $[A_1\partial_1, Z_j]$. For $j > 1$, $[A_1\partial_1, Z_j] = (Z_jA_1)\partial_1$. Write $A_1(x_1, x') = A_1(0, x') + x_1G(x_1, x')$ with G of class $C^{s-1,1}$. Then

$$(Z_jA_1)\partial_1 = \big(\partial_jA_1(0, x')\big)\partial_1 + (\partial_jG)Z_1$$

with ∂_jG of class $C^{s-2,1}$ (resp. L^∞) if $s > 1$ (resp. $s = 1$). Because of the special form (19) we have

$$\partial_jA_1(0, x') = H(0, x')A_1(0, x')$$

with H of class $C^{s-1,1}$. Thus,

$$\big(\partial_jA_1(0, x')\big)\partial_1 = HA_1\partial_1 - HGZ_1.$$

Now $A_1\partial_1 = L - \Sigma_{j=2}^{\nu}A_jZ_j - B$, so $[A_1\partial_1, Z_j]$ has the desired form for $j \geqslant 2$. For $j = 1$, $[A_1\partial_1, Z_1] = A_1\partial_1 - (\partial_1A_1)Z_1$. Replacing $A_1\partial_1$ as above completes the proof of (20).

We would like to apply the energy inequality to Z_lu_i. Dropping the subscripts we have

$$(L + \lambda)Zu = Z(L + \lambda)u + [L, Z]u.$$

For $f \in H^1_{\tan}$ the middle term lies in $\mathscr{L}^2$ as does the last term by virtue of the commutator identity (20). Thus $Z_lu_i \in \mathscr{H}_L$. We need to know that $Z_lu_i \in N_0$ on $x_1 = 0$. Since $u_i \in N_0$ and N_0 is independent of x' this is obvious on the formal level. For proof consider $J_\varepsilon u_i$ as in the proof of Theorem 4. Using the lemma from that proof we see (subscripts dropped) that $J_\varepsilon u \to u$ in $H^1_{\tan}$ and $LJ_\varepsilon u \to Lu$ in $H^1_{\tan}$. It follows that $ZJ_\varepsilon u \to Zu$ in $\mathscr{H}_L$ and therefore the equivalence classes $ZJ_\varepsilon u$ mod N_0 converge to Zu mod N_0 in $H^{-1/2}(\mathbf{R}^{\nu-1})$. Thus, it suffices to show that $ZJ_\varepsilon u \in N_0$ at $x_1 = 0$. Since N_0 is independent of x' a simple calculation shows that for $u \in C^\infty_{(0)}(\overline{\mathbf{R}^\nu_+})$,

$$ZJ_\varepsilon u \bmod N = \gamma_\varepsilon(x') * (u \bmod N_0),$$

where $\gamma_\varepsilon = -\varepsilon^{-\nu}\gamma(x'/\varepsilon)$ and $\gamma(x') = \int_{-\infty}^{\infty}(Z_lj)(x_1, x')\,dx_1$. This identity extends by continuity to all $u \in \mathscr{H}_L$ with equality in $H^{-1/2}(\mathbf{R}^{\nu-1})$. In particular, if $u \in N_0$, then $ZJ_\varepsilon u \in N_0$.

The preliminaries complete, we apply the energy identity to Z_lu_i to find

$$(\operatorname{Re}\lambda - c_0)\|Z_lu_i\|^2_{\mathscr{L}^2(\mathbf{R}^\nu_+)} \leqslant c_1\|(L_i + \lambda)Z_lu_i\|_{\mathscr{L}^2(\mathbf{R}^\nu_+)}\|Z_lu_i\|_{\mathscr{L}^2(\mathbf{R}^\nu_+)}.$$

Write $(L_i + \lambda)Zu = Z(L_i + \lambda)u + [L, Z]u$ and use (20) for the commutator. For the L_iu term which arises write $L_iu = (L_i + \lambda)u - \lambda u$. This yields

$$\|(L_i + \lambda)Z_lu_i\|_{\mathscr{L}^2(\mathbf{R}^\nu_+)} \leqslant c\Big(\|(L_i + \lambda)u\|_{H^1_{\tan}} + \|u\|_{H^1_{\tan}} + |\lambda|\,\|u\|_{\mathscr{L}^2}\Big).$$

Plugging in and summing over l and i yields

$$(\operatorname{Re}\lambda - c_2)\|u\|^2_{H^1_{\tan}} \leqslant c_3\|u\|_{H^1_{\tan}}\Big(\|f\|_{H^1_{\tan}} + \|u\|_{H^1_{\tan}} + |\lambda|\,\|u\|_{\mathscr{L}^2}\Big).$$

This is the desired estimate (14) for $s = 1$.

For higher s, one needs higher order commutators. Using (20) one proves by induction on $|\alpha|$ that for each α there are matrices $\Gamma_{\alpha,\beta}$, $\Psi_{\alpha,\beta}$ so that

$$[L_i, Z^\alpha] = \sum_{|\beta|\leqslant|\alpha|} \Gamma_{\alpha,\beta} Z^\beta + \sum_{|\beta|<|\alpha|} \Psi_{\alpha,\beta} Z^\beta L_i,$$

$$\Gamma_{\alpha,\beta}, \Psi_{\alpha,\beta} \text{ of class } \begin{cases} C^{s-|\alpha|-1,1} & \text{if } s > |\alpha|, \\ L^\infty & \text{if } s = |\alpha|. \end{cases}$$

Then reasoning as above, one proves the a priori estimate (14) by induction on s.

We next turn to the proof of regularity. Assuming numbers λ_s, c_s have been found so that (14) holds and that $f \in H^s_{\tan}$ and $\operatorname{Re}\lambda > \lambda_s$ we must show that the solution u of $(L+\lambda)u = f$, $u \in N$ at $\partial\Omega$, lies in $H^s_{\tan}(\Omega)$. The proof proceeds in two steps. First we produce a number Λ_s so that the conclusion holds provided $\operatorname{Re}\lambda > \Lambda_s$. The continuity method then yields the desired result for $\operatorname{Re}\lambda > \lambda_s$.

The first step proceeds by *noncharacteristic regularization*. Choose $n(x)$ as an extension of the unit outward normal to a C^{s+1} vector field on $\overline{\Omega}$. Extend $N(x)$ to a $C^{s,1}$ map defined on an open neighborhood of $\partial\Omega$, and choose $\phi \in C^\infty(\overline{\Omega})$ supported in the domain of definition of N and equal to one at $\partial\Omega$. Let $\pi_{N(x)}$ be the orthogonal projection of $\mathbf{C}^k$ onto $N(x)$, and set

$$L^\varepsilon \equiv L + \varepsilon\phi\pi_N \sum_{L=1}^{\nu} n_i \partial_i.$$

For L^ε with ε small, $\partial\Omega$ is noncharacteristic and N is a maximal positive boundary space. Actually, N is *strictly* positive in the sense that (10) holds with strict inequality, $\langle A_n^\varepsilon v, v\rangle \geqslant \varepsilon|v|^2$ for $v \in N(x)$. A straightforward argument shows that there is an ω_0 so that $L^\varepsilon + \lambda$ is positive for all $\operatorname{Re}\lambda > \omega_0$ and $0 < \varepsilon < 1$. Let u^ε be the solution to $(L^\varepsilon + \lambda)u^\varepsilon = f$, $u^\varepsilon \in N$ at $\partial\Omega$. Then it is easy to see that as $\varepsilon \to 0$, u^ε converges to u in $\mathscr{L}^2(\Omega)$. In addition, if one retraces the derivation of the a priori $H^s_{\tan}$ estimate one finds ω_s, c_s so that $\omega_0 \leqslant \omega_1 \leqslant \cdots$ and for all $u \in H^s_{\tan}$ with $Lu^\varepsilon \in H^s_{\tan}$ and $u \in N$ at $\partial\Omega$,

$$(21) \qquad (\operatorname{Re}\lambda - \omega_s)\|u\|_{H^1_{\tan}} \leqslant c_s\Big(\|(L^\varepsilon + \lambda)u\|_{H^1_{\tan}} + |\lambda|\,\|u\|_{H^{s-1}_{\tan}}\Big).$$

From this and the fact that $\partial\Omega$ is noncharacteristic, we find the H^s estimate

$$(\operatorname{Re}\lambda - \omega_s)\|u\|_{H^s(\Omega)} \leqslant \frac{\tilde{c}_s}{\varepsilon}\Big(\|(L^\varepsilon + \lambda)u\|_{H^s_{\tan}(\Omega)} + |\lambda|\,\|u\|_{H^{s-1}_{\tan}}\Big).$$

LEMMA. *There is an* $\varepsilon_s > 0$ *so that if* $\operatorname{Re}\lambda > \omega_s$, $0 < \varepsilon < \varepsilon_s$, *and* $f \in H^s(\Omega)$, *the unique solution to* $(L^\varepsilon + \lambda)u = f$, $u \in N$ *at* $\partial\Omega$, *lies in* $H^s(\Omega)$.

REMARK. There are two reasons why we cannot merely apply the result of Tartakoff [**23**]. First, the coefficients of L^ε and $\partial\Omega$ are not sufficiently regular and second, Tartakoff provides a $\Lambda_{s,\varepsilon}$ so that the regularity holds for $\operatorname{Re}\lambda > \Lambda_{s,\varepsilon}$. We need a constant Λ independent of ε.

We postpone the proof of the lemma. Given the lemma and $\operatorname{Re}\lambda > \omega_s$, estimate (21) allows one to prove inductively that $\{u^\varepsilon\}$ is bounded in $H^\sigma_{\tan}$ for $\sigma = 0, 1, 2, \ldots, s$. Since $u^\varepsilon \to u$ in $\mathscr{L}^2(\Omega)$, we conclude that $u \in H^s_{\tan}(\Omega)$.

Finally, we must prove the same conclusion assuming $\operatorname{Re}\lambda > \lambda_s$, where λ_s appearing in (14) may be smaller than ω_s. Define a closed operator L on $H^s_{\tan}(\Omega)$ by

$$\mathscr{D}(L) = \{u \in H^s_{\tan}(\Omega) | Lu \in H^s_{\tan}(\Omega) \text{ and } u \in N \text{ at } \partial\Omega\}.$$

We have proved that $\{\operatorname{Re}\lambda > \omega_s\}$ lies in the resolvent set $\rho(L)$ of L. We must show that $\mathscr{O} = \{\operatorname{Re}\lambda > \lambda_s\}$ lies in $\rho(L)$. Now $\mathscr{O}$ is connected and $\mathscr{O} \cap \rho(L)$ is open. Thus, it suffices to show that $\mathscr{O} \cap \rho(L)$ is a closed subset of $\mathscr{O}$. Suppose $\mu_k \in \mathscr{O} \cap \rho(L)$ and $\mu_k \to \mu$ in $\mathscr{O}$. We must show that $\mu \in \rho(L)$. Inequality (14) implies that $Z(x) \geqslant (\operatorname{Re}\mu - \lambda_s)I$ for all $x \in \Omega$ where $Z(x)$ is the matrix appearing in (9). Thus, for any $f \in H^s_{\tan}(\Omega)$, there is a unique $u \in \mathscr{L}^2(\Omega)$ with $(L + \mu)u = f$, $u \in N$ at $\partial\Omega$. We need to show that $u \in H^s_{\tan}(\Omega)$. Now

$$u = \mathscr{L}^2 - \lim u^k, \qquad u^k \equiv (L + \mu_k)^{-1} f.$$

By hypothesis $u^k \in H^s_{\tan}(\Omega)$. The a priori estimate (21) shows that $\{u^k\}$ is bounded in $H^s_{\tan}$. It follows that $u \in H^s_{\tan}(\Omega)$, the desired conclusion.

REMARK. Instead of noncharacteristic regularization one could prove tangential regularity directly using our mollifiers J_ε in a proof imitating that of Tartakoff. Given Tartakoff's theorem the present path is shorter.

PROOF OF LEMMA. Fix ε. We want to apply Tartakoff's theorem to L^ε, but the coefficients $\partial\Omega$ and N are not smooth enough. Let A_j, B be the coefficients of L^ε. Choose A^k_j, $B^k \in C^\infty(\overline{\Omega})$, A^k_j symmetric so that as $k \to \infty$, $A^k_j \to A_j$, and $B^k_j \to B$ uniformly with $\{A^k_j\}$ and $\{B^k\}$ bounded in $C^{s,1}(\overline{\Omega})$ and $C^{s-1,1}(\overline{\Omega})$, respectively. This yields operators $L^{\varepsilon,k}$ converging to L^ε.

Next choose $\Omega^k \subset \Omega$, increasing to Ω with $\partial\Omega^k \to \partial\Omega$ in the C^{s+1} topology.

Finally choose boundary spaces N^k defined in $\partial\Omega^k$, smooth and converging to N in the $C^{s,1}$ topology (which makes sense in a unique way). We may choose N^k strictly positive and so that the strictly positive smooth problems $L^{\varepsilon,k}$, Ω^k, N^k satisfy $H^s(\Omega^k)$ estimates uniformly in k. That is, there are constants $\tilde{\omega}_s$ and $\tilde{C}_s$ so that for all k and $u \in H^s(\mathbf{R}^\nu)$

$$(\operatorname{Re}\lambda - \tilde{\omega}_s)\|u\|_{H^s(\Omega^k)} \leqslant \tilde{C}_s\left(\|(L^{\varepsilon,k} + \lambda)u\|_{H^s(\Omega^k)} + |\lambda|\,\|u\|_{H^{s-1}(\Omega^k)}\right).$$

For ε, k fixed, the proof of Tartakoff's theorem provides a $\Lambda_{\varepsilon,k}$ so that if $\operatorname{Re}\lambda > \Lambda_{\varepsilon,k}$, then the solution $u^{\varepsilon,k}$ to $(L^{\varepsilon,k} + \lambda)u^{\varepsilon,k} = f$, $u^{\varepsilon,k} \in N^k$ at $\partial\Omega^k$, satisfies $u^{\varepsilon,k} \in H^s(\Omega^k)$. A continuity argument as the end of the proof of Theorem 6 implies that the same conclusion holds for $\operatorname{Re}\lambda > \tilde{\omega}_s$.

Now suppose that $\operatorname{Re}\lambda > \tilde{\omega}_s$. Then $\|u^{\varepsilon,k}\|_{H^s(\Omega^k)}$ is bounded independent of k. Choose extensions $u^{\varepsilon,k}_{\text{ext}}$ uniformly bounded in $H^s(\mathbf{R}^\nu)$. Let $u^\varepsilon \in H^s(\mathbf{R}^\nu)$ be a weak limit point. One shows easily that $u^\varepsilon|_\Omega$ satifies the boundary value problem of the lemma. Again a continuity argument yields the same conclusion for all λ with $\operatorname{Re}\lambda > \omega_s$. This completes the proof of the lemma and consequently of tangential regularity. □

4. Mixed initial boundary value problems. We are interested in solving

$$(22)\qquad \begin{cases} Lu = F(t, x) & \text{in } (0, T) \times \Omega, \\ u(0, \cdot) = g & \text{in } \Omega, \\ U(t, x) \in N(t, x) & \text{for } (t, x) \in [0, T] \times \partial\Omega, \end{cases}$$

where $N(t, x)$ is a Lipshitz continuous map from $[0, T] \times \partial\Omega$ to the subspaces of $\mathbf{C}^k$,

$$L = \partial_t + \sum_{j=1}^{v} A_j(t, x)\partial_j + B(t, x),$$

$$A_j \in \operatorname{Lip}\big([0, T] \times \bar{\Omega}\big), B \in L^\infty([0, T] \times \Omega),$$

and $\partial\Omega$ is assumed to be of class C^1. We introduce the notations $I \equiv (0, T)$ $\mathscr{O} \equiv (0, T) \times \Omega$ and $\Gamma \equiv (0, T) \times \partial\Omega$.

Though $\partial\mathscr{O}$ has corners it is Lipshitzean so that $H^\sigma(\partial\mathscr{O})$ is well defined for all $|\sigma| \leqslant 1$. Hilbert spaces $\mathscr{K}_L$, $\mathscr{H}_L$, $\mathscr{K}_{L^*}$, $\mathscr{H}_{L^*}$ are defined as in §1 with $\mathscr{O}$ replacing Ω. Again $C^1(\bar{\mathscr{O}})$ is dense in each. The next result is the analogue of Theorems 1 and 2. The proof is exactly as before.

THEOREM 7. *The map*

$$C^1(\bar{\mathscr{O}}) \ni u \mapsto \tau \in \mathscr{L}^\infty(\partial\mathscr{O}),$$

$$\tau = \begin{cases} u & \text{on } \{T\} \times \Omega, \\ -u & \text{on } \{0\} \times \Omega, \\ A_n u & \text{on } \Gamma \end{cases}$$

extends uniquely to a continuous map from $\mathscr{K}_L$ *to* $H^{1/2}(\partial\mathscr{O})'$. *The map*

$$C^1(\bar{\mathscr{O}}) \times C^1(\bar{\mathscr{O}}) \ni (u, v) \mapsto \rho \in \mathscr{L}^\infty(\partial\mathscr{O}),$$

$$\rho = \begin{cases} \langle u, v\rangle & \text{on } \{T\} \times \Omega, \\ -\langle u, v\rangle & \text{on } \{0\} \times \Omega, \\ \langle A_n u, v\rangle & \text{on } \Gamma \end{cases}$$

extends uniquely to a continuous map from $\mathscr{H}_L \times \mathscr{H}_{L^*}$ *to* $\operatorname{Lip}(\partial\mathscr{O})'$.

For $u \in \mathscr{K}_L$, $A_n u|_\Gamma$ is a distribution on Γ which has an extension to an element of $H^{-1/2}(\partial\mathscr{O}) \equiv H^{1/2}(\partial\mathscr{O})'$. It follows that $A_n u \in H^{1/2}(\Gamma)' \equiv \mathring{H}^{-1/2}(\Gamma)$. If $N \supset \ker A_n$ we then find that $u \bmod N$ is a well defined element of $H^{1/2}(\Gamma : \mathbf{C}^k/N)'$. When $u \bmod N$ vanishes we say that $u \in N$ on Γ. In the same way the restrictions of u to $\{t = 0\} \times \Omega$ and $\{t = T\} \times \Omega$ are well defined elements of $H^{1/2}(\Omega)'$. Next, we impose the hypothesis that Γ is characteristic of constant multiplicity in the same sense that *dim ker* A_n *is constant on each component of* Γ.

THEOREM 8. *If* $\partial\Gamma$ *is characteristic of constant multiplicity,* $u \in \mathscr{K}_L$ *(resp.* $\mathscr{H}_L$*) and* $u \in N$ *on* Γ, *then there is a sequence* $u_k \in C^1(\bar{\mathscr{O}})$ *such that* $u_k \in N$ *on* Γ *and* $u_k \to u$ *in* $\mathscr{K}_L$ *(resp.* $\mathscr{H}_L$*). In addition, if* $u = 0$ *on* $\{t = 0\} \times \Omega$, *then the* u_k *may be chosen with* $u_k|_{t=0} = 0$ *on* Ω.

OUTLINE OF PROOF. The proof is by mollification as in Theorem 4 but one must mollify in x *and* t. By a partition of unity, one reduces to u of small support. If u vanishes near $\{t = 0\} \times \bar{\Omega}$ and $\{t = T\} \times \bar{\Omega}$, one mollifies in x as before and then in t. For u supported near $\{t = 0\}$ the mollification in time is performed with a kernel j supported in $t > 0$. Near $t = T$ one takes $\operatorname{supp} j \subset \{t < 0\}$. See [**9**, §4] for a similar calculation.

For the second part of the theorem, let $F \equiv Lu$. Extend the coefficients of L and the boundary space N to $(-\infty, T] \times \bar{\Omega}$ by taking them independent of t for $t < 0$. Extend u and F to be equal to zero for $t < 0$. Denote with a subscript e the extended quantities. Since $u = 0$ when $t = 0$, we find $L_e u_e = F_e$.

Viewing u_e as an element of $\mathscr{H}_{L_e}((-\infty, T) \times \Omega)$ we see that it can be approximated by elements in $C^1((-\infty, T] \times \bar{\Omega})$ obtained by mollification. The main point is that no special attention must be payed at $\{t = 0\}$. One may mollify in x and then in t with *any* kernel. Choosing a kernel j supported in $t > 0$ for the contribution near $\{t = 0\}$ yields approximations supported in $\{t > 0\}$ □

We next suppose that N is *maximal positive*, that is, (10) and (11) hold on $\bar{\Gamma}$. With

$$Z \equiv \frac{B + B^*}{2} - \sum_{j=1}^{\nu} \partial_j A_j \in L^\infty(\mathcal{O})$$

we have for $u \in C^1(\bar{\mathcal{O}})$, $F \equiv Lu$,

$$\frac{d}{dt}\|u(t)\|^2_{\mathscr{L}^2(\Omega)} + 2(u, Zu)_\Omega = 2\operatorname{Re}(u, F) - 2\int_{\partial\Omega} \langle A_n u, u\rangle \, d\sigma.$$

If $u \in N$ on Γ the boundary integral is positive. With $c = \|Z\|_{\mathscr{L}^\infty(\mathcal{O})}$ and $\phi(t) \equiv \|u(t)\|_{\mathscr{L}^2(\Omega)}$,

$$\frac{d}{dt}\phi^2(t) \leqslant 2\phi(t)\big(\|F(t)\|_{\mathscr{L}^2(\Omega)} + c\phi(t)\big).$$

It follows that

$$\sup_{0 \leqslant t \leqslant T} \phi(t) \leqslant c\|F\|_{\mathscr{L}^1(I : \mathscr{L}^2(\Omega))} + \phi(0), \tag{23}$$

$$\phi(t_2) - \phi(t_1) \leqslant \int_{t_1}^{t_2} \|F(\sigma)\|_{\mathscr{L}^2(\Omega)} + c\phi(\sigma)\, d\sigma \tag{24}$$

with new constants c independent of u. These estimates suggest the following theorem.

THEOREM 9 ($\mathscr{L}^2$ WELL-POSEDNESS). *For any* $F \in \mathscr{L}^1(I : \mathscr{L}^2(\Omega))$, $g \in \mathscr{L}^2(\Omega)$, *there is a unique* $u \in \mathscr{L}^2(\mathcal{O})$ *satisfying* (22). *In addition*, $u \in C(I : \mathscr{L}^2(\Omega))$ *and with* $\phi(t) \equiv \|u(t)\|_{\mathscr{L}^2(\Omega)}$, *estimates* (23) *and* (24) *hold.*

PROOF OF UNIQUENESS. The difference δ of two solutions lies in $\mathscr{H}_L$, satisfies the boundary condition on Γ, and vanishes on $\{t = 0\}$. Use the second part of Theorem 8 to construct approximations $\delta_k \in C^1(\bar{\mathcal{O}})$ converging to δ in $\mathscr{H}_L$. Estimate (23) implies that $\delta_k \to 0$ in $\mathscr{L}^2(\mathcal{O})$, hence $\delta = 0$.

PROOF OF EXISTENCE. A simple approximation argument shows that it suffices to prove the existence of solutions u when $g \in C_0^\infty(\Omega)$. For such g, subtracting a smooth function from u reduces to the case $g = 0$. If we can construct a solution to this problem with $u \in \mathscr{H}_L$, then the fact that $u \in C(I : \mathscr{L}^2(\Omega))$ and satisfies (23), (24) follows by a simple approximation argument using the second part of Theorem 8. Thus, it suffices to construct $u \in \mathscr{H}_L$ with $Lu = F$, $u|_{t=0} = 0$ and $u \in N$ on Γ.

Fix $F \in \mathscr{L}^2(\mathscr{O})$. The adjoint boundary space $N^* \equiv A_n(N)^\perp$ is Lipshitzean since Γ is characteristic of constant multiplicity. Let

$$\mathscr{B} \equiv \left\{ v \in \operatorname{Lip}(\bar{\mathscr{O}}) | v \in N^* \text{ on } \Gamma \text{ and } v|_{t=T} = 0 \right\}.$$

Let $\mathscr{R} \equiv L^*(\mathscr{B})$. As N^* is also maximal positive an inequality analogous to (23) shows that L^* is one-to-one on $\mathscr{B}$ and that $\|w\|_{\mathscr{L}^2(\Omega)} \leqslant c\|L^*w\|_{\mathscr{L}^2(\mathscr{O})}$ with c independent of $w \in \mathscr{B}$. Define $l: \mathscr{R} \to \mathbf{C}$ by

$$\mathscr{R} \ni L^*w \mapsto \int_0 \langle w, F \rangle \equiv l(L^*w).$$

Then for any $r = L^*w \in \mathscr{R}$ we have

$$|l(r)| \leqslant \|w\|_{\mathscr{L}^2(\mathscr{O})}\|F\|_{\mathscr{L}^2(\mathscr{O})} \leqslant c\|r\|_{\mathscr{L}^2(\mathscr{O})}\|F\|_{\mathscr{L}^2(\mathscr{O})}.$$

It follows that there is a $u \in \mathscr{L}^2(\mathscr{O})$ such that $l(r) = (r, u)_{\mathscr{L}^2(\mathscr{O})}$ for all $r \in \mathscr{R}$. Choosing $r = L^*\psi$ with $\psi \in C_0^\infty(\mathscr{O})$ we see that $Lu = F$ in $\mathscr{O}$ so $u \in \mathscr{H}_L$. Let $\tau \in \operatorname{Lip}(\partial\theta)'$ be the distribution which is formally equal to u on $t = T$, $-u$ on $t = 0$ and $\langle A_n u, u\rangle$ on Γ. The identity satisfied by u shows that $\tau(v|_{\partial\mathscr{O}}) = 0$ for all $v \in \mathscr{B}$. Thus, $\tau(f) = 0$ for all $f \in \operatorname{Lip}(\partial\mathscr{O})$ which vanish on $t = T$ and lie in N^* on Γ. Choosing f supported in $\{t = 0\}$ we see that $\operatorname{supp} u|_{t=0} \subset \partial\Omega$. Similarly choosing f supported in Γ we find that $\operatorname{supp}(u \bmod N) \subset \partial\Gamma$. Now $u|_{t=0}$ is an element of $H^{-1/2}(\Omega)$, and $u \bmod N$ is an element of $H^{-1/2}(\Gamma : \mathbf{C}^k/N)$. The only such distributions supported on the boundaries are the zero elements. This well-known fact is proved by localizing and then applying the following lemma.

LEMMA. *If* $S \in \mathscr{E}'(\mathbf{R}^\nu) \cap H^{-1/2}(\mathbf{R}^\nu)$ *with* $\operatorname{supp} \mathscr{S} \subset \{x_1 = 0\}$, *then* $\mathscr{S} = 0$.

PROOF. Let $m \geqslant 0$ be the order of S and introduce the notation $x_1 = (x_1, x')$, $\xi = (\xi_1, \xi')$. Then

$$\langle S, e^{-ix\cdot\xi}\rangle = \left\langle S, e^{-x'\cdot\xi'} \sum_{j=0}^{m} \frac{(ix_1\xi_1)^j}{j!} \right\rangle.$$

Thus $\hat{S} = \sum_{j=0}^m \xi_1^j f_j(\xi')$, where $f_j \in C^\infty(\mathbf{R}^{\nu-1})$ is given by

$$f_j(\xi') = \left\langle S, \frac{(ix_1)^j}{j!} e^{-ix'\cdot\xi'} \right\rangle.$$

One easily shows that if $f_m(\overline{\xi'}) \neq 0$, then for any $r > 0$

$$\int_0^\infty \int_{|\xi' - \overline{\xi'}| < r} |\hat{S}(\xi)|^2 \langle\xi\rangle^{2s}\, d\xi'\, d\xi_1 = \infty,$$

Unless $2s + 2m < -1$. By hypothesis the integral is finite for $s = -1/2$ so f_m must vanish identically. A simple recursion then yields $f_{m-1} = f_{m-2} = \cdots = f_0 = 0$, whence $S = 0$. □

We next study the regularity of solutions. For $s \in \mathbf{Z}_+$ we seek solutions with derivatives up to order s. Toward this end we suppose that A_j, N, and $\partial\Omega$ are of Lipshitz class $C^{s,1}$ and B is of class $C^{s-1,1}$. One localizes as in §2 so that N and $\ker A_1$ are independent of x', then applies the standard energy method to $(\partial_t, x_1\partial x_1, \partial x_2, \ldots, \partial x_n)^\alpha u$ with commutators controlled as in §2. Letting

$$\phi_s(t) \equiv \sum_{j=0}^{s} \left\| \partial_t^j u(t) \right\|_{H_{\tan}^{s-j}(\Omega)}$$

one finds

$$\sup_{0 \leqslant t \leqslant T} \phi_s(t) \leqslant c\left(\sum_{j=0}^{s} \left\| \partial_t^j F \right\|_{\mathscr{L}^1(I: H_{\tan}^{s-j}(\Omega))} + \phi_s(0) \right), \tag{25}$$

$$\phi_s(t_2) - \phi_s(t_1) \leqslant c\int_{t_1}^{t_2} \phi_s(t) + \sum_{j=0}^{s} \left\| \partial_t^j F(t) \right\|_{H_{\tan}^{s-j}(\Omega)} dt. \tag{26}$$

In addition to regularity of F, g we must impose compatibility conditions at the corner $\{t = 0\} \times \partial\Omega$. These conditions are computed in the usual fashion. For $(t, x) \in \bar{\Gamma}$, let $\pi(t, x)$ be the orthogonal projection of $\mathbf{C}^k$ onto $N(t, x)^\perp$. The compatibility condition of order j comes from expressing $\partial_t^j(\pi u)$ at $\{t = 0\} \times \partial\Omega$ in terms of g and F and requiring that the resulting expression vanishes. For example, for $j = 0, 1$ we find $t = 0$

$$\partial_t^0(\pi u) = \pi g, \qquad \partial_t^1(\pi u) = \pi(F(0, \cdot) - Gg) + \pi_t g,$$

where $L = \partial_t + G$. The compatibility conditions of order zero and one are

$$\pi g = 0 \quad \text{on } \partial\Omega, \qquad \pi(F(0, \cdot) - Gg) + \pi_t g = 0 \quad \text{on } \partial\Omega.$$

There is a subtle problem with the compatibility conditions. We illustrate this by considering the condition of order zero. If one seeks $u \in \bigcap_{j=0}^1 C^j(\bar{I} : H_{\tan}^{1-j}(\Omega))$ the a priori estimates suggest

$$g \in H_{\tan}^1(\Omega), \qquad F \in \bigcap_{j=0}^{1} \mathscr{L}^1(I : H_{\tan}^{1-j}(\Omega)), \qquad \pi g = 0 \quad \text{in } \partial\Omega$$

as the natural conditions on the data. However, for data as above, the trace $\pi g|_{\partial\Omega}$ is not defined. This is easily seen with the example on $\overline{\mathbf{R}}_+$ given by the function $\psi(x)\ln x$, $\psi \in C_{(0)}^\infty(\overline{\mathbf{R}}_+)$, $\psi(0) \neq 0$, which lies in $\bigcap_s H_{\tan}^s(\overline{\mathbf{R}}_+)$ and has no trace. On the other hand if $u \in \bigcap_{j=0}^1 C^j(I : H_{\tan}^{1-j}(\Omega))$ satisfies $Lu = F$, where F has the regularity suggested above, it follows that $\pi u(0, \cdot)|_{\partial\Omega}$ is well-defined element of $H^{1/2}(\partial\Omega)$. Postponing the demonstration for a moment, we see that it is necessary to require more than $g \in H_{\tan}^1(\Omega)$. To prove that $\pi u(0, \cdot)$ has a trace, one localizes with a partition of unity and changes and independent variable, reducing to the case $[0, T] \times \mathbf{R}_+^\nu$. Then one finds $A_1 u \in C(\bar{I} : H_{\tan}^1)$ and, from the differential equation, $\partial_1(A_1 u) \in C(\bar{I} : \mathscr{L}^2)$ since $F \in W^{1,1}(I : \mathscr{L}^2(\Omega))$. Thus

$$A_\nu u|_{\partial\Omega} \in C(\bar{I} : H^{1/2}(\partial\Omega)),$$

and it follows that $\pi u(0, \cdot)|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$.

We are not able to describe the precise set of data g, F leading to solutions in $\bigcap_{j=0}^{s} C^j(\bar{I} : H^{s-j}_{\tan}(\Omega))$. However, we give an important subclass by assuming H^s regularity in place of $H^s_{\tan}$ regularity near $t = 0$.

THEOREM 10 (TANGENTIAL REGULARITY). *Suppose $s \geqslant 1$ is an integer, A_j, N, $\partial\Omega$ are of class $C^{s,1}$ and B is of class $C^{s-1,1}$. Suppose the data $g \in H^s$ and $\partial_t^j F \in \mathscr{L}^1(I : H^{s-j}_{\tan}(\Omega))$ for $0 \leqslant j \leqslant s$ and in addition there is a $0 < T' \leqslant T$ such that $\partial_t^j F \in \mathscr{L}^1([0, T'] : H^{s-j}(\Omega))$, $0 \leqslant j \leqslant s$. If the data satisfy the compatibility conditions up to order $s - 1$, then the solution u to* (22) *lies in $\bigcap_{j=0}^{s} C^j(\bar{I} : H^{s-j}_{\tan}(\Omega))$ and satisfies the estimates* (25), (26).

REMARK. With g, F as in the theorem the compatibility conditions make sense. We check the cases $s = 0, 1$. For $s = 0$, the condition is $\pi_{N^\perp} g|_{\partial\Omega} = 0$ and we have $g \in H^1(\Omega)$ so $\pi_{N^\perp} g|_{\partial\Omega}$ is a well-defined element of $H^{1/2}(\partial\Omega)$. For $s = 1$, we need the condition of order 1 which makes sense provided $F(0, \cdot)|_{\partial\Omega}$ and $Gg|_{\partial\Omega}$ make sense. Here $g \in H^2(\Omega)$ so the second term is ok. For the first we observe that

$$F \in W^{1,1}(I : H^1(\Omega)) \subset C(I : H^1(\Omega))$$

so the trace of F on $\{t = 0\} \times \partial\Omega$ lies in $H^{1/2}(\partial\Omega)$. The higher order conditions are similar.

PROOF OF THEOREM 10. As in the proof of Theorem 6 we make a noncharacteristic regularization, replacing L by

$$L^\varepsilon = L + \varepsilon\phi\pi_N \Sigma n_j \frac{\partial}{\partial xj}.$$

For this operator $\partial\Omega$ is noncharacteristic and the boundary space is strictly positive. One then approximates L^ε by operators $\tilde{L}^\varepsilon$, with smooth coefficients, Ω by smooth domains Ω^ε, and N by smooth strictly positive boundary spaces N^ε so that estimates hold uniformly in ε. For the mixed problem we encounter a new difficulty. The data F, g will not, in general, satisfy the compatibility condition for the regularized problem. In addition, F is not smooth enough to apply directly the results in the literature. To solve the second dilemma choose $F^\varepsilon \in C^\infty_{(0)}(\bar{\mathscr{O}})$ such that

$$\partial_t^j F^\varepsilon \to \partial_t^j F \quad \text{in } L^1([0, T'] : H^{s-j}(\Omega)) \cap L^1([0, T] : H^{s-j}_{\tan}(\Omega))$$

for $0 \leqslant j \leqslant s$. The final approximation $g^\varepsilon \in C^\infty_{(0)}(\Omega)$ must be done with care so as to ensure that the compatibility conditions are satisfied.

LEMMA. *One can choose $g^\varepsilon \in C^\infty_{(0)}(\bar{\Omega})$ so that $g^\varepsilon \to g$ in $H^s(\Omega)$ and the compatibility conditions up to order $s - 1$ are satisfied by $\tilde{L}^\varepsilon$, N^ε, Ω^ε, F^ε, g^ε.*

PROOF. The construction is local. We localize then introduce coordinates in $\mathbf{R}^\nu$ and $\mathbf{C}^k$ so that $\Omega^\varepsilon = \{x_1 < \varepsilon\}$, $N^\varepsilon = \{u_{l+1} = \cdots = u_k = 0\}$. Then since $N^\varepsilon \supset \ker A_1^\varepsilon$, the last $k - l$ rows of A_1^ε form a matrix of rank $k - l$. The compatibility conditions for g are

$$g \equiv 0 \bmod N,$$
$$Gg + F \equiv 0 \bmod N, \qquad \text{on } \partial\Omega.$$
$$G^2 g + [GF + F_t + G_t u] \equiv 0 \bmod N$$

If one approximates g by g_1^ε in H^s one finds that

$$\left.\begin{aligned} g_1^\varepsilon &\equiv \phi_0^\varepsilon \bmod N^\varepsilon \\ Gg_1^\varepsilon + F^\varepsilon &\equiv \phi_1^\varepsilon \bmod N^\varepsilon \end{aligned}\right\} \text{ on } \partial\Omega$$

with $\phi_0^\varepsilon \in C_i^\infty(\partial\Omega^\varepsilon)$, $\phi_i^\varepsilon = o(1)$ in $H^{s-1/2-i}(\partial\Omega^\varepsilon)$ as $\varepsilon \to 0$. We will choose $g^\varepsilon = g_1^\varepsilon + g_2^\varepsilon$, g_2^ε chosen with its first l components identically zero. The first compatibility condition for g^ε requires $g_2^\varepsilon \equiv -\phi_0^\varepsilon \bmod N^\varepsilon$ which determines the trace of g_2^ε on $\partial\Omega^\varepsilon$. Given this trace the second compatibility condition determines the trace of $\partial g_2^\varepsilon/\partial x_1$ since the last rows of A_1 are of maximal rank. Continuing we see that to satisfy the compatibility conditions we must choose g_2^ε so that

$$\left(\frac{\partial}{\partial x_1}\right)^j g_2^\varepsilon \equiv \psi_j^\varepsilon \bmod N^\varepsilon \quad \text{on } \partial\Omega^\varepsilon$$

with $\psi_j^\varepsilon \in C_0^\infty(\partial\Omega^\varepsilon)$, $\psi_j^\varepsilon = o(1)$ in $H^{s-j-1/2}$, $0 \leqslant j \leqslant s-1$. This can be done with $g_2^\varepsilon \in C_{(0)}^\infty(\Omega^\varepsilon)$, $g_2^\varepsilon = o(1)$ in $H^s(\Omega)$ and the lemma is proved. □

We now complete the proof of the theorem. The results of Rauch-Massey [**21**] imply that the solution of $\tilde{L}^\varepsilon u^\varepsilon = F^\varepsilon$, $u^\varepsilon(0) = g^\varepsilon$, $u^\varepsilon \in N^\varepsilon$ on Γ^ε, lies in

$$\bigcap_{j=0}^{s} C^j\left(\bar{I} : H^{s-j}(\Omega^\varepsilon)\right).$$

As ε tends to zero the H^s norm of u^ε need not stay bounded, however, the tangential estimate (25) shows that $\partial_t^j u^\varepsilon$ is bounded in $C(\bar{I} : H_{\tan}^{s-j}(\Omega^\varepsilon))$. Passing to a weak star convergent subsequence yields a solution u to (22) with $\partial_t^j u \in \mathscr{L}^\infty(\bar{I} : H_{\tan}^{s-j}(\Omega))$. Estimate (26) applied to the convergent subsequence shows that $\partial_t^j u \in C(\bar{I} : H_{\tan}^{s-j}(\Omega))$ and itself satisfies (25) and (26). By uniqueness this u is *the* solution. □

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