Symmetric Product Codes

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Coding: From Practice to Theory
Simons Institute
UC Berkeley

Prologue

- Let $\mathcal{C}$ be an (n, k, d) linear code over $\mathbb{F}$
  - generator / parity-check matrix: $G \in \mathbb{F}^{k\times n}$ / $H \in \mathbb{F}^{(n-k)\times n}$
  - product code given by $n\times n$ arrays with rows/columns in $\mathcal{C}$:

$$\mathcal{P} = \left\{ G^\top U G \,|\, U \in \mathbb{F}^{k\times k} \right\}$$

  - well-known that $\mathcal{P}$ is an (n^2, k^2, d^2) linear code

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  - well-known that $\mathcal{P}$ is an (n^2, k^2, d^2) linear code
- Let $\mathcal{U}$ be the symmetric subcode of $\mathcal{P}$:

$$\mathcal{U} = \left\{X \in \mathcal{P} \,\middle|\, X^\top = X\right\}$$

  - if char$(\mathbb{F}) \neq 2$, then $\mathcal{U} = \left\{2^{-1}(X^\top + X) \,|\, X \in \mathcal{P}\right\}$
  - puncturing the lower triangle gives $\left(\binom{n+1}{2}, \binom{k+1}{2}, \binom{d+1}{2}\right)$ code

Prologue (2)

Product Code

$$
\begin{matrix}
X_{0,0} & X_{0,1} & X_{0,2} & X_{0,3} & X_{0,4} & X_{0,5} & X_{0,6} \\
X_{1,0} & X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} & X_{1,5} & X_{1,6} \\
X_{2,0} & X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} & X_{2,5} & X_{2,6} \\
X_{3,0} & X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} & X_{3,5} & X_{3,6} \\
X_{4,0} & X_{4,1} & X_{4,2} & X_{4,3} & X_{4,4} & X_{4,5} & X_{4,6} \\
X_{5,0} & X_{5,1} & X_{5,2} & X_{5,3} & X_{5,4} & X_{5,5} & X_{5,6} \\
X_{6,0} & X_{6,1} & X_{6,2} & X_{6,3} & X_{6,4} & X_{6,5} & X_{6,6}
\end{matrix}
$$

Prologue (2)

Symmetric Subcode

$$\begin{matrix}
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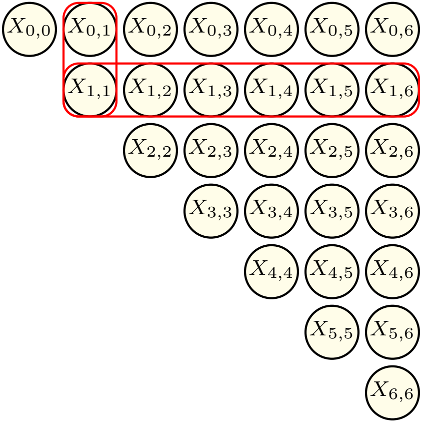
X_{0,2} & X_{1,2} & X_{2,2} & X_{2,3} & X_{2,4} & X_{2,5} & X_{2,6} \\
X_{0,3} & X_{1,3} & X_{2,3} & X_{3,3} & X_{3,4} & X_{3,5} & X_{3,6} \\
X_{0,4} & X_{1,4} & X_{2,4} & X_{3,4} & X_{4,4} & X_{4,5} & X_{4,6} \\
X_{0,5} & X_{1,5} & X_{2,5} & X_{3,5} & X_{4,5} & X_{5,5} & X_{5,6} \\
X_{0,6} & X_{1,6} & X_{2,6} & X_{3,6} & X_{4,6} & X_{5,6} & X_{6,6}
\end{matrix}$$

Prologue (2)

Punctured Symmetric Subcode



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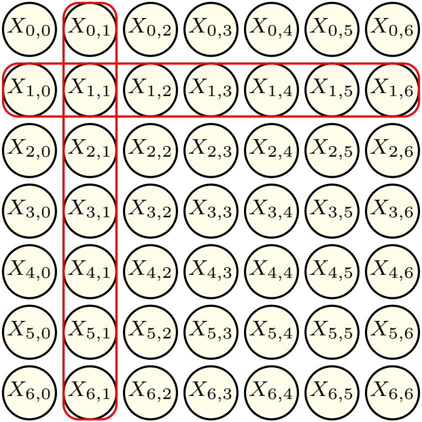
$$\mathcal{V} = \left\{ X \in \mathcal{P} \,|\, X^\top = -X,\ \mathsf{diag}(X) = 0 \right\}$$

  - if $\mathsf{char}(\mathbb{F}) \neq 2$, then $\mathcal{V} = \left\{ 2^{-1}(X^\top - X) \,|\, X \in \mathcal{P} \right\}$
  - Justesen suggested puncturing the lower triangle to get an

$$\left(\binom{n}{2}, \binom{k}{2}, D \right) \quad \text{Half-Product Code } \mathcal{H}$$

Product Code



Prologue (4)

Anti-Symmetric Subcode

$$
\begin{matrix}
0 & X_{0,1} & X_{0,2} & X_{0,3} & X_{0,4} & X_{0,5} & X_{0,6} \\
X_{0,1} & 0 & X_{1,2} & X_{1,3} & X_{1,4} & X_{1,5} & X_{1,6} \\
X_{0,2} & X_{1,2} & 0 & X_{2,3} & X_{2,4} & X_{2,5} & X_{2,6} \\
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\end{matrix}
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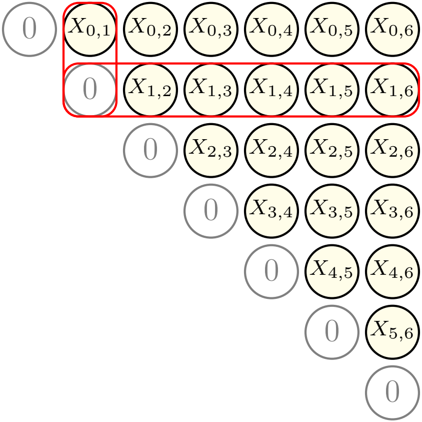
Prologue (4)

Punctured Anti-Symmetric Subcode



Outline

- Background
- Applications
- Half-Product Codes
- Symmetric Product Codes

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- Product Codes
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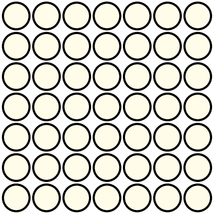
  - hard-decision "cascade decoding" by Abramson in 1968
  - "GLDPC" introduced by Tanner in 1981

- Example: 2-error-correcting codes, bounded distance decoding



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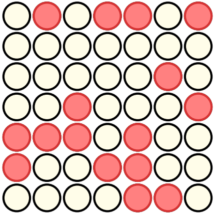

Received block

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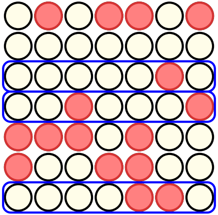

Row decoding

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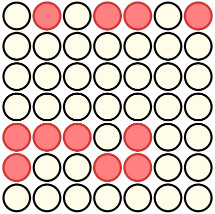

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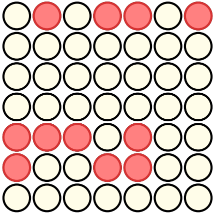

Column decoding

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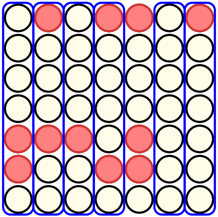

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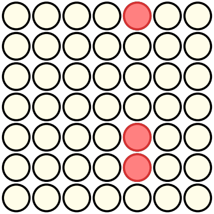

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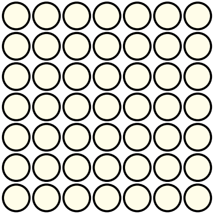

Decoding successful

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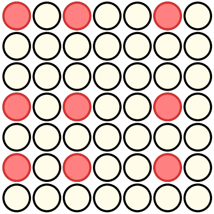

Or trapped in a stopping set

Applications

- Applications
  - recent interest for high speed optical communication
  - focus on 100 Gb/s with 7% redundancy (i.e., $1-\frac{239}{255} \approx 0.07$)
  - high-rate generalized product codes with BCH component codes and iterative algebraic hard-decision
  - many designs appeared in ITU 975.1 in 2004
  - Justesen recognized the potential in 2010
- Decoding
  - decoding complexity much lower than comparable LDPC codes
  - for hard-decision channels, BER performance is comparable

A Note on Decoding

- Syndrome-Based Iterative Algebraic Decoding
  - Initialization
    - compute and store the syndrome for each row and column
  - Iteration
    - run algebraic decoding on each row using syndromes
    - correct errors by updating the column syndromes
    - run algebraic decoding on each column using syndromes
    - correct errors by updating the row syndromes
- Memory to store syndromes is $2n(n-k) = 2n^2(1-R)$ vs. n^2
- $(1023, 993)$ BCH vs. $n = 1023^2$ LDPC: factor 50 less memory
- Well-known trick in industry for many years...

Symmetric Product Codes

- What are they?
  - subclass of generalized product codes that use symmetry to reduce the block length while using the same component code
  - one example, dubbed half-product codes (HPCs) in 2011 by Justesen, based on work by Tanner in 1981
  - the minimum distance is also larger than expected
- Match the length and rate between product and HPC
  - PC is (n_0^2, k_0^2) and HPC is $\approx (n_1^2/2, k_1^2/2)$
    - $n_1 \approx \sqrt{2} n_0$, $k_1 \approx \sqrt{2} k_0$, and $n_1 - k_1 \approx \sqrt{2}(n_0 - k_0)$
  - HPC component code has n and t larger by factor $\sqrt{2}$!

Minimum Distance (1)

- Support Sets and Generalized Hamming Weights
  - let $\mathrm{supp}(x) \triangleq \{i \in [n] \mid [x]_i \neq 0\}$ denote the support set of x
  - the 2nd generalized Hamming weight [HKY92] is

$$\begin{aligned} d_2 &= \min_{\substack{x_1, x_2 \in \mathcal{C} \setminus \{0\} \\ x_1 \neq x_2}} |\mathrm{supp}(x_1) \cup \mathrm{supp}(x_2)| \\ &\geq \lceil 3 d_{\min} / 2 \rceil \end{aligned}$$

  - measures minimal total support of two codewords
  - Bound: if d_2 smaller than $\lceil 3 d_{\min} / 2 \rceil$, then sum violates $d_{\min}$

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  - First, note $X \in \mathcal{P}$ because $HX = (Hx_1^T)x_2 = 0$
  - But, $\text{diag}(X) = 0$ for $X \in \mathcal{V}$ and, thus, $[x_1]_i \, [x_2]_i = 0$ for all i
    - implies $\text{supp}(x_1) \cap \text{supp}(x_2) = \emptyset$
    - and $X_{i,j} = [x_1]_i \, [x_2]_j \neq 0$ implies $X_{j,i} = [x_1]_j \, [x_2]_i = 0$
    - Thus, $X^\top \neq -X$ and $X \notin \mathcal{V}$

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    - Thus, $X^\top \neq -X$ and $X \notin \mathcal{V}$
- Thus, no $X \in \mathcal{V}$ where n.z. rows are scalar multiples of a c.w.

Minimum Distance (3)

- No $X \in \mathcal{V}$ where n.z. rows are scalar multiples of a c.w.
  - n.z. codeword in $\mathcal{V}$ must have ≥ 2 distinct non-zero rows
  - Minimum number of n.z. columns is lower bounded by d_2
  - Likewise, each column must have at least d non-zero elements
  - So, minimum distance of $\mathcal{V}$ must be $\geq d_2 d \geq \lceil 3d/2 \rceil d$
  - Puncturing lower triangle gives $\mathcal{H}$
    - implies $D \geq \lceil 3d/2 \rceil d/2$
    - Or $D \geq 3d^2/4$ if d even

Minimum Distance (4)

- $\mathcal{H}$ is an (N, K, D) code with $N = \binom{n}{2}$, $K = \binom{k}{2}$, and

$$D \geq \begin{cases} \frac{3d^2}{4} & \text{if } d \text{ even} \\ \frac{(3d+1)d}{4} & \text{if } d \bmod 4 = 1 \\ \frac{(3d+1)d+2}{4} & \text{if } d \bmod 4 = 3 \end{cases}$$

  - Also have matching upper bound if d is even and there are minimum distance codewords achieving the minimum for d_2
  - **Basic Idea**: Zeros on diagonal prevent standard square pattern codewords. Thus, support in one dimension must contain at least 2 distinct codewords. Thus, there are d_2 non-zero rows (or columns) each with weight at least d and $D \geq d_2 d$.

Minimum Distance (5)

- Example: If $\mathcal{C}$ is an (8,4,4) extended Hamming code
  - then $d = 4$, $d_2 = \lceil 3d/2 \rceil = 6$, and $D \geq 12$
  - there exists $x_1, x_2 \in \mathcal{C}$ such that $|\mathrm{supp}(x_1) \cup \mathrm{supp}(x_2)| = 6$ and $w(x_1) = w(x_2) = 4$
- Half-product code is a $(28, 6, 12)$ binary linear code
  - no $(28, 6)$ binary linear code with larger $d_{\min}$ exists

Iterative Decoding Analysis (1)

- Peeling Decoder for Generalized Product Codes
  - received symbols corrected sequentially without mistakes
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- Peeling Decoder for Generalized Product Codes
  - received symbols corrected sequentially without mistakes
  - for the BEC and, if a genie prevents miscorrection, the BSC
- Based on "error graph":
  - vertices are code constraints
  - edges connect code constraints containing same symbol
  - initial observations remove fraction $1 - p$ edges
  - decoder peels any code constraint with t or fewer errors/edges
  - always reaches stopping set after finite number of iterations

Iterative Decoding Analysis (2)

- Asymptotic Results for Half-Product Codes
  - t-error-correcting components w/bounded distance decoding
  - complete graph, edges removed i.i.d. prob. $1-p$
- Assume $n \to \infty$ with fixed t and $p_n = \frac{\lambda}{n}$
  - decoding threshold λ^* via k-core problem in graph theory
  - observed in 2007 by Justesen and Høholdt
  - thresholds for $t = 2,\ 3,\ 4$ are $\lambda^* = 3.35\,, 5.14\,, 6.81$
    - information about finite length via $\lambda^* = \lim_{n\to\infty} np_n^*$

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- First Example
  - product code from $(170, 154, 5)$ shortened binary BCH code
    - $(N', K', D') = (28900, 23716, 25)$, rate ≈ 0.82, $s_{\min} = 9$
  - half-product code from $(255, 231, 7)$ binary BCH code
    - $(N, K, D) = (32385, 26565, 40)$, rate ≈ 0.82, $s_{\min} = 10$

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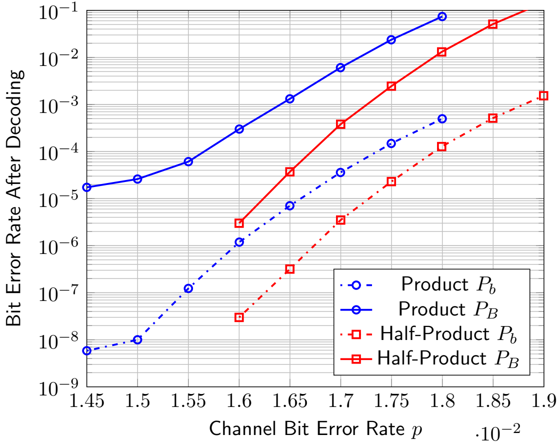
- Iterative decoding assuming genie to prevent miscorrection
  - connection to k-core problem allows "threshold" estimates
  - For the product code, $p^* \approx 3.35/170 = 0.0197$
  - For the half-product code, $p^* \approx 5.14/255 = 0.0201$

Simulation Results (2)



- DE predicts better HPC threshold because $5.14/3.35 > 3/2$
- Stopping set analysis predicts better HPC error floor

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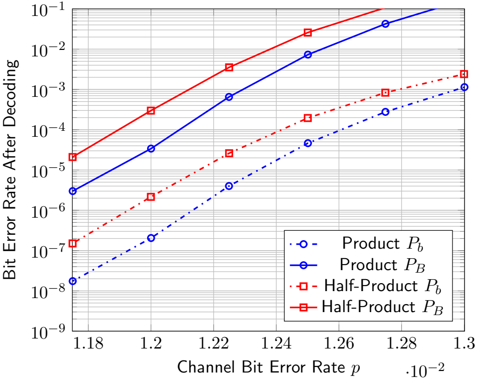

- product code from $(383, 356, 7)$ shortened binary BCH code
  - $(146689, 126736, 49)$ code, rate ≈ 0.86, $s_{\text{min}} = 16$
- half-product code from $(511, 475, 9)$ binary BCH code
  - $(130305, 112575, 65)$ code, rate ≈ 0.86, $s_{\text{min}} = 15$
- DE predicts worse HPC threshold because $6.81/5.14 < 4/3$

Conclusions

- Half-product codes
  - Length and dimension reduced by half with same component
  - Normalized minimum distance improved by $3/2$
  - For same blocklength and rate, one can increase t by $\sqrt{2}$
  - Changing $t = 2$ to $t = 3$ generally improves performance
  - More comprehensive simulations are needed
- Symmetric product codes (see ITA 2015 paper)
  - Natural extension to m-dimensional product codes
  - Length and dimension reduced roughly by m factorial
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  - Minimum distance improves
  - By how much is an open problem...